Completeness Proof by Semantic Diagrams for Transitive Closure of Accessibility Relation

Ryo Kashima

Department of Mathematical and Computing Sciences, Tokyo Institute of Technology.
Ookayama, Meguro, Tokyo 152-8552, Japan.
kashima@is.titech.ac.jp

Abstract

We treat the smallest normal modal propositional logic with two modal operators $\Box$ and $\Box^+$. While $\Box$ is interpreted in Kripke models by the accessibility relation R, $\Box^+$ is interpreted by the transitive closure of R. Intuitively the formula $\Box^+\varphi$ means the infinite conjunction $\Box\varphi \wedge \Box\Box\varphi \wedge \Box\Box\Box\varphi \wedge \cdots$. There is a Hilbert style axiomatization of this logic (a characteristic axiom is $\Box\varphi \wedge \Box^+(\varphi \to \Box\varphi) \to \Box^+\varphi$, called "induction axiom"), and its completeness with respect to finite models was shown by the canonical model method. This paper gives an alternative proof of this completeness. We use the method of "semantic diagram", which is a variant of semantic tableaux, as follows. Given an unprovable formula φ, we first make a small model (consisting of one world that forces φ to be false); then we add worlds step by step using the Hilbert system as an oracle, and finally we get a finite countermodel for φ. The point is how to handle $\Box^+$ in this construction.

Keywords: completeness of modal logic, transitive closure of accessibility relation, semantic diagram

1 Introduction

In Kripke models, the modal operator $\Box$ is interpreted as

$$w \models \Box\varphi \iff x \models \varphi \text{ for any } x \text{ such that } wRx$$

where w and x are possible worlds and R is the accessibility relation. Then we introduce a new modal operator $\Box^+$ by

$$w \models \Box^+\varphi \iff x \models \varphi \text{ for any } x \text{ such that } wR^+x$$

where R^+ is the transitive closure of R. Intuitively $\Box^+\varphi$ means the infinite conjunction as follows:

$$\Box^+\varphi \quad \leftrightarrow \quad \Box\varphi \wedge \Box\Box\varphi \wedge \Box\Box\Box\varphi \wedge \cdots .$$

This paper treats the smallest normal modal propositional logic with the operators $\Box$ and $\Box^+$ as above. This logic will be called K^+.

The relationship between $\Box$ and $\Box^+$ in K^+ is equal to that between the operators E ("everyone knows") and C ("common knowledge") in the common knowledge logic, since

$$C\varphi \quad \leftrightarrow \quad E\varphi \wedge EE\varphi \wedge EEE\varphi \wedge \cdots .$$

Moreover the relationship is similar to that between the operators X ("next time") and G ("globally") in temporal logic, since

$$G\varphi \quad \leftrightarrow \quad \varphi \wedge X\varphi \wedge XX\varphi \wedge \cdots .$$

There are Hilbert style systems for the common knowledge logic and the temporal logic, and the completeness with respect to finite models (i.e., a formula is provable in a system if it is true in every finite model) was proved by using canonical models and filtrations (see, e.g., [2] and [4]). Of course the argument can be applied to K^+ — there is a Hilbert system, which we will call HK^+ (a characteristic axiom is the *induction axiom*: $\Box\varphi \wedge \Box^+(\varphi \to \Box\varphi) \to \Box^+\varphi$), and the completeness with respect to finite models can be shown by using canonical models and filtrations.

The purpose of this paper is to give a new proof for the completeness of HK^+. We use the method of "semantic diagram", which is a variant of semantic tableaux, as follows. Given an unprovable formula α_0, we first make a small model (consisting of one world that forces α_0 to be false); then we add worlds step by step using HK^+ as an oracle, and finally we get a finite countermodel for α_0.

Here we give an informal explanation of the point of our method. It is well known that the finite set $\mathrm{Sub}^\pm(\alpha_0) = \{\varphi, \neg\varphi \mid \varphi \text{ is a subformula of } \alpha_0\}$ is sufficient for the construction of a countermodel for α_0. Then the point of our method is how to make the witness of $\Diamond^+\varphi$. If $\Diamond^+\varphi \in \Gamma$ and a world $\boxed{\Gamma}$ (this means all the elements of Γ are true at this world) is in a Kripke model, then we may consider a path to the witness $\boxed{\varphi}$ to be of the form

$$\boxed{\Gamma} \xrightarrow{R} \boxed{\Gamma'} \xrightarrow{R} \cdots \xrightarrow{R} \boxed{\Gamma''} \xrightarrow{R} \boxed{\varphi} \qquad (1)$$

where $\Gamma, \Gamma', \ldots, \Gamma''$ are *mutually distinct* subsets of $\mathrm{Sub}^\pm(\alpha_0)$. For example, suppose imaginarily that the powerset $\mathcal{P}(\mathrm{Sub}^\pm(\alpha_0))$ consists of just three sets Γ, Δ and Λ; then the candidates of paths to the witness can be limited to the five paths:

$$\begin{array}{l}
\boxed{\Gamma} \xrightarrow{R} \boxed{\varphi} \\
\boxed{\Gamma} \xrightarrow{R} \boxed{\Delta} \xrightarrow{R} \boxed{\varphi} \\
\boxed{\Gamma} \xrightarrow{R} \boxed{\Lambda} \xrightarrow{R} \boxed{\varphi} \\
\boxed{\Gamma} \xrightarrow{R} \boxed{\Delta} \xrightarrow{R} \boxed{\Lambda} \xrightarrow{R} \boxed{\varphi} \\
\boxed{\Gamma} \xrightarrow{R} \boxed{\Lambda} \xrightarrow{R} \boxed{\Delta} \xrightarrow{R} \boxed{\varphi}
\end{array} \tag{2}$$

This limitation is justified by the following argument. Given a long path from Σ_1 $(= \Gamma)$ to φ as

$$\boxed{\Sigma_1} \xrightarrow{R} \boxed{\Sigma_2} \xrightarrow{R} \cdots \xrightarrow{R} \boxed{\Sigma_k} \xrightarrow{R} \boxed{\varphi}, \tag{3}$$

we can extract a *skipping path* $(\Sigma_{a_1}, \Sigma_{a_2}, \ldots, \Sigma_{a_m})$ such that

- Σ_{a_1} is *the last* Σ_1 *before* φ; that is, Σ_{a_1} is the same set as Σ_1, and none of $\Sigma_{a_1+1}, \Sigma_{a_1+2}, \ldots, \Sigma_k$ are the same set as Σ_1;
- Σ_{a_2} is the last Σ_{a_1+1} before φ;

  $\vdots$
- $\Sigma_{a_m} = \Sigma_k$ is the last $\Sigma_{a_{m-1}+1}$ before φ.

Then

$$\boxed{\Sigma_{a_1}} \xrightarrow{R} \boxed{\Sigma_{a_2}} \xrightarrow{R} \cdots \xrightarrow{R} \boxed{\Sigma_{a_m}} \xrightarrow{R} \boxed{\varphi}$$

is the very path denoted by (1), of length $\leq |\mathcal{P}(\mathrm{Sub}^{\pm}(\alpha_0))|$.

This principle of extraction (of length-limited paths from unlimited paths) is the core of our method. While such a principle was used in Brünnler and Lange [1] and Gaintzarain et al. [3] for temporal logics, the originality of this paper is that our method *does not need the until operator*. If a binary operator U' is available as

$$w \models \alpha\, U' \beta \iff \exists w_1, \ldots, \exists w_n \Big(wRw_1R \cdots Rw_n, w_i \models \alpha \text{ for } i < n, \text{ and } w_n \models \beta \Big),$$

then the condition "Σ_{a_1} is the last Σ_1 before φ" can be easily described by putting

$$\Sigma_{a_1} = \Sigma_1 \cup \{(\neg \Sigma_1)\, U' \varphi\}.$$

(Brünnler and Lange [1] and Gaintzarain et al. [3] introduced similar description as an inference rule of sequent calculi, and proved the completeness of the calculi.) However our K^{+} does not have the until operator; hence we realize the extraction by *explicit enumeration of all the possible candidates of paths to the witness*, like (2) above.

2 Axiomatization

Formulas are constructed from the following symbols: propositional variables (the set of propositional variables is called **Prop**); logical connectives $\wedge$ and $\neg$; and modal operators $\Box$ and $\Box^+$. We will use letters $p, q, \ldots$ to denote propositional variables, and letters $\alpha, \beta, \ldots \varphi, \psi, \ldots$ to denote formulas. Other symbols ($\bot, \top, \to, \vee, \Diamond, \Diamond^+, \ldots$) are defined by the usual abbreviations. Parentheses are omitted by the convention that the unary operators $\neg$, $\Box$, $\Box^+$, $\Diamond$, and $\Diamond^+$ bind stronger than other connectives, $\wedge$ and $\vee$ bind stronger than $\to$, and that $\alpha_1 \to \alpha_2 \to \cdots \to \alpha_n = \alpha_1 \to (\alpha_2 \to (\cdots \to (\alpha_{n-1} \to \alpha_n)\cdots))$. For example, the axiom scheme (A2) below is $(\Box(\alpha \to \beta)) \to ((\Box\alpha) \to \Box\beta)$, and $\neg\alpha \wedge \beta \to \Box^+\gamma \vee \delta = ((\neg\alpha) \wedge \beta) \to ((\Box^+\gamma) \vee \delta)$.

A *Kripke model* is a triple $M = \langle W, R, V \rangle$ where W is a nonempty set (the set of *possible worlds*), R is a binary relation on W (the *accessibility relation*), and V is a function from $W \times \mathbf{Prop}$ to $\{\mathsf{True}, \mathsf{False}\}$. M is said to be *finite* if W is a finite set. The transitive closure of R is denoted by R^+; that is, xR^+y holds if and only if $x = a_0Ra_1R\cdots Ra_n = y$ for some $a_0, a_1, \ldots, a_n$ $(n \geq 1)$. The notion "a formula φ is true at a world w in M", written by "$M, w \models \varphi$" (or "$w \models \varphi$" for short), is defined as usual: $w \models p \iff V(w,p) = \mathsf{True}$; $w \models \alpha \wedge \beta \iff w \models \alpha$ and $w \models \beta$; $w \models \neg\alpha \iff w \not\models \alpha$; $w \models \Box\alpha \iff x \models \alpha$ for any x such that wRx; and $w \models \Box^+\alpha \iff x \models \alpha$ for any x such that wR^+x. We say that a formula φ is *valid in* M if and only if $M, x \models \varphi$ for any world x.

The system HK^+ is defined as follows (cf. the axiomatization of linear temporal logic in [4, §9]). The axiom schemata are

(A1) instances of classical tautologies,

(A2) $\Box(\alpha \to \beta) \to \Box\alpha \to \Box\beta$ ('K axiom' for $\Box$),

(A3) $\Box^+(\alpha \to \beta) \to \Box^+\alpha \to \Box^+\beta$ ('K axiom' for $\Box^+$),

(A4) $\Box^+\alpha \to \Box\alpha \wedge \Box\Box^+\alpha$, and

(A5) $\Box\alpha \wedge \Box^+(\alpha \to \Box\alpha) \to \Box^+\alpha$ (induction axiom)

and the inference rules are

(R1) $\dfrac{\alpha \to \beta \quad \alpha}{\beta}$ (modus ponens), and

(R2) $\dfrac{\alpha}{\Box^+\alpha}$ (generalization for $\Box^+$).

Note that the 'transitive axiom' $\Box^+\alpha \to \Box^+\Box^+\alpha$ is derivable using (A4) and the instance $\Box\Box^+\alpha \wedge \Box^+(\Box^+\alpha \to \Box\Box^+\alpha) \to \Box^+\Box^+\alpha$ of induction axiom. The generalization rule for $\Box$ is also derivable using (R2) and (A4).

By "$\vdash \varphi$", we mean "φ is provable in HK^+". The purpose of this paper is to give a new proof of the completeness of HK^+ with respect to finite models, which states "if α_0 is valid in any finite model, then $\vdash \alpha_0$" or equivalently "if $\not\vdash \alpha_0$, then there is a finite countermodel for α_0". The soundness (converse of the completeness) can be easily shown as usual.

3 Special formulas

In this section, we show provability of certain formulas which will be used in the next section.

Two formulas α and β are said to be *provably equivalent* when $\vdash (\alpha \to \beta) \wedge (\beta \to \alpha)$. If $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a finite set of formulas, then "$\vdash \Gamma \Rightarrow \varphi$" means "$\vdash (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) \to \varphi$". Note that we do not mind permutations or duplications in Γ because, for example, $((\gamma_1 \wedge \gamma_2) \wedge \gamma_3) \to \varphi$ and $((\gamma_2 \wedge \gamma_1) \wedge (\gamma_3 \wedge \gamma_1)) \to \varphi$ are provably equivalent.

Lemma 3.1 *(1) If $\vdash \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \Rightarrow \psi$, then $\vdash \{\varphi_1 \vee \rho, \varphi_2 \vee \rho, \ldots, \varphi_n \vee \rho\} \Rightarrow \psi \vee \rho$.*

(2) If $\vdash \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \Rightarrow \psi$, then $\vdash \{\rho \to \varphi_1, \rho \to \varphi_2, \ldots, \rho \to \varphi_n\} \Rightarrow \rho \to \psi$.

(3) If $\vdash \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \Rightarrow \psi$, then $\vdash \{\Box\varphi_1, \Box\varphi_2, \ldots, \Box\varphi_n\} \Rightarrow \Box\psi$.

(4) If $\vdash \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \Rightarrow \psi$, then $\vdash \{\Box^+\varphi_1, \Box^+\varphi_2, \ldots, \Box^+\varphi_n\} \Rightarrow \Box^+\psi$.

Proof. (1) and (2) are properties of classical logic. (3) and (4) are properties of normal modal logics. □

Lemma 3.2 *If formulas $\sigma, \sigma', \tau, \tau'$ and ω satisfy the conditions (a) $\vdash \sigma \to \Box\tau$, (b) $\vdash \sigma' \to \Box\tau'$, and (c) $\vdash \neg\sigma' \to \Box\tau$; then we have $\vdash \{\sigma \to \Box^+(\tau \to \omega),\ \sigma \to \Box^+(\tau \to \sigma' \to \Box^+(\tau' \to \omega))\} \Rightarrow \sigma \to \Box^+\omega$.*

Proof. See Appendix A. □

In the rest of this section, a natural number $N \geq 2$ and formulas ω, σ_i, τ_i ($i = 1, 2, \ldots, N$) are fixed. A formula is called *special* if and only if it is of the form

$$\sigma_{f(1)} \to \Box^+\Big(\tau_{f(1)} \to \sigma_{f(2)} \to \Box^+\Big(\tau_{f(2)} \to \cdots \to \sigma_{f(m)} \to \Box^+(\tau_{f(m)} \to \omega) \cdots\Big)\Big)$$

for some natural number m and some function f that satisfy the following conditions.

- $1 \leq m \leq N$.
- f is an injection (one-to-one) from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, N\}$.
- $f(1) = 1$.

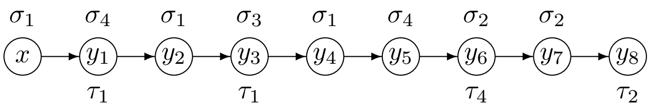
The set of special formulas is called **SP**, which is a finite set. For example, if $N = 3$, then

$$\begin{aligned} \mathbf{SP} = \{\ & \sigma_1 \to \Box^+(\tau_1 \to \omega), \\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_2 \to \omega)), \\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \omega)), \\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_2 \to \sigma_3 \to \Box^+(\tau_3 \to \omega))), \\ & \sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \sigma_2 \to \Box^+(\tau_2 \to \omega)))\ \}. \end{aligned} \tag{4}$$

Note that the shapes of these formulas are same as the paths (2) in Section 1. If $N = 4$, then **SP** consists of sixteen formulas.

Theorem 3.3 (Main theorem on special formulas) *Suppose that*

Fig. 1.



(i) $\vdash \sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_N$, *and*

(ii) $\vdash \sigma_i \to \Box\tau_i$, *for* $i = 1, 2, \dots, N$,

where $N \geq 2$. *Then we have* $\vdash \mathbf{SP} \Rightarrow (\sigma_1 \to \Box^+\omega)$.

Before the proof, we give a semantical explanation of this theorem; that is, we show the formula $\Box^+\omega$ is true at a world x in a model $M = \langle W, R, V\rangle$ on the assumption that (I) $\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_N$ is valid, (II) $\sigma_i \to \Box\tau_i$ ($i = 1, 2, \dots, N$) are all valid, (III) all the special formulas are true at x, and (IV) σ_1 is true at x. For example, let us assume that Figure 1 describes some worlds around x, where $xRy_1Ry_2R\cdots Ry_8$ and the displayed σ_i is true there ($\because$ (I),(IV)). We can verify that ω is true at all y_i ($i = 1, 2, \dots, 8$). For example, ω is true at y_3 because $x \models \sigma_1 \to \Box^+(\tau_1 \to \omega)$ ($\because$ (III)), $x \models \sigma_1$ ($\because$ (IV)), and $y_3 \models \tau_1$ ($\because y_2 \models \sigma_1 \to \Box\tau_1$ by (II)); and ω is true at y_8 because $x \models \sigma_1 \to \Box^+(\tau_1 \to \sigma_4 \to \Box^+(\tau_4 \to \sigma_2 \to \Box^+(\tau_2 \to \omega)))$ ($\because$ (III)), $x \models \sigma_1$ ($\because$ (IV)), $y_1 \models \tau_1 \wedge \sigma_4$ ($\because x \models \sigma_1 \to \Box\tau_1$ by (II)), $y_6 \models \tau_4 \wedge \sigma_2$ ($\because y_5 \models \sigma_4 \to \Box\tau_4$ by (II)), and $y_8 \models \tau_2$ ($\because y_7 \models \sigma_2 \to \Box\tau_2$ by (II)).

Now we start proving Theorem 3.3. If $N = 2$, this can be done by simple application of Lemma 3.2 (by $\sigma = \sigma_1$, $\sigma' = \sigma_2$, $\tau = \tau_1$, $\tau' = \tau_2$). However, we need a more complicated proof when $N > 2$. For this, we introduce an extra notion of *key formulas*.

A formula is called a *key formula of type I* if and only if it is of the form

$$\sigma_{f(1)} \to \Box^+\Big(\tau_{g(1)} \to \sigma_{f(2)} \to \Box^+\Big(\tau_{g(2)} \to \cdots \to \sigma_{f(m)} \to \Box^+(\underline{\tau_{g(m)} \to} \,\omega)\cdots\Big)\Big) \qquad (5)$$

(the underline will be used later) for some natural number m and some functions f and g that satisfy the following conditions.

- $1 \leq m \leq N$.
- f is an injection from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, N\}$.
- g is a function (not limited to injection) from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, N\}$.
- $f(1) = g(1) = 1$.

♡ $(\forall i \in \{1, \dots, m\})(\exists j \leq i)(f(j) = g(i))$.

The set of key formulas of type I is called **KeyI**, which is a finite superset of **SP**. For example, if $N = 3$, then **KeyI** is the union of **SP** (see (4)) and

$$\begin{aligned}\{\ &\sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_1 \to \omega)),\\ &\sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_1 \to \omega)),\\ &\sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_i \to \omega)))\ (i = 1,2,3),\\ &\sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_2 \to \sigma_3 \to \Box^+(\tau_j \to \omega)))\ (j = 1,2)^\dagger,\\ &\sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_i \to \omega)))\ (i = 1,2,3),\\ &\sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \sigma_2 \to \Box^+(\tau_k \to \omega)))\ (k = 1,3)^\dagger\}.\end{aligned}$$

(† This is a special formula if $j = 3$ or $k = 2$.)

A formula φ is called a *key formula of type II* if and only if there is a formula ψ that satisfies the following conditions.

- ψ is a key formula of type I as (5) where $m \le (N-1)$.
- φ is obtained from ψ by deleting the underlined '$\tau_{g(m)} \to$' in (5).

The natural number m is called the *depth of* φ. For example, if $N = 3$, then there are just three key formulas of type II:

$$\begin{aligned}&\sigma_1 \to \Box^+\omega. && \text{(depth} = 1)\\ &\sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+\omega). && \text{(depth} = 2)\\ &\sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+\omega). && \text{(depth} = 2)\end{aligned}$$

The set of key formulas of type II is called **KeyII**.

The target formula $\sigma_1 \to \Box^+\omega$ of Theorem 3.3 is the shortest element of **KeyII**, and the other elements will be used in the inductive proof of Lemma 3.5 below.

Lemma 3.4 $\vdash$ **SP** $\Rightarrow \varphi$, *for any* $\varphi \in$ **KeyI**.

Lemma 3.5 *Suppose that*

(i) $\vdash \sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_N$, *and*

(ii) $\vdash \sigma_i \to \Box\tau_i$, *for* $i = 1, 2, \ldots, N$,

where $N \ge 2$. *Then* $\vdash$ **KeyI** $\Rightarrow \varphi$, *for any* $\varphi \in$ **KeyII**.

These two lemmas straightforwardly imply the Main Theorem 3.3. So the rest of this section is devoted to proving these lemmas.

Proof of Lemma 3.4. For any key formula φ of type I, there is a special formula $\varphi*$ embedded in φ such that $\vdash \{\varphi*\} \Rightarrow \varphi$. For example, if φ is

$$\sigma_1 \to \Box^+(\tau_1 \to \sigma_2 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \sigma_4 \to \Box^+(\tau_3 \to \sigma_5 \to \Box^+(\tau_1 \to \sigma_6 \to \Box^+(\tau_5 \to \omega)))))),$$

then $\varphi*$ is

$$\sigma_1 \to \Box^+(\tau_1 \to \sigma_3 \to \Box^+(\tau_3 \to \sigma_5 \to \Box^+(\tau_5 \to \omega)))),$$

which is embedded in φ as

$$\underline{\sigma_1 \to} \Box^+(\tau_1 \to \sigma_2 \to \underline{\Box^+(\tau_1 \to \sigma_3 \to} \Box^+(\tau_3 \to \sigma_4 \to \underline{\Box^+(\tau_3 \to \sigma_5 \to} \Box^+(\tau_1 \to \sigma_6 \to \underline{\Box^+(\tau_5 \to \omega)}))))).$$

In general, $\varphi*$ is defined as follows. Let φ be the formula as (5). Without loss of generality, we suppose $f(i) = i$ for all i. Then, by the property $\heartsuit$, we have

$$g(i) \leq i. \tag{$\heartsuit'$}$$

Now we define a sequence $a_1, a_2, \ldots$ of natural numbers by

$$a_1 = g(m), \quad a_{x+1} = g(a_x - 1) \text{ for } x = 1, 2, \ldots$$

By ($\heartsuit'$), this sequence is strictly decreasing, and $\varphi*$ is

$$\sigma_{a_z} \to \Box^+\Big(\tau_{a_z} \to \sigma_{a_{z-1}} \to \Box^+\Big(\tau_{a_{z-1}} \to \cdots \to \sigma_{a_1} \to \Box^+(\tau_{a_1} \to \omega) \cdots\Big)\Big)$$

where $a_z = 1$. The fact $\vdash \{\varphi*\} \Rightarrow \varphi$ is obtained from $\vdash \{\Box^+(\tau_{g(m)} \to \omega)\} \Rightarrow \Box^+(\tau_{g(m)} \to \omega)$ by appropriate applications of Lemma 3.1(2), 3.1(4) and the fact "$\vdash \{\Box^+\alpha\} \Rightarrow \Box^+\beta$ implies $\vdash \{\Box^+\alpha\} \Rightarrow \Box^+(\tau \to \sigma \to \Box^+\beta)$". $\Box$

Proof of Lemma 3.5. The key formula φ of type II is of the form

$$\sigma_{f(1)} \to \Box^+\Big(\tau_{g(1)} \to \cdots \to \sigma_{f(m-1)} \to \Box^+\Big(\tau_{g(m-1)} \to \sigma_{f(m)} \to \Box^+\omega\Big) \cdots\Big).$$

We will abbreviate this to

$$\bullet \to \sigma_{f(m)} \to \Box^+\omega.$$

That is, "$\bullet$" denotes the context " $\sigma_{f(1)} \to \Box^+(\tau_{g(1)} \to \cdots \to \sigma_{f(m-1)} \to \Box^+(\tau_{g(m-1)} \to$ ". Therefore, for example, $\bullet \to \sigma_{f(m)} \to \Box^+(\tau_{g(m)} \to \omega)$ is the formula (5), and $\bullet \to \sigma_1 \to \Box^+\omega$ is just $\sigma_1 \to \Box^+\omega$ when $m = 1$.

We define a set U of natural numbers by

$$U = \{1, 2, \ldots, N\} - \{f(1), f(2), \ldots, f(m)\}.$$

U is not empty because of the definition of key formula of type II. We prove Lemma 3.5 by induction on $|U|$; in other words, we prove this lemma for any φ of depth $(N-1)$, any φ of depth $(N-2)$, ..., any φ of depth 1, successively.

(Case 1: $|U| = 1$; depth of φ is $N - 1$.) For any $i \in \{1, \ldots, m\}$, the formula

$$\bullet \to \sigma_{f(m)} \to \Box^+(\tau_{f(i)} \to \omega)$$

is a key formula of type I. Therefore we have

$$\vdash \mathbf{KeyI} \Rightarrow \ \bullet \to \sigma_{f(m)} \to \Box^+((\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)}) \to \omega) \tag{6}$$

because of the fact

$$\vdash \{\tau_{f(1)} \to \omega,\ \tau_{f(2)} \to \omega,\ \ldots,\ \tau_{f(m)} \to \omega\} \Rightarrow (\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)}) \to \omega$$

and Lemma 3.1(4) and 3.1(2). Let u be the only element of U. Similarly to (6), we have

$$\vdash \mathbf{KeyI} \Rightarrow \ \bullet \to \sigma_{f(m)} \to \Box^{+}\big((\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)}) \to \sigma_u \to \Box^{+}(\tau_u \to \omega)\big) \quad (7)$$

because the formula

$$\bullet \to \sigma_{f(m)} \to \Box^{+}(\tau_{f(i)} \to \sigma_u \to \Box^{+}(\tau_u \to \omega))$$

is a key formula of type I for any $i \in \{1, \ldots, m\}$. On the other hand, by Lemma 3.2 ($\sigma = \sigma_{f(m)}$, $\sigma' = \sigma_u$, $\tau = (\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)})$, $\tau' = \tau_u$), we get

$$\begin{aligned} \vdash \{\ & \sigma_{f(m)} \to \Box^{+}((\tau_{f(1)} \vee \cdots \vee \tau_{f(m)}) \to \omega), \\ & \sigma_{f(m)} \to \Box^{+}((\tau_{f(1)} \vee \cdots \vee \tau_{f(m)}) \to \sigma_u \to \Box^{+}(\tau_u \to \omega))\ \} \Rightarrow \sigma_{f(m)} \to \Box^{+}\omega. \end{aligned} \quad (8)$$

Note that the hypotheses (a), (b), and (c) of Lemma 3.2 are shown by the hypotheses (i) and (ii) of this Lemma 3.5. Then (6), (7), (8) and Lemma 3.1 imply

$$\vdash \mathbf{KeyI} \Rightarrow \ \bullet \to \sigma_{f(m)} \to \Box^{+}\omega, \quad (9)$$

which is the required formula.

(Case 2: $|U| > 1$; depth of φ is less then $N-1$.) By the same argument as (6), we obtain

$$\vdash \mathbf{KeyI} \Rightarrow \ \bullet \to \sigma_{f(m)} \to \Box^{+}((\tau_{f(1)} \vee \tau_{f(2)} \vee \cdots \vee \tau_{f(m)}) \to \omega). \quad (10)$$

On the other hand, for any $i \in \{1, \ldots, m\}$ and any $u \in U$, the formula

$$\bullet \to \sigma_{f(m)} \to \Box^{+}(\tau_{f(i)} \to \sigma_u \to \Box^{+}\omega))$$

is a key formula of type II with greater depth. Therefore by the induction hypothesis,

$$\vdash \mathbf{KeyI} \Rightarrow \ \bullet \to \sigma_{f(m)} \to \Box^{+}(\tau_{f(i)} \to \sigma_u \to \Box^{+}\omega)),$$

and then

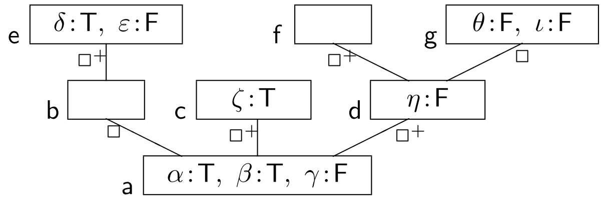
$$\vdash \mathbf{KeyI} \Rightarrow \ \bullet \to \sigma_{f(m)} \to \Box^{+}\big((\tau_{f(1)} \vee \cdots \vee \tau_{f(m)}) \to (\sigma_{u_1} \vee \cdots \vee \sigma_{u_k}) \to \Box^{+}(\top \to \omega)\big) \quad (11)$$

where $U = \{u_1, \ldots, u_k\}$. Now (10), (11), and Lemma 3.2 ($\sigma = \sigma_{f(m)}$, $\sigma' = (\sigma_{u_1} \vee \cdots \vee \sigma_{u_k})$, $\tau = (\tau_{f(1)} \vee \cdots \vee \tau_{f(m)})$, $\tau' = \top$) imply

$$\vdash \mathbf{KeyI} \Rightarrow \ \bullet \to \sigma_{f(m)} \to \Box^{+}\omega$$

similarly to (9). □

Fig. 2. A semantic diagram.



4 Making a countermodel

If φ is a formula, then the expressions $\varphi : \mathsf{T}$ and $\varphi : \mathsf{F}$ are called *signed formulas*. A *semantic diagram* is a finite tree whose nodes are associated with finite sets of signed formulas and whose edges are labeled by $\Box$ or $\Box^+$. Set(a) denotes the set of signed formulas that is associated with the node a. If a node b is a $\Box$-successor (or $\Box^+$-successor) of a node a, then we write $\mathsf{a}<^{\Box}\mathsf{b}$ (or $\mathsf{a}<^{\Box^+}\mathsf{b}$, respectively). Moreover we write $\mathsf{a} < \mathsf{b}$ if and only if $\mathsf{a}<^{\Box}\mathsf{b}$ or $\mathsf{a}<^{\Box^+}\mathsf{b}$. The transitive closure of $<$ is written by $\lll$. Figure 2 is an example of a semantic diagram, in which $\mathrm{Set}(\mathsf{a}) = \{\alpha:\mathsf{T}, \beta:\mathsf{T}, \gamma:\mathsf{F}\}$, $\mathrm{Set}(\mathsf{b}) = \emptyset$, $\mathsf{a}<^{\Box}\mathsf{b}$, $\mathsf{b}<^{\Box^+}\mathsf{e}$, $\mathsf{a} < \mathsf{b}$, $\mathsf{b} < \mathsf{e}$, $\mathsf{a} \not< \mathsf{e}$, $\mathsf{a} \lll \mathsf{b}$, $\mathsf{a} \lll \mathsf{e}$, and $\mathsf{a} \not\lll \mathsf{a}$ hold. In the following, $\Gamma, \Delta, \ldots$ will denote sets of signed formulas, $\mathcal{S}, \mathcal{T}, \ldots$ will denote semantic diagrams, and $\mathsf{a}, \mathsf{b}, \ldots$ will denote nodes of diagrams. By "$\varphi \in_{\mathsf{T}} \mathsf{x}$" (or "$\varphi \in_{\mathsf{F}} \mathsf{x}$"), we mean "$(\varphi:\mathsf{T}) \in \mathrm{Set}(\mathsf{x})$" (or "$(\varphi:\mathsf{F}) \in \mathrm{Set}(\mathsf{x})$", respectively).

For each diagram $\mathcal{S}$, we define a formula $\mathrm{Neg}(\mathcal{S})$ (called the *negation of* $\mathcal{S}$) inductively as follows. If a set $\{\varphi_1:\mathsf{T}, \varphi_2:\mathsf{T}, \ldots, \varphi_m:\mathsf{T},\ \psi_1:\mathsf{F},\ \psi_2:\mathsf{F}, \ldots, \psi_n:\mathsf{F}\}$ is associated with the root of $\mathcal{S}$, and subdiagrams $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ are connected with the root by $\Box$-edges and $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_l$ are connected with the root by $\Box^+$-edges, then $\mathrm{Neg}(\mathcal{S})$ is the formula

$$\begin{aligned}\bot \vee \neg\varphi_1 \vee \neg\varphi_2 \vee \cdots \vee \neg\varphi_m \vee \psi_1 \vee \psi_2 \vee \cdots \vee \psi_n \vee \\ \Box(\mathrm{Neg}(\mathcal{S}_1)) \vee \Box(\mathrm{Neg}(\mathcal{S}_2)) \vee \cdots \vee \Box(\mathrm{Neg}(\mathcal{S}_k)) \vee \\ \Box^+(\mathrm{Neg}(\mathcal{T}_1)) \vee \Box^+(\mathrm{Neg}(\mathcal{T}_2)) \vee \cdots \vee \Box^+(\mathrm{Neg}(\mathcal{T}_l)).\end{aligned}$$

For example, the negation of the diagram of Figure 2 is provably equivalent to the formula $\neg\alpha \vee \neg\beta \vee \gamma \vee \Box\Box^+(\neg\delta \vee \varepsilon) \vee \Box^+\neg\zeta \vee \Box^+\big(\eta \vee \Box^+\bot \vee \Box(\theta \vee \iota)\big)$. A diagram $\mathcal{S}$ is said to be *HK*$^+$*-consistent* if and only if $\nvdash \mathrm{Neg}(\mathcal{S})$.

Let $\mathcal{S}$ and $\mathcal{T}$ be semantic diagrams and a be a node of $\mathcal{S}$. By $\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{T}$, we mean the diagram obtained by joining $\mathcal{S}$ and $\mathcal{T}$, in which a and the root of $\mathcal{T}$ are merged into one node. Figure 3 describes an example.

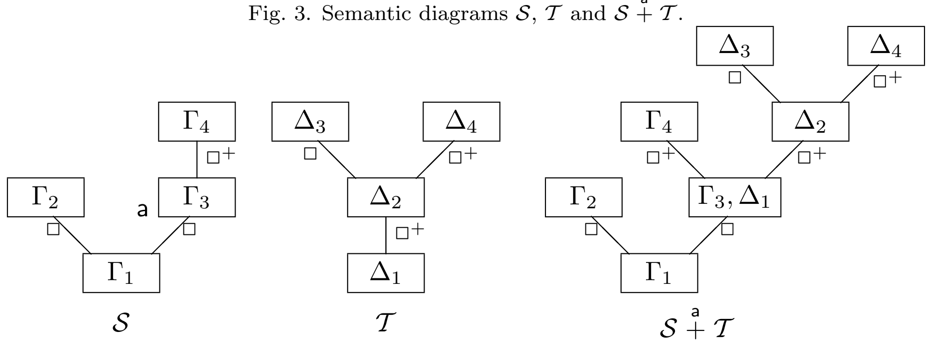
Let $\mathbb{L}$ be a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ of formulas. We say that a set Λ of signed formulas is a *valuation of* $\mathbb{L}$ if Λ is $\{\lambda_1 : \bullet_1, \lambda_2 : \bullet_2, \ldots, \lambda_k : \bullet_k\}$ ($\bullet_i$ is T or F). There are 2^k distinct valuations of $\mathbb{L}$.

For a set Γ of signed formulas, we define a formula $\langle\Gamma\rangle$ and a set $\Gamma^{\mathsf{T}}_{\Box}$ of signed formulas as follows.

$$\langle\Gamma\rangle = \bigwedge\{\varphi \mid (\varphi:\mathsf{T}) \in \Gamma\} \wedge \bigwedge\{\neg\varphi \mid (\varphi:\mathsf{F}) \in \Gamma\}.$$

$$\Gamma^{\mathsf{T}}_{\Box} = \{\varphi:\mathsf{T} \mid (\Box\varphi:\mathsf{T}) \in \Gamma\}.$$

Fig. 3. Semantic diagrams $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{T}$.



For example, if $\Gamma = \{\Box\varphi_1 : \mathsf{T},\ \Box\Box\varphi_2 : \mathsf{T},\ \neg\neg\Box\varphi_3 : \mathsf{T},\ \Box^+\varphi_4 : \mathsf{T},\ \Box\varphi_5 : \mathsf{F}\}$, then $\langle\Gamma\rangle$ is $\Box\varphi_1 \wedge \Box\Box\varphi_2 \wedge \neg\neg\Box\varphi_3 \wedge \Box^+\varphi_4 \wedge \neg\Box\varphi_5$, and $\Gamma^{\mathsf{T}}_{\Box}$ is $\{\varphi_1 : \mathsf{T},\ \Box\varphi_2 : \mathsf{T}\}$. Note that

$$\vdash \langle\Lambda\rangle \to \Box\langle\Lambda^{\mathsf{T}}_{\Box}\rangle \tag{12}$$

holds for any set Λ; for example, $\vdash (\Box\varphi_1 \wedge \Box\Box\varphi_2 \wedge \neg\neg\Box\varphi_3 \wedge \Box^+\varphi_4 \wedge \neg\Box\varphi_5) \to \Box(\varphi_1 \wedge \Box\varphi_2)$ if Λ is the above Γ.

We give some basic lemmas on diagrams.

Lemma 4.1 *Let $\mathcal{S}, \mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ be semantic diagrams ($n \geq 0$) and* a *be a node of $\mathcal{S}$. If*

$$\vdash \{\mathrm{Neg}(\mathcal{T}_1),\ \mathrm{Neg}(\mathcal{T}_2),\ \ldots,\ \mathrm{Neg}(\mathcal{T}_n)\} \Rightarrow \mathrm{Neg}(\mathcal{T}),$$

then

$$\vdash \{\mathrm{Neg}(\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{T}_1),\ \mathrm{Neg}(\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{T}_2),\ \ldots,\ \mathrm{Neg}(\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{T}_n)\} \Rightarrow \mathrm{Neg}(\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{T}).$$

Proof. By Lemma 3.1 and the definition of Neg(). □

Lemma 4.2 (Maximalization) *Let $\mathbb{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ ($k \geq 1$) be a finite set of formulas. If a semantic diagram $\mathcal{S}$ is HK$^+$-consistent and* a *is a node of it, then there exists a valuation Λ of $\mathbb{L}$ such that the diagram $\mathcal{S} \overset{\mathsf{a}}{+} \Lambda$ (i.e., the diagram obtained from $\mathcal{S}$ by adding Λ to the node* a*) is HK$^+$-consistent. The process of making $\mathcal{S} \overset{\mathsf{a}}{+} \Lambda$ from $\mathcal{S}$ will be called "maximalization for* a *with respect to $\mathbb{L}$".*

Proof. Since $\vdash \{\neg\lambda_i, \lambda_i\} \Rightarrow \bot$, one of the diagrams $\mathcal{S} \overset{\mathsf{a}}{+} \{\lambda_i : \mathsf{T}\}$ and $\mathcal{S} \overset{\mathsf{a}}{+} \{\lambda_i : \mathsf{F}\}$ is HK$^+$-consistent (otherwise $\vdash \mathrm{Neg}(\mathcal{S})$ by Lemma 4.1). By iterating this argument, we can chose $\bullet_1, \bullet_2, \ldots, \bullet_k$ ($\bullet_i \in \{\mathsf{T}, \mathsf{F}\}$) such that $\mathcal{S} \overset{\mathsf{a}}{+} \{\lambda_1 : \bullet_1, \lambda_2 : \bullet_2, \ldots, \lambda_k : \bullet_k\}$ is HK$^+$-consistent. □

Lemma 4.3 (Fulfillment of $\Box$) *If a diagram $\mathcal{S}$ of Figure 4 is HK$^+$-consistent, then also the diagram $\mathcal{T}$ of Figure 4 is HK$^+$-consistent. (In the Figure, $\mathcal{U}$, $\mathcal{V}_1$, $\mathcal{V}_2$, …, $\mathcal{V}_n$ are subdiagrams, where $\mathcal{U}$ may be null and $n \geq 0$ — this means that the node* a *may be the root or a leaf.) The process of making $\mathcal{T}$ from $\mathcal{S}$ will be called "fulfillment of $\Box\varphi : \mathsf{F}$ for* a*", and the added node* b *will be called the "witness node".*

Fig. 4. Diagrams $\mathcal{S}$ and $\mathcal{T}$ of Lemma 4.3.

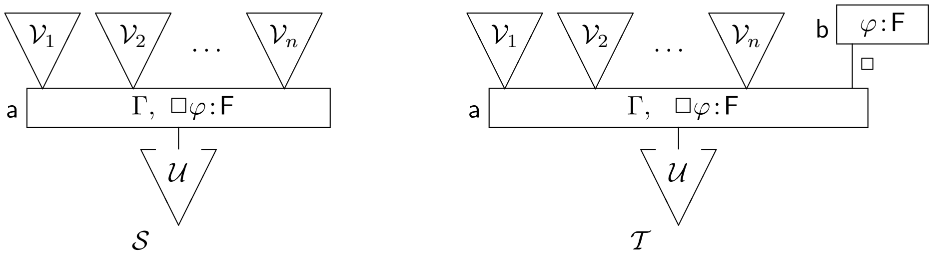
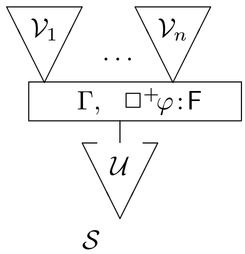


Fig. 5. Diagram $\mathcal{S}$ of Proposition 4.4.



Proof. Neg($\mathcal{S}$) and Neg($\mathcal{T}$) are provably equivalent. □

Let us explain the outline and the point of our completeness proof.

The goal is to construct a finite countermodel for a given unprovable formula α_0. When α_0 does not contain the operator $\Box^+$, the argument is equivalent to the well-known completeness proof for the smallest normal modal logic K, and it is arranged as follows. If $\not\vdash \alpha_0$, then the one-node diagram $\{\alpha_0 : \mathsf{F}\}$ is HK$^+$-consistent. We extend it by iterated applications of maximalization (Lemma 4.2) and fulfillment of $\Box$ (Lemma 4.3), and we eventually get a "saturated diagram" $\mathcal{T}$. Then a model $M = \langle W, R, V\rangle$ is defined by: W is the set of nodes in $\mathcal{T}$; $R = <^{\Box}$; and $V(\mathsf{a}, p) = \mathsf{True} \iff p \in_{\mathsf{T}} \mathsf{a}$. This is the required countermodel, because "$\varphi \in_{\mathsf{T}} \mathsf{a} \Rightarrow M, \mathsf{a} \models \varphi$" and "$\varphi \in_{\mathsf{F}} \mathsf{a} \Rightarrow M, \mathsf{a} \not\models \varphi$" hold for any φ, and the root contains $\alpha_0 : \mathsf{F}$.

When α_0 contains both the operators $\Box$ and $\Box^+$, we need additional constructions to fulfill $\Box^+\varphi : \mathsf{F}$. There are two naive and unsuccessful ways for this. After showing these *bad* ways, we will present our *good* way, which enables us to make a witness of $\Box^+\varphi : \mathsf{F}$ in an HK$^+$-consistent diagram.

The first way uses the following proposition corresponding to Lemma 4.3.

Proposition 4.4 *If a diagram $\mathcal{S}$ of Figure 5 is HK$^+$-consistent, then at least one of the diagram $\mathcal{T}_i$ of Figure 6 is HK$^+$-consistent. Note that Figure 6 contains infinitely many diagrams.*

In this way, we are faced with a difficulty in proving Proposition 4.4. Of course we can prove this proposition using the soundness and completeness of HK$^+$; however, we

Fig. 6. Diagrams $\mathcal{T}_1, \mathcal{T}_2, \ldots$ of Proposition 4.4.

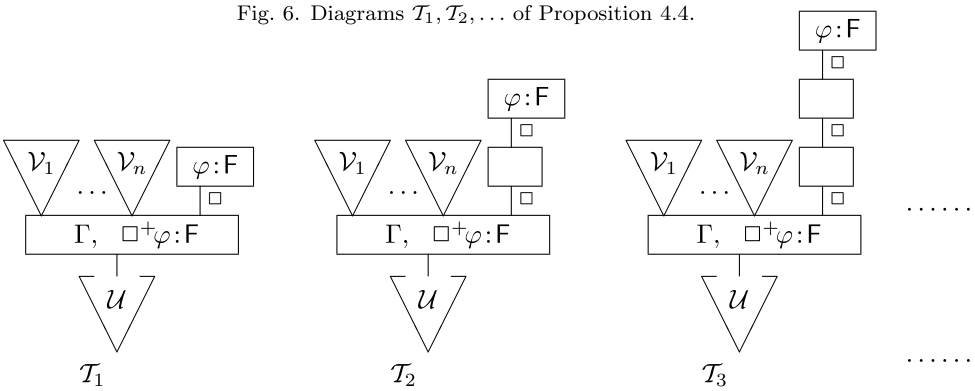
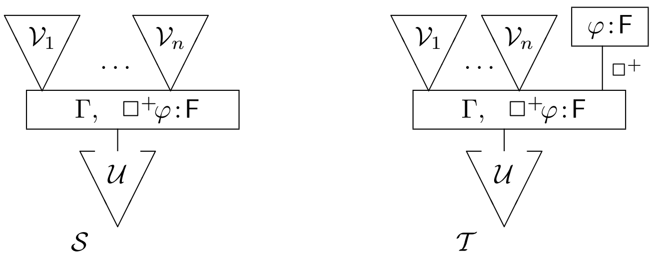


Fig. 7. Diagrams $\mathcal{S}$ and $\mathcal{T}$ of Proposition 4.5.



are now in course of proving completeness theorem.

The second way uses the following proposition.

Proposition 4.5 *If a diagram $\mathcal{S}$ of Figure 7 is HK$^+$-consistent, then also the diagram $\mathcal{T}$ of Figure 7 is HK$^+$-consistent.*

This proposition is easily proved in contrast to Proposition 4.4; however, we are faced with another difficulty in making a countermodel — we cannot define a well-behaved accessibility relation on the saturated diagram based on Proposition 4.5.

Then the following lemma is the third and successful way, which is the main contribution of this paper. This is done by enumerating all possible candidates of paths to the witness (called "special paths" below), as (2) in Section 1.

Lemma 4.6 (Fulfillment of $\Box^+$) *Let $\mathbb{L} = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ ($k \geq 1$) be a finite set of formulas. If a diagram $\mathcal{S}$ of Figure 8 is HK$^+$-consistent and Γ_1 is a valuation of $\mathbb{L}$, then there exist valuations $\Gamma_2, \Gamma_3, \ldots, \Gamma_m$ of $\mathbb{L}$ for some $m \geq 1$ such that the diagram $\mathcal{T}$ of Figure 8 is HK$^+$-consistent. The process of making $\mathcal{T}$ from $\mathcal{S}$ will be called "fulfillment of $\Box^+\varphi$:* F *for* a *with respect to $\mathbb{L}$", and the top node* b *will be called the "witness node".*

Proof. We say that a diagram is a *special path from Γ_1 to φ*:F if and only if it is of the form as in Figure 9 for some valuations $\Gamma_2, \ldots, \Gamma_m$ of $\mathbb{L}$ ($m \geq 1$) such that $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are mutually distinct. There are finitely many distinct valuations of $\mathbb{L}$, say $\Lambda_1, \Lambda_2 \ldots, \Lambda_N$

Fig. 8. Diagrams $\mathcal{S}$ and $\mathcal{T}$ of Lemma 4.6

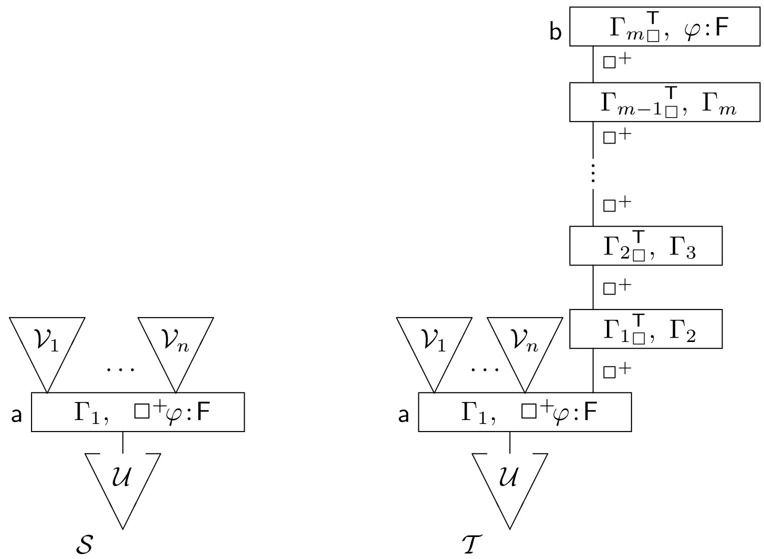
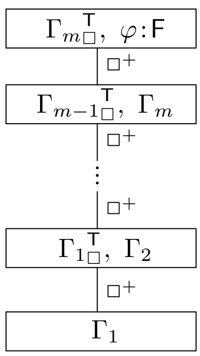


Fig. 9. Special path from Γ_1 to $\varphi{:}\mathsf{F}$



($N = 2^k \geq 2$ because $k \geq 1$); therefore the number of all special paths from Γ_1 to $\varphi{:}\mathsf{F}$ is also finite. Then let $\{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_P\}$ be the set of special paths. Now we will show

$$\vdash \{\mathrm{Neg}(\mathcal{W}_1), \mathrm{Neg}(\mathcal{W}_2), \ldots, \mathrm{Neg}(\mathcal{W}_P)\} \Rightarrow \mathrm{Neg}(\Gamma_1, \Box^+\varphi{:}\mathsf{F}). \tag{13}$$

The negation of a special path is provably equivalent to the formula

$$\langle\Gamma_1\rangle \to \Box^+\Big(\langle\Gamma_{1\Box}^{\mathsf{T}}\rangle \to \langle\Gamma_2\rangle \to \Box^+\Big(\cdots \to \Box^+\Big(\langle\Gamma_{m-1\Box}^{\mathsf{T}}\rangle \to \langle\Gamma_m\rangle \to \Box^+\Big(\langle\Gamma_{m\Box}^{\mathsf{T}}\rangle \to \varphi\Big)\Big)\Big)\Big),$$

and the formula $\mathrm{Neg}(\Gamma_1, \Box^+\varphi{:}\mathsf{F})$ is provably equivalent to

$$\langle\Gamma_1\rangle \to \Box^+\varphi.$$

Moreover we have

$$\vdash \langle\Lambda_1\rangle \vee \langle\Lambda_2\rangle \vee \cdots \vee \langle\Lambda_N\rangle$$

because this formula is a tautology. Using these facts and (12) (before Lemma 4.1), we can apply Theorem 3.3 ($\{\sigma_1, \sigma_2, \ldots, \sigma_N\} = \{\langle\Lambda_1\rangle, \langle\Lambda_2\rangle, \ldots, \langle\Lambda_N\rangle\}$, $\sigma_1 = \langle\Gamma_1\rangle$, $\sigma_{f(i)} = \langle\Gamma_i\rangle$, $\tau_{f(i)} = \langle\Gamma_{i\Box}^{\mathsf{T}}\rangle$, $\omega = \varphi$), and we get (13). Now the HK$^+$-consistent diagram $\mathcal{S}$ of Figure 8 is equivalent to $\mathcal{S} \overset{\mathsf{a}}{+} \{\Gamma_1, \Box^+\varphi : \mathsf{F}\}$. Then (13) and Lemma 4.1 imply that there is a special path $\mathcal{W}$ such that $\mathcal{S} \overset{\mathsf{a}}{+} \mathcal{W}$ is HK$^+$-consistent. □

Remarks on Lemma 4.6.
(1) If the set $\mathbb{L}$ is closed under subformulas and $\mathcal{T}$ is HK$^+$-consistent, then it must be the case that $\Gamma_{i\Box}^{\mathsf{T}} \subseteq \Gamma_{i+1}$ in the node $\{\Gamma_{i\Box}^{\mathsf{T}}, \Gamma_{i+1}\}$.
(2) Special paths consist of not $\Box$-edges, but $\Box^+$-edges, since the origin of a $\Box^+$-edge is the R^+-edge between $\{\Sigma_{a_i}\}$ and $\{\Sigma_{a_{(i+1)}}\}$ in the long path (3) from Section 1. On the other hand, the $\Box^+$-edges will become not R^+-edges but R-edges in the countermodel below. This one-step reachability is justified by the connection between Γ_i and $\Gamma_{i\Box}^{\mathsf{T}}$.

Now let us fix a formula α_0, for which we are going to construct a countermodel. The set of subformulas of α_0 is called Sub(α_0). We define some conditions on a node a of semantic diagrams as follows.

[Sub(α_0)**-maximality**] $\varphi \in \text{Sub}(\alpha_0) \iff (\varphi \in_{\mathsf{T}} \mathsf{a} \text{ or } \varphi \in_{\mathsf{F}} \mathsf{a})$.

[**□-correctness**] If $\Box\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} < \mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$.

[**□-witness property**] If $\Box\varphi \in_{\mathsf{F}} \mathsf{a}$, then the following condition holds.

$$\exists \mathsf{b}\big(\mathsf{a} <^{\Box} \mathsf{b} \text{ and } \varphi \in_{\mathsf{F}} \mathsf{b}\big). \tag{♠}$$

[**□$^+$-witness property**] If $\Box^+\varphi \in_{\mathsf{F}} \mathsf{a}$, then the following condition holds.

$$\exists m \geq 1, \exists \mathsf{b}_1, \exists \mathsf{b}_2, \ldots, \exists \mathsf{b}_m\big(\mathsf{a} <^{\Box^+} \mathsf{b}_1 <^{\Box^+} \mathsf{b}_2 <^{\Box^+} \cdots <^{\Box^+} \mathsf{b}_m \text{ and } \varphi \in_{\mathsf{F}} \mathsf{b}_m\big). \tag{♣}$$

We say that a node x is *set-fresh* if and only if the condition ($\mathsf{y} \ll \mathsf{x} \Rightarrow \text{Set}(\mathsf{y}) \neq \text{Set}(\mathsf{x})$) holds for any node y. The following is called the *diagram-model condition for $\mathcal{T}$ with respect to* Sub(α_0), which is the key notion of our completeness proof.

- $\mathcal{T}$ is HK$^+$-consistent;
- all nodes of $\mathcal{T}$ are Sub(α_0)-maximal and □-correct; and
- all set-fresh nodes of $\mathcal{T}$ satisfy □-witness and □$^+$-witness properties.

Lemma 4.7 *If $\not\vdash \alpha_0$, then there exists a semantic diagram $\mathcal{T}$ such that the diagram-model condition holds with respect to* Sub(α_0) *and the root contains the signed formula* $\alpha_0 : \mathsf{F}$.

Proof. We define a procedure to construct semantic diagrams $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2 \ldots$, such that $\mathcal{T}_i$ is HK$^+$-consistent and all the nodes of $\mathcal{T}_i$ are Sub(α_0)-maximal and □-correct.

[Construction of $\mathcal{T}_0$]
The one-node diagram $\{\alpha_0 : \mathsf{F}\}$ is HK$^+$-consistent because $\nvdash \alpha_0$. We apply the maximalization with respect to $\mathrm{Sub}(\alpha_0)$ (Lemma 4.2). Then we obtain a diagram whose only node is $\mathrm{Sub}(\alpha_0)$-maximal and contains $\alpha_0 : \mathsf{F}$. This is the diagram $\mathcal{T}_0$.

[Construction of $\mathcal{T}_{i+1}$ from $\mathcal{T}_i$]
If $\mathcal{T}_i$ satisfies the diagram-model condition with respect to $\mathrm{Sub}(\alpha_0)$, then we stop the procedure and we get the required diagram. Otherwise there is a node, say a, which is set-fresh, but the $\Box$-witness (or $\Box^+$-witness) property fails; that is, there is a formula $\Box\varphi$ (or $\Box^+\varphi$) $\in_{\mathsf{F}} \mathsf{a}$ such that the condition ♠ (or ♣) does not hold. Then we apply the fulfillment (Lemma 4.3 or 4.6) of $\Box\varphi : \mathsf{F}$ (or $\Box^+\varphi : \mathsf{F}$) for a and maximalization (Lemma 4.2) with respect to $\mathrm{Sub}(\alpha_0)$ for the witness node (the other nodes are already maximal), and the resulting diagram is $\mathcal{T}_{i+1}$. The node a will be called a *growing point*. Note that all the nodes of $\mathcal{T}_{i+1}$ satisfy $\Box$-correctness; here we show some cases: (Case 1) If $\Box\psi : \mathsf{T}$ is in the growing point of fulfillment of $\Box\varphi : \mathsf{F}$, then $\psi : \mathsf{T}$ must be in the maximalized witness node, say b; otherwise $(\psi : \mathsf{F}) \in \mathsf{b}$ and the diagram would be HK$^+$-inconsistent because $\vdash \neg\Box\psi \vee \Box(\psi \vee \cdots)$. (Case 2) If $\Box\psi : \mathsf{T}$ is in a node $\{\Gamma_{j}{}^{\mathsf{T}}_{\Box}, \Gamma_{j+1}\}$ in the special path of fulfillment of $\Box^+\varphi : \mathsf{F}$, then $(\Box\psi : \mathsf{T}) \in \Gamma_{j+1}$ (otherwise $(\Box\psi : \mathsf{F}) \in \Gamma_{j+1}$ and the diagram would be HK$^+$-inconsistent), and then $\psi : \mathsf{T}$ is in the next node $\{\Gamma_{j+1}{}^{\mathsf{T}}_{\Box}, \Gamma_{j+2}\}$.

We show that the above procedure must terminate, and hence we eventually get the required diagram. In fact, otherwise an infinite sequence $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2 \ldots$ is produced. Then consider the infinite diagram $\bigcup_{i=0}^{\infty} \mathcal{T}_i$. This infinite tree is finite branching because we can apply at most p times fulfillment for each growing point where p is the number of $\Box$- or $\Box^+$- formulas in $\mathrm{Sub}(\alpha_0)$. Therefore there is an infinite path which contains infinitely many growing points; however this is impossible because each growing point must be set-fresh and the number of set-fresh nodes in one path cannot be greater than $2^{|\mathrm{Sub}(\alpha_0)|}$. □

Lemma 4.8 *If a semantic diagram $\mathcal{T}$ satisfies the diagram-model condition with respect to* $\mathrm{Sub}(\alpha_0)$*, then the following hold for any node* a *of* $\mathcal{T}$*. (1) If $\varphi \in_{\mathsf{F}} \mathsf{a}$, then $\varphi \notin_{\mathsf{T}} \mathsf{a}$. (2) If $\varphi \wedge \psi \in_{\mathsf{T}} \mathsf{a}$, then $\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\psi \in_{\mathsf{T}} \mathsf{a}$. (3) If $\varphi \wedge \psi \in_{\mathsf{F}} \mathsf{a}$, then $\varphi \in_{\mathsf{F}} \mathsf{a}$ or $\psi \in_{\mathsf{F}} \mathsf{a}$. (4) If $\neg\varphi \in_{\mathsf{T}} \mathsf{a}$, then $\varphi \in_{\mathsf{F}} \mathsf{a}$. (5) If $\neg\varphi \in_{\mathsf{F}} \mathsf{a}$, then $\varphi \in_{\mathsf{T}} \mathsf{a}$. (6) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} < \mathsf{b}$, then $\Box^+\varphi \in_{\mathsf{T}} \mathsf{b}$ and $\varphi \in_{\mathsf{T}} \mathsf{b}$.*

Proof. We check only the clause (6), which is divided into the following four: (6-1) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} <^{\Box} \mathsf{b}$, then $\Box^+\varphi \in_{\mathsf{T}} \mathsf{b}$. (6-2) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} <^{\Box} \mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$. (6-3) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} <^{\Box^+} \mathsf{b}$, then $\Box^+\varphi \in_{\mathsf{T}} \mathsf{b}$. (6-4) If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a} <^{\Box^+} \mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$. The clause (6-1) is verified as follows. If $\Box^+\varphi \in_{\mathsf{T}} \mathsf{a}$, $\mathsf{a} <^{\Box} \mathsf{b}$, and $\Box^+\varphi \notin_{\mathsf{T}} \mathsf{b}$, then $\Box^+\varphi \in_{\mathsf{F}} \mathsf{b}$ by $\mathrm{Sub}(\varphi_0)$-maximality, and then $\mathcal{T}$ would be HK$^+$-inconsistent because $\vdash \neg\Box^+\varphi \vee \Box(\Box^+\varphi \vee \cdots)$ ($\because \vdash \Box^+\varphi \to \Box\Box^+\varphi$). The clauses (6-2), (6-3) and (6-4) are considered similarly using the facts $\vdash \neg\Box^+\varphi \vee \Box(\varphi \vee \cdots)$ ($\because \vdash \Box^+\varphi \to \Box\varphi$), $\vdash \neg\Box^+\varphi \vee \Box^+(\Box^+\varphi \vee \cdots)$ ($\because \vdash \Box^+\varphi \to \Box^+\Box^+\varphi$), and $\vdash \neg\Box^+\varphi \vee \Box^+(\varphi \vee \cdots)$ ($\because \vdash \Box^+\varphi \to \Box^+\varphi$). □

Theorem 4.9 (Completeness of HK$^+$ with respect to finite models) *If* $\nvdash \alpha_0$, *then there exists a finite model M such that $M, x \not\models \alpha_0$ for some world x.*

Proof. Let $\mathcal{T}$ be the diagram obtained by Lemma 4.7. We define $M = \langle W, R, V \rangle$ as follows.

- W is the set of nodes in $\mathcal{T}$.
- $\mathsf{a}R\mathsf{b} \iff \mathsf{a} < \mathsf{b}$ or $\exists \mathsf{a}_0 \big(\mathsf{a}_0 \ll \mathsf{a},\ \mathrm{Set}(\mathsf{a}_0) = \mathrm{Set}(\mathsf{a}),\ \text{and } \mathsf{a}_0 < \mathsf{b}\big)$.
- $V(\mathsf{a}, p) = \mathsf{True} \iff p \in_{\mathsf{T}} \mathsf{a}$.

Using the diagram-model condition of $\mathcal{T}$ and (6) of Lemma 4.8, we can show the following:

(i) If $\Box\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a}R\mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$.

(ii) If $\Box\varphi \in_{\mathsf{F}} \mathsf{a}$, then there is a node b such that $\mathsf{a}R\mathsf{b}$ and $\varphi \in_{\mathsf{F}} \mathsf{b}$.

(iii) If $\Box^{+}\varphi \in_{\mathsf{T}} \mathsf{a}$ and $\mathsf{a}R^{+}\mathsf{b}$, then $\varphi \in_{\mathsf{T}} \mathsf{b}$.

(iv) If $\Box^{+}\varphi \in_{\mathsf{F}} \mathsf{a}$, then there is a node b such that $\mathsf{a}R^{+}\mathsf{b}$ and $\varphi \in_{\mathsf{F}} \mathsf{b}$.

Then we have "$\varphi \in_{\mathsf{T}} \mathsf{a} \Rightarrow M, \mathsf{a} \models \varphi$" and "$\varphi \in_{\mathsf{F}} \mathsf{a} \Rightarrow M, \mathsf{a} \not\models \varphi$", which are proved by induction on φ using (1)–(5) of Lemma 4.8 and (i)–(iv) above. M is the required model because the root of $\mathcal{T}$ contains $\alpha_0 : \mathsf{F}$. □

5 Concluding remarks

This paper gives a new proof of the completeness theorem for the Hilbert style system of the propositional modal logic with two operators $\Box$ and $\Box^{+}$. Our method is "semantic diagram", and the point is how to construct the witness of $\neg\Box^{+}\varphi$. We enumerate all the possible candidates of paths to the witness ("special paths"), and search them using the Hilbert system as an oracle. The two-facedness of $\Box^{+}$-edges (Remark (2) on Lemma 4.6) is also remarkable.

A feature of our method is that we do not need extra operators other than $\Box$ and $\Box^{+}$. If the 'until' operator is allowed, there may be another possible way as in Brünnler and Lange [1] and Gaintzarain et al. [3]. Although our method seems to be ineffective for more complex logics like modal μ-calculus, it may be useful for certain logics without the 'until' operator, for example, epistemic logics.

Acknowledgement

I give great thanks to anonymous reviewers for their valuable comments on an earlier version of this paper.

References

[1] Kai Brünnler and Martin Lange. Cut-free sequent systems for temporal logic. *J. Log. Algebr. Program.*, 76(2):216–225, 2008.

[2] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning about Knowledge*. MIT Press, 1995.

[3] Joxe Gaintzarain, Montserrat Hermo, Paqui Lucio, Marisa Navarro, and Fernando Orejas. A cut-free and invariant-free sequent calculus for PLTL. In Jacques Duparc and Thomas A. Henzinger, editors, *CSL*, volume 4646 of *Lecture Notes in Computer Science*, pages 481–495. Springer, 2007.

[4] Robert Goldblatt. *Logics of Time and Computation.* Number 7 in CSLI Lecture Notes. Center for the Study of Language and Information, Stanford, CA, 2. edition, 1992.

A Proof of Lemma 3.2

Define formulas $\alpha, \beta, \gamma, \delta, \delta', \delta'', \varepsilon, \varepsilon', \varepsilon'', \zeta, \zeta'$:

$\alpha = \Box\tau.$ $\quad \beta = \Box^+(\sigma' \to \Box\tau').$ $\quad \gamma = \Box^+(\neg\sigma' \to \Box\tau).$
$\delta = \Box^+(\tau \to \omega).$ $\quad \delta' = \Box^+\Box(\tau \to \omega).$ $\quad \delta'' = \Box(\tau \to \omega).$
$\zeta = \sigma' \to \Box^+(\tau' \to \omega).$ $\quad \zeta' = \sigma' \to \Box(\tau' \to \omega).$
$\varepsilon = \Box^+(\tau \to \zeta).$ $\quad \varepsilon' = \Box(\tau \to \zeta).$ $\quad \varepsilon'' = \Box^+(\Box\tau \to \Box\zeta).$

An outline of the proof is as follows.

(i) $\vdash \Box^+(\tau' \to \omega) \to \Box\Box^+(\tau' \to \omega).$ $\quad (\because$ A4)

(ii) $\vdash \Box^+(\tau' \to \omega) \to \Box\zeta.$ $\quad (\because$ i)

(iii) $\vdash \Box^+(\Box^+(\tau' \to \omega) \to \Box\zeta).$ $\quad (\because$ ii, R2),

(iv) $\vdash \{\alpha,\ \varepsilon'\} \Rightarrow \Box\zeta.$ $\quad (\because$ A2)

(v) $\vdash \{\gamma,\ \varepsilon''\} \Rightarrow \Box^+(\neg\sigma' \to \Box\zeta).$ $\quad (\because$ Lemma 3.1(4))

(vi) $\vdash \{\gamma,\ \varepsilon''\} \Rightarrow \Box^+(\neg\sigma' \vee \Box^+(\tau' \to \omega) \to \Box\zeta).$ $\quad (\because$ v, iii)

(vii) $\vdash \{\gamma,\ \varepsilon''\} \Rightarrow \Box^+(\zeta \to \Box\zeta).$ $\quad (\because$ vi)

(viii) $\vdash \{\alpha,\ \gamma,\ \varepsilon',\ \varepsilon''\} \Rightarrow \Box^+\zeta.$ $\quad (\because$ iv, vii, A5)

(ix) $\vdash \{\alpha,\ \delta''\} \Rightarrow \Box\omega.$ $\quad (\because$ A2)

(x) $\vdash \{\Box\tau',\ \Box(\tau' \to \omega)\} \Rightarrow \Box\omega.$ $\quad (\because$ A2)

(xi) $\vdash \{\beta,\ \Box^+\zeta'\} \Rightarrow \Box^+(\sigma' \to \Box\omega).$ $\quad (\because$ x, Lemma 3.1(2,4))

(xii) $\vdash \{\Box\tau,\ \Box(\tau \to \omega)\} \Rightarrow \Box\omega.$ $\quad (\because$ A2)

(xiii) $\vdash \{\gamma,\ \delta'\} \Rightarrow \Box^+(\neg\sigma' \to \Box\omega).$ $\quad (\because$ xii, Lemma 3.1(2,4))

(xiv) $\vdash \{\beta,\ \gamma,\ \delta',\ \Box^+\zeta'\} \Rightarrow \Box^+\Box\omega.$ $\quad (\because$ xi, xiii)

(xv) $\vdash \{\alpha,\ \beta,\ \gamma,\ \delta',\ \delta'',\ \Box^+\zeta'\} \Rightarrow \Box^+\omega.$ $\quad (\because$ ix, xiv, A5)

(xvi) $\vdash \delta \to \delta', \quad \vdash \delta \to \delta'', \quad \vdash \varepsilon \to \varepsilon', \quad \vdash \varepsilon \to \varepsilon'', \quad \vdash \zeta \to \zeta'.$

(xvii) $\vdash \{\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon\} \Rightarrow \Box^+\omega.$ $\quad (\because$ viii, xv, xvi)

(xviii) $\vdash \Box^+(\sigma' \to \Box\tau').$ $\quad (\because$ (b), R2)

(xix) $\vdash \Box^+(\neg\sigma' \to \Box\tau).$ $\quad (\because$ (c), R2)

(xx) $\vdash \{\alpha,\ \delta,\ \varepsilon\} \Rightarrow \Box^+\omega.$ $\quad (\because$ xvii, xviii, xix)

(xxi) $\vdash \{\sigma \to \alpha,\ \sigma \to \delta,\ \sigma \to \varepsilon\} \Rightarrow \sigma \to \Box^+\omega.$ $\quad (\because$ xx, Lemma 3.1(2))

(xxii) $\vdash \{\sigma \to \Box^+(\tau \to \omega),\ \sigma \to \Box^+(\tau \to \sigma' \to \Box^+(\tau' \to \omega))\} \Rightarrow \sigma \to \Box^+\omega.$
$(\because$ (a), xxi)