Completeness of idempotent semifields and a generalization of the linear extension theorem in max-plus algebra

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Completeness of Idempotent Semifields and a Generalization of the Linear Extension Theorem in Max-plus Algebra

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Abstract

Idempotent semifield is a fundamental algebraic structure in max-plus algebra. It differs from field in that the addition is idempotent and there is no inverse of addition. We discuss the completeness and complete extension of idempotent semifields, which is achieved through a max-plus version of Dedekind cuts. Subsequently, we use our results to obtain a generalization of the linear extension theorem, an important result in max-plus algebra and systems theory. This theorem had been proved valid for finitely generated semimodules, and we generalize it to countably generated semimodules.

1 Introduction

Max-plus algebra (e.g. [1]) plays a central role in the study of timing aspects of some special classes of discrete event systems. Although max-plus algebra is very similar to conventional linear algebra, it differs from the latter in that the addition is idempotent. This difference has prevented straightforward translation of results between the two areas.

The field is the basic structure in conventional linear algebra: vectors, matrices, and linear spaces are all defined with respect to a field. The counterpart of field in max-plus algebra is the *idempotent semifield*, and many important problems of max-plus algebra are closely related to the properties of idempotent semifield. In this paper we first study the completeness of totally ordered idempotent semifields (TOIS). A TOIS may be complete or not, while an incomplete TOIS can be extended to a complete one by introducing a max-plus version of Dedekind cuts, if it satisfies some conditions.

We then proceed to the generalization of linear extension theorem, on which many useful results in max-plus linear systems theory are based. The linear extension theorem was proved in [4, 3], and we generalize it to countably generated semimodules. The generalization is helpful because there are some important (possibly infinite) countably generated semimodules in the max-plus systems theory, such as the reachability and observability semimodules of a max-plus linear system.

2 Basic definitions

A semiring is an algebraic structure equiped with addition $\oplus$ and multiplication $\otimes$, and $\forall a, b$ and c,

$$\begin{aligned}(a \oplus b) \oplus c &= a \oplus (b \oplus c) && (1)\\ a \oplus b &= b \oplus a && (2)\\ a \oplus \varepsilon &= a && (3)\\ (ab)c &= a(bc) && (4)\\ ae = ea &= a && (5)\\ a(b \oplus c) &= ab \oplus ac && (6)\\ (b \oplus c)a &= ba \oplus ca && (7)\\ \varepsilon &= \varepsilon a = a\varepsilon && (8)\end{aligned}$$

where ε is the zero element and e is the identity element.

If a semiring is equipped with commutative and invertible multiplication, it is called a semifield:

$$\begin{aligned}ab &= ba && (9)\\ \forall a \neq \varepsilon, \exists a^{-1}, aa^{-1} &= e && (10)\end{aligned}$$

If a semiring is equipped with idempotent addition, it is called a dioid:

$$a \oplus a = a \qquad (11)$$

An algebraic structure is called an idempotent semifield, if it is simultaneously a semifield and a dioid.

A dioid induces a natural order relation: $a \leq b \Leftrightarrow a \oplus b = b$, and we say a dioid $\mathcal{D}$ is distributive if the induced lattice $(\mathcal{D}, \leq)$ is distributive. Distributivity is required for the linear extension theorem to hold. In this paper, we impose stricter restriction on idempotent semifield: we require the idempotent semifield to be totally ordered:

$$\forall a, b,\ a \leq b \text{ or } b \leq a \tag{12}$$

For convenience, we will use the abbreviation TOIS to stand for *totally ordered idempotent semifield*, and the discussions in the rest of this paper will be based on TOIS. This does not imply much loss of generality, because the most important max-plus algebraic structure $\mathbb{R}_{max}$ is a TOIS.

It is easy to show that the only finite TOIS is the boolean algebra $\{\varepsilon, e\}$, therefore all nontrivial TOIS' are infinite.

Definition 1 *An infinite TOIS S is said to be discrete, if there is a minimal element in the set $\{x \in S | x > e\}$; S is said to be dense otherwise.*

Definition 2 *A TOIS S is said to be archimedean, if for arbitrary $a, b \in S$ where $a > e$, there exists $n \in \mathbb{N}$ such that $a^n > b$.*

It is not difficult to show that our definition of denseness is equivalent to the common understanding: for arbitrary $a < b$, there is a c such that $a < c < b$. Also note that our definition of "archimedean" is translated from the archimedean axiom of real number theory, and it should not be confused with other definitions of archimedeanness in the literature.

Proposition 1 *Any infinite discrete archimedean TOIS is isomorphic to $(\mathbb{Z} \cup \{-\infty\}, max, +)$.*

Proof: Left to the reader. ∎

3 Completeness and complete extension of TOIS

Definition 3 *A TOIS S is said to be complete, if any upper-bounded nonempty subset of S has a supremum in S.*

This definition is directly borrowed from the real number theory, and is weaker than the definition of dioid completeness given in [2, 3], which does not require the subset to be upper-bounded. For a dioid to be complete in that sense, a maximal element $\top$ must be introduced, which is called "top completion" of dioid. With the introduction of $\top$, some elegant properties of semifield are no longer satisfied. In our sense of completeness, the top completion is avoided. Later on we will develop on our sense of completeness.

Definition 3 of completeness also ensures infimum-completeness, that is, any nonempty subset of a complete TOIS has an infimum. The proof is left to the reader. We also have the following lemma, whose proof is also omitted.

Lemma 1 *Let S be a TOIS, $a \in S$, $B \subseteq S$, denote $\{ab | b \in B\}$ by aB, we have:*

1. *If $sup(B)$ exists, then $sup(aB) = asup(B)$.*
2. *If $inf(B)$ exists, then $inf(aB) = ainf(B)$.*

Definition 4 *Let S be a TOIS, a complete TOIS $\bar{S}$ is a called a complete extension of S, if S is a sub-TOIS of $\bar{S}$.*

Theorem 1 *Any non-archimedean TOIS does not admit a complete extension.*

Proof: Let S be a non-archimedean TOIS. By Definition 2, there exist $a, b \in S$, $a > e$, and for any $n \in \mathbb{N}$, $a^n \leq b$. Let $A = \{a^n | n \in \mathbb{N}\}$, then b is an upper-bound of A. Suppose S has a complete extension $\bar{S}$, then A has a supremum T in $\bar{S}$. By Lemma 1, $sup(aA) = aT$, and since $aA \subset A$, it must be true that $aT \leq T$, hence $a \leq e$, which yields a contradiction. ∎

In the sequel we will try to construct a complete extension of a dense archimedean TOIS. This is done through a process similar to the Dedekind complete extension of rational numbers.

Definition 5 *Let S be a dense archimedean TOIS, a set $X \subset S$ is called a Dedekind cut of S if the following conditions are satisfied:*

1. *$X \neq S$;*
2. *$\forall x \in X$, if $y \in S$ and $y \leq x$, then $y \in X$;*
3. *$\forall x \in X$, $\exists z \in X$, such that $x < z$.*

Lemma 2 *Let S be a dense archimedean TOIS, A, B are Dedekind cuts of S, then either $A \subseteq B$ or $A \supseteq B$.*

Proof: Suppose not $A \subseteq B$, there must exist a such that $a \in A$ and $a \notin B$. We must have $\forall b \in B$, $b < a$: if otherwise $a \leq b$, since $b \in B$, then $a \in B$, which is a contradiction. Now since $a \in A$ and $b < a$, it must be true that $b \in A$, and therefore $B \subseteq A$. ∎

Lemma 3 *Let S be a dense archimedean TOIS, A, B are Dedekind cuts of S. If we define $A \oplus B = A \cup B$ and $AB = \{ab | a \in A, b \in B\}$, then $A \oplus B$ and AB are also Dedekind cuts of S.*

Proof: (1) *Closure of addition.* By Lemma 2, $A \oplus B$ is identical to either A or B, therefore $A \oplus B$ must be a Dedekind cut of S. (2) *Closure of multiplication.* (2.1) Suppose there are $\alpha \notin A$ and $\beta \notin B$, then $\forall a \in A$, $a < \alpha$ and $\forall b \in B$, $b < \beta$, consequently $\forall c \in AB$, $c < \alpha\beta$, and hence $\alpha\beta \notin AB$, $AB \neq S$. (2.2) If $\varepsilon = c \in AB$ then $\forall d \leq c$, $d = c$ and $d \in AB$. Now assume $\varepsilon \neq c = ab \in AB$ where $a \in A$ and $b \in B$, and $\forall d \in S$ such that $d \leq c$, we have $a^{-1}d \leq a^{-1}c = b$, so $a^{-1}d \in B$, therefore $d = a \cdot a^{-1}d \in AB$. (2.3) $\forall c \in AB$, $c = ab$, where $a \in A$ and $b \in B$, there must be $\alpha \in A$ and $\beta \in B$ such that $a < \alpha$ and $b < \beta$, therefore $c < \alpha\beta \in AB$. By Definition 5, AB is a Dedekind cut of S. ∎

Lemma 4 *Let S be a dense archimedean TOIS, $X \neq \emptyset$ is a Dedekind cut of S, then $\forall r < e$, $\exists t \in X$, such that $\forall x \in X$, $t > rx$.*

Proof: Because $r < e$, $r^{-1} > e$. Let $M = \{n \in \mathbb{Z} | \forall x \in X, r^{-n} \geq x\}$. By Definition 5 we have that X is upper-bounded, and by the archimedeanness of S, $M \neq \emptyset$. Conversely, since $X \neq \emptyset$, by Definition 5 there must exist an $\varepsilon \neq a \in X$. Again by the archimedeanness of S, there exists a $k \in \mathbb{Z}$ such that $r^{-k} > a^{-1}$ i.e. $r^k < a$, hence $\forall n \leq -k$, $n \notin M$ and M is lower-bounded. Therefore, $m = min(M)$ is well-defined. Now $\exists t \in X$, $t > r^{-(m-1)}$ and $\forall x \in X$, $x \leq r^{-m}$. Therefore, $\exists t \in X, \forall x \in X$, $t > r^{-(m-1)} = rr^{-m} \geq rx$. ∎

Lemma 5 *Let S be a dense archimedean TOIS and $a, b \in S$, then $\forall z \in S$, $z < ab \Leftrightarrow \exists x < a \exists y < b$ such that $z = xy$.*

Proof: $\Rightarrow$: The proof is trivial for the case $ab = \varepsilon$, so we can assume $ab \neq \varepsilon$. Let $r = (ab)^{-1}z$, then $r < e$, and since S is dense, there exists s such that $r < s < e$. Let $t = rs^{-1}$, then $t < e$. Let $x = as$, $y = bt$, then $xy = abst = abr = z$. $\Leftarrow$ is trivial. ∎

Lemma 6 *Let S be a dense archimedean TOIS, and $A, B \subset S$, $a, b \in S$, if $a = sup(A)$ and $b = sup(B)$, then $ab = sup(AB)$.*

Proof: The case $ab = \varepsilon$ is trivial, so we can assume $ab \neq \varepsilon$. It is trivial to show that ab is an upper-bound of AB. For any $c < ab$, let $r = (ab)^{-1}c$, then $r < e$. Because S is dense, there exists s such that $r < s < e$, let $t = rs^{-1}$, then $t < e$. Since $a = sup(A)$, there exists $x \in A$ such that $x > as$; and since $b = sup(B)$, there exists $y \in B$ such that $y > bt$. So $xy > abst = abr = c$, which implies that ab is the minimal upper-bound of AB, i.e. $ab = sup(AB)$. ∎

In the sequel we denote the set of all Dedekind cuts of S by $\bar{S}$, and denote two special Dedekind cuts $\emptyset$ and $\{x | x < e\}$ by $\bar{\varepsilon}$ and $\bar{e}$, respectively.

Theorem 2 *Let S be a dense archimedean TOIS, then $(\bar{S}, \oplus, \cdot, \bar{\varepsilon}, \bar{e})$ is a complete extension of S, and S is dense in $\bar{S}$.*

Proof: By Lemma 3, $\bar{S}$ is closed under operations $\oplus$ and $\cdot$, and it is easy to check that $(\bar{S}, \oplus, \cdot, \bar{\varepsilon}, \bar{e})$ satisfies the axioms required of TOIS except axioms 10 and 12. Lemma 2 has ensured that $\bar{S}$ satisfies axiom 12 (total order). Now it remains to show that axiom 10 is satisfied.

$\forall X \in \bar{S}$ and $X \neq \bar{\varepsilon} = \emptyset$, let $X^{-1} = \{y \in S | \exists r < e, \forall x \in X, xy < r\}$. It is easy to verify that $X^{-1} \in \bar{S}$, and $XX^{-1} \subseteq \bar{e}$. Now for any $r < e$, because S is dense, there is $s \in S$, such that $r < s < e$. By Lemma 4, there exists $t \in X$ such that $\forall x \in X, t > sx$. Thus $\forall x \in X$, $x \cdot rt^{-1} < rs^{-1} < e$, which implies $rt^{-1} \in X^{-1}$. Hence $XX^{-1} \supseteq \bar{e}$. Therefore, $XX^{-1} = \bar{e}$.

Now we show $\bar{S}$ is complete. Suppose $\Delta \subset \bar{S}$ and Δ is nonempty and upper-bounded. This means that there exists $B \in \bar{S}$, such that $\forall X \in \Delta$, $X \subseteq B$. Let $T = \bigcup_{X \in \Delta} X$. It is easy to check that T is a Dedekind cut of S, i.e. $T \in \bar{S}$, and T is the supremum of Δ. Therefore, $\bar{S}$ is complete.

Because S is dense, we can define an injective map $f : S \mapsto \bar{S}$, $f(a) = \{x | x < a\}$. We can also prove that $f(x \oplus y) = f(x) \oplus f(y)$ and $f(xy) = f(x)f(y)$ by use of Lemma 5. Hence we can regard S as a sub-TOIS of $\bar{S}$.

It remains to show that S is dense in $\bar{S}$. For any $X, Y \in \bar{S}$, $X < Y$, there must be $a \in Y$ and $a \notin X$. By the definition of Dedekind cut, there exists $b > a$, $b \in Y$, and apparently $b \notin X$. Let $Z = \{z | z < b\}$. Since $b \in Y$, there must exist $c > b$, $c \in Y$, but $c \notin Z$ by definition, thus $Z < Y$. Conversely, because S is dense, $\exists d$, $a < d < b$, hence $d \in Z$, but $d \notin X$, and thus $X < Z$. Therefore we have $X < Z < Y$ and $Z \in S$ by definition, S is dense in $\bar{S}$. ∎

From the above results, we can infer that a TOIS admits a complete extension iff it is archimedean.

Theorem 3 *Let S be a dense archimedean TOIS, then any complete extension of S is isomorphic to the*

Dedekind complete extension $\bar{S}$, and each element in S is invariant under the isomorphism.

Proof: Let U denote any complete extension of S. Define map: $f : \bar{S} \mapsto U$, $f(\emptyset) = \varepsilon_U$ and $f(X) = sup_U(X)$ for $X \neq \emptyset$, where $sup_U(X)$ denotes the supremum of X in U. For arbitrary $X, Y \in \bar{S}$, if $X < Y$, then $\exists y \in Y, y \notin X$, and also $\exists z \in Y$, $z > y$, therefore $sup_U(Y) \geq z > y \geq sup_U(X)$, i.e. $f(X) < f(Y)$. This means that f is injective.

It is trivial to show $f(X \oplus Y) = f(X) \oplus f(Y)$. By Lemma 6, we have $f(XY) = f(X)f(Y)$. Hence $\bar{S}$ is isomorphic to $Imf \subset U$, and $\bar{S}$ can be considered as a sub-TOIS of U. Therefore we have $S \subseteq \bar{S} \subseteq U$.

$\forall \lambda \in U$, let $A = \{x \in \bar{S} | x \leq \lambda\}$, and $B = \{x \in \bar{S} | x \geq \lambda\}$. $A, B \neq \emptyset$ because U is archimedean. Let $a = sup_{\bar{S}}(A)$, $b = inf_{\bar{S}}(B)$. We have $a, b \in \bar{S}$ and $a \leq \lambda \leq b$. Let us suppose $a < b$. Since $\bar{S}$ is dense, there exists $\mu \in \bar{S}$ such that $a < \mu < b$. We cannot have $\mu \leq \lambda$, since it contradicts $a = sup_{\bar{S}}(A)$; we also cannot have $\mu \geq \lambda$, since it contradicts $b = sup_{\bar{S}}(B)$. Therefore, it must be true that $a = b = \lambda$, thus $\lambda \in \bar{S}$, and we have $\bar{S} = U$.

An element $a \in S$ is equivalent to $\bar{a} = \{x | x < a\} \in \bar{S}$. It is obvious that $f(\bar{a}) = sup_U(\bar{a}) = a$. Therefore, every element in S is invariant under the isomorphism f. ■

We can also obtain the following result, the proof is left to the reader.

Theorem 4 *Any dense complete TOIS is isomorphic to $\mathbb{R}_{max}$, and there are only three possible structures of complete TOIS: Boolean algebra, $\mathbb{Z}_{max}$ and $\mathbb{R}_{max}$.*

4 A generalization of the linear extension theorem

4.1 Some preparatory results

For notational simplicity we denote $sup\{a(k)|k \in \mathbb{N}\}$ and $inf\{a(k)|k \in \mathbb{N}\}$ by $\oplus_{k=1}^{\infty} a(k)$, and $\wedge_{k=1}^{\infty} a(k)$, respectively.

Lemma 7 *Let S be a complete TOIS. $A \in S^{m \times n}$, $\forall k \in \mathbb{N}$, $x(k) \in S^n$, and x is a uniformly upper-bounded ascending sequence, i.e. there exists $\beta \in S^n$, such that $\forall k \in \mathbb{N}$, $x(k) \leq \beta$, $x(k) \leq x(k+1)$, then $\oplus_{k=1}^{\infty} Ax(k) = A \oplus_{k=1}^{\infty} x(k)$.*

Proof: (1) For the case S is finite or discrete, it is easy to show that $\exists K \in \mathbb{N}$, $x(k) = x(\infty)$. Therefore, $\oplus_{k=1}^{\infty} Ax(k) = \oplus_{k=1}^{K} Ax(k) = A \oplus_{k=1}^{K} x(k) = A \oplus_{k=1}^{\infty} x(k)$. (2) For the case S is dense, it suffices to prove for the case $m = 1$. Let $x(\infty) = \oplus_{k=1}^{\infty} x(k)$. Apparently $Ax(k) \leq Ax(\infty)$ for $k \in \mathbb{N}$, and $\forall r < e \in S$, $\exists K \in \mathbb{N}$, such that $x(k) \geq rx(\infty)$, hence $Ax(k) \geq rAx(\infty)$ for $k > K$. It then follows that $\oplus_{k=1}^{\infty} Ax(k) = Ax(\infty) = A \oplus_{k=1}^{\infty} x(k)$. ■

Lemma 8 *Let S be a complete TOIS. $A \in S^{m \times n}$, $\forall k \in \mathbb{N}$, $x(k) \in S^n$, and x is a descending sequence, i.e. $\forall k \in \mathbb{N}$, $x(k) \geq x(k+1)$, then $\wedge_{k=1}^{\infty} Ax(k) = A \wedge_{k=1}^{\infty} x(k)$.*

Proof: (1) For the case S is finite or discrete, the proof is nearly the same as Lemma 7. (2) For the case S is dense, it suffices to prove for the case $m = 1$. Let $A = [a_1, \cdots, a_n]$, $x(\infty) = \wedge_{k=1}^{\infty} x(k)$, and $E = \{i \in \{1, \cdots, n\} | a_i x_i(\infty) = \varepsilon\}$. If $E = \{1, \cdots, n\}$, it is obvious that $\wedge_{k=1}^{\infty} Ax(k) = \varepsilon = A \wedge_{k=1}^{\infty} x(k)$. If $E \neq \{1, \cdots, n\}$, then $\oplus_{i \notin E} a_i x_i(k) \geq a_j x_j(\infty)$, where $j \notin E$. There must be $K \in \mathbb{N}$, such that for $k > K$, $\oplus_{i \in E} a_i x_i(k) < a_j x_j(\infty)$, and hence $Ax(k) = \oplus_{i \notin E} a_i x_i(k)$. Therefore, we can disregard the components in E, and we can assume without loss of generality that $x_i(\infty) \neq \varepsilon$, $i = 1, \cdots, n$. Apparently $Ax(k) \geq Ax(\infty)$ for $k \in \mathbb{N}$, and $\forall r > e \in S$, $\exists K \in \mathbb{N}$, such that $x(k) \leq rx(\infty)$, hence $Ax(k) \leq rAx(\infty)$ for $k > K$. It follows that $\wedge_{k=1}^{\infty} Ax(k) = Ax(\infty) = A \wedge_{k=1}^{\infty} x(k)$. ■

Definition 6 *Let S be a complete TOIS, a set $X \subseteq S^n$ is said to be closed, if*

1. *$\forall x, y \in X$, $x \oplus y \in X$;*
2. *For any upper-bounded ascending sequence $\{x_k\}_{k=1}^{\infty}$ in X, we have $\oplus_{k=1}^{\infty} x_k \in X$;*
3. *For any descending sequence $\{x_k\}_{k=1}^{\infty}$ in X, we have $\wedge_{k=1}^{\infty} x_k \in X$.*

Theorem 5 *Let S be a complete TOIS. $\forall k \in \mathbb{N}$, $\emptyset \neq X_k \subseteq S^n$, $X_k \supseteq X_{k+1}$, X_k is closed, and X_1 is upper-bounded, then $\bigcap_{k=1}^{\infty} X_k \neq \emptyset$.*

Proof: Since for any $k \in \mathbb{N}$, $X_k \neq \emptyset$, we can select $x_k \in X_k$. For any $l \leq p \in \mathbb{N}$, let $y_{l,p} = \oplus_{k=l}^{p} x_k$, and $z_l = \oplus_{p=l}^{\infty} y_{l,p}$. Because X_l is closed and $X_p \subseteq X_l$, we have $y_{l,p} \in X_l$ for any $p \geq l$. Also because sequence $y_{l,p}$ with index p is an upper-bounded ascending sequence, we have $z_l \in X_l$. It is also apparent that $z_l = \oplus_{k=l}^{\infty} x_k$, which means that sequence z_l is descending. Let $z = \wedge_{l=1}^{\infty} z_l$, then $z = \wedge_{l=k}^{\infty} z_l$ for any $k \in \mathbb{N}$. Since $z_l \in X_k$ for $l \geq k$ and X_k is closed, $\wedge_{l=k}^{\infty} z_l \in X_k$, i.e. $z \in X_k$. Therefore, $\bigcap_{k=1}^{\infty} X_k \neq \emptyset$. ■

4.2 Linear extension theorem and its generalization to countably generated semimodules

Linear Extension Theorem([4, 3]): Let S be a distributive idempotent semifield. Let $\mathcal{F}$,$\mathcal{G}$ denote two free finitely generated S-semimodule, and let $\mathcal{H} \subset \mathcal{G}$ be a finitely generated subsemimodule. For all $F \in Hom(\mathcal{H}, \mathcal{F})$, there exists $G \in Hom(\mathcal{G}, \mathcal{F})$ such that $\forall x \in \mathcal{H}$, $G(x) = F(x)$.

We give here an equivalent form of the linear extension theorem in matrix language.

Linear Extension Theorem: Let S be a distributive idempotent semifield. $A \in S^{m\times n}$, $b \in S^{m\times 1}$, and $\forall u, v \in S^{1\times m}$, $uA = vA \Rightarrow ub = vb$, then there exists $x \in S^{n\times 1}$ such that $Ax = b$.

Theorem 6 *Let S be a complete TOIS. $A \in S^{\infty\times n}$, $b \in S^{\infty\times 1}$, and $\forall u, v \in S^{1\times\infty}$, where u, v have finite support (finite number of coordinates $\neq \varepsilon$), $uA = vA \Rightarrow ub = vb$, then there exists $x \in S^{n\times 1}$ such that $Ax = b$.*

Proof: We say a column in a matrix empty if all its entries are ε. It suffices to prove for the case that there is no empty columns in A. If otherwise, we can eliminate the empty columns in A and corresponding coordinates in b and x.

Let $A^{(k)}$ and $b^{(k)}$ denote the sub-matrix or sub-vector formed by the first k rows of A and b, respectively. Since we assume there are no empty columns in A, there exists k_0 such that each column in $A^{(k_0)}$ has at least one non-ε entry. Let $X_k = \{x \in S^{n\times 1} | A^{(k)}x = b^{(k)}\}$, then X_{k_0} is upper-bounded, $X_k \supseteq X_{k+1}$, and by Definition 6, Lemma 7 and 8, X_k is closed. By the linear extension theorem, $X_k \neq \emptyset$. Thus by Theorem 5, there exists $x \in \bigcap_{k=k_0}^{\infty} X_k = \bigcap_{k=1}^{\infty} X_k$, which means $\forall k \in \mathbb{N}$, $A^{(k)}x = b^{(k)}$, i.e. $Ax = b$. ∎

This generalization can be further relaxed to archimedean TOIS' which are not necessarily complete.

Theorem 7 *Let S be an archimedean TOIS. $A \in S^{\infty\times n}$, $b \in S^{\infty\times 1}$, and $\forall u, v \in S^{1\times\infty}$, where u, v have finite support, $uA = vA \Rightarrow ub = vb$, then there exists $x \in S^{n\times 1}$ such that $Ax = b$.*

Proof: Since S is archimedean, there must exist a complete extension $\bar{S}$ of S. Obviously $A \in \bar{S}^{\infty\times n}$, $b \in \bar{S}^{\infty\times 1}$. $\forall m \in \mathbb{N}$, let $A^{(m)}$ and $b^{(m)}$ denote the sub-matrix or sub-vector formed by the first m rows of A and b, respectively. According to the given condition, we have $\forall u, v \in S^{1\times m}$, $uA^{(m)} = vA^{(m)} \Rightarrow ub^{(m)} = vb^{(m)}$, thus due to the linear extension theorem, there exists $x^{(m)} \in S^{n\times 1}$ such that $A^{(m)}x^{(m)} = b^{(m)}$. Thus $\forall \bar{u}, \bar{v} \in \bar{S}^{1\times m}$ such that $\bar{u}A^{(m)} = \bar{v}A^{(m)}$, it must hold true that $\bar{u}b^{(m)} = \bar{u}A^{(m)}x^{(m)} = \bar{v}A^{(m)}x^{(m)} = \bar{v}b^{(m)}$. Therefore, $\forall \bar{u}, \bar{v} \in \bar{S}^{1\times\infty}$, where $\bar{u}, \bar{v}$ have finite support, $\bar{u}A = \bar{v}A \Rightarrow \bar{u}b = \bar{v}b$. By Theorem 6, $\exists \bar{x} \in \bar{S}^{n\times 1}$ such that $A\bar{x} = b$.

Let $P = \{j \in \{1, \cdots, n\} | \bar{x}_j \in S\}$, and $Q = \{1, \cdots, n\} \backslash P$. Suppose there exists $i \in \mathbb{N}$ such that $b_i \neq \oplus_{j\in P} a_{ij}\bar{x}_j$, then $b_i \neq \varepsilon$ and $\exists j \in Q$ such that $b_i = a_{ij}\bar{x}_j$, thus $b_i \notin S$, a contradiction. Therefore it must be true that $b_i = \oplus_{j\in P} a_{ij}\bar{x}_j$ for any $i \in \mathbb{N}$. Let x be such that $x_j = \bar{x}_j$ for $j \in P$ and $x_j = \varepsilon$ otherwise. It is trivial to see that $x \in S^{n\times 1}$ and $Ax = b$. ∎

It is easily seen that if translated into geometric language, Theorem 7 is the generalization of the linear extension theorem from finitely to countably generated semimodules, as long as the concerned semifield is archimedean. This generalization may provide a tool for attacking the realization problems in max-plus systems theory, because the reachability and observability semimodules are (possibly infinite) countably generated.

5 Conclusions

We have studied the completeness of totally ordered idempotent semifields and obtained some interesting results. They are helpful in understanding the "analytical" structure of idempotent semifields. Further, we used our completeness results to derive a generalization of the linear extension theorem. We hope the generalization will be of help in solving the realization problems in max-plus systems theory.

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