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Completing the eclectic flavor scheme of the $\mathbb{Z}_2$ orbifold

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Abstract: We present a detailed analysis of the eclectic flavor structure of the two-dimensional $\mathbb{Z}_2$ orbifold with its two unconstrained moduli T and U as well as $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ modular symmetry. This provides a thorough understanding of mirror symmetry as well as the R-symmetries that appear as a consequence of the automorphy factors of modular transformations. It leads to a complete picture of local flavor unification in the (T, U) modulus landscape. In view of applications towards the flavor structure of particle physics models, we are led to top-down constructions with high predictive power. The first reason is the very limited availability of flavor representations of twisted matter fields as well as their (fixed) modular weights. This is followed by severe restrictions from traditional and (finite) modular flavor symmetries, mirror symmetry, $\mathcal{CP}$ and R-symmetries on the superpotential and Kähler potential of the theory.

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Contents

1 Introduction

In this paper we extend our previous discussion [1] of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold. $\mathbb{T}^2/\mathbb{Z}_2$ is the only two-dimensional orbifold with two unconstrained moduli T, U that transform under $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ and under mirror symmetry, which interchanges T and U. Hence, it can serve as a building block for the discussion of six-dimensional orbifolds.[1] In our previous study, ref. [1], we had identified the traditional flavor symmetries and the finite modular symmetries Γ_N for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold. The groups Γ_N (for small N) are isomorphic to groups like S_3, A_4, S_4 and A_5 that could be suitable for a description of discrete flavor symmetries in particle physics [7–9]. Modular symmetries, however, require more than just a discussion of the finite modular groups Γ_N. In addition, we have to include automorphy factors corresponding to the explicit modular weights of matter fields. In the present paper, we discuss the implications of these automorphy factors in the case of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold. Once they are taken into account, we find an extension of the finite modular flavor symmetry in form of an R-symmetry, which implies further restrictions to the superpotential and Kähler potential of the theory. This is one of the reasons why a modular flavor symmetry has more predictive power than traditional flavor symmetries. In the top-down approach (which we adopt here), this extension of the symmetry reflects the symmetries of the underlying string theory, which restrict the modular weights to well-defined specific values.[2] In the bottom-up approach to modular flavor symmetries, the choice of the modular weights of matter fields is part of model building and can be used to obtain so-called "shaping symmetries" that appear as additional accidental symmetries for specific choices of the modular weights [10, 11].

The main results of the paper can be summarized as follows:

- We identify the full eclectic flavor symmetry [12] of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold to be

$$[(D_8 \times D_8)/\mathbb{Z}_2] \;\cup\; \left[(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}}\right] \;\cup\; \mathbb{Z}_4^R\,. \tag{1.1}$$

  It includes a $\mathbb{Z}_4^R$ R-symmetry that originates from the discussion of the automorphy factors and extends the order of the eclectic flavor group from 2304 to 4608. With $\mathcal{CP}$, the order of the eclectic flavor symmetry is further enhanced to a group of order 9216.

[1] For a general discussion of $\mathbb{T}^2/\mathbb{Z}_K$ including $K > 2$, we refer to refs. [2, 3] and [4–6] for $K = 3$.

[2] For a previous discussion on the modular weights of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold, see refs. [2, 3, 6]. Although in this case the modulus U is fixed, there remains a discrete R-symmetry from $\mathrm{SL}(2,\mathbb{Z})_U$.

- We provide a discussion of the *landscape* of flavor symmetries in (T, U)-moduli space and identify the local unified flavor groups at specific points and lines in this moduli space. The results are given in figure 3, accompanied by an explicit discussion of the flavor symmetries in the cases of two specific geometrical shapes (the tetrahedron and the squared raviolo) as well as $T \leftrightarrow U$ mirror symmetry in section 4.

- We observe a specific relation between mirror symmetry and the allowed values of modular weights of matter fields (discussed explicitly in section 3).

- The additional R-symmetry is closely related to the modular symmetry and leads to further constraints on the allowed values of modular weights of matter fields. Hence, it further restricts the form of superpotential and Kähler potential, as explicitly discussed in section 5.

- We discover the appearance of continuous gauge symmetries for specific configurations in moduli space.

The paper is structured as follows. In section 2, we recall the results of our previous study [1]. Section 3 discusses the automorphy factors and modular weights of matter fields. We identify the additional R-symmetry and the extended eclectic flavor group accordingly. This includes a discussion of the interplay of the modular weights with both, $T \leftrightarrow U$ mirror symmetry and the R-symmetry. In section 4 we analyze the unified local flavor groups that appear at specific points, lines and other hyper-surfaces in moduli space. The results including $\mathcal{CP}$ are illustrated in figure 3. Section 5 is devoted to the discussion of the superpotential and Kähler potential. We observe the appearance of continuous gauge symmetries for certain configurations of the moduli (naïvely, they might appear as accidental symmetries, but they are consequences of underlying symmetries in string theory). In section 6 we give conclusions and outlook. Technical details are relegated to several appendices that complete the general discussion of ref. [1].

2 What do we know already?

Technical details of the eclectic flavor symmetries of $\mathbb{T}^2/\mathbb{Z}_K$ orbifolds ($K = 2, 3, 4, 6$) have been given in section 2 of ref. [3]. In the cases $K > 2$, the complex structure modulus U has to be fixed to allow for the orbifold twist. For $\mathbb{T}^2/\mathbb{Z}_2$, in contrast, we have two unconstrained moduli T and U with the corresponding modular transformations $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$. For generic values of the moduli, we find the traditional flavor symmetry (we use the Small Group notation from `GAP` [13])

$$(D_8 \times D_8)/\mathbb{Z}_2 \;\cong\; [32, 49] \tag{2.1}$$

as the result of geometry and string selection rules (see refs. [14, 15]) or, equivalently, as a result of the outer automorphisms of the (Narain) space group that describes the orbifold [4, 5]. Furthermore, the finite modular symmetry for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold is shown to be the multiplicative closure of $\Gamma_2^T \times \Gamma_2^U = S_3^T \times S_3^U$ and mirror symmetry (which exchanges T and U), as discussed in ref. [1]. The full mirror symmetry acting on the matter fields turns out to be $\mathbb{Z}_4^{\hat{M}}$ (which acts on the moduli as $\mathbb{Z}_2$, cf. [16]). This leads to the finite modular group [144, 115]. If we include a $\mathcal{CP}$-like symmetry acting on the moduli as $T \to -\overline{T}$ and $U \to -\overline{U}$, the finite modular group enhances to

$$\left[(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}}\right] \times \mathbb{Z}_2^{\mathcal{CP}} \cong [288, 880]\,. \tag{2.2}$$

In combination with the generators of the traditional flavor symmetry [32, 49], we obtained an eclectic flavor group with 4608 elements (2304 without $\mathcal{CP}$).

Only some of the eclectic flavor symmetries are linearly realized. For generic values of the moduli just the traditional flavor group [32, 49] remains unbroken. For specific "geometrical" configurations, this symmetry is enhanced to a larger subgroup of the eclectic flavor group (via the so-called stabilizer subgroups). The generators of the unbroken groups are displayed explicitly in figure 7 of ref. [1]. Relevant values correspond to the moduli $\langle U\rangle = \mathrm{i}$ (the squared raviolo) and $\langle U\rangle = \exp(\pi\mathrm{i}/3)$ (the tetrahedron) as well as the line $\langle T\rangle = \langle U\rangle$ as a consequence of mirror symmetry. At $\langle T\rangle = \langle U\rangle$, we find the enhancement of [32, 49] to [64, 257]. For the tetrahedron, the group [32, 49] is enhanced to [96, 204] as discussed in section 4.2 of ref. [1], while for the raviolo, we shall see here in section 4 that [32, 49] is enhanced to [128, 523]. If we include the $\mathcal{CP}$-like transformation, we gain a further enhancement of the number of elements by a factor of two. The largest linearly realized subgroup of the eclectic flavor group (including $\mathcal{CP}$) was found (in ref. [1]) to be [1152, 157463] at $\langle T\rangle = \langle U\rangle = \exp(\pi\mathrm{i}/3)$.

So far the results are based on the finite modular groups. A full discussion of modular symmetries should, however, not only include the finite symmetries Γ_N (here $\Gamma_2^T \times \Gamma_2^U = S_3^T \times S_3^U$), but also the so-called automorphy factors that arise from the non-trivial (fractional) modular weights (n_T, n_U) of $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$. This leads to further restrictions on the action (given by Kähler and superpotential) of the theory with an enhancement of the symmetries. As discussed in refs. [2, 3], these automorphy factors lead to discrete phases resulting in R-symmetries. In our previous paper [1], for the sake of clarity and simplicity, we had not included these automorphy factors in our discussion. We shall include them in the following in full detail.

3 Discrete R-symmetries and mirror symmetry

In this section, we show that a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector gives rise to a $\mathbb{Z}_4^R$ symmetry that originates from modular transformations, where the automorphy factors of certain modular transformations give rise to the R-charges. As the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector is equipped with two moduli, T and U, there exists a modular group for each of them, $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$, each associated with a modular weight (n_T, n_U). Since R-charges can be defined in terms of both modular groups, these modular weights are highly constrained. Furthermore, we give a detailed discussion about the action of mirror symmetry on matter fields and discover a new relation between mirror symmetry and the R-symmetry.

3.1 Automorphy factors of modular transformations

Let us consider a general matter field $\Phi_{(n_T,n_U)}$ originating from string theory with modular weights n_T and n_U corresponding to $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$. Then, under a (non $\mathcal{CP}$-like) modular transformation $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$, the field transforms as

$$\Phi_{(n_T,n_U)} \xmapsto{\hat{\Sigma}} j^{(n_T,n_U)}(\hat{\Sigma},T,U)\, \rho_{\boldsymbol{r}}(\hat{\Sigma})\, \Phi_{(n_T,n_U)}\,. \tag{3.1}$$

Here, $j^{(n_T,n_U)}(\hat{\Sigma},T,U)$ is the automorphy factor of the modular transformation and $\rho_{\boldsymbol{r}}(\hat{\Sigma})$ is the representation matrix of $\hat{\Sigma}$ that forms a representation $\boldsymbol{r}$ of the finite modular group, as derived in appendix D.

The modular weights of matter fields can be computed in string theory, as reviewed in appendix B, and it turns out that, apart from $n_T = n_U$, there is also the possibility

$$n_T \neq n_U \tag{3.2}$$

in string theory. In order to determine the automorphy factor $j^{(n_T,n_U)}(\hat{\Sigma},T,U)$, we might use as a first step the analogy to Siegel modular forms based on $\mathrm{Sp}(4,\mathbb{Z})$. However, Siegel modular forms are defined for parallel weights $n := n_T = n_U$ only. In this case, following refs. [16, 17], we have

$$j^{(n)}(M,T,U) = \big(\det(C\,\Omega + D)\big)^n \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \text{ and } \Omega = \begin{pmatrix} U & 0 \\ 0 & T \end{pmatrix}. \tag{3.3}$$

$\mathrm{Sp}(4,\mathbb{Z})$ contains $\mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{SL}(2,\mathbb{Z})_U$ via the element $M_{(\gamma_T,\gamma_U)} \in \mathrm{Sp}(4,\mathbb{Z})$, where

$$\gamma_T := \begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_T \quad \text{and} \quad \gamma_U := \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_U\,, \tag{3.4}$$

as defined in ref. [18]. Then, eq. (3.3) yields

$$j^{(n)}(M_{(\gamma_T,\gamma_U)},T,U) = (c_T\,T + d_T)^n\,(c_U\,U + d_U)^n\,. \tag{3.5}$$

Using the dictionary [18] that relates $\mathrm{Sp}(4,\mathbb{Z})$ with the modular group $\mathrm{O}_{\hat{\eta}}(2,2{+}16,\mathbb{Z})$ of our string setting, the $\mathrm{Sp}(4,\mathbb{Z})$ element $M_{(\gamma_T,\gamma_U)}$ is equivalent to $\hat{\Sigma}_{(\gamma_T,\gamma_U)} \in \mathrm{O}_{\hat{\eta}}(2,2+16,\mathbb{Z})$ defined in appendix A.1. This is consistent with the string setup discussed in ref. [19], resulting in

$$j^{(n_T,n_U)}\left(\hat{\Sigma}_{(\gamma_T,\gamma_U)},T,U\right) := (c_T\,T + d_T)^{n_T}\,(c_U\,U + d_U)^{n_U}\,, \tag{3.6}$$

also for the case $n_T \neq n_U$. It turns out that for the other cases $\hat{\Sigma} \neq \hat{\Sigma}_{(\gamma_T,\gamma_U)}$ one can use the automorphy factor eq. (3.3) for the element $M \in \mathrm{Sp}(4,\mathbb{Z})$ that corresponds to $\hat{\Sigma}$ using again the dictionary of ref. [18]. It is important to note that the resulting automorphy factor will be independent of the specific choice $n = n_T$ or $n = n_U$, since $n_T = n_U \mod 2$. In the following, we will see this explicitly at some examples.

	bulk matter		twisted matter				
	$\Phi_{(0,0)}$	$\Phi_{(-1,-1)}$	$\Phi_{(-1/2,-1/2)}$	$\Phi_{(-3/2,\,1/2)}$	$\Phi_{(\,1/2,-3/2)}$	$Y^{(2)}_{\mathbf{4}_3}$	$\mathcal{W}$
traditional	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{4}$	$\mathbf{4}$	$\mathbf{4}$	$\mathbf{1}_0$	$\mathbf{1}_0$
modular	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{4}_1$	$(\mathbf{4}_1 \oplus \mathbf{4}_1)$		$\mathbf{4}_3$	$\mathbf{1}_0$
n_T	0	-1	$-1/2$	$-3/2$	$1/2$	2	-1
n_U	0	-1	$-1/2$	$1/2$	$-3/2$	2	-1
R-charge	0	2	3	1	1	0	$2 \mod 4$

Table 1. All admissible modular weights of massless matter fields $\Phi_{(n_T,n_U)}$ as well as their representations under the flavor symmetries and $\mathbb{Z}_4^R$ of a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, see appendix D. The $\mathbb{Z}_4^R$ R-charges are normalized to be integer for matter superfields $\Phi_{(n_T,n_U)}$. The traditional flavor group is $[64, 266]$ and the modular flavor group is $[144, 115]$.

3.2 Discrete R-symmetry

In the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, a $\mathbb{Z}_2$ sublattice rotation is given by

$$\hat{\Theta}_{(2)} := \hat{C}_{\mathrm{S}}^2 = \hat{K}_{\mathrm{S}}^2 = -\mathbb{1}_4 \;\in\; \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})\,, \tag{3.7}$$

i.e. in the Narain formulation, $\hat{\Theta}_{(2)} = -\mathbb{1}_4$ is a left-right symmetric 180° rotation in the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector that leaves the orthogonal compact dimensions invariant, see refs. [2, 3]. It is an outer automorphism of the full Narain space group of the six-dimensional orbifold and, hence, it is a symmetry of the theory. Using the definition of $\hat{\Sigma}_{(\gamma_T,\gamma_U)}$ given in eq. (A.19) of appendix A.1, this $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ transformation can be expressed in two ways as

$$\hat{\Theta}_{(2)} = \hat{\Sigma}_{(\mathrm{R},\mathbb{1}_2)} = \hat{\Sigma}_{(\mathbb{1}_2,\mathrm{R})}\,, \quad \text{where} \quad \mathrm{R} = \mathrm{S}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{Z})\,. \tag{3.8}$$

As the generalized metric $\mathcal{H}(T,U)$ is invariant under a transformation (A.29) with $\hat{\Theta}_{(2)}$, this $\mathbb{Z}_2$ sublattice rotation leaves T and U invariant. Hence, the sublattice rotation $\hat{\Theta}_{(2)}$ corresponds to a traditional flavor symmetry. Still, the transformation with $\hat{\Theta}_{(2)}$ originates from a modular transformation, $\hat{\Theta}_{(2)} = \hat{C}_{\mathrm{S}}^2 = \hat{K}_{\mathrm{S}}^2$. So, we expect the appearance of an automorphy factor. Since $\mathrm{R} \in \mathrm{SL}(2,\mathbb{Z})_T$ and $\mathrm{R} \in \mathrm{SL}(2,\mathbb{Z})_U$ are identified in $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$, we have to ensure that we compute the automorphy factor correctly: we can use either the factor $(c_T T + d_T)^{n_T} = (-1)^{n_T}$ or $(c_U U + d_U)^{n_U} = (-1)^{n_U}$ for the transformation R. Yet, the resulting automorphy factor must coincide in both cases, $(-1)^{n_T} = (-1)^{n_U}$. Hence, we see that

$$n_U \overset{!}{=} n_T \mod 2\,. \tag{3.9}$$

This relation is satisfied for all (massless) matter from the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, as one can see from table 1. Moreover, eq. (3.9) also holds for all massive strings, as shown in appendix B. Consequently, having control over the automorphy factor, we can choose $\mathrm{R} \in \mathrm{SL}(2,\mathbb{Z})_U$ and the modular weight n_U in the following.

The superpotential $\mathcal{W}$ transforms under $\hat{\Theta}_{(2)}$ as

$$\mathcal{W} \xmapsto{\hat{\Theta}_{(2)}} -\mathcal{W}\,, \tag{3.10}$$

due to the automorphy factor $(c_U U + d_U)^{-1} = -1$ evaluated for R given in eq. (3.8). Thus, the transformation $\hat{\Theta}_{(2)}$ generates a discrete R-symmetry [2, 3].

The action of $\hat{\Theta}_{(2)}$ on matter fields $\Phi_{(n_T,n_U)}$ with $\mathrm{SL}(2,\mathbb{Z})_U$ modular weights $n_U \in \{0, -1, -1/2, 1/2, -3/2\}$, as listed in table 1, is given by

$$\Phi_{(n_T,n_U)} \xmapsto{\hat{\Theta}_{(2)}} (-1)^{n_U}\, \rho_{\boldsymbol{r}}(\hat{\Theta}_{(2)})\, \Phi_{(n_T,n_U)} = (-1)^{n_U}\, \Phi_{(n_T,n_U)} =: \exp\left(2\pi\mathrm{i}\, R/4\right)\, \Phi_{(n_T,n_U)}\,. \tag{3.11}$$

Here, we used that $\rho_{\boldsymbol{r}}(\hat{\Theta}_{(2)}) = \rho_{\boldsymbol{r}}(\hat{K}_\mathrm{S})^2 = \rho_{\boldsymbol{r}}(\hat{C}_\mathrm{S})^2 = \mathbb{1}$. For the allowed modular weights $n_U \in \{0, -1, -1/2, 1/2, -3/2\}$ the multivalued phase factor gives rise to $\mathbb{Z}_4^R$ R-charges $R \in \{0, 2, 3, 1, 1\}$, respectively. Thus, for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector we find that the R-charge R is given in terms of the modular weight n_U (or n_T) as

$$R = 2\, n_U \mod 4 = 2\, n_T \mod 4\,, \tag{3.12}$$

cf. ref. [20]. Note that due to the fractional modular weights n_U, $(\hat{\Theta}_{(2)})^2$ gives a non-trivial transformation with charges $2R = 4n_U \mod 4$ that turns out to be equivalent to the point group selection rule of eq. (D.24a). Since the R-symmetry transformation acts trivially on all moduli, it belongs to the traditional flavor symmetry, which gets enhanced to

$$\frac{(D_8 \times D_8)\,/\mathbb{Z}_2\, \times\, \mathbb{Z}_4^R}{\mathbb{Z}_2} \;\cong\; [64, 266]\,, \tag{3.13}$$

where the $\mathbb{Z}_2$ in the latter quotient identifies the point group selection rule of $\mathbb{T}^2/\mathbb{Z}_2$ contained in both the $\mathbb{Z}_4^R$ and the traditional symmetry $(D_8 \times D_8)\,/\mathbb{Z}_2$. In string theory, modular symmetries are anomaly-free (see e.g. [19, 21] for details on anomaly cancellation for modular symmetries). Hence, since the $\mathbb{Z}_4^R$ R-symmetry arises from modular symmetries, it is anomaly-free.

Due to the $\mathbb{Z}_4^R$ R-symmetry, the eclectic flavor group of a $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector gets extended to

$$[(D_8 \times D_8)/\mathbb{Z}_2] \;\cup\; \left[(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}}\right] \;\cup\; \mathbb{Z}_4^R\,, \tag{3.14}$$

which results in a group of order 4608. Including a $\mathcal{CP}$-like transformation, the order of the eclectic flavor group is further enhanced to a group of order 9216.

3.3 The action of mirror symmetry on matter fields

In order to analyze the action of mirror symmetry $\hat{M} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ on matter fields in our string setup, we have to determine the automorphy factor first. Using the results of section 3.1, we consider the mirror element in $\mathrm{Sp}(4,\mathbb{Z})$, which reads

$$M_\times = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \;\in\; \mathrm{Sp}(4,\mathbb{Z})\,. \tag{3.15}$$

Thus, the automorphy factor (3.3) of a mirror transformation is given by $(-1)^n$. Since $n_T = n_U \bmod 2$ as derived in eq. (3.9), one can assign $n = n_U$ and the automorphy factor of a mirror transformation $\hat{M}$ is given as $(-1)^{n_U}$, without loss of generality. Moreover, note that the $\mathbb{Z}_4^R$ R-charges given in eq. (3.11) are analogously defined. Thus, the automorphy factor of $\hat{M}$ can be removed using the $\mathbb{Z}_4^R$ R-symmetry, as we will do in the following.

Now, let us assume that under a mirror transformation $\hat{M}$ we have

$$\Phi_{(n_T,n_U)} \xmapsto{\hat{M}} \rho_{\boldsymbol{r}}(\hat{M})\,\Phi_{(n_T,n_U)}\,, \tag{3.16}$$

for a matter field $\Phi_{(n_T,n_U)}$ that transforms in the representation $\boldsymbol{r}$ of the finite modular group $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144, 115]$ (where we have absorbed the automorphy factor $(-1)^{n_U}$ using $\mathbb{Z}_4^R$). In the following, we will see from a bottom-up perspective that the transformation (3.16) is correct for $n_T = n_U$, but must be modified in the case $n_T \neq n_U$. To do so, let us consider the following chain of transformations

$$\begin{aligned}
\Phi_{(n_T,n_U)} &\xmapsto{\hat{M}} \rho_{\boldsymbol{r}}(\hat{M})\,\Phi_{(n_T,n_U)} && \text{(3.17a)}\\
&\xmapsto{\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)}} (c_U\,U + d_U)^{n_U}\,\rho_{\boldsymbol{r}}(\hat{M})\,\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)})\,\Phi_{(n_T,n_U)} && \text{(3.17b)}\\
&\xmapsto{\hat{M}^{-1}} (c_U\,T + d_U)^{n_U}\,\rho_{\boldsymbol{r}}(\hat{M})\,\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)})\,\rho_{\boldsymbol{r}}(\hat{M})^{-1}\,\Phi_{(n_T,n_U)}\,, && \text{(3.17c)}
\end{aligned}$$

under the assumption that eq. (3.16) were correct. However, a mirror transformation maps an element $\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)} \in \mathrm{SL}(2,\mathbb{Z})_U$ to an element $\hat{\Sigma}_{(\gamma_U,\mathbb{1}_2)} \in \mathrm{SL}(2,\mathbb{Z})_T$, see eq. (A.25). Thus, eq. (3.17c) must be equal to

$$\Phi_{(n_T,n_U)} \xmapsto{\hat{\Sigma}_{(\gamma_U,\mathbb{1}_2)}} (c_U\,T + d_U)^{n_T}\,\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\gamma_U,\mathbb{1}_2)})\,\Phi_{(n_T,n_U)}\,, \tag{3.18}$$

where the 2×2 matrix $\gamma_T \in \mathrm{SL}(2,\mathbb{Z})_T$ in $\hat{\Sigma}_{(\gamma_T,\mathbb{1}_2)}$ has to be equal to the matrix γ_U used in eq. (3.17b). Now, in the case of so-called parallel weights (i.e. if $n_T = n_U$) the representation matrices $\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\gamma_U,\mathbb{1}_2)})$ and $\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)})$ have to be related as follows

$$\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\gamma_U,\mathbb{1}_2)}) = \rho_{\boldsymbol{r}}(\hat{M})\,\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)})\,\rho_{\boldsymbol{r}}(\hat{M})^{-1}\,, \tag{3.19}$$

and eq. (3.17c) coincides with eq. (3.18). In contrast, eqs. (3.17c) and (3.18) are inconsistent if $n_T \neq n_U$. Consequently, the (preliminary) chain of transformations given in eq. (3.17) has to be modified. The only possibility turns out to be

$$\begin{aligned}
\Phi_{(n_T,n_U)} &\xmapsto{\hat{M}} \rho_{\boldsymbol{r}}(\hat{M})\,\Phi_{(n_U,n_T)} && \text{(3.20a)}\\
&\xmapsto{\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)}} (c_U\,U + d_U)^{n_T}\,\rho_{\boldsymbol{r}}(\hat{M})\,\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)})\,\Phi_{(n_U,n_T)} && \text{(3.20b)}\\
&\xmapsto{\hat{M}^{-1}} (c_U\,T + d_U)^{n_T}\,\rho_{\boldsymbol{r}}(\hat{M})\,\rho_{\boldsymbol{r}}(\hat{\Sigma}_{(\mathbb{1}_2,\gamma_U)})\,\rho_{\boldsymbol{r}}(\hat{M})^{-1}\,\Phi_{(n_T,n_U)}\,. && \text{(3.20c)}
\end{aligned}$$

Then, we have to impose condition (3.19) and, consequently, eq. (3.20c) coincides with eq. (3.18) using $\hat{\Sigma}_{(\gamma_T,\mathbb{1}_2)}$ with γ_T equal to γ_U.

To summarize, for each matter field $\Phi_{(n_T,n_U)}$ with $n_T \neq n_U$ (satisfying the constraint (3.9)) there must exist a partner field, denoted by $\Phi_{(n_U,n_T)}$, which coincides in all properties with $\Phi_{(n_T,n_U)}$ but has interchanged modular weights. Then, a mirror transformation has to act on matter fields $(\Phi_{(n_T,n_U)}, \Phi_{(n_U,n_T)})$ as

$$\begin{pmatrix}\Phi_{(n_T,n_U)}\\ \Phi_{(n_U,n_T)}\end{pmatrix} \xmapsto{\hat{M}} \begin{pmatrix}0 & \rho_{\boldsymbol{r}}(\hat{M})\\ \rho_{\boldsymbol{r}}(\hat{M}) & 0\end{pmatrix}\begin{pmatrix}\Phi_{(n_T,n_U)}\\ \Phi_{(n_U,n_T)}\end{pmatrix} \quad \text{if} \quad n_T \neq n_U\,, \tag{3.21}$$

and eq. (3.19) has to hold. On the other hand, the transformations $\hat{K}_\mathrm{S}$, $\hat{K}_\mathrm{T}$, $\hat{C}_\mathrm{S}$ and $\hat{C}_\mathrm{T}$ act diagonally on $(\Phi_{(n_T,n_U)}, \Phi_{(n_U,n_T)})$.

Finally, this bottom-up derivation of eq. (3.21) is confirmed in our string setting. In appendix B we show for $\mathbb{Z}_2$ orbifolds that a matter field $\Phi_{(n_T,n_U)}$ with $n_T \neq n_U$ is always accompanied by a mirror partner $\Phi_{(n_U,n_T)}$ such that a mirror transformation interchanges $\Phi_{(n_T,n_U)}$ and $\Phi_{(n_U,n_T)}$. In addition, we follow ref. [22] stating that all twisted strings from the same twisted sector differ only in their modular weights but otherwise share the same transformation properties with respect to the finite modular group, see section D.3. Hence, one can perform an appropriate basis change and check using the character table 4 that the twisted matter fields $(\Phi_{(-3/2,\,1/2)}, \Phi_{(\,1/2,-3/2)})$ transform in the representations $\mathbf{4}_1 \oplus \mathbf{4}_1$ of the finite modular group $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144, 115]$ (see table 1).

4 Local flavor unification

The full eclectic flavor group of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector is a group of order 4608 that consists of the enhanced traditional flavor symmetry $(D_8 \times D_8)/\mathbb{Z}_2 \cup \mathbb{Z}_4^R \cong [64, 266]$, and the finite modular symmetry $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144, 115]$. Adding the $\mathcal{CP}$-like generator $\hat{\Sigma}_*$ (see eq. (A.24)) enhances the finite modular symmetry to $[288, 880]$ and the eclectic flavor group with $\mathcal{CP}$ has order 9216. However, the full eclectic flavor group gets broken spontaneously by non-vanishing vevs of the moduli (T, U). In this section, we complete the analysis of ref. [1] of the unbroken groups at various points $(\langle T\rangle, \langle U\rangle)$ in moduli space.

The couplings of interest among matter fields are governed by the modular forms of (parallel) weight $(2, 2)$. They can be spanned by

$$\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U) = \begin{pmatrix}\hat{Y}_1(T,U)\\ \hat{Y}_2(T,U)\\ \hat{Y}_3(T,U)\\ \hat{Y}_4(T,U)\end{pmatrix} := \begin{pmatrix}\hat{Y}_1(T)\,\hat{Y}_1(U)\\ \hat{Y}_2(T)\,\hat{Y}_1(U)\\ \hat{Y}_1(T)\,\hat{Y}_2(U)\\ \hat{Y}_2(T)\,\hat{Y}_2(U)\end{pmatrix}\,, \tag{4.1}$$

where $\hat{Y}_1(\tau)$ and $\hat{Y}_2(\tau)$ are the S_3 modular forms of weight 2, see for example refs. [16, 23]. $\hat{Y}_1(\tau)$ and $\hat{Y}_2(\tau)$ can be written as

$$\hat{Y}_1(\tau) := \frac{1}{16}\left((\vartheta_{00}(0;\tau))^4 + (\vartheta_{01}(0;\tau))^4\right)\,, \tag{4.2a}$$

$$\hat{Y}_2(\tau) := \frac{\sqrt{3}}{16}\left((\vartheta_{00}(0;\tau))^4 - (\vartheta_{01}(0;\tau))^4\right)\,, \tag{4.2b}$$

in terms of the Jacobi theta function

$$\vartheta(z,\tau) := \sum_{m\in\mathbb{Z}} \exp\left(\pi \mathrm{i}\tau m^2 + 2\pi\mathrm{i} m z\right)\,, \tag{4.3}$$

where we have defined $\vartheta_{00}(z,\tau) := \vartheta(z,\tau)$ and $\vartheta_{01}(z,\tau) := \vartheta(z+1/2,\tau)$. The modular form $\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)$ transforms under modular transformations according to eq. (E.1) as a $\mathbf{4}_3$ of the finite modular group [144, 115], but is invariant under the traditional flavor group. Further details are given in appendix E.1, see also table 1.

At special points in moduli space, i.e. for fixed vacuum expectation values $(\langle T\rangle, \langle U\rangle)$, some of the modular transformations are left unbroken in the vacuum, i.e. they leave invariant the moduli, building the so-called stabilizer subgroup,

$$H_{(\langle T\rangle,\langle U\rangle)} := \langle\ \gamma \mid \gamma\ \in\ \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) \quad \text{with} \quad \gamma(\langle T\rangle) = \langle T\rangle \ \text{ and } \ \gamma(\langle U\rangle) = \langle U\rangle\ \rangle\,. \tag{4.4}$$

Here, $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$, generated by $\{\hat{K}_{\mathrm{S}}, \hat{K}_{\mathrm{T}}, \hat{C}_{\mathrm{S}}, \hat{C}_{\mathrm{T}}, \hat{\Sigma}_*, \hat{M}\}$, is the modular group of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, see appendix A.1.

At these points, the couplings $\hat{Y}^{(2)}_{\mathbf{4}_3}(\langle T\rangle, \langle U\rangle)$ are left invariant under stabilizer elements $\gamma \in H_{(\langle T\rangle,\langle U\rangle)}$. If γ takes the form given in eq. (A.19), i.e. if it is an $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ element, its action on the couplings, eq. (E.1), becomes the eigenvalue equation

$$\rho_{\mathbf{4}_3}(\gamma)\, \hat{Y}^{(2)}_{\mathbf{4}_3}(\langle T\rangle, \langle U\rangle) \ \stackrel{!}{=} (c_T\langle T\rangle + d_T)^{-2}\,(c_U\langle U\rangle + d_U)^{-2}\, \hat{Y}^{(2)}_{\mathbf{4}_3}(\langle T\rangle, \langle U\rangle)\,, \tag{4.5}$$

where c_T, d_T, c_U, d_U are integers that define γ. On the other hand, if the stabilizer element is given by the mirror symmetry $\hat{M}$ or the $\mathcal{CP}$-like generator $\hat{\Sigma}_*$, the couplings satisfy the relations

$$\gamma = \hat{M} \quad \text{at } \langle T\rangle = \langle U\rangle \quad : \quad \hat{Y}^{(2)}_{\mathbf{4}_3}(\langle U\rangle, \langle T\rangle) \ \stackrel{!}{=} \hat{Y}^{(2)}_{\mathbf{4}_3}(\langle T\rangle, \langle U\rangle)\,, \tag{4.6a}$$

$$\gamma = \hat{\Sigma}_* \quad \text{at } \mathrm{Re}\langle T\rangle = \mathrm{Re}\langle U\rangle = 0 \quad : \quad \left(\hat{Y}^{(2)}_{\mathbf{4}_3}(\langle T\rangle, \langle U\rangle)\right)^* \ \stackrel{!}{=} \hat{Y}^{(2)}_{\mathbf{4}_3}(\langle T\rangle, \langle U\rangle)\,. \tag{4.6b}$$

Note that eq. (4.6a) is consistent with the automorphy factor $(-1)^n$ discussed in section 3.3 using $n = 2$. Further, eq. (4.6b) is shown as follows: the $\mathcal{CP}$ transformation acting on the moduli results in complex conjugation for the explicit expressions eq. (4.2) of $\hat{Y}_1$ and $\hat{Y}_2$. Since the representation $\rho_{\mathbf{4}_3}(\gamma)$ of the finite modular group is unitary, its eigenvalues and hence the automorphy factors must be phases, see also ref. [3, section 6]. Note that eq. (4.5) corresponds to the mechanism of flavon alignment in the context of modular flavor symmetries, as also discussed in e.g. refs. [24–26].

A consequence of the automorphy factors being phases at $(\langle T\rangle, \langle U\rangle)$ is that the modular transformations from $H_{(\langle T\rangle,\langle U\rangle)}$ act linearly on matter fields. Hence, the stabilizer enhances the traditional flavor symmetry to the multiplicative closure of the traditional flavor group and the stabilizer modular subgroup, i.e. to the so-called unified flavor group

$$[(D_8 \times D_8)/\mathbb{Z}_2 \times \mathbb{Z}_4^R]/\mathbb{Z}_2\ \cup\ H_{(\langle T\rangle,\langle U\rangle)}\,. \tag{4.7}$$

Explicitly, from eqs. (3.1) and (3.6), the action of a (non-$\mathcal{CP}$-like) stabilizer element $\gamma \in H_{(\langle T\rangle,\langle U\rangle)}$ on a field $\Phi_{(n_T,n_U)}$ with modular weights (n_T, n_U) is given by

$$\Phi_{(n_T,n_U)} \ \stackrel{\gamma}{\longmapsto}\ \rho_{\boldsymbol{t},(\langle T\rangle,\langle U\rangle)}\Phi_{(n_T,n_U)} := (c_T\langle T\rangle + d_T)^{n_T}(c_U\langle U\rangle + d_U)^{n_U}\rho_{\boldsymbol{r}}(\gamma)\Phi_{(n_T,n_U)}\,, \tag{4.8}$$

where $\rho_{\boldsymbol{t},(\langle T\rangle,\langle U\rangle)}$ is a t-dimensional representation $\boldsymbol{t}$ of the unified flavor group, whereas $\boldsymbol{r}$ is a representation of the finite modular group [144, 115]. We stress here the presence of the automorphy factors in the transformation (4.8), which can enhance the order of the unbroken transformations due to the possibility of fractional weights of matter fields.

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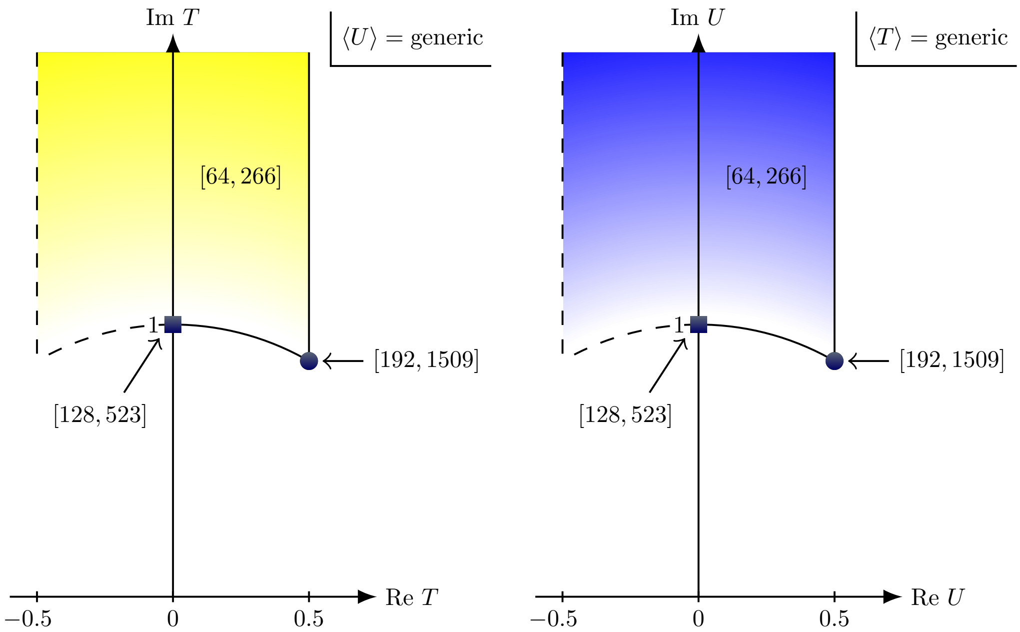


(a) U fixed at a generic value $\langle U\rangle \neq \mathrm{i}, e^{\pi \mathrm{i}/3}$.   **(b)** T fixed at a generic value $\langle T\rangle \neq \mathrm{i}, e^{\pi \mathrm{i}/3}$.

Figure 1. Unified flavor groups at special points in moduli space, including the $\mathbb{Z}_4^R$ symmetry. (a) For generic $\langle U\rangle$, only at $\langle T\rangle = \mathrm{i}$ (square) and $\langle T\rangle = e^{\pi \mathrm{i}/3}$ (bullet) the traditional flavor symmetry is enhanced by $H_{(\mathrm{i},\langle U\rangle)} = \langle \hat{K}_\mathrm{S}\rangle$ and $H_{(e^{\pi\mathrm{i}/3},\langle U\rangle)} = \langle \hat{K}_\mathrm{T}\hat{K}_\mathrm{S}\rangle$, respectively. (b) For generic $\langle T\rangle$, the results are equivalent due to mirror symmetry $\hat{M}$, which exchanges $T \leftrightarrow U$, $\hat{K}_\mathrm{T} \leftrightarrow \hat{C}_\mathrm{T}$ and $\hat{K}_\mathrm{S} \leftrightarrow \hat{C}_\mathrm{S}$.

4.1 Unified flavor groups at generic points in moduli space

Even for generic values of the moduli, the traditional flavor group is enhanced for $\langle T\rangle = \langle U\rangle$. In this case, the mirror transformation $\hat{M}$ is left unbroken. Considering its $\mathbb{Z}_4^{\hat{M}}$ action on matter fields, as given by eqs. (3.16) or (3.21), we find that the unified flavor group in this case is given by [128, 2316].

We can consider also the case that one of the moduli has a generic value while the second modulus is fixed at one of the special points, i or $e^{\pi \mathrm{i}/3}$. In figure 1, we display the different unified flavor groups achieved by incorporating the unbroken modular transformations at those special points in moduli space. Let us consider the results for generic $\langle U\rangle$, as presented in figure 1a. At $\langle T\rangle = \mathrm{i}$, we know (cf. ref. [1]) that the stabilizer subgroup is generated by $\hat{K}_\mathrm{S}$, which becomes a $\mathbb{Z}_8$ generator, considering the admissible automorphy factors $(-\mathrm{i})^{n_T} = e^{-\pi\mathrm{i}\, n_T/2}$ for matter fields with $n_T \in \{0, -1, \pm 1/2, -3/2\}$. However, the traditional flavor group [64, 266] is only enhanced to [128, 523] because $(\hat{K}_\mathrm{S})^2 = (\hat{C}_\mathrm{S})^2$ amounts to the $\mathbb{Z}_4^R$ symmetry already contained in the traditional symmetry. Further, at $\langle T\rangle = e^{\pi\mathrm{i}/3}$, the stabilizer is generated by $\hat{K}_\mathrm{T}\hat{K}_\mathrm{S}$. Taking the automorphy factor $(-e^{\pi\mathrm{i}/3})^{n_T} = e^{-2\pi\mathrm{i}\, n_T/3}$, it corresponds to a $\mathbb{Z}_6$ symmetry, such that $(\hat{K}_\mathrm{T}\hat{K}_\mathrm{S})^3$ is equivalent to the point group of $\mathbb{T}^2/\mathbb{Z}_2$. Thus, the order of the traditional flavor group is only enhanced by a factor of three, leading to the unified flavor group [192, 1509]. Because of the mirror symmetry $\hat{M}$, it suffices to consider generic $\langle U\rangle$ to learn what happens also for generic $\langle T\rangle$, as we notice by comparing figures 1b and 1a. For example, for generic value of the Kähler modulus and $\langle U\rangle = \mathrm{i}$, the stabilizer is generated by $\hat{C}_\mathrm{S}$, which enhances the traditional flavor group $[64, 266]$ to the unified flavor group $[128, 523]$.

4.2 Unified flavor groups of the raviolo

At $\langle U\rangle = \mathrm{i}$ (cf. ref. [11]), the geometry of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold adopts the form of a square raviolo, where the corners correspond to the singularities of the orbifold and the edges are perpendicular and have the same length. As just mentioned, in this case the traditional flavor group is enhanced by $\hat{C}_\mathrm{S}$ to the unified flavor group $[128, 523]$, considering the $\mathbb{Z}_8$ phases $(-\mathrm{i})^{n_U} = e^{-\pi\mathrm{i}\,n_U/2}$ associated with the automorphy factors of the matter fields. Since $\hat{C}_\mathrm{S}$ acts on the compact space as a $\pi/2$ rotation [3, eq. (63a)], the unified flavor group contains a remnant of the Lorentz symmetry in higher dimensions and is hence a discrete R-symmetry.

There are two special points in moduli space for the raviolo, where further enhancements occur if the $\mathcal{CP}$-like modular transformation $\hat{\Sigma}_*$ is considered. First, at $\langle T\rangle = \mathrm{i}$ we find the stabilizer $H_{(\mathrm{i},\mathrm{i})} = \langle \hat{C}_\mathrm{S}, \hat{M}, \hat{\Sigma}_*\rangle$, which enhances the traditional flavor symmetry to

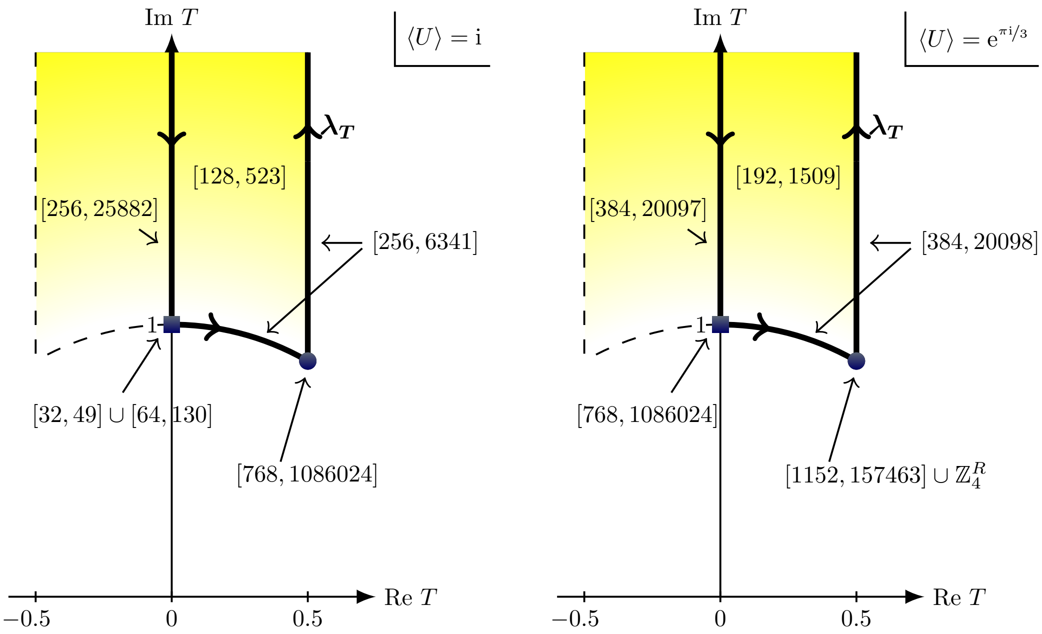
$$\frac{D_8 \times D_8}{\mathbb{Z}_2} \;\cup\; (D_8 \rtimes D_8) \;\cong\; \frac{[32, 49] \,\times\, [64, 130]}{\mathbb{Z}_2}\,, \tag{4.9}$$

where $(D_8 \times D_8)/\mathbb{Z}_2 \cong [32, 49]$ is the traditional flavor group without $\mathbb{Z}_4^R$, $D_8 \rtimes D_8 \cong [64, 130]$ is the modular group corresponding to $H_{(\mathrm{i},\mathrm{i})}$ including the automorphy factors, and the $\mathbb{Z}_2$ in the quotient on the right-hand side identifies the point group selection rule of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector present in both groups. The unified flavor group (4.9) has thus order 1024, which corresponds to the maximal enhancement of the traditional flavor symmetry for $\langle U\rangle = \mathrm{i}$. Secondly, at $\langle T\rangle = e^{\pi\mathrm{i}/3}$, the traditional flavor group is enhanced by the subgroup of modular transformations $H_{(e^{\pi\mathrm{i}/3},\mathrm{i})} = \langle \hat{C}_\mathrm{S}, \hat{K}_\mathrm{T}\hat{K}_\mathrm{S}, \hat{K}_\mathrm{S}\hat{\Sigma}_*\rangle$ to the group $[768, 1086024]$. These enhancements are illustrated in figure 2a.

Additionally, on the line in moduli space described by the boundary λ_T of the fundamental domain, depicted in figure 2a, we find two more enhancements. We see that the Kähler modulus fixed at $\langle T\rangle = \mathrm{i}\,\mathrm{Im}\langle T\rangle > \mathrm{i}$ is left invariant by $\langle \hat{C}_\mathrm{S}, \hat{\Sigma}_*\rangle$. Acting on matter fields along with their corresponding automorphy factors, this yields the unified flavor group $[256, 25882]$. Furthermore, the traditional flavor symmetry is enhanced to $[256, 6341]$ along the regions of the locus λ_T where $\langle T\rangle = e^{\mathrm{i}\varphi}$ with ${}^{\pi}\!/\!{}_{3} < \varphi < {}^{\pi}\!/\!{}_{2}$, and $\langle T\rangle = {}^1\!/\!{}_2 + \mathrm{i}\,\mathrm{Im}\langle T\rangle$ with $\mathrm{Im}\langle T\rangle > {}^{\sqrt{3}}\!/\!{}_2$. In these regions, the stabilizers are given by $\langle \hat{C}_\mathrm{S}, \hat{K}_\mathrm{S}\hat{\Sigma}_*\rangle$ and $\langle \hat{C}_\mathrm{S}, \hat{K}_\mathrm{T}\hat{\Sigma}_*\rangle$, respectively.

4.3 Unified flavor groups of the tetrahedron

When the complex structure is stabilized at $\langle U\rangle = e^{\pi\mathrm{i}/3}$, the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector has the shape of a tetrahedron, cf. ref. [1, figure 5]. As can be read off from figure 1b, in the tetrahedron with a generic value for T, the $\hat{C}_\mathrm{T}\hat{C}_\mathrm{S}$ modular transformation leaves the moduli invariant and enhances the traditional flavor group to the unified flavor group $[192, 1509]$, the generic flavor symmetry of the tetrahedron. This contains discrete R-symmetries due to the inclusion of the discrete rotation in the compact space generated by $\hat{C}_\mathrm{T}\hat{C}_\mathrm{S}$.



(a) $\mathbb{T}^2/\mathbb{Z}_2$ with $\langle U\rangle = \mathrm{i}$ **(b)** $\mathbb{T}^2/\mathbb{Z}_2$ with $\langle U\rangle = e^{\pi\mathrm{i}/3}$

Figure 2. Unified flavor groups at different points $\langle T\rangle$ and the boundary λ_T of the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})_T$ for two special vevs $\langle U\rangle$ of the complex structure modulus. We use the shorthands $[32,49]\cup[64,130] \cong \big[[32,49]\times[64,130]\big]/\mathbb{Z}_2$ and $[1152,157463]\cup\mathbb{Z}_4^R \cong \big[[1152,157463]\times\mathbb{Z}_4^R\big]/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ identifies the point group selection rule of $\mathbb{T}^2/\mathbb{Z}_2$ contained in the multiplied groups.

The symmetry of the tetrahedron can be further enlarged if the Kähler modulus is fixed e.g. at the special values $\langle T\rangle = e^{\pi\mathrm{i}/3}$ or $\langle T\rangle = \mathrm{i}$. At these points, the respective stabilizer subgroups are $H_{(e^{\pi\mathrm{i}/3},e^{\pi\mathrm{i}/3})} = \langle \hat{C}_\mathrm{T}\hat{C}_\mathrm{S}, \hat{C}_\mathrm{T}\hat{K}_\mathrm{T}\hat{\Sigma}_*, \hat{M}\rangle$ and $H_{(\mathrm{i},e^{\pi\mathrm{i}/3})} = \langle \hat{C}_\mathrm{T}\hat{C}_\mathrm{S}, \hat{C}_\mathrm{S}\hat{\Sigma}_*, \hat{K}_\mathrm{S}\rangle$, which include the $\mathcal{CP}$-like modular transformation $\hat{\Sigma}_*$. Considering the action of the stabilizer elements in each case, including their automorphy factors, we find the enhancements shown in figure 2b. Note that, like in the case of the raviolo, the point at which both moduli values coincide, i.e. at $(\langle T\rangle,\langle U\rangle) = (e^{\pi\mathrm{i}/3}, e^{\pi\mathrm{i}/3})$, is endowed with the largest possible linear enhancement of the traditional flavor symmetry in the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, the group

$$\frac{[1152,157463]\,\times\,\mathbb{Z}_4^R}{\mathbb{Z}_2}\,, \tag{4.10}$$

which is of order 2304. Here, as before, the $\mathbb{Z}_2$ corresponds to the identification of the point group selection rule of this orbifold sector, which appears in both groups, [1152, 157463] and $\mathbb{Z}_4^R$.

Along the locus λ_T, there are two more enhancements of the traditional flavor group. For $\langle T\rangle = \mathrm{i}\,\mathrm{Im}\langle T\rangle > \mathrm{i}$, not only $\hat{C}_\mathrm{T}\hat{C}_\mathrm{S}$ leaves the moduli invariant but also $\hat{C}_\mathrm{T}\hat{\Sigma}_*$. This implies that the flavor symmetry of the tetrahedron [192, 1509] is enhanced to the unified flavor group [384, 20097]. Besides, for $\langle T\rangle = e^{\mathrm{i}\varphi}$ with $\pi/3 < \varphi < \pi/2$, the stabilizer is $\langle \hat{C}_\mathrm{T}\hat{C}_\mathrm{S}, \hat{K}_\mathrm{S}\hat{C}_\mathrm{S}\hat{\Sigma}_*\rangle$ and leads to the unified flavor group [384, 20098]. The same flavor enhancement is obtained if the Kähler modulus sits at $\langle T\rangle = 1/2 + \mathrm{i}\,\mathrm{Im}\langle T\rangle$ with $\mathrm{Im}\langle T\rangle > \sqrt{3}/2$, where the stabilizer is generated by $\hat{C}_\mathrm{T}\hat{C}_\mathrm{S}$ and $\hat{K}_\mathrm{T}\hat{C}_\mathrm{S}\hat{\Sigma}_*$.

4.3.1 A_4 flavor symmetry from the tetrahedron

Let us turn back to the case of the tetrahedron, $\langle U\rangle = e^{\pi \mathrm{i}/3}$, with a generic value for the Kähler modulus. In this case, according to eq. (4.8), the stabilizer modular generator $\hat{C}_{\mathrm{T}}\hat{C}_{\mathrm{S}}$ acts on matter fields as

$$\Phi_{(n_T,n_U)} \xmapsto{\hat{C}_{\mathrm{T}}\,\hat{C}_{\mathrm{S}}} \rho_{\boldsymbol{r},(\langle T\rangle,e^{\pi\mathrm{i}/3})}(\hat{C}_{\mathrm{T}}\,\hat{C}_{\mathrm{S}})\,\Phi_{(n_T,n_U)}\,, \tag{4.11}$$

where due to the automorphy factor $(c_U\,\langle U\rangle + d_U)^{n_U}$ we get

$$\rho_{\boldsymbol{r},(\langle T\rangle,e^{\pi\mathrm{i}/3})}(\hat{C}_{\mathrm{T}}\,\hat{C}_{\mathrm{S}}) := (\exp\left(-2\pi\mathrm{i}/3\right))^{n_U}\,\rho_{\boldsymbol{r}}(\hat{C}_{\mathrm{T}})\rho_{\boldsymbol{r}}(\hat{C}_{\mathrm{S}})\,. \tag{4.12}$$

The admissible modular weights of massless matter fields are $n_U \in \{0,-1\}$ for bulk matter, and $n_U \in \{-3/2,-1/2,1/2\}$ for twisted matter. Hence, eq. (4.11) describes a $\mathbb{Z}_6$ transformation that we can write as $\mathbb{Z}_2\times\mathbb{Z}_3$, generated by

$$\mathbb{Z}_2^{(\mathrm{PG})}:\qquad \left(\rho_{\boldsymbol{r},(\langle T\rangle,e^{\pi\mathrm{i}/3})}(\hat{C}_{\mathrm{T}}\,\hat{C}_{\mathrm{S}})\right)^3 = \exp\left(-2\pi\mathrm{i}n_U\right)\,\mathbb{1}\,, \tag{4.13a}$$

$$\mathbb{Z}_3^{R}:\qquad \left(\rho_{\boldsymbol{r},(\langle T\rangle,e^{\pi\mathrm{i}/3})}(\hat{C}_{\mathrm{T}}\,\hat{C}_{\mathrm{S}})\right)^2 = \exp\left(-4\pi\mathrm{i}n_U/3\right)\,\left(\rho_{\boldsymbol{r}}(\hat{C}_{\mathrm{T}})\rho_{\boldsymbol{r}}(\hat{C}_{\mathrm{S}})\right)^2\,, \tag{4.13b}$$

respectively. Using the admissible modular weights, we note that the $\mathbb{Z}_2$ factor in eq. (4.13a) corresponds to the $\mathbb{Z}_2^{(\mathrm{PG})}$ point group selection rule of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector. Moreover, the $\mathbb{Z}_3$ factor eq. (4.13b) acts on the superpotential $\mathcal{W}$ as

$$\mathcal{W} \xmapsto{(\hat{C}_{\mathrm{T}}\,\hat{C}_{\mathrm{S}})^2} \omega^2\,\mathcal{W}\,, \tag{4.14}$$

which is a discrete $\mathbb{Z}_3^R$ R-symmetry using the definition $\omega := \exp\left(2\pi\mathrm{i}/3\right)$.

The group generated by the traditional flavor group elements $\rho_{\mathbf{4}}(h_1)$, $\rho_{\mathbf{4}}(h_2)$ from eq. (D.21) together with the $\mathbb{Z}_2^{(\mathrm{PG})}$ factor from eq. (4.13a) and the $\mathbb{Z}_3^R$ factor from eq. (4.13b) turns out to be

$$A_4^R\;\times\;\mathbb{Z}_2^{(\mathrm{PG})}\;\cong\;[24,13]\,. \tag{4.15}$$

Here, the alternating group A_4^R is a non-Abelian R-symmetry as it arises from $\rho_{\mathbf{4}}(h_1)$, $\rho_{\mathbf{4}}(h_2)$ and the $\mathbb{Z}_3^R$ R-symmetry, cf. ref. [27] for a general discussion on non-Abelian R-symmetries. The matter fields and the superpotential $\mathcal{W}$ build the following representations of $A_4^R\times\mathbb{Z}_2^{(\mathrm{PG})}$

$$\Phi_{(0,0)}\,:\,(\mathbf{1},\mathbf{1}_0)\,, \tag{4.16a}$$

$$\Phi_{(-1,-1)}\,:\,(\mathbf{1}',\mathbf{1}_0)\,, \tag{4.16b}$$

$$\Phi_{(-1/2,-1/2)}\,:\,(\mathbf{3},\mathbf{1}_1)\;\oplus\;(\mathbf{1}'',\mathbf{1}_1)\,, \tag{4.16c}$$

$$\mathcal{W}\,:\,(\mathbf{1}',\mathbf{1}_0)\,, \tag{4.16d}$$

where we denote the irreducible representations of $\mathbb{Z}_2$ by $\mathbf{1}_0$ and $\mathbf{1}_1$ and the ones of A_4^R by $\mathbf{3}$, $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$, see appendix E.4.

Combined with the $\mathbb{Z}_4^R$ symmetry associated with the sublattice rotation $\hat{\Theta}_{(2)}$ and the $\mathbb{Z}_2\times\mathbb{Z}_2$ generators $\rho_{\mathbf{4}}(h_3)$ and $\rho_{\mathbf{4}}(h_4)$ of the traditional flavor group, associated with the space group selection rule of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, the group $A_4^R \times \mathbb{Z}_2$ enhances to the unified flavor group of the tetrahedron, $[192, 1509]$, see figure 2b.

Compared to the literature (see e.g. refs. [28–30]), we see that in the consistent string approach the naïve A_4 symmetry obtained by the compactification of two extra dimensions on a tetrahedron has to be extended in two ways: first, the $\mathbb{Z}_3$ generator of A_4^R turns out to be an R-symmetry. This can be understood equivalently either as a discrete remnant of the extra-dimensional Lorentz symmetry or as a discrete remnant of an $\mathrm{SL}(2,\mathbb{Z})_U$ modular symmetry. In addition, A_4^R is enhanced in the full string approach by stringy selection rules to $[192, 1509]$ of order 192, which still contrasts with previous results [14].

4.4 Other $\mathcal{CP}$-enhanced unified flavor groups

Figure 3 displays all unified flavor groups of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector that include i) the $\mathbb{Z}_4^R$ symmetry arising from embedding the orbifold sector in higher dimensions, ii) the $\mathcal{CP}$-like modular transformation $\hat{\Sigma}_*$, and iii) the phases associated with the automorphy factors of modular transformations acting on matter fields $\Phi_{(n_T,n_U)}$. We use as reference axes the straightened lines describing the boundaries λ_T and λ_U of the T and U moduli spaces, respectively. For example, λ_T is illustrated in figure 2. Note that the flavor enhancements along the horizontal lines in figure 3 have already been discussed in sections 4.2 (bottom line with $\langle U\rangle = \mathrm{i}$) and 4.3 (upper line with $\langle U\rangle = e^{\pi \mathrm{i}/3}$).

Mirror symmetry $\hat{M}$ acts on the moduli and the modular generators as $U \leftrightarrow T$, $\hat{C}_\mathrm{S} \leftrightarrow \hat{K}_\mathrm{S}$, and $\hat{C}_\mathrm{T} \leftrightarrow \hat{K}_\mathrm{T}$. As a first consequence, the points along the diagonal in figure 3, defined by $\langle T\rangle = \langle U\rangle$, are left invariant by $\hat{M}$. Furthermore, the points below and above this diagonal are connected by a mirror transformation. It then follows that $\hat{M}$ identifies the unified flavor groups in these two sectors of moduli space.

Focusing on the lower half of the plane, below the diagonal of figure 3, we see that the sole enhancements that have not been discussed in the preceding subsections are those that lie at the diagonal, and those that are valid in the squared and triangular regions of figure 3. Let us consider two examples. In the lowest part of the diagonal, the stabilizer modular subgroup is $H_{(\mathrm{i}x,\mathrm{i}x)} = \langle \hat{M}, \hat{\Sigma}_*\rangle$, with $x > 1$. Considering the associated transformations on matter fields with their corresponding automorphy factors, as given in appendix D.3.2, we find that the traditional flavor group $[64, 266]$ is in this case enhanced to $[256, 56079]$. Similarly, in the bottom triangle of the figure, the stabilizer is given just by $H_{(\mathrm{i}x,\mathrm{i}y)} = \langle \hat{\Sigma}_*\rangle$, with $x, y > 1$ and $y > x$, which yields the unified flavor group $[128, 2326]$. Other cases can be easily determined by using the proper stabilizer subgroups provided in our previous work [1, figure 7].

5 Effective field theory of the $\mathbb{Z}_2$ orbifold

In this section, we focus on four **4**-plets of twisted matter fields $\Phi^i_{(n_T,n_U)}$, for $i \in \{1,2,3,4\}$, with modular weights $n_T = n_U = -1/2$, see table 1. Each **4**-plet contains four fields $\phi^i_{(n_1,n_2)}$ that we label by the winding numbers $(n_1, n_2) \in \{(0,0),(1,0),(0,1),(1,1)\}$ and *not* by the modular weights n_T and n_U. Hence, each twisted matter field $\phi^i_{(n_1,n_2)}$ is localized at one of the four fixed points of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, see appendix A and ref. [1, figure 1]. In

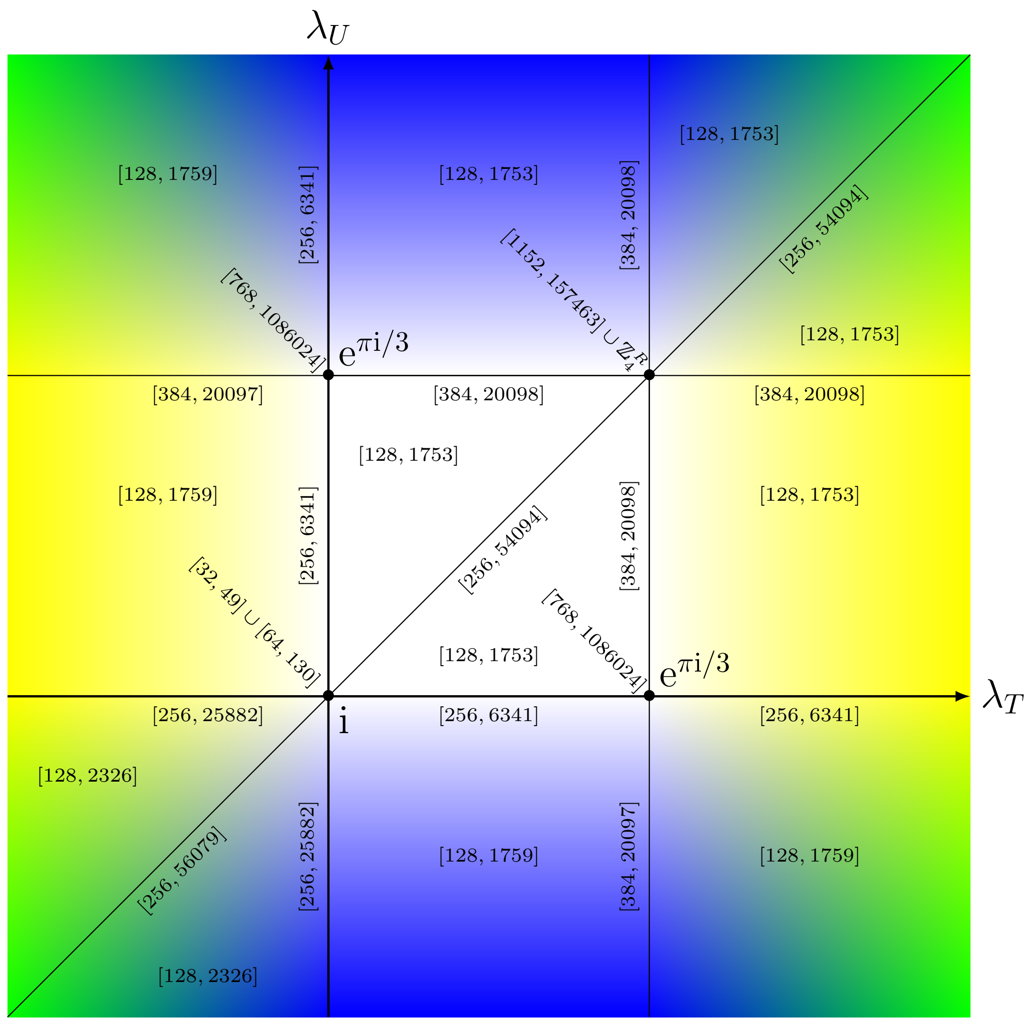


Figure 3. Unified flavor groups with $\mathcal{CP}$ at $\langle T\rangle \in \lambda_T$ and $\langle U\rangle \in \lambda_U$. We use the shorthands $[32,49]\cup[64,130] \cong \big[[32,49]\times[64,130]\big]/\mathbb{Z}_2$ and $[1152,157463]\cup\mathbb{Z}_4^R \cong \big[[1152,157463]\times\mathbb{Z}_4^R\big]/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ corresponds to the point group selection rule of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, contained in both groups in the product. The axes λ_T and λ_U describe the boundaries of the fundamental domains of T and U, see e.g. figure 2 for λ_T. The diagonal depicts the hypersurface where $\langle U\rangle = \langle T\rangle$ on the curves λ_T and λ_U. The unified flavor groups above and below the diagonal are related by mirror symmetry $\hat{M}$.

other words, we consider four **4**-plets $\Phi^i_{(-1/2,-1/2)} := (\phi^i_{(0,0)}, \phi^i_{(1,0)}, \phi^i_{(0,1)}, \phi^i_{(1,1)})^{\mathrm{T}}$. We assume that the **4**-plets differ in some additional charges, for example with respect to the unbroken gauge group from $\mathrm{E}_8\times\mathrm{E}_8$ (or $\mathrm{SO}(32)$). Then, we use the eclectic flavor symmetry to write down the most general Kähler and superpotential to lowest order in these fields.

5.1 The Kähler potential

The Hermitian Kähler potential K of a single twisted matter field $\Phi_{(-1/2,-1/2)}$ reads to leading order [31]

$$K \supset (-\mathrm{i}T + \mathrm{i}\bar{T})^{-1/2}\,(-\mathrm{i}U + \mathrm{i}\bar{U})^{-1/2}\,\tilde{K}\,, \tag{5.1}$$

where $\tilde{K}$ is an Hermitian bilinear polynomial of the form $\tilde{K} = c_{n_1 n_2 n_1' n_2'} \phi_{(n_1,n_2)}\, \bar{\phi}_{(n_1',n_2')}$. In the following, we constrain $\tilde{K}$ by imposing the traditional flavor symmetry step by step. First, the space and point group selection rules (D.24) enforce $n_1' = n_1$ and $n_2' = n_2$, resulting in $\tilde{K} = c_{n_1 n_2} \phi_{(n_1,n_2)}\, \bar{\phi}_{(n_1,n_2)}$. Then, invariance under eq. (D.21a) forces all coefficients $c_{n_1 n_2}$ to be equal (and we normalize them to 1). Hence,

$$\tilde{K} = \left(|\phi_{(0,0)}|^2 + |\phi_{(1,0)}|^2 + |\phi_{(0,1)}|^2 + |\phi_{(1,1)}|^2\right) . \tag{5.2}$$

Now, we can generalize this easily to four **4**-plets of twisted matter fields $\Phi^i_{(-1/2,-1/2)}$, for $i \in \{1,2,3,4\}$. Due to our assumption of additional (gauge) charges that distinguish between $\phi^i_{(n_1,n_2)}$ and $\phi^j_{(n_1,n_2)}$ for $i \neq j$, we obtain

$$\tilde{K} = \sum_{i=1}^{4} \left(|\phi^i_{(0,0)}|^2 + |\phi^i_{(1,0)}|^2 + |\phi^i_{(0,1)}|^2 + |\phi^i_{(1,1)}|^2\right) . \tag{5.3}$$

Consequently, for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector the traditional flavor symmetry already enforces the Kähler potential to be diagonal in twisted matter fields. Hence, this diagonal structure can not be changed by the full eclectic flavor symmetry: since additional terms involving modular forms $\hat{Y}(T,U)$ (as suggested by ref. [32]) are singlets of the traditional flavor group, the Kähler potential must remain diagonal, cf. ref. [6]. Yet, additional corrections to the Kähler potential that involve flavons are still possible, cf. ref. [33].

5.2 The superpotential

To lowest order in twisted matter fields $\Phi^i_{(-1/2,-1/2)}$, the superpotential reads schematically

$$\mathcal{W} \supset \hat{Y}^{(0)}(T,U)\, \Phi_{(0,0)}\, \Phi^i_{(-1/2,-1/2)}\, \Phi^j_{(-1/2,-1/2)} \tag{5.4a}$$

$$+\, \hat{Y}^{(2)}(T,U)\, \Phi_{(-1,-1)}\, \Phi^1_{(-1/2,-1/2)}\, \Phi^2_{(-1/2,-1/2)}\, \Phi^3_{(-1/2,-1/2)}\, \Phi^4_{(-1/2,-1/2)} . \tag{5.4b}$$

Here, we have imposed the $\mathbb{Z}_4^R$ R-symmetry and the fact that the modular weights of matter fields and couplings have to add up to $(-1,-1)$ for the superpotential, see table 1. Thus, $\hat{Y}^{(0)}(T,U)$ has to carry modular weights $(0,0)$, while the modular form $\hat{Y}^{(2)}(T,U)$ has modular weights $(2,2)$. There exists a unique modular form of weight $(2,2)$, which we denote by $\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)$ in the following (see eq. (4.1)). In addition, the superpotential has to be covariant under the full eclectic flavor symmetry, i.e. it has to be invariant simultaneously under the traditional non-R symmetries and the finite modular flavor symmetry but transform with the appropriate phases (automorphy factors) under the R-symmetry (modular symmetry). Note that $\Phi_{(0,0)}$ is required in eq. (5.4a) because the lowest-order possible couplings of massless fields are trilinear. This can be seen as a result of additional discrete R-symmetries that appear in a full 6D orbifold compactification [20, 34]. From a more phenomenological point of view, this becomes clear from the fact that the strings under consideration are massless by construction such that a bilinear term would give a contradiction. Note additionally that the twisted fields $\Phi^i_{(-1/2,-1/2)}$ in eq. (5.4b) need not be all different.

5.2.1 Constraints from the traditional flavor symmetry

Let us start with invariance under the traditional flavor symmetry $(D_8 \times D_8)/\mathbb{Z}_2 \cong [32, 49]$. First, we consider the product of two twisted matter fields $\Phi^i_{(-1/2,-1/2)}\,\Phi^j_{(-1/2,-1/2)}$ needed for the terms (5.4a) in the superpotential $\mathcal{W}$. The fields $\Phi^i_{(-1/2,-1/2)}$ transform as irreducible **4**-plets of the traditional flavor group [32, 49]. Hence, we need to consider the tensor product

$$\mathbf{4} \otimes \mathbf{4} = \bigoplus_{\alpha,\beta,\gamma,\delta\, \in\, \{-,+\}} \mathbf{1}_{\alpha\beta\gamma\delta}\,. \tag{5.5}$$

This tensor product contains one trivial singlet $\mathbf{1}_{++++}$, which corresponds to the terms

$$\mathcal{I}_0^{ij} = \phi^i_{(0,0)}\phi^j_{(0,0)} + \phi^i_{(1,0)}\phi^j_{(1,0)} + \phi^i_{(0,1)}\phi^j_{(0,1)} + \phi^i_{(1,1)}\phi^j_{(1,1)}\,, \tag{5.6}$$

for $i, j \in \{1, 2, 3, 4\}$. The total R-charge is 2 as one can see easily from table 1. As a remark, one can check the invariance of the terms $\mathcal{I}_0^{ij}$ explicitly using the orthogonality of the representation matrices given in eq. (D.21). Since $\Phi_{(0,0)}$ is a trivial singlet $\mathbf{1}_{++++}$ of [32, 49] with R-charge 0, the terms $\Phi_{(0,0)}\,\mathcal{I}_0^{ij} \subset \mathcal{W}$ are allowed by both, the traditional flavor symmetry and $\mathbb{Z}_4^R$.

Next, we study the product of four twisted matter fields in order to construct the superpotential terms in eq. (5.4b). Since

$$\mathbf{1}_{\alpha\beta\gamma\delta} \otimes \mathbf{1}_{\alpha\beta\gamma\delta} = \mathbf{1}_{++++}\,, \tag{5.7}$$

we know from eq. (5.5) that there are 16 invariant combinations $\mathcal{I}_i$, $i \in \{1, \ldots, 16\}$. We list them in appendix E.2. Consequently, out of the $4^4 = 256$ possible terms from $\Phi^1_{(-1/2,-1/2)}\,\Phi^2_{(-1/2,-1/2)}\,\Phi^3_{(-1/2,-1/2)}\,\Phi^4_{(-1/2,-1/2)}$, invariance under the traditional flavor symmetry [32, 49] allows only 16.

5.2.2 Constraints from the modular symmetry

As explained in ref. [1] and derived in appendix D, the modular symmetry $(\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U) \rtimes \mathbb{Z}_2$ of the Kähler and complex structure modulus T and U, respectively, is realized as finite modular group $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144, 115]$ on twisted matter fields.

Using the matrix representation eq. (D.26) of the twisted matter fields $\Phi^i_{(-1/2,-1/2)}$, we see that the terms $\hat{Y}^{(0)}(T,U)\,\Phi_{(0,0)}\,\mathcal{I}_0^{ij}$ in eq. (5.4a) are invariant under the finite modular symmetry. In contrast, the terms $\mathcal{I}_i$, $i \in \{1, \ldots, 16\}$, defined in appendix E.2, do not transform trivially under the generators $\hat{C}_\mathrm{S}$, $\hat{C}_\mathrm{T}$ and $\hat{M}$ of the finite modular symmetry,

$$\begin{pmatrix} \mathcal{I}_1 \\ \vdots \\ \mathcal{I}_{16} \end{pmatrix} \xmapsto{\hat{\Sigma}} \left(j^{(-1/2)}(\hat{\Sigma}, T, U)\right)^4 R(\hat{\Sigma}) \begin{pmatrix} \mathcal{I}_1 \\ \vdots \\ \mathcal{I}_{16} \end{pmatrix}. \tag{5.8}$$

We list the 16×16 matrices $R(\hat{\Sigma})$ in appendix E.2. Comparing the traces of $R(\hat{\Sigma})$ to the character table of [144, 115] given in appendix E.3, we find that $(\mathcal{I}_1, \ldots, \mathcal{I}_{16})^\mathrm{T}$ decomposes into the irreducible representations

$$\mathbf{1}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_0 \oplus \mathbf{1}_1 \oplus \mathbf{4}_3 \oplus \mathbf{4}_3 \oplus \mathbf{4}_5 \tag{5.9}$$

of the finite modular group [144, 115]. Only the two $\mathbf{4}_3$ representations yield invariant terms when they are combined with the modular form $\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)$ defined in eq. (4.1). Hence, we identify the two $\mathbf{4}_3$ representations from the tensor product eq. (5.9). We obtain

$$Q_1 \;:=\; \begin{pmatrix} 2\,\mathcal{I}_2 - 2\,\mathcal{I}_3 + \mathcal{I}_6 + \mathcal{I}_9 - \mathcal{I}_5 - \mathcal{I}_8 \\ \sqrt{3}\,(\,\mathcal{I}_{12} + \mathcal{I}_{14} - \mathcal{I}_{11} - \mathcal{I}_{13}\,) \\ \sqrt{3}\,(\,\mathcal{I}_6 + \mathcal{I}_8 - \mathcal{I}_5 - \mathcal{I}_9\,) \\ 2\,\mathcal{I}_{15} - 2\,\mathcal{I}_{16} + \mathcal{I}_{12} + \mathcal{I}_{13} - \mathcal{I}_{11} - \mathcal{I}_{14} \end{pmatrix}, \tag{5.10a}$$

$$Q_2 \;:=\; \begin{pmatrix} 2\,\mathcal{I}_3 - 2\,\mathcal{I}_4 + \mathcal{I}_7 + \mathcal{I}_{10} - \mathcal{I}_6 - \mathcal{I}_9 \\ \sqrt{3}\,(\,\mathcal{I}_{15} + \mathcal{I}_{16} - \mathcal{I}_{12} - \mathcal{I}_{14}\,) \\ \sqrt{3}\,(\,\mathcal{I}_7 + \mathcal{I}_9 - \mathcal{I}_6 - \mathcal{I}_{10}\,) \\ 2\,\mathcal{I}_{11} - 2\,\mathcal{I}_{13} + \mathcal{I}_{12} + \mathcal{I}_{16} - \mathcal{I}_{15} - \mathcal{I}_{14} \end{pmatrix}. \tag{5.10b}$$

Note that the quartic polynomial Q_1 is antisymmetric when $\Phi^1_{(-1/2,-1/2)} \leftrightarrow \Phi^4_{(-1/2,-1/2)}$ or $\Phi^2_{(-1/2,-1/2)} \leftrightarrow \Phi^3_{(-1/2,-1/2)}$ are interchanged. Analogously, Q_2 is antisymmetric under $\Phi^1_{(-1/2,-1/2)} \leftrightarrow \Phi^2_{(-1/2,-1/2)}$ or $\Phi^3_{(-1/2,-1/2)} \leftrightarrow \Phi^4_{(-1/2,-1/2)}$. As a consequence, the quartic term in $\Phi^i_{(-1/2,-1/2)}$ in the superpotential (5.4) vanishes if all twisted matter fields are equal. The invariant terms in the superpotential then read

$$\mathcal{W}(T,U,\phi) \supset c_1\left(\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)\cdot Q_1\right)\Phi_{(-1,-1)} + c_2\left(\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)\cdot Q_2\right)\Phi_{(-1,-1)}\,. \tag{5.11}$$

Hence, the number of unfixed superpotential parameters in eq. (5.4b) is reduced from 16 in the case of imposing only the traditional flavor symmetry to 2 (i.e. c_1 and c_2) when we include the constraints from the full eclectic flavor symmetry. This is in contrast to the leading order Kähler potential, see eq. (5.3), where the finite modular symmetry did not yield additional constraints compared to the traditional flavor symmetry. Finally, the superpotential of eq. (5.4) is thus explicitly given by

$$\mathcal{W}(T,U,\phi) = c_0^{ij}\,\hat{Y}^{(0)}_{\mathbf{1}_0}(T,U)\,\mathcal{I}_0^{ij}\,\Phi_{(0,0)} \tag{5.12a}$$
$$\qquad + c_1\left(\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)\cdot Q_1\right)\Phi_{(-1,-1)} + c_2\left(\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)\cdot Q_2\right)\Phi_{(-1,-1)}\,, \tag{5.12b}$$

where Q_1 and Q_2 are the quartic polynomials in the twisted matter fields $\Phi^i_{(-1/2,-1/2)}$, given in eq. (5.10).

5.3 Gauge symmetry enhancement in moduli space

Let us analyze the "accidental" continuous symmetries of the superpotential eq. (5.12) that appear at special points in (T,U) moduli space, cf. ref. [3] for the analogous discussion in the case of the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold sector. We assume that the four twisted matter fields $\Phi^i_{(-1/2,-1/2)}$ transform identically under the enhanced symmetry. To identify continuous symmetries, we define a general U(4) transformation that leaves the Kähler potential of $\Phi^i_{(-1/2,-1/2)}$ invariant. Infinitesimally, it reads

$$\Phi^i_{(-1/2,-1/2)} \xmapsto{\mathrm{U}(4)} \left(\mathbb{1}_4 + \mathrm{i}\,\alpha_a\,\mathrm{T}_a\right)\,\Phi^i_{(-1/2,-1/2)}\,, \tag{5.13}$$

where summation over $a = 1,\ldots,16$ is implied and the 4×4 matrices T_a denote the 16 Hermitian generators of the U(4) Lie algebra. Even though the superpotential has no

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point in moduli space	alignment of coupling $\hat{Y}^{(2)}_{\mathbf{4}_3}(\langle T\rangle,\langle U\rangle)$	unbroken Lie algebra t	basis change M_{g}	enhanced gauge symmetry	charges or representations of matter $\Phi^i_{(-1/2,-1/2),\mathrm{g}}$
$\langle T\rangle=\langle U\rangle$	$\begin{pmatrix}\hat{Y}_1(\langle T\rangle,\langle T\rangle)\\ \hat{Y}_2(\langle T\rangle,\langle T\rangle)\\ \hat{Y}_2(\langle T\rangle,\langle T\rangle)\\ \hat{Y}_4(\langle T\rangle,\langle T\rangle)\end{pmatrix}$	$\begin{pmatrix}0&-\mathrm{i}&0&0\\ \mathrm{i}&0&0&0\\ 0&0&0&-\mathrm{i}\\ 0&0&\mathrm{i}&0\end{pmatrix}$	$\dfrac{1}{\sqrt{2}}\begin{pmatrix}1&-\mathrm{i}&0&0\\ 0&0&-\mathrm{i}&1\\ 0&0&1&-\mathrm{i}\\ \mathrm{i}&-1&0&0\end{pmatrix}$	$\mathrm{U}(1)$	$+1$ -1 $+1$ -1
$\langle T\rangle=\langle U\rangle=\mathrm{i}$	$c_{\mathrm{i}}\begin{pmatrix}1\\ 1/\sqrt{3}\\ 1/\sqrt{3}\\ 1/3\end{pmatrix}$ $c_{\mathrm{i}}:=\hat{Y}_1(\mathrm{i},\mathrm{i})$ ≈ 0.01706	$\begin{pmatrix}0&-\mathrm{i}&0&0\\ \mathrm{i}&0&0&0\\ 0&0&0&-\mathrm{i}\\ 0&0&\mathrm{i}&0\end{pmatrix}$ $\begin{pmatrix}0&0&-\mathrm{i}&0\\ 0&0&0&-\mathrm{i}\\ \mathrm{i}&0&0&0\\ 0&\mathrm{i}&0&0\end{pmatrix}$	$\dfrac{1}{2}\begin{pmatrix}-1&\mathrm{i}&\mathrm{i}&1\\ 1&-\mathrm{i}&\mathrm{i}&1\\ 1&\mathrm{i}&-\mathrm{i}&1\\ -1&-\mathrm{i}&-\mathrm{i}&1\end{pmatrix}$	$\mathrm{U}(1)^2$	$(+1,+1)$ $(+1,-1)$ $(-1,+1)$ $(-1,-1)$
$\langle T\rangle=\langle U\rangle=\omega$	$c_\omega\begin{pmatrix}1\\ -\mathrm{i}\\ -\mathrm{i}\\ -1\end{pmatrix}$ $c_\omega:=\hat{Y}_1(\omega,\omega)$ ≈ 0.01256	$\begin{pmatrix}0&0&-\mathrm{i}&0\\ 0&0&0&\mathrm{i}\\ \mathrm{i}&0&0&0\\ 0&-\mathrm{i}&0&0\end{pmatrix}$ $\begin{pmatrix}0&0&0&-\mathrm{i}\\ 0&0&-\mathrm{i}&0\\ 0&\mathrm{i}&0&0\\ \mathrm{i}&0&0&0\end{pmatrix}$ $\begin{pmatrix}0&-\mathrm{i}&0&0\\ \mathrm{i}&0&0&0\\ 0&0&0&-\mathrm{i}\\ 0&0&\mathrm{i}&0\end{pmatrix}$	$\dfrac{1}{\sqrt{2}}\begin{pmatrix}1&-\mathrm{i}&0&0\\ 0&0&-\mathrm{i}&1\\ 0&0&1&-\mathrm{i}\\ \mathrm{i}&-1&0&0\end{pmatrix}$	$\mathrm{SU}(2)$	$\mathbf{2}\oplus\mathbf{2}$

Table 2. Gauge symmetry enhancements by Lie algebra elements t in the case of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector at special points in moduli space, uncovered as "accidental" symmetries of the superpotential due to the alignment of couplings in flavor space. Note that the field basis of twisted matter fields $\Phi^i_{(-1/2,-1/2),\mathrm{g}}$ with well-defined gauge charges differs from the field basis of localized twisted strings $\Phi^i_{(-1/2,-1/2)}$ using the basis change M_{g} in eq. (5.14).

continuous symmetries for general values of the moduli (T,U), we do observe subgroups of $\mathrm{U}(4)$ being unbroken at special values of the moduli. We discuss such cases in the following.

Note that from the top-down perspective of string theory, the appearance of continuous symmetries is expected. As discussed in appendix C, these "accidental" symmetries are actually gauged: at special points in moduli space some winding strings become massless, giving rise to the gauge bosons of enhanced gauge symmetries. Consequently, the enhanced symmetries that we uncover in this section are exact symmetries to all orders in the superpotential.

In the following, we briefly discuss the results for three special configurations: i) $\langle T\rangle=\langle U\rangle$ for generic $\langle U\rangle$, ii) $\langle T\rangle=\langle U\rangle=\omega$, and iii) $\langle T\rangle=\langle U\rangle=\mathrm{i}$. In each case, we first evaluate the superpotential eq. (5.12) in the respective vev configuration by analyzing the

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alignment of the couplings $\hat{Y}^{(2)}_{\mathbf{4}_3}(\langle T\rangle,\langle U\rangle)$ in flavor space. Then, we identify the unbroken Lie algebra elements $\mathrm{t} := \alpha_a \mathrm{T}_a$ from the transformation (5.13). Afterwards, we perform a (unitary) basis change

$$\Phi^i_{(-1/2,-1/2),\mathrm{g}} := M_\mathrm{g}\, \Phi^i_{(-1/2,-1/2)}\,, \tag{5.14}$$

such that the unbroken Lie algebra elements $\mathrm{t}_\mathrm{g} := M_\mathrm{g}\,\mathrm{t}\,M_\mathrm{g}^{-1}$ are (block-)diagonalized. Finally, we identify the continuous symmetry and the charges (or representations) of the twisted matter fields $\Phi^i_{(-1/2,-1/2),\mathrm{g}}$. The results are summarized in table 2.

At $\langle T\rangle = \langle U\rangle$, there appears an enhanced U(1) symmetry. Note that the traditional flavor subgroup $\mathbb{Z}_4 \subset (D_8\times D_8)/\mathbb{Z}_2$ generated by $h_1\,h_3$ is a subgroup of this U(1). Hence, one can verify that the traditional flavor symmetry gets enhanced to

$$G_{\text{enhanced traditional}} = \frac{(\mathrm{U}(1)\rtimes\mathbb{Z}_2)\times D_8}{\mathbb{Z}_2} \quad \text{for} \quad \langle T\rangle = \langle U\rangle\,, \tag{5.15}$$

where the $\mathbb{Z}_2$ quotient identifies $(h_1\,h_3)^2$ from the left factor with $(h_2\,h_4)^2$ from the right factor.

At $\langle T\rangle = \langle U\rangle = \mathrm{i}$ in moduli space, there is an enhanced $\mathrm{U}(1)^2$ symmetry. In this case, the traditional flavor subgroup

$$\frac{\mathbb{Z}_4\times\mathbb{Z}_4}{\mathbb{Z}_2} \;\cong\; \mathbb{Z}_4\times\mathbb{Z}_2 \;\subset\; \frac{D_8\times D_8}{\mathbb{Z}_2} \tag{5.16}$$

generated by the order 4 elements $h_1\,h_3$ and $h_2\,h_4$ is a subgroup of this $\mathrm{U}(1)^2$ symmetry. Hence, the traditional flavor symmetry is enhanced to

$$G_{\text{enhanced traditional}} = \frac{(\mathrm{U}(1)\rtimes\mathbb{Z}_2)\times(\mathrm{U}(1)\rtimes\mathbb{Z}_2)}{\mathbb{Z}_2} \quad \text{for} \quad \langle T\rangle = \langle U\rangle = \mathrm{i}\,, \tag{5.17}$$

where the $\mathbb{Z}_2$ quotient identifies $(h_1\,h_3)^2$ with $(h_2\,h_4)^2$, as before.

Finally, in the case $\langle T\rangle = \langle U\rangle = \omega$, an enhanced SU(2) symmetry emerges. Note that the traditional flavor group $[32,49]$ can be written as

$$[32,49] \;\cong\; \frac{D_8\times D_8}{\mathbb{Z}_2} \;\cong\; \frac{Q_8\times Q_8}{\mathbb{Z}_2}\,, \tag{5.18}$$

where the first $Q_8 \cong [8,4]$ is generated by the elements $h_1\,h_3$ and $h_1\,h_2\,h_4$, the second Q_8 is generated by $h_1h_3h_2$ and $h_1h_3h_4$, and the $\mathbb{Z}_2$ identifies the elements $-\mathbb{1}$ of both groups in each product. Now, it turns out that the first Q_8 factor is contained in SU(2). Then, we can write the enhanced traditional flavor symmetry at $T = U = \omega$ as

$$G_{\text{enhanced traditional}} = \frac{\mathrm{SU}(2)\times Q_8}{\mathbb{Z}_2} \quad \text{for} \quad \langle T\rangle = \langle U\rangle = \omega\,. \tag{5.19}$$

6 Conclusions and outlook

We performed a detailed analysis of the modular symmetries of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold which (among others) might be relevant for the (discrete) flavor symmetries of string compactifications with an elliptic fibration. The $\mathbb{T}^2/\mathbb{Z}_2$ case has two unconstrained moduli with

$\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ modular symmetry and allows contact with previous bottom-up constructions that have more than one modulus [16, 17, 35–37]. In the present paper, we completed the discussion of our earlier work [1] now including the automorphy factors of modular symmetry. This leads to an additional R-symmetry $\mathbb{Z}_4^R$ (for the given modular weights of matter fields) that plays the role of a so-called "shaping symmetry" and extends the discrete flavor symmetry. In more detail, the traditional flavor symmetry of ref. [1] is extended from [32,49] via $\mathbb{Z}_4^R$ to [64,216], see eq. (3.13). Together with the finite modular flavor symmetry [144,115] (and [288,880] including $\mathcal{CP}$) discussed in section 3.1 and appendix D, this leads to an eclectic flavor symmetry of order 4608 (and order 9216 with $\mathcal{CP}$).

This picture reveals the fact that the top-down discussion of modular flavor symmetry constitutes an extremely restrictive scenario, which is confirmed in other top-down scenarios [38–42]. As in the case of the bottom-up discussion, firstly the role of (otherwise freely chosen) flavons is played by the moduli T and U, and secondly we arrive at a specific finite modular group, being $\Gamma_2^T \times \Gamma_2^U = S_3^T \times S_3^U$ for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold. In addition, we have to consider the restrictions from the automorphy factors with modular weights fixed from string theory (in contrast to the bottom-up case where these values can be chosen freely). Moreover, in addition to the finite modular symmetry, string theory provides a traditional flavor symmetry, which gives severe restrictions for Kähler- and superpotential of the theory (discussed in section 5). Finally, the representations of the relevant matter fields of the traditional and modular flavor symmetries are determined by the theory. We summarize them (along with the corresponding modular weights) in table 1.

Compared to the earlier discussions [4, 5] where one modulus was frozen, the two-modulus case allows a full understanding of mirror symmetry (as discussed in section 3 and appendix B), including the situation of matter fields whose modular weights n_T and n_U differ from each other, see eq. (3.9). In this case, mirror symmetry requires the presence of matching representations where n_T and n_U are interchanged.

We observe enhancements of the traditional flavor group at specific locations in moduli space. These unified flavor groups are discussed in section 4 and summarized in figure 3. The largest group is located at $T = U = \exp(\pi \mathrm{i}/3)$ and has 2304 elements (including $\mathcal{CP}$). We also provide a detailed discussion of the tetrahedral $\mathbb{T}^2/\mathbb{Z}_2$ orbifold, which leads to the group $[192, 1509]$ as extension of the traditional flavor group. It includes a "geometrical" A_4 as an R-symmetry, where twisted matter fields transform as $\mathbf{3} \oplus \mathbf{1}''$ of this A_4^R, as explained in section 4.3.1.

The restrictions on Kähler- and superpotential are discussed in section 5. The traditional flavor symmetry is extremely powerful towards the restrictions on the Kähler potential. As in the $\mathbb{T}^2/\mathbb{Z}_3$ discussed earlier [6], the traditional flavor group restricts the Kähler potential to its trivial diagonal form (5.3): a fact that seems to hold in full generality. In contrast, both symmetries are relevant for the form of the superpotential given in eq. (5.12). The traditional flavor symmetry reduces the 256 terms in eq. (5.4b) down to 16, and the modular flavor symmetry reduces the remaining 16 to 2, see eq. (5.12b).

A further special feature of string theory is the possible appearance of continuous gauge (flavor) symmetries in moduli space. At special points in moduli space, winding modes of the string can become massless and are candidates for the gauge bosons, as discussed in section 5.3 (see table 2) and appendix C (with table 3). These symmetries, of course, reflect themselves in the symmetries of the superpotential. From a bottom-up perspective they might appear as accidental symmetries, but from the top-down point of view they correspond to continuous gauge symmetries of string theory.

Together with our earlier discussion [3] of the $\mathbb{T}^2/\mathbb{Z}_K$ orbifolds with $K = 3, 4, 6$, we now have uncovered the basic properties of the flavor symmetries of two-dimensional orbifold compactifications for the case of up to two unconstrained moduli. One might expect that some of these properties will generalize from the case of toroidal orbifolds to more general string compactifications with an elliptic fibration. Moreover, from the string theory point of view, the next step would be the consideration of orbifolds with Wilson lines as additional moduli. This would require the embedding of $\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U$ and mirror symmetry in the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$, as discussed in ref. [18], see also refs. [16, 17], where bottom-up model building based on $\mathrm{Sp}(4,\mathbb{Z})$ has been initiated.

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A Strings on orbifolds

Geometrically, two spatial extra dimensions are compactified on a $\mathbb{T}^2/\mathbb{Z}_K$ orbifold by identifying points y in $\mathbb{R}^2$ if they are related by the orbifold action [43–45]

$$y \sim \theta^k\, y + e\, n\,, \tag{A.1}$$

where $k \in \{0, \ldots, K-1\}$ enumerates the twisted sectors and the orbifold twist θ is of order K, i.e. $\theta^K = \mathbb{1}_2$. In addition, the column vectors e_i, $i \in \{1, 2\}$, of the geometrical vielbein e span the two-torus $\mathbb{T}^2$ and $n = (n_1, n_2)^{\mathrm{T}} \in \mathbb{Z}^2$ are called winding numbers. We focus on the case $\theta = -\mathbb{1}_2$ of order $K = 2$ that generates a $\mathbb{Z}_2$ point group. The $\mathbb{T}^2/\mathbb{Z}_2$ orbifold has four inequivalent fixed points where twisted strings are localized. We denote the corresponding matter fields by $(\phi_{(0,0)}, \phi_{(1,0)}, \phi_{(0,1)}, \phi_{(1,1)})^{\mathrm{T}}$, where $\phi_{(n_1,n_2)}$ is localized at the fixed point $y_{\mathrm{f}} = {}^1\!/_2(n_1 e_1 + n_2 e_2)$ satisfying identically the fixed point condition $y_{\mathrm{f}} \stackrel{!}{=} \theta\, y_{\mathrm{f}} + e\, n$ from eq. (A.1).

We focus here on the case without background Wilson lines. In the Narain formulation of the heterotic string [46–48], the string coordinate in extra-dimensional space y is split into right- and left-moving string modes y_{R} and y_{L}, respectively. Then, we define

$$Y := \begin{pmatrix} y_{\mathrm{R}} \\ y_{\mathrm{L}} \end{pmatrix} \tag{A.2}$$

and eq. (A.1) is extended to

$$Y \sim \Theta^k\, Y + E\, \hat{N}\,, \quad \text{where} \quad \Theta = \begin{pmatrix} \theta_{\mathrm{R}} & 0 \\ 0 & \theta_{\mathrm{L}} \end{pmatrix} \quad \text{and} \quad \hat{N} = \begin{pmatrix} n \\ m \end{pmatrix} \ \in\ \mathbb{Z}^4\,. \tag{A.3}$$

Here, $k \in \{0, \ldots, K-1\}$ for a Narain twist Θ, with θ_{R}, $\theta_{\mathrm{L}} \in \mathrm{SO}(2)$, that is of order K, i.e. $\Theta^K = \mathbb{1}_4$. The Narain twist generates the Narain point group $P_{\text{Narain}} \cong \mathbb{Z}_K$ and the orbifold action $Y \mapsto \Theta^k Y + E\hat{N}$ defines the so-called Narain space group S_{Narain},

$$Y \;\mapsto\; g\,Y := \Theta^k\,Y + E\,\hat{N}\,, \quad \text{where} \quad g = \left(\Theta^k, E\,\hat{N}\right) \;\in\; S_{\text{Narain}}\,. \tag{A.4}$$

Each (conjugacy class) $g \in S_{\text{Narain}}$ defines a closed string and, therefore, we call g the constructing element. We focus on symmetric orbifolds by setting $\theta_{\mathrm{R}} = \theta_{\mathrm{L}} = \theta$ and choose $\theta = -\mathbb{1}_2$ for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold. Furthermore, $\hat{N} = (n, m)^{\mathrm{T}}$ contains winding numbers $n \in \mathbb{Z}^2$ and Kaluza-Klein numbers $m \in \mathbb{Z}^2$, so that the vectors $E\,\hat{N}$ give rise to a four-dimensional auxiliary lattice $\Gamma = \{E\hat{N} \mid \hat{N} \in \mathbb{Z}^4\}$, called the Narain lattice. The 4×4 matrix E is called the Narain vielbein. Due to worldsheet modular invariance of the one-loop string partition function, E has to satisfy the condition

$$E^{\mathrm{T}}\eta\,E = \hat{\eta} := \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}\,, \quad \text{where} \quad \eta := \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}\,. \tag{A.5}$$

Therefore, the Narain lattice Γ is an even, integer, self-dual lattice of signature $(2, 2)$. In the absence of Wilson lines, the Narain vielbein E can be parameterized in terms of the geometrical vielbein e, its inverse transposed $e^{-\mathrm{T}}$, the geometrical metric $G = e^{\mathrm{T}}e$ and the B-field background B,

$$E := \begin{pmatrix} \frac{1}{\sqrt{2\alpha'}}e^{-\mathrm{T}}(G+B) & -\sqrt{\frac{\alpha'}{2}}e^{-\mathrm{T}} \\ \frac{1}{\sqrt{2\alpha'}}e^{-\mathrm{T}}(G-B) & \sqrt{\frac{\alpha'}{2}}e^{-\mathrm{T}} \end{pmatrix}\,, \tag{A.6}$$

see for example refs. [49, 50] (where we changed the convention from B to $-B$). Then, a two-torus compactification can be parameterized by a Kähler modulus T and a complex structure modulus U, defined as

$$T := \frac{1}{\alpha'}\left(B_{12} + \mathrm{i}\sqrt{\det G}\right)\,, \tag{A.7a}$$

$$U := \frac{1}{G_{11}}\left(G_{12} + \mathrm{i}\sqrt{\det G}\right) = \frac{e_2}{e_1}\,. \tag{A.7b}$$

In the last equation of U, we have taken both two-dimensional column vectors e_i of the geometrical vielbein e to be complex numbers, $e_i \in \mathbb{C}$, so that e_2/e_1 is defined. Note that T determines the strength of the B-field and the area of the extra-dimensional two-torus, while U specifies the shape of the two-torus. It is convenient to associate a generalized metric $\mathcal{H}$ to the Narain vielbein E and express $\mathcal{H}$ in terms of the moduli T and U,

$$\mathcal{H}(T,U) = E^{\mathrm{T}}E = \frac{1}{\operatorname{Im}T\operatorname{Im}U}\begin{pmatrix} |T|^2 & |T|^2\operatorname{Re}U & -\operatorname{Re}T\operatorname{Re}U & \operatorname{Re}T \\ |T|^2\operatorname{Re}U & |T\,U|^2 & -|U|^2\operatorname{Re}T & \operatorname{Re}T\operatorname{Re}U \\ -\operatorname{Re}T\operatorname{Re}U & -|U|^2\operatorname{Re}T & |U|^2 & -\operatorname{Re}U \\ \operatorname{Re}T & \operatorname{Re}T\operatorname{Re}U & -\operatorname{Re}U & 1 \end{pmatrix}, \tag{A.8}$$

see for example ref. [3]. Furthermore, we define the Narain twist in the lattice basis as

$$\hat{\Theta} := E^{-1}\Theta\,E \;\in\; \mathrm{GL}(4,\mathbb{Z})\,, \tag{A.9}$$

such that it maps the Narain lattice Γ to itself. $\hat{\Theta}$ generates the Narain point group in the lattice basis $\hat{P}_{\text{Narain}}$. Moreover, due to the left-right structure of $\hat{\Theta}$ given in eq. (A.3), $\hat{\Theta}$ has to satisfy the conditions

$$\hat{\Theta}^{\mathrm{T}}\hat{\eta}\,\hat{\Theta} = \hat{\eta} \quad \text{and} \quad \hat{\Theta}^{\mathrm{T}}\mathcal{H}(T,U)\,\hat{\Theta} = \mathcal{H}(T,U)\,. \tag{A.10}$$

In general, the condition that involves the generalized metric $\mathcal{H}(T,U)$ can stabilize T and/or U. However, for the symmetric $\mathbb{T}^2/\mathbb{Z}_2$ orbifold under consideration, both moduli remain unconstrained.

Let us focus in the following on bulk strings, i.e. on strings that close under the identification eq. (A.3) with constructing element $(\mathbb{1}_4, E\hat{N}) \in S_{\text{Narain}}$. Then, right- and left-moving momenta p_{R} and p_{L} of a string have to be quantized, because the extra dimensions are compact. As the Narain lattice Γ is self-dual, p_{R} and p_{L} must belong to Γ, too. Hence,

$$P := \begin{pmatrix} p_{\mathrm{R}} \\ p_{\mathrm{L}} \end{pmatrix} = E\,\hat{N} \;\in\; \Gamma\,. \tag{A.11}$$

Then, one can see easily using eqs. (A.5) and (A.8) that

$$-\,(p_{\mathrm{R}})^2 + (p_{\mathrm{L}})^2 = \hat{N}^{\mathrm{T}}\hat{\eta}\,\hat{N} = 2\,n^{\mathrm{T}}m \qquad \text{and} \qquad (p_{\mathrm{R}})^2 + (p_{\mathrm{L}})^2 = \hat{N}^{\mathrm{T}}\mathcal{H}(T,U)\,\hat{N}\,. \tag{A.12}$$

In order to identify (massless) string states from the bulk, one has to consider the right- and left-moving mass equations

$$\frac{\alpha'}{2}M_{\mathrm{R}}^2 = q^2 + (p_{\mathrm{R}})^2 + 2\left(N_{\mathrm{R}} - \frac{1}{2}\right)\,, \tag{A.13a}$$

$$\frac{\alpha'}{2}M_{\mathrm{L}}^2 = p^2 + (p_{\mathrm{L}})^2 + 2\left(N_{\mathrm{L}} - 1\right)\,, \tag{A.13b}$$

where $N_{\mathrm{R}} \geq 0$ and $N_{\mathrm{L}} \geq 0$ count the number of right- and left-moving oscillator excitations. In addition, the mass equations (A.13) are subject to level-matching $M_{\mathrm{R}}^2 = M_{\mathrm{L}}^2$. Note that the total mass $M^2(\hat{N};T,U)$ of a bulk string with winding and KK numbers $\hat{N} \in \mathbb{Z}^4$ depends on the moduli T and U via

$$M^2(\hat{N};T,U) := \frac{\alpha'}{2}\left(M_{\mathrm{R}}^2 + M_{\mathrm{L}}^2\right) = \hat{N}^{\mathrm{T}}\mathcal{H}(T,U)\,\hat{N} + q^2 + p^2 + 2\left(N_{\mathrm{R}} + N_{\mathrm{L}} - \frac{3}{2}\right)\,. \tag{A.14}$$

In eq. (A.13a), $q = (q_0, q_1, q_2, q_3)$ denotes the bosonized momentum of the right-moving worldsheet fermions. It is called the H-momentum. q has to be an element of one of the following weight lattices of SO(8): either the vector lattice $\mathbf{8}_{\mathrm{v}}$ or the spinor lattice $\mathbf{8}_{\mathrm{s}}$, see for example ref. [51]. The shortest H-momenta q satisfy $q^2 = 1$, i.e.

$$q \;\in\; \left\{ \left(\underline{\pm 1, 0, 0, 0}\right)\,,\; \left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right) \right\}. \tag{A.15}$$

Here, in the first case ($\mathbf{8}_{\mathrm{v}}$), the underline denotes all permutations and, in the second case ($\mathbf{8}_{\mathrm{s}}$), the number of plus-signs must be even. The first component q_0 of q defines the four-dimensional chirality. For example, $q_0 = 0$ yields a scalar. Note that in the four-dimensional effective quantum field theory, we use the convention that the scalar components of left-chiral superfields ϕ from the bulk are associated with string states having $q \in \{\left(0, \underline{+1, 0, 0}\right)\}$, such that string states with $q \in \{\left(0, \underline{-1, 0, 0}\right)\}$ give rise to their CPT partners.

A.1 The origin of modular symmetries

The rotational outer automorphisms of the $(2,2)$-dimensional Narain lattice Γ are given by those transformations Σ that satisfy for all $E\hat{N} \in \Gamma$

$$\left(\mathbb{1}_4, E\,\hat{N}\right) \;\mapsto\; (\Sigma,0)^{-1}\left(\mathbb{1}_4, E\,\hat{N}\right)(\Sigma,0) = \left(\mathbb{1}_4, E\,\hat{\Sigma}^{-1}\hat{N}\right) \;\stackrel{!}{=}\; \left(\mathbb{1}_4, E\,\hat{N}'\right)\,, \tag{A.16}$$

where we defined $\hat{\Sigma} := E^{-1}\Sigma E$ and $\hat{N}' = \hat{\Sigma}^{-1}\hat{N} \in \mathbb{Z}^4$ so that $E\hat{N}' \in \Gamma$. Hence, $\hat{\Sigma} \in \mathrm{GL}(4,\mathbb{Z})$. Furthermore, we demand that Σ leaves the metric η invariant,

$$\Sigma^{\mathrm{T}}\eta\,\Sigma \;\stackrel{!}{=}\; \eta \qquad\Leftrightarrow\qquad \hat{\Sigma}^{\mathrm{T}}\hat{\eta}\,\hat{\Sigma} \;\stackrel{!}{=}\; \hat{\eta}\,. \tag{A.17}$$

In other words, we demand that Σ leaves any Narain scalar product $P^{\mathrm{T}}\eta P'$ for $P, P' \in \Gamma$ invariant. The resulting transformations $\hat{\Sigma}$ form a group

$$\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) := \langle\; \hat{\Sigma} \mid \hat{\Sigma} \;\in\; \mathrm{GL}(4,\mathbb{Z}) \quad\text{with}\quad \hat{\Sigma}^{\mathrm{T}}\hat{\eta}\,\hat{\Sigma} = \hat{\eta} \;\rangle\,, \tag{A.18}$$

the so-called modular group of the Narain lattice Γ. It is easy to see that $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ contains two factors of $\mathrm{SL}(2,\mathbb{Z})$, i.e. we can define

$$\hat{\Sigma}_{(\gamma_T,\gamma_U)} := \begin{pmatrix} d_T\,a_U & -d_T\,b_U & -c_T\,b_U & -c_T\,a_U \\ -d_T\,c_U & d_T\,d_U & c_T\,d_U & c_T\,c_U \\ -b_T\,c_U & b_T\,d_U & a_T\,d_U & a_T\,c_U \\ -b_T\,a_U & b_T\,b_U & a_T\,b_U & a_T\,a_U \end{pmatrix}. \tag{A.19}$$

Then, $\hat{\Sigma}_{(\gamma_T,\gamma_U)} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ if

$$\gamma_T := \begin{pmatrix} a_T & b_T \\ c_T & d_T \end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{Z})_T \quad\text{and}\quad \gamma_U := \begin{pmatrix} a_U & b_U \\ c_U & d_U \end{pmatrix} \;\in\; \mathrm{SL}(2,\mathbb{Z})_U\,. \tag{A.20}$$

As a remark, $\hat{\Sigma}_{(\gamma_T,\gamma_U)}$ satisfies the property of being a representation,

$$\hat{\Sigma}_{(\gamma_T,\gamma_U)}\,\hat{\Sigma}_{(\delta_T,\delta_U)} = \hat{\Sigma}_{(\gamma_T\,\delta_T,\gamma_U\,\delta_U)}\,, \tag{A.21}$$

for all $\gamma_T, \delta_T \in \mathrm{SL}(2,\mathbb{Z})_T$ and $\gamma_U, \delta_U \in \mathrm{SL}(2,\mathbb{Z})_U$. The generators S and T of the modular group $\mathrm{SL}(2,\mathbb{Z})$ can be represented by the 2×2 matrices

$$\mathrm{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad\text{and}\quad \mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,, \tag{A.22}$$

respectively. Then, we can define

$$\hat{K}_{\mathrm{S}} := \hat{\Sigma}_{(\mathrm{S},\mathbb{1}_2)}\,,\quad \hat{K}_{\mathrm{T}} := \hat{\Sigma}_{(\mathrm{T},\mathbb{1}_2)} \quad\text{and}\quad \hat{C}_{\mathrm{S}} := \hat{\Sigma}_{(\mathbb{1}_2,\mathrm{S})}\,,\quad \hat{C}_{\mathrm{T}} := \hat{\Sigma}_{(\mathbb{1}_2,\mathrm{T})}\,, \tag{A.23}$$

where $\hat{K}_{\mathrm{S}}$ and $\hat{K}_{\mathrm{T}}$ generate the $\mathrm{SL}(2,\mathbb{Z})_T$ factor associated with the Kähler modulus T and $\hat{C}_{\mathrm{S}}$ and $\hat{C}_{\mathrm{T}}$ generate the $\mathrm{SL}(2,\mathbb{Z})_U$ factor associated with the complex structure modulus U. The two remaining generators of $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ are[3]

$$\hat{M} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad\text{and}\quad \hat{\Sigma}_* := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,, \tag{A.24}$$

[3] Following ref. [18], we have changed the conventions compared to ref. [5] by redefining $\hat{K}_{\mathrm{S}}$, $\hat{C}_{\mathrm{S}}$ and $\hat{M}$.

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where one can easily show that mirror symmetry $\hat{M}$ interchanges the $\mathrm{SL}(2,\mathbb{Z})$ factors,

$$\hat{M}\,\hat{\Sigma}_{(\gamma_T,\gamma_U)}\,\hat{M}^{-1} = \hat{\Sigma}_{(\gamma_U,\gamma_T)}\,. \tag{A.25}$$

Having identified the modular symmetries of a toroidal compactification, the modular symmetries of an orbifold are given by the rotational outer automorphisms of the Narain space group S_{Narain} that preserve the Narain metric η. They can be understood as those modular transformations $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(D,D,\mathbb{Z})$ (with $D = 2$ in the present case) that are also from the normalizer of the Narain point group,

$$\hat{\Sigma}\,\hat{P}_{\text{Narain}}\,\hat{\Sigma}^{-1} = \hat{P}_{\text{Narain}}\,. \tag{A.26}$$

Note that the Narain twist is not an outer automorphism, but an inner automorphism of S_{Narain}. For example, for the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold we have $\hat{\Theta} = -\mathbb{1}_4$ and $\hat{P}_{\text{Narain}} \cong \mathbb{Z}_2$. Hence, $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})/\mathbb{Z}_2$ is the modular group of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold. However, we consider the two-dimensional $\mathbb{T}^2/\mathbb{Z}_2$ orbifold to be contained in a full six-dimensional orbifold. Hence, we assume that the underlying six-dimensional torus is factorized as $\mathbb{T}^6 = \mathbb{T}^2 \oplus \mathbb{T}^2 \oplus \mathbb{T}^2$ and that the Narain twist of the $(6,6)$-dimensional Narain lattice takes the form

$$\hat{\Theta} = \hat{\Theta}_{(2)} \;\oplus\; \hat{\Theta}_{(K_2)} \;\oplus\; \hat{\Theta}_{(K_3)}\,. \tag{A.27}$$

Here, $\hat{\Theta}_{(K_i)}$ denotes an order K_i Narain twist of the i-th $(2,2)$-dimensional Narain sublattice for $i \in \{1,2,3\}$, where $K_1 = 2$ and $\hat{\Theta}_{(2)} = -\mathbb{1}_4$. Then, we can define a so-called sublattice rotation $\hat{\Theta}_{(2)} \oplus \mathbb{1}_4 \oplus \mathbb{1}_4$ which is an outer automorphism of the Narain space group of the full six-dimensional orbifold. Consequently, the modular group in the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector is $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$.

A.2 Transformation of bulk fields under modular symmetries

In this section, we analyze the action of modular transformations from $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ on those fields of the effective four-dimensional theory that originate from the bulk of the extra dimensions. The transformation of twisted matter fields will be discussed later in appendix D.

First, we discuss the moduli (i.e. the Kähler modulus T and the complex structure modulus U, see eq. (A.7)). According to eq. (A.16), a modular transformation $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ acts as

$$E \;\xmapsto{\hat{\Sigma}}\; E\,\hat{\Sigma}^{-1} \tag{A.28}$$

on the Narain vielbein E. Consequently, we can use the generalized metric to compute the transformation of the moduli,

$$\mathcal{H}(T,U) \;\xmapsto{\hat{\Sigma}}\; \hat{\Sigma}^{-\mathrm{T}}\mathcal{H}(T,U)\hat{\Sigma}^{-1} =: \mathcal{H}(T',U')\,. \tag{A.29}$$

This can be used to show that $\hat{\Sigma}_{(\gamma_T,\gamma_U)}$ from eq. (A.19) acts on the moduli as

$$T \;\mapsto\; \frac{a_T\,T + b_T}{c_T\,T + d_T} \quad\text{and}\quad U \;\mapsto\; \frac{a_U\,U + b_U}{c_U\,U + d_U}\,. \tag{A.30}$$

Moreover, using eq. (A.29) we can confirm that the mirror transformation $\hat{M}$ interchanges the moduli, $T \leftrightarrow U$, while the $\mathcal{CP}$-like transformation $\hat{\Sigma}_*$ acts as

$$T \xmapsto{\hat{\Sigma}_*} -\bar{T} \quad \text{and} \quad U \xmapsto{\hat{\Sigma}_*} -\bar{U}\,. \tag{A.31}$$

As a remark, we can now understand the conditions (A.10) on a Narain twist as follows: a Narain twist $\hat{\Theta} \in \hat{P}_{\text{Narain}}$ must be a modular transformation ($\hat{\Theta} \in \text{O}_{\hat{\eta}}(2,2,\mathbb{Z})$) that leaves the moduli invariant (compare eq. (A.29) to eq. (A.10)).

Next, we consider a general (massive) bulk field $\phi_{(\hat{N})}$ labeled by its winding and KK numbers $\hat{N} \in \mathbb{Z}^4$ that corresponds to a closed string with boundary condition (A.4) given by the constructing element $(\mathbb{1}_4, E\,\hat{N})$. Its total mass $M^2(\hat{N};T,U)$ is moduli dependent via $\hat{N}^{\text{T}}\mathcal{H}(T,U)\hat{N}$, as shown in eq. (A.14). Then, the corresponding mass terms in the superpotential read schematically

$$\mathcal{W} \supset \sum_{\hat{N}\in\mathbb{Z}^4} M^2(\hat{N};T,U)\,\left(\phi_{(\hat{N})}\right)^2\,. \tag{A.32}$$

Under a (non-$\mathcal{CP}$-like) modular transformation $\hat{\Sigma} \in \text{O}_{\hat{\eta}}(2,2,\mathbb{Z})$, moduli and bulk fields transform as

$$T \xmapsto{\hat{\Sigma}} T'\,, \quad U \xmapsto{\hat{\Sigma}} U' \quad \text{and} \quad \phi_{(\hat{N})} \xmapsto{\hat{\Sigma}} \phi'_{(\hat{N})} = \pm\phi_{(\hat{N}')}\,, \tag{A.33}$$

where we suppress the automorphy factor for $\phi'_{(\hat{N})}$. In addition, we have $\hat{N}' = \hat{\Sigma}^{-1}\hat{N}$ as shown in eq. (A.16) and the factor ±1 of $\phi_{(\hat{N}')}$ will be derived later in eq. (D.15). Then, due to its moduli-dependence, the total string mass $M^2(\hat{N};T,U)$ transforms as

$$M^2(\hat{N};T,U) \xmapsto{\hat{\Sigma}} M^2(\hat{N};T',U') = M^2(\hat{N}';T,U)\,, \tag{A.34}$$

using eq. (A.29). Thus, we obtain

$$M^2(\hat{N};T,U)\,\left(\phi_{(\hat{N})}\right)^2 \xmapsto{\hat{\Sigma}} M^2(\hat{N};T',U')\,\left(\phi'_{(\hat{N})}\right)^2 = M^2(\hat{N}';T,U)\,\left(\phi_{(\hat{N}')}\right)^2\,, \tag{A.35}$$

as expected for the superpotential $\mathcal{W}$.

B Action of mirror symmetry on strings with $n_T \neq n_U$

In the string construction of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector (without Wilson lines), a string state reads

$$|p_{\text{R}};\,q_{\text{sh}}\rangle_{\text{R}} \;\otimes\; \prod_f \left(\tilde{\alpha}^i_f\right)^{N^i_f}\,\left(\tilde{\alpha}^{\bar{i}}_f\right)^{\bar{N}^{\bar{i}}_f}\,|p_{\text{L}};\,p_{\text{sh}}\rangle_{\text{L}}\,, \tag{B.1}$$

where $q_{\text{sh}} := q + kv$ is the so-called shifted right-moving H-momentum and $p_{\text{sh}} := p + kV$ denotes the shifted left-moving gauge momentum. In addition, the string state (B.1) is excited by $N^i_f \in \mathbb{N}_0$ left-moving bosonic oscillators $\tilde{\alpha}^i_f$ of "frequency" $f < 0$ in the complex direction $z^i := y^{2i-1}_{\text{L}} + \text{i}\,y^{2i}_{\text{L}}$ for $i \in \{1,2,3\}$ (i.e. holomorphic oscillators), while $\bar{N}^{\bar{i}}_f \in \mathbb{N}_0$

counts the (independent) anti-holomorphic oscillators in the direction $\bar{z}^{\bar{i}}$. Moreover, $k \in \{0,1\}$ corresponds to an untwisted or twisted string, respectively. The right- and left-moving momenta $(p_\mathrm{R}, p_\mathrm{L})$ are given in eq. (A.11), q is from an SO(8) weight lattice and p specifies the gauge charges as p is from the $\mathrm{E}_8\times\mathrm{E}_8$ (or Spin(32)/$\mathbb{Z}_2$) weight lattice. Finally, we assume an orbifold twist θ with twist vector $v = (0, {}^1\!/\!_2, v_2, v_3)$, such that the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector is in the first out of three complex extra dimensions, while the so-called shift vector V determines the gauge embedding of v. Then, the modular weights (n_T, n_U) of the string state eq. (B.1) are defined as [19, 31]

$$\text{if } q^1_\mathrm{sh} \in \{0,1\} \;:\; n_T := -q^1_\mathrm{sh} \quad,\quad n_U := -q^1_\mathrm{sh}\,, \tag{B.2a}$$

$$\text{if } q^1_\mathrm{sh} \notin \{0,1\} \;:\; n_T := -1 + q^1_\mathrm{sh} - \Delta N^1 \quad,\quad n_U := -1 + q^1_\mathrm{sh} + \Delta N^1\,, \tag{B.2b}$$

where $\Delta N^i := N^i - \bar{N}^{\bar{i}} \in \mathbb{N}_0$ is the total number of holomorphic minus anti-holomorphic oscillators with internal index $i = 1$ or $\bar{i} = \bar{1}$ in the direction of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector, i.e. $N^i = \sum_f N^i_f$ and $\bar{N}^{\bar{i}} = \sum_f \bar{N}^{\bar{i}}_f$. Hence,

$$n_U - n_T = 0 \bmod 2\,, \tag{B.3}$$

and n_T and n_U coincide if the associated string state carries no oscillator excitations. For example, a massless matter field from the bulk ($k = 0$) has no oscillators and $q \in \{\big(0, \underline{+1, 0, 0}\big)\}$, so that $n_T = n_U \in \{0, -1\}$, while a massless twisted matter field ($k = 1$) without oscillators has $q^1_\mathrm{sh} = \frac{1}{2}$ and $n_T = n_U = -{}^1\!/\!_2$, see table 1. In addition, there are twisted string states (massless or massive) that are excited by oscillators. According to eq. (B.2), the modular weights are increased/decreased by adding oscillator excitations,

$$\text{add holomorphic oscillator}: \quad n_T \mapsto n_T - 1\,, \quad n_U \mapsto n_U + 1 \tag{B.4a}$$

$$\text{add anti-holomorphic oscillator}: \quad n_T \mapsto n_T + 1\,, \quad n_U \mapsto n_U - 1\,. \tag{B.4b}$$

In both cases, the resulting string states transform identically under the $\mathbb{Z}_2$ orbifold projection (this is a special property of $\mathbb{Z}_2$ orbifolds and not true for $\mathbb{Z}_K$ orbifolds with $K \neq 2$). Hence, for each matter field $\Phi_{(n_T,n_U)}$ with $n_T \neq n_U$, there exists a partner with exactly the same mass and identical quantum numbers except for interchanged weights, i.e. $\Phi_{(n_T,n_U)}$ has a partner $\Phi_{(n_U,n_T)}$ if $n_T \neq n_U$, cf. table 1.

Mirror symmetry interchanges holomorphic and anti-holomorphic left-moving oscillators. In order to see this, we rewrite $\hat{M}$ (given in eq. (A.24)) into the left-right coordinate basis $(y_\mathrm{R}, y_\mathrm{L})$ at $T = U$ in moduli space. This results in

$$M := E\,\hat{M}\,E^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,, \quad \text{such that} \quad \begin{pmatrix} y^1_\mathrm{R} \\ y^2_\mathrm{R} \\ y^1_\mathrm{L} \\ y^2_\mathrm{L} \end{pmatrix} \xmapsto{M} M \begin{pmatrix} y^1_\mathrm{R} \\ y^2_\mathrm{R} \\ y^1_\mathrm{L} \\ y^2_\mathrm{L} \end{pmatrix}\,. \tag{B.5}$$

Recall that a general transformation $\Sigma := E\,\hat{\Sigma}\,E^{-1}$ acts on the coordinate Y eq. (A.2) as $Y \xmapsto{\Sigma} \Sigma\,Y$ [5, appendix A.2]. Hence, the mirror transformation M acts on the complex left-moving string coordinate z^1 in the direction of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector as

$$z^1 := y^1_\mathrm{L} + \mathrm{i}\,y^2_\mathrm{L} \xmapsto{M} y^1_\mathrm{L} - \mathrm{i}\,y^2_\mathrm{L} = \bar{z}^{\bar{1}}\,. \tag{B.6}$$

Hence, a mirror transformation interchanges holomorphic and anti-holomorphic oscillators resulting in eq. (3.21).

C Gauge symmetry enhancement

It is a well-known feature of string theory that at special points (T, U) in moduli space, additional gauge symmetries arise whose gauge bosons are associated with massless winding strings. These massless strings become massive by moving in moduli space away from the special points. Hence, the enhanced gauge symmetry gets broken spontaneously by the moduli vevs. In order to identify the enhanced gauge symmetries, we look for additional massless strings from the orbifold bulk that become massless only at certain points in moduli space. We do this in two steps: first, we construct the massless string states on the torus $\mathbb{T}^2$ and then move on to the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold by projecting the massless torus states onto $\mathbb{Z}_2$-invariant states.

In general, a massless string has to satisfy $M_{\mathrm{R}}^2 = M_{\mathrm{L}}^2 = 0$. Then, from eq. (A.13a) together with $q^2 = 1$ it follows that $N_{\mathrm{R}} = 0$ and $p_{\mathrm{R}} = 0$. Hence, for $p_{\mathrm{R}} = 0$ eqs. (A.11) and (A.12) yield

$$p_{\mathrm{L}} = \sqrt{\frac{2}{\alpha'}}\, e\, n \quad \text{and} \quad (p_{\mathrm{L}})^2 = \hat{N}^{\mathrm{T}} \mathcal{H}(T, U)\, \hat{N} = 2\, n^{\mathrm{T}} m\,. \tag{C.1}$$

For generic points (T, U) in moduli space, eq. (C.1) is satisfied only for $\hat{N} = (0^4)$, i.e. massless strings carry in general neither KK numbers m nor winding numbers n along the compactified dimensions. However, for special points (T, U) in moduli space, additional massless strings can originate from specific solutions of the left-moving mass equation (A.13b),

$$N_{\mathrm{L}} = 0\,, \qquad p = 0\,, \qquad (p_{\mathrm{L}})^2 = 2 \Rightarrow M_{\mathrm{L}}^2 = 0\,. \tag{C.2}$$

Consequently, we can find additional massless strings if the following conditions are satisfied

$$\hat{N}^{\mathrm{T}} \mathcal{H}(T, U)\, \hat{N} = 2 \quad \text{and} \quad n^{\mathrm{T}} m = 1 \quad \text{for} \quad \hat{N} = \begin{pmatrix} n \\ m \end{pmatrix} \in \mathbb{Z}^4\,. \tag{C.3}$$

We denote the set of all solutions $\{\hat{N}\}$ of eq. (C.3) at (T, U) in moduli space by $N_{\mathrm{g}}(T, U) \subset \mathbb{Z}^4$.

Note that if $\hat{N} \in N_{\mathrm{g}}(T, U)$, also $-\hat{N} \in N_{\mathrm{g}}(T, U)$ is a solution to eq. (C.3). This statement can be generalized as follows: assume that there is a transformation $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2, 2, \mathbb{Z})$, such that a specific point (T, U) in moduli space is invariant under $\hat{\Sigma}$ (using the transformation of the generalized metric given in eq. (A.29)). Now, take a massless string solution $\hat{N} \in N_{\mathrm{g}}(T, U)$. Then, define $\hat{N}' := \hat{\Sigma}\, \hat{N}$ and consider

$$\hat{N}'^{\mathrm{T}} \hat{\eta}\, \hat{N}' = \hat{N}^{\mathrm{T}} \hat{\Sigma}^{\mathrm{T}} \hat{\eta}\, \hat{\Sigma}\, \hat{N} = \hat{N}^{\mathrm{T}} \hat{\eta}\, \hat{N} = 2 \Rightarrow n'^{\mathrm{T}} m' = 1\,, \tag{C.4}$$

and

$$\hat{N}'^{\mathrm{T}} \mathcal{H}(T, U)\, \hat{N}' = \hat{N}^{\mathrm{T}} \hat{\Sigma}^{\mathrm{T}} \mathcal{H}(T, U)\, \hat{\Sigma}\, \hat{N} = \hat{N}^{\mathrm{T}} \mathcal{H}(T, U)\, \hat{N} = 2\,. \tag{C.5}$$

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Hence, we see from eq. (C.3) that also $\hat{N}' \in N_{\mathrm{g}}(T,U)$ corresponds to a massless string. The set of transformations $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ that leave the specific point (T,U) in moduli space invariant is defined as the stabilizer subgroup $H_{(T,U)}$ of $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ at (T,U).

The set of solutions $N_{\mathrm{g}}(T,U)$ gives rise to additional massless string states with non-trivial left-moving momenta p_{L},

$$|0;\, q_{\mathrm{g}}\rangle_{\mathrm{R}} \;\otimes\; |p_{\mathrm{L}};\, 0\rangle_{\mathrm{L}} \quad \text{with} \quad p_{\mathrm{L}} = \sqrt{\frac{2}{\alpha'}}\, e\, n \quad \text{for} \quad \hat{N} = \begin{pmatrix} n \\ m \end{pmatrix} \;\in\; N_{\mathrm{g}}(T,U)\,, \tag{C.6}$$

see eq. (B.1) (with $k = 0$ for the bulk). As we are especially interested in the massless gauge bosons we choose $q_{\mathrm{g}} := (\pm 1, 0, 0, 0)$ in eq. (A.15). In addition, there are two massless gauge bosons with $p_{\mathrm{R}} = p_{\mathrm{L}} = (0^2)$, $p = (0^{16})$, $q_{\mathrm{g}} = (\pm 1, 0, 0, 0)$ and $N_{\mathrm{L}} = 1$. The associated string states read

$$|0;\, q_{\mathrm{g}}\rangle_{\mathrm{R}} \;\otimes\; \tilde{\alpha}^{1}_{-1}|0;\, 0\rangle_{\mathrm{L}} \quad \text{and} \quad |0;\, q_{\mathrm{g}}\rangle_{\mathrm{R}} \;\otimes\; \tilde{\alpha}^{\bar{1}}_{-1}|0;\, 0\rangle_{\mathrm{L}} \tag{C.7}$$

where the indices $i = 1$ and $\bar{i} = \bar{1}$ lie in the two-torus that will be orbifolded by the $\mathbb{Z}_2$ action.

Note that the string states (C.7) correspond to the Cartan generators, while the string states (C.6) correspond to raising operators (with $+\hat{N} \in N_{\mathrm{g}}(T,U)$) and lowering operators (with $-\hat{N} \in N_{\mathrm{g}}(T,U)$) of some non-Abelian, enhanced gauge symmetry. The root lattice of this symmetry group is spanned by the left-moving momenta p_{L} that correspond to the solutions $\hat{N} \in N_{\mathrm{g}}(T,U)$ using eqs. (A.6) and (A.11). Thus, the stabilizer subgroup $H_{(T,U)}$ of $\mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})$ at (T,U) in moduli space gives rise to the rotational symmetries of the lattice spanned by $N_{\mathrm{g}}(T,U)$ and, as such, contains the Weyl group W of the resulting gauge symmetry.

Under the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold, the gauge bosons of the Cartan generators (C.7) are projected out (since $\tilde{\alpha}^{1}_{-1} \to -\tilde{\alpha}^{1}_{-1}$ and $\tilde{\alpha}^{\bar{1}}_{-1} \to -\tilde{\alpha}^{\bar{1}}_{-1}$ under the $\mathbb{Z}_2$ orbifold), while the gauge bosons of the raising and lowering operators get combined to $\mathbb{Z}_2$-invariant linear combinations

$$|0;\, q_{\mathrm{g}}\rangle_{\mathrm{R}} \;\otimes\; |+p_{\mathrm{L}};\, 0\rangle_{\mathrm{L}} + |0;\, q_{\mathrm{g}}\rangle_{\mathrm{R}} \;\otimes\; |-p_{\mathrm{L}};\, 0\rangle_{\mathrm{L}}\,, \tag{C.8}$$

where $\pm p_{\mathrm{L}}$ is given by $\pm\hat{N} \in N_{\mathrm{g}}(T,U)$, respectively.

We analyze three special points in moduli space:[4] i) $T = U$, ii) $T = U = \mathrm{i}$ and iii) $T = U = \omega$ and summarize the results in table 3. Consequently, the enhanced continuous symmetries identified in section 5.3 are actually gauge symmetries.

D Vertex operators of the $\mathbb{Z}_2$ Narain orbifold

The spectrum of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector includes untwisted strings, associated with constructing elements $(\mathbb{1}, \hat{N}) \in \hat{S}_{\text{Narain}}$, and twisted strings constructed by elements $(\hat{\Theta}, \hat{N}) \in \hat{S}_{\text{Narain}}$. In this appendix, we study how the symmetries of the theory act on these strings by inspecting the transformations of their corresponding vertex operators.

[4] Note that here, in contrast with section 4, we choose $T = U = \omega$ instead of $T = U = e^{\pi \mathrm{i}/3}$. However, they correspond to equivalent points.

point in moduli space	massless strings on $\mathbb{T}^2$ $N_{\mathrm{g}}(T,U)$	stabilizer $H_{(T,U)}$	gauge symmetry for $\mathbb{T}^2$	gauge symmetry for $\mathbb{T}^2/\mathbb{Z}_2$
$T = U$	$(1,0,1,0)^{\mathrm{T}}$, $(-1,0,-1,0)^{\mathrm{T}}$	$\langle -\mathbb{1}_4, \hat{M}\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathrm{SU}(2)\times\mathrm{U}(1)$	$\mathrm{U}(1)$
$T = U = \mathrm{i}$	$(1,0,1,0)^{\mathrm{T}}$, $(-1,0,-1,0)^{\mathrm{T}}$, $(0,1,0,1)^{\mathrm{T}}$, $(0,-1,0,-1)^{\mathrm{T}}$	$\langle \hat{M}, \hat{C}_{\mathrm{S}}, \hat{\Sigma}_*\rangle \cong [32,27] \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_2^4$	$\mathrm{SU}(2)^2$	$\mathrm{U}(1)^2$
$T = U = \omega$	$(1,0,1,0)^{\mathrm{T}}$, $(-1,0,-1,0)^{\mathrm{T}}$, $(0,1,-1,1)^{\mathrm{T}}$, $(0,-1,1,-1)^{\mathrm{T}}$, $(1,1,0,1)^{\mathrm{T}}$, $(-1,-1,0,-1)^{\mathrm{T}}$	$\langle -\mathbb{1}_4, \hat{M}, \hat{C}_{\mathrm{S}}\hat{C}_{\mathrm{T}}, \hat{\Sigma}_*\hat{C}_{\mathrm{T}}\hat{K}_{\mathrm{T}}\rangle \cong [72,46] \cong S_3 \times S_3 \times \mathbb{Z}_2$	$\mathrm{SU}(3)$	$\mathrm{SU}(2)$

Table 3. Gauge symmetry enhancements of the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold sector at special points in moduli space. We use $\omega := e^{2\pi\mathrm{i}/3}$.

D.1 Untwisted vertex operators

The zero-mode vertex operator corresponding to a bosonic string on a toroidal background with winding and Kaluza-Klein numbers $\hat{N} = (n,m)^{\mathrm{T}} \in \mathbb{Z}^4$ is given by [52, eq. (3.41)]

$$V(\hat{N}) = \mathrm{e}^{-\pi\mathrm{i}/4\,\hat{N}^{\mathrm{T}}\hat{\eta}\hat{N}}\,\mathrm{e}^{2\pi\mathrm{i}\,\hat{N}^{\mathrm{T}}\hat{\eta}\,\boldsymbol{Y}}\,, \tag{D.1}$$

where the string coordinate operator $\boldsymbol{Y}$ results from promoting $E^{-1}Y$ to an operator (see eq. (A.2)). $\boldsymbol{Y}$ satisfies the commutation relations[5] (derived from the action of the sigma model)

$$\left[\boldsymbol{Y}, \boldsymbol{Y}^{\mathrm{T}}\right] = \frac{\mathrm{i}}{4\pi}\,\hat{\omega}\,, \qquad \text{where} \quad \hat{\omega} = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} \tag{D.2}$$

is the symplectic structure in the Narain basis. The nonzero value of the commutator (D.2) is a result of intrinsic non-commutative effects of closed strings [52]. The zero-mode vertex operators (D.1) in combination with the commutator (D.2) are subject to the so-called Weyl quantization relation

$$V(\hat{N}_1)\,V(\hat{N}_2) = \mathrm{e}^{\pi\mathrm{i}/2\,\hat{N}_1^{\mathrm{T}}(\hat{\eta}+\hat{\omega})\,\hat{N}_2}\,V(\hat{N}_1+\hat{N}_2)\,. \tag{D.3}$$

According to ref. [53], this relation is instrumental to evaluate the time ordering of operators as required in the computation of scattering amplitudes. The quantization relation (D.3) must hold independently of whether the vertex operators have been affected by modular transformations. As we shall shortly see (cf. eqs. (D.13)), this helps determine the phases required for the modular generators to act consistently on twisted vertex operators [54].

Using eq. (D.1), one finds that the $\mathbb{Z}_2$ orbifold-invariant untwisted vertex operators are given by

$$V(\hat{N})^{\text{orb.}} := \frac{1}{\sqrt{2}}\left(V(\hat{N}) + V(-\hat{N})\right)\,. \tag{D.4}$$

[5] The matrix components of the commutators (D.2) are such that $[\boldsymbol{Y}, \boldsymbol{Y}^{\mathrm{T}}]_{IJ} = [\boldsymbol{Y}_I, \boldsymbol{Y}_J] = \mathrm{i}\hat{\omega}_{IJ}/4\pi$.

These untwisted vertex operators can be arranged into 16 classes (corresponding to the 16 conjugacy classes $[(\mathbb{1}_4, \hat{N}^0)]$),

$$V^{\hat{N}^0} := V^{(n^0,m^0)^\mathrm{T}} = \sum_{\hat{N}\in\mathbb{Z}^4} \mathrm{e}^{\pi\mathrm{i}\,\hat{N}^\mathrm{T}\hat{\eta}\hat{N}^0}\; V(\hat{N}^0 + 2\hat{N})^{\mathrm{orb.}}\,. \tag{D.5}$$

They are characterized by a representative winding number n^0 and a representative KK number m^0, also called charges and collected in $\hat{N}^0 := (n^0, m^0)^\mathrm{T}$, with $n^0, m^0 \in \{(0,0)^\mathrm{T}, (0,1)^\mathrm{T}, (1,0)^\mathrm{T}, (1,1)^\mathrm{T}\}$. Note that the phases in eq. (D.5) let one establish a relation between the representative vertex operator $V^{\hat{N}^0}$ of a class and any other member of the class through

$$V^{\hat{N}^0+2\hat{N}'} = \mathrm{e}^{-\pi\mathrm{i}\,\hat{N}'^\mathrm{T}\hat{\eta}\,\hat{N}^0} V^{\hat{N}^0}\,, \qquad \text{with} \quad \hat{N}' \in \mathbb{Z}^4\,. \tag{D.6}$$

The phases further ensure that each class holds uniform properties, particularly in couplings and under outer automorphisms.

Let us now focus on the transformations of untwisted vertex operators $V^{\hat{N}^0}$ under the action of general outer automorphisms of the Narain orbifold space group,

$$\mathrm{Out}(\hat{S}_{\mathrm{Narain}}) = \left\langle (\hat{\Sigma}, 0), \hat{h}_i := (\mathbb{1}_4, T_i) \;|\; \hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z})\,,\; T_i \in \frac{1}{2}\mathbb{Z}^4 \right\rangle\,. \tag{D.7}$$

As discussed in appendix A.1, the rotational outer automorphisms defined by $\hat{\Sigma} \in \mathrm{O}_{\hat{\eta}}(2,2,\mathbb{Z}) = \langle \hat{K}_\mathrm{S}, \hat{K}_\mathrm{T}, \hat{C}_\mathrm{S}, \hat{C}_\mathrm{T}, \hat{M}, \hat{\Sigma}_* \rangle$ can be interpreted as modular transformations. In addition, the translational outer automorphisms $\hat{h}_i$, $i = 1,2,3,4$, are defined by the shift vectors T_i whose components in the lattice basis are $T_i{}^j = \frac{1}{2}\delta_i{}^j$, cf. ref. [1, appendix A].

To determine the transformation of $V^{\hat{N}^0}$ under a translation $\hat{h}_i$, we observe that

$$V(\hat{N}) \;\xmapsto{\hat{h}_i}\; \mathrm{e}^{2\pi\mathrm{i}\,T_i^\mathrm{T}\hat{\eta}\,\hat{N}}\; V(\hat{N})\,. \tag{D.8}$$

This implies that $V(\hat{N})$ acquires a $\mathbb{Z}_2$ phase, which is identical for $V(-\hat{N})$. Consequently, the orbifold invariant vertex operator (D.4) inherits the same phase. It thus follows that the untwisted vertex operator class $V^{\hat{N}^0}$ gets a $\mathbb{Z}_2$ phase too,

$$V^{\hat{N}^0} \;\xmapsto{\hat{h}_i}\; \mathrm{e}^{2\pi\mathrm{i}\,T_i^\mathrm{T}\hat{\eta}\,\hat{N}^0}\; V^{\hat{N}^0}\,. \tag{D.9}$$

Under a rotational outer automorphism $\hat{\Sigma}$, $\hat{N}$ transforms to $\hat{\Sigma}^{-1}\hat{N}$. Then, we expect that the vertex operator $V(\hat{N})$ transforms according to

$$V(\hat{N}) \;\xmapsto{\hat{\Sigma}}\; \varphi_{\hat{\Sigma}}(\hat{N})\, V(\hat{\Sigma}^{-1}\hat{N})\,. \tag{D.10}$$

Here, we propose, due to the nontrivial commutation relations (D.2), a phase $\varphi_{\hat{\Sigma}}(\hat{N})$ that is given by the ansatz

$$\varphi_{\hat{\Sigma}}(\hat{N}) = \mathrm{e}^{\pi\mathrm{i}\,\hat{N}^\mathrm{T} A_{\hat{\Sigma}}\hat{N} + \pi\mathrm{i}\,C_{\hat{\Sigma}}^\mathrm{T}\hat{N}}\,. \tag{D.11}$$

The 4×4 matrix $A_{\hat{\Sigma}}$ (with only half-integral off-diagonal entries) and the vector $C_{\hat{\Sigma}} \in \mathbb{Z}^4$ will be determined next. Note that, with these conditions, $\varphi_{\hat{\Sigma}}(\hat{N})$ can only be a $\mathbb{Z}_2$ phase.

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By demanding that the Weyl quantization relation (D.3) be preserved by $\hat{\Sigma}$ and using the abbreviation $\hat{\mu} = \frac{1}{2}(\hat{\eta} + \hat{\omega})$, one arrives at

$$A_{\hat{\Sigma}} = \frac{1}{2}\left(\hat{\mu} - \hat{\Sigma}^{-T}\hat{\mu}\,\hat{\Sigma}^{-1}\right) \mod 2\,. \tag{D.12}$$

In contrast, $C_{\hat{\Sigma}}$ cannot be constrained by the quantization condition. However, the effect of $C_{\hat{\Sigma}}$ is equivalent to the one of a translation $\hat{h}_i$ in the Narain lattice, given in eq. (D.9). These translations generate the traditional flavor symmetry, which is unbroken independently of the moduli. Therefore, the traditional flavor symmetry allows for a free choice of the vector $C_{\hat{\Sigma}}$. We choose $C_{\hat{K}_\mathrm{T}} = (1,1,0,0)^\mathrm{T}$, $C_{\hat{M}} = (1,1,1,1)^\mathrm{T}$, and $C_{\hat{\Sigma}} = 0$ for $\hat{\Sigma} \notin \{\hat{K}_\mathrm{T}, \hat{M}\}$ such that the transformations (D.10) generate only the finite modular group with a minimal amount of traditional flavor transformations. Thus, we are led to the phases

$$\varphi_{\hat{K}_\mathrm{S}}(\hat{N}) = \mathrm{e}^{\pi\mathrm{i}\,(m_1n_1+m_2n_2)}\,, \qquad \varphi_{\hat{K}_\mathrm{T}}(\hat{N}) = \mathrm{e}^{\pi\mathrm{i}\,(n_1n_2+n_1+n_2)}\,, \tag{D.13a}$$

$$\varphi_{\hat{C}_\mathrm{S}}(\hat{N}) = 1\,, \qquad \varphi_{\hat{C}_\mathrm{T}}(\hat{N}) = 1\,, \tag{D.13b}$$

$$\varphi_{\hat{M}}(\hat{N}) = \mathrm{e}^{\pi\mathrm{i}\,(m_1n_1+n_1+n_2+m_1+m_2)}\,, \tag{D.13c}$$

where $\hat{N} = (n,m)^\mathrm{T} = (n_1,n_2,m_1,m_2)^\mathrm{T}$. We do not determine the phase corresponding to the $\mathcal{CP}$-like generator $\hat{\Sigma}_*$ by the previous procedure because the result would be trivial. Instead, we fix its value by demanding that the transformations of the untwisted vertex operators $V^{\hat{N}^0}$ be compatible with the transformation $\phi_n \xmapsto{\hat{\Sigma}_*} \bar{\phi}_n$ of twisted states in the operator product expansions (OPEs) of twisted fields discussed in section D.2. We then find

$$\varphi_{\hat{\Sigma}_*}(\hat{N}) := \mathrm{e}^{\pi\mathrm{i}\,(m_1n_1+m_2n_2)}\,. \tag{D.14}$$

Noticing that the $\mathbb{Z}_2$ phases (D.13) and (D.14) coincide for $\hat{N}$ and $-\hat{N}$, we find that the orbifold invariant vertex operators $V(\hat{N})^{\text{orb.}}$ transform just as

$$V(\hat{N})^{\text{orb.}} \xmapsto{\hat{\Sigma}} \varphi_{\hat{\Sigma}}(\hat{N})\; V(\hat{\Sigma}^{-1}\hat{N})^{\text{orb.}}\,. \tag{D.15}$$

Therefore, a rotational outer automorphism $\hat{\Sigma}$ acts as

$$V^{\hat{N}^0} \xmapsto{\hat{\Sigma}} \varphi_{\hat{\Sigma}}(\hat{N}^0)\, V^{\hat{\Sigma}^{-1}\hat{N}^0}\,. \tag{D.16}$$

Since $\hat{\Sigma}^{-1}\hat{N}^0$ does not always take the form $(n^{0\prime}, m^{0\prime})^\mathrm{T}$ with $n^{0\prime}, m^{0\prime} \in \{(0,0)^\mathrm{T}, (0,1)^\mathrm{T}, (1,0)^\mathrm{T}, (1,1)^\mathrm{T}\}$, expressing $V^{\hat{\Sigma}^{-1}\hat{N}^0}$ in terms of the vertex operator classes (D.5) through eq. (D.6) introduces an extra $\mathbb{Z}_2$ phase in the transformation (D.16).

D.2 Operator product expansions of twisted vertex operators

Even though vertex operators of twisted string states are more involved than untwisted vertex operators, OPEs of two twisted states in two-dimensional orbifolds are known. Let us consider the twisted vertex operators ϕ_{n^a} and ϕ_{n^b} of twisted strings localized at orbifold fixed points given by the winding numbers n^a, $n^b \in \{(0,0)^\mathrm{T}, (0,1)^\mathrm{T}, (1,0)^\mathrm{T}, (1,1)^\mathrm{T}\}$. Up to a constant overall factor, they satisfy the OPE [55]

$$\bar{\phi}_{n^a}\,\phi_{n^b} = \sum_{m^0} C(n^a, n^b; n^0, m^0)\, V^{\hat{N}^0}\,, \qquad \text{where} \quad n^0 = \left(n^b - n^a\right) \mod 2\,. \tag{D.17}$$

$V^{\hat{N}^0}$ with $\hat{N}^0 = (n^0, m^0)^{\mathrm{T}}$ are the untwisted vertex operator classes (D.5), the bar on ϕ_{n^a} denotes conjugation, and $C(n^a, n^b\,; n^0, m^0) \in \mathbb{C}$ are known as coupling constants. For the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold, they read

$$C(n^a, n^b\,; n^0, m^0) = \mathrm{e}^{\pi \mathrm{i}\,(n^b)^{\mathrm{T}} m^0}\ \mathrm{e}^{\pi \mathrm{i}/2\,(n^0)^{\mathrm{T}} m^0}\,. \tag{D.18}$$

By inverting eq. (D.17), we can express the classes of untwisted string states $V^{\hat{N}^0}$ in terms of combinations of different OPEs of twisted states $\bar{\phi}_{n^a}\phi_{n^b}$. Explicitly, one finds

$$V^{(0,0,0,0)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(0,0)} + \bar{\phi}_{(1,0)}\phi_{(1,0)} + \bar{\phi}_{(0,1)}\phi_{(0,1)} + \bar{\phi}_{(1,1)}\phi_{(1,1)}\,, \tag{D.19a}$$
$$V^{(0,0,1,0)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(0,0)} - \bar{\phi}_{(1,0)}\phi_{(1,0)} + \bar{\phi}_{(0,1)}\phi_{(0,1)} - \bar{\phi}_{(1,1)}\phi_{(1,1)}\,, \tag{D.19b}$$
$$V^{(0,0,0,1)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(0,0)} + \bar{\phi}_{(1,0)}\phi_{(1,0)} - \bar{\phi}_{(0,1)}\phi_{(0,1)} - \bar{\phi}_{(1,1)}\phi_{(1,1)}\,, \tag{D.19c}$$
$$V^{(0,0,1,1)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(0,0)} - \bar{\phi}_{(1,0)}\phi_{(1,0)} - \bar{\phi}_{(0,1)}\phi_{(0,1)} + \bar{\phi}_{(1,1)}\phi_{(1,1)}\,, \tag{D.19d}$$
$$V^{(1,0,0,0)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(1,0)} + \bar{\phi}_{(1,0)}\phi_{(0,0)} + \bar{\phi}_{(0,1)}\phi_{(1,1)} + \bar{\phi}_{(1,1)}\phi_{(0,1)}\,, \tag{D.19e}$$
$$V^{(1,0,1,0)^{\mathrm{T}}} = \mathrm{i}\,\bar{\phi}_{(0,0)}\phi_{(1,0)} - \mathrm{i}\,\bar{\phi}_{(1,0)}\phi_{(0,0)} + \mathrm{i}\,\bar{\phi}_{(0,1)}\phi_{(1,1)} - \mathrm{i}\,\bar{\phi}_{(1,1)}\phi_{(0,1)}\,, \tag{D.19f}$$
$$V^{(1,0,0,1)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(1,0)} + \bar{\phi}_{(1,0)}\phi_{(0,0)} - \bar{\phi}_{(0,1)}\phi_{(1,1)} - \bar{\phi}_{(1,1)}\phi_{(0,1)}\,, \tag{D.19g}$$
$$V^{(1,0,1,1)^{\mathrm{T}}} = \mathrm{i}\,\bar{\phi}_{(0,0)}\phi_{(1,0)} - \mathrm{i}\,\bar{\phi}_{(1,0)}\phi_{(0,0)} - \mathrm{i}\,\bar{\phi}_{(0,1)}\phi_{(1,1)} + \mathrm{i}\,\bar{\phi}_{(1,1)}\phi_{(0,1)}\,, \tag{D.19h}$$
$$V^{(0,1,0,0)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(0,1)} + \bar{\phi}_{(1,0)}\phi_{(1,1)} + \bar{\phi}_{(0,1)}\phi_{(0,0)} + \bar{\phi}_{(1,1)}\phi_{(1,0)}\,, \tag{D.19i}$$
$$V^{(0,1,1,0)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(0,1)} - \bar{\phi}_{(1,0)}\phi_{(1,1)} + \bar{\phi}_{(0,1)}\phi_{(0,0)} - \bar{\phi}_{(1,1)}\phi_{(1,0)}\,, \tag{D.19j}$$
$$V^{(0,1,0,1)^{\mathrm{T}}} = \mathrm{i}\,\bar{\phi}_{(0,0)}\phi_{(0,1)} + \mathrm{i}\,\bar{\phi}_{(1,0)}\phi_{(1,1)} - \mathrm{i}\,\bar{\phi}_{(0,1)}\phi_{(0,0)} - \mathrm{i}\,\bar{\phi}_{(1,1)}\phi_{(1,0)}\,, \tag{D.19k}$$
$$V^{(0,1,1,1)^{\mathrm{T}}} = \mathrm{i}\,\bar{\phi}_{(0,0)}\phi_{(0,1)} - \mathrm{i}\,\bar{\phi}_{(1,0)}\phi_{(1,1)} - \mathrm{i}\,\bar{\phi}_{(0,1)}\phi_{(0,0)} + \mathrm{i}\,\bar{\phi}_{(1,1)}\phi_{(1,0)}\,, \tag{D.19l}$$
$$V^{(1,1,0,0)^{\mathrm{T}}} = \bar{\phi}_{(0,0)}\phi_{(1,1)} + \bar{\phi}_{(1,0)}\phi_{(0,1)} + \bar{\phi}_{(0,1)}\phi_{(1,0)} + \bar{\phi}_{(1,1)}\phi_{(0,0)}\,, \tag{D.19m}$$
$$V^{(1,1,1,0)^{\mathrm{T}}} = \mathrm{i}\,\bar{\phi}_{(0,0)}\phi_{(1,1)} - \mathrm{i}\,\bar{\phi}_{(1,0)}\phi_{(0,1)} + \mathrm{i}\,\bar{\phi}_{(0,1)}\phi_{(1,0)} - \mathrm{i}\,\bar{\phi}_{(1,1)}\phi_{(0,0)}\,, \tag{D.19n}$$
$$V^{(1,1,0,1)^{\mathrm{T}}} = \mathrm{i}\,\bar{\phi}_{(0,0)}\phi_{(1,1)} + \mathrm{i}\,\bar{\phi}_{(1,0)}\phi_{(0,1)} - \mathrm{i}\,\bar{\phi}_{(0,1)}\phi_{(1,0)} - \mathrm{i}\,\bar{\phi}_{(1,1)}\phi_{(0,0)}\,, \tag{D.19o}$$
$$V^{(1,1,1,1)^{\mathrm{T}}} = -\,\bar{\phi}_{(0,0)}\phi_{(1,1)} + \bar{\phi}_{(1,0)}\phi_{(0,1)} + \bar{\phi}_{(0,1)}\phi_{(1,0)} - \bar{\phi}_{(1,1)}\phi_{(0,0)}\,. \tag{D.19p}$$

These expressions together with the transformation properties of untwisted operators, eqs. (D.9) and (D.16), can lead to the corresponding transformations of the twisted vertex operators ϕ_n, as we now discuss.

D.3 Transformations of twisted vertex operators

With the help of the explicit relations (D.19) between the OPEs of twisted string states and the untwisted states, and the transformations (D.9) and (D.16) of the latter, we can deduce the action of $\mathrm{Out}(\hat{S}_{\text{Narain}})$ on $\bar{\phi}_{n^a}\,\phi_{n^b}$. One can then infer the action of those transformations on the single twisted operators, arranged in a twisted multiplet $(\phi_{(0,0)}, \phi_{(1,0)}, \phi_{(0,1)}, \phi_{(1,1)})^{\mathrm{T}}$. Note that, since no oscillator excitation is present in these twisted string states, this multiplet corresponds to the components of the twisted matter field $\Phi_{(-1/2,-1/2)}$, i.e. with $n_T = n_U = -1/2$. However, as shown in ref. [22] in the case of the $\mathbb{Z}_3$ orbifold, oscillator excitations only affect the modular weights and not the representation matrices. Hence, the results discussed here equally apply to the matter fields $\Phi_{(-3/2,1/2)}$ and $\Phi_{(1/2,-3/2)}$. In these terms, the transformations of twisted states will be encoded in transformation matrices $\rho_{\boldsymbol{r}}(\hat{\Sigma})$ or $\rho_{\boldsymbol{r}}(h_i)$, which denote r-dimensional representations of the outer automorphisms $\hat{\Sigma}$ and $\hat{h}_i$. Our goal here is to present those transformation matrices.

D.3.1 Traditional flavor group

Let us first inspect the action of the translational outer automorphisms $\hat{h}_i$, defined in eq. (D.7). Applying the transformations (D.9) on the untwisted operators (D.19) and interpreting for the twisted operators that build $\Phi_{(-1/2,-1/2)}$, we find that the twisted multiplet transforms under a translational outer automorphism as

$$\Phi_{(-1/2,-1/2)} \xmapsto{\hat{h}_i} \rho_{\boldsymbol{4}}(h_i)\,\Phi_{(-1/2,-1/2)}\,, \tag{D.20}$$

where the representation matrices are given by

$$\rho_{\boldsymbol{4}}(h_1) = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}, \qquad \rho_{\boldsymbol{4}}(h_2) = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}, \tag{D.21a}$$

$$\rho_{\boldsymbol{4}}(h_3) = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix}, \qquad \rho_{\boldsymbol{4}}(h_4) = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}. \tag{D.21b}$$

They generate the traditional flavor group

$$(D_8 \times D_8)\,/\mathbb{Z}_2 \;\cong\; [32, 49]\,. \tag{D.22}$$

The irreducible representations of this group are: one four-dimensional representation $\boldsymbol{4}$ and 16 one-dimensional representations $\boldsymbol{1}_{\alpha\beta\gamma\delta}$, such that

$$\boldsymbol{r} = \boldsymbol{1}_{\alpha\beta\gamma\delta} \quad \text{defined by} \quad \rho_{\boldsymbol{r}}(h_1) = \alpha\,,\ \rho_{\boldsymbol{r}}(h_2) = \beta\,,\ \rho_{\boldsymbol{r}}(h_3) = \gamma\,,\ \rho_{\boldsymbol{r}}(h_4) = \delta\,, \tag{D.23}$$

where $\alpha, \beta, \gamma, \delta \in \{-1, +1\}$ and $\boldsymbol{1}_0 := \boldsymbol{1}_{++++}$ is the trivial singlet.

The traditional flavor group eq. (D.22) contains the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ space and point group (PG) selection rules [56]. For example, twisted matter strings transform as follows

$$\mathbb{Z}_2^{\mathrm{PG}}\ :\quad \phi_{(n_1,n_2)} \xmapsto{(h_1 h_3)^2} -\phi_{(n_1,n_2)}\,, \tag{D.24a}$$

$$\mathbb{Z}_2^{e_1}\ :\quad \phi_{(n_1,n_2)} \xmapsto{h_3} (-1)^{n_1}\,\phi_{(n_1,n_2)}\,, \tag{D.24b}$$

$$\mathbb{Z}_2^{e_2}\ :\quad \phi_{(n_1,n_2)} \xmapsto{h_4} (-1)^{n_2}\,\phi_{(n_1,n_2)}\,, \tag{D.24c}$$

while the bulk strings $\Phi_{(0,0)}$ and $\Phi_{(-1,-1)}$ transform as $\boldsymbol{1}_0$. Hence, they are invariant under the transformations (D.24).

Since twisted strings transform in the (real) representation $\mathbf{4}$ of $[32,49]$, the OPEs $\bar{\phi}_{n^a}\phi_{n^b}$ can be associated with the tensor product

$$\mathbf{4}\ \otimes\ \mathbf{4} = \bigoplus_{\alpha,\beta,\gamma,\delta\ \in\ \{+,-\}} \mathbf{1}_{\alpha\beta\gamma\delta}\,. \tag{D.25}$$

Hence, eq. (D.19) implies that the 16 classes of untwisted vertex operators $V^{\hat{N}^0}$ build the 16 one-dimensional representations $\mathbf{1}_{\alpha\beta\gamma\delta}$ of the traditional flavor symmetry. Further, the representation $\mathbf{4}\otimes\mathbf{4}$ is not faithful. Thus, taking only the untwisted vertex operators $V^{\hat{N}^0}$ into account, the traditional flavor symmetry is only $[16,14] \cong (\mathbb{Z}_2)^4$. This can be confirmed by the explicit one-dimensional representations (D.23).

Finally, since oscillator excitations are not affected by the transformations associated with $\hat{h}_i$, all other twisted matter fields $\Phi_{(n_T,n_U)}$ (see table 1) must transform in the same 4-dimensional representation defined by eq. (D.21).

D.3.2 Modular flavor group

Let us consider the twisted matter field $\Phi_{(-1/2,-1/2)}$, which builds a $\mathbf{4}$-plet of the traditional flavor group, as seen in the previous section. Its transformations under rotational automorphisms $\hat{\Sigma}$ are governed by eq. (3.1), where the automorphy factors $j^{(n_T,n_U)}(\hat{\Sigma},T,U)$ are given by eqs. (3.3) or (3.6), depending on $\hat{\Sigma}$. Further, the 4-dimensional matrix representation of these transformations acting on the multiplet $\Phi_{(-1/2,-1/2)}$ are identified using the OPEs (D.19), resulting in

$$\rho_{\mathbf{4}_1}(\hat{K}_\mathrm{S}) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}, \qquad \rho_{\mathbf{4}_1}(\hat{K}_\mathrm{T}) = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{D.26a}$$

$$\rho_{\mathbf{4}_1}(\hat{C}_\mathrm{S}) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \qquad \rho_{\mathbf{4}_1}(\hat{C}_\mathrm{T}) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}, \tag{D.26b}$$

$$\rho_{\mathbf{4}_1}(\hat{M}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\\ 1 & -1 & 0 & 0\\ -1 & -1 & 0 & 0\end{pmatrix}. \tag{D.26c}$$

They build the representation $\mathbf{4}_1$ of the finite modular flavor group $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144,115]$ without $\mathcal{CP}$ (see appendix E.3).

Note that, as in the traditional flavor group, the tensor product $\mathbf{4}_1 \otimes \mathbf{4}_1$ of twisted vertex operators does not behave as a faithful representation. Hence, we learn that the classes of untwisted vertex operators $V^{\hat{N}^0}$ transform only under the subgroup $[72,40]$ of the full finite modular symmetry.

The finite modular flavor group can be extended by $\mathcal{CP}$. Since the $\mathcal{CP}$-like transformation $\hat{\Sigma}_*$ interchanges matter fields with their conjugates, the 4-dimensional representation

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of twisted matter strings transforms as $\Phi_{(-1/2,-1/2)} \leftrightarrow \bar{\Phi}_{(-1/2,-1/2)}$. So, the representation of $\hat{\Sigma}_*$ acts on $(\Phi_{(-1/2,-1/2)}, \bar{\Phi}_{(-1/2,-1/2)})^{\mathrm{T}}$. Furthermore, to determine the corresponding automorphy factor we follow the discussion in section 3.1 and consider the associated $\mathrm{GSp}(4,\mathbb{Z})$ element $M_* = \mathrm{diag}(-1,-1,1,1)$ [18, eq. (39)]. This implies $A = -\mathbb{1}_2$, $B = 0$, $C = 0$ and $D = \mathbb{1}_2$ in the context of eq. (3.3). It follows that the automorphy factor of $\mathcal{CP}$ is trivial. Hence, we find that $\hat{\Sigma}_*$ acts on the twisted multiplet as

$$\begin{pmatrix} \Phi_{(-1/2,-1/2)} \\ \bar{\Phi}_{(-1/2,-1/2)} \end{pmatrix} \xmapsto{\hat{\Sigma}_*} \rho_{\mathbf{4}_1 \oplus \mathbf{4}_1}(\hat{\Sigma}_*) \begin{pmatrix} \Phi_{(-1/2,-1/2)} \\ \bar{\Phi}_{(-1/2,-1/2)} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{pmatrix} \begin{pmatrix} \Phi_{(-1/2,-1/2)} \\ \bar{\Phi}_{(-1/2,-1/2)} \end{pmatrix} . \tag{D.27}$$

This enhances the finite modular flavor group to $\left[(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}}\right] \times \mathbb{Z}_2^{\mathcal{CP}} \cong [288, 880]$.

E Details on the superpotential

E.1 Representation of modular forms

The four-dimensional multiplet of modular forms $\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)$ has been given in eq. (4.1). It transforms under modular transformations with a $\mathbf{4}_3$ representation of the finite modular group $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144, 115]$. In detail, for a modular transformation $\hat{\Sigma}$ we find

$$\hat{Y}^{(2)}_{\mathbf{4}_3}(T,U) \xmapsto{\hat{\Sigma}} j^{(2)}(\hat{\Sigma},T,U)\; \rho_{\mathbf{4}_3}(\hat{\Sigma})\, \hat{Y}^{(2)}_{\mathbf{4}_3}(T,U)\,, \tag{E.1}$$

where $j^{(2)}(\hat{\Sigma},T,U)$ is the automorphy factor, cf. section 3.1, and the representations $\rho_{\mathbf{4}_3}(\hat{\Sigma})$ read

$$\rho_{\mathbf{4}_3}(\hat{K}_{\mathrm{S}}) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \qquad \rho_{\mathbf{4}_3}(\hat{K}_{\mathrm{T}}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{E.2a}$$

$$\rho_{\mathbf{4}_3}(\hat{C}_{\mathrm{S}}) = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}, \qquad \rho_{\mathbf{4}_3}(\hat{C}_{\mathrm{T}}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{E.2b}$$

$$\rho_{\mathbf{4}_3}(\hat{M}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{E.2c}$$

Note however that this is an unfaithful representation, i.e. $\rho_{\mathbf{4}_3}(\hat{\Sigma})$ only spans the group $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_2 \cong [72, 40]$.

E.2 Components of the superpotential

There are 16 terms in the product of four twisted matter fields $\Phi^i_{(-1/2,-1/2)} = (\phi^i_{(0,0)}, \phi^i_{(1,0)}, \phi^i_{(0,1)}, \phi^i_{(1,1)})^{\mathrm{T}}$, $i \in \{1,2,3,4\}$, that are invariant under the traditional flavor symmetry $(D_8 \times D_8)/\mathbb{Z}_2 \cong [32,49]$. They read

$$\begin{aligned}\mathcal{I}_1 = \phi^1_{(0,0)}\phi^2_{(0,0)}\phi^3_{(0,0)}\phi^4_{(0,0)} + \phi^1_{(1,0)}\phi^2_{(1,0)}\phi^3_{(1,0)}\phi^4_{(1,0)} \\ + \phi^1_{(0,1)}\phi^2_{(0,1)}\phi^3_{(0,1)}\phi^4_{(0,1)} + \phi^1_{(1,1)}\phi^2_{(1,1)}\phi^3_{(1,1)}\phi^4_{(1,1)}\,,\end{aligned} \tag{E.3}$$

and

$$\mathcal{I}_2 = \Delta_1\left(\phi_{(0,0)}, \phi_{(1,0)}, \phi_{(0,1)}, \phi_{(1,1)}\right), \quad \mathcal{I}_{11} = \Xi\left(\phi_{(0,0)}, \phi_{(1,0)}, \phi_{(0,1)}, \phi_{(1,1)}\right), \tag{E.4a}$$

$$\mathcal{I}_3 = \Delta_2\left(\phi_{(0,0)}, \phi_{(1,0)}, \phi_{(0,1)}, \phi_{(1,1)}\right), \quad \mathcal{I}_{12} = \Xi\left(\phi_{(0,0)}, \phi_{(0,1)}, \phi_{(1,0)}, \phi_{(1,1)}\right), \tag{E.4b}$$

$$\mathcal{I}_4 = \Delta_3\left(\phi_{(0,0)}, \phi_{(1,0)}, \phi_{(0,1)}, \phi_{(1,1)}\right), \quad \mathcal{I}_{13} = \Xi\left(\phi_{(0,0)}, \phi_{(1,0)}, \phi_{(1,1)}, \phi_{(0,1)}\right), \tag{E.4c}$$

$$\mathcal{I}_5 = \Delta_1\left(\phi_{(0,0)}, \phi_{(0,1)}, \phi_{(1,0)}, \phi_{(1,1)}\right), \quad \mathcal{I}_{14} = \Xi\left(\phi_{(0,0)}, \phi_{(1,1)}, \phi_{(1,0)}, \phi_{(0,1)}\right), \tag{E.4d}$$

$$\mathcal{I}_6 = \Delta_2\left(\phi_{(0,0)}, \phi_{(0,1)}, \phi_{(1,0)}, \phi_{(1,1)}\right), \quad \mathcal{I}_{15} = \Xi\left(\phi_{(0,0)}, \phi_{(0,1)}, \phi_{(1,1)}, \phi_{(1,0)}\right), \tag{E.4e}$$

$$\mathcal{I}_7 = \Delta_3\left(\phi_{(0,0)}, \phi_{(0,1)}, \phi_{(1,0)}, \phi_{(1,1)}\right), \quad \mathcal{I}_{16} = \Xi\left(\phi_{(0,0)}, \phi_{(1,1)}, \phi_{(0,1)}, \phi_{(1,0)}\right), \tag{E.4f}$$

$$\mathcal{I}_8 = \Delta_1\left(\phi_{(0,0)}, \phi_{(1,1)}, \phi_{(1,0)}, \phi_{(0,1)}\right), \tag{E.4g}$$

$$\mathcal{I}_9 = \Delta_2\left(\phi_{(0,0)}, \phi_{(1,1)}, \phi_{(1,0)}, \phi_{(0,1)}\right), \tag{E.4h}$$

$$\mathcal{I}_{10} = \Delta_3\left(\phi_{(0,0)}, \phi_{(1,1)}, \phi_{(1,0)}, \phi_{(0,1)}\right), \tag{E.4i}$$

where we have used the following abbreviations:

$$\Xi(A,B,C,D) = A^1\,B^2\,C^3\,D^4 + A^2\,B^1\,C^4\,D^3 + A^3\,B^4\,C^1\,D^2 + A^4\,B^3\,C^2\,D^1\,, \tag{E.5a}$$

$$\Delta_i(A,B,C,D) = \tilde{\Delta}_i(A,B) + \tilde{\Delta}_i(C,D)\,, \tag{E.5b}$$

with

$$\tilde{\Delta}_1(A,B) = A^1\,A^2\,B^3\,B^4 + A^3\,A^4\,B^1\,B^2\,, \tag{E.6a}$$

$$\tilde{\Delta}_2(A,B) = A^1\,A^3\,B^2\,B^4 + A^2\,A^4\,B^1\,B^3\,, \tag{E.6b}$$

$$\tilde{\Delta}_3(A,B) = A^1\,A^4\,B^2\,B^3 + A^2\,A^3\,B^1\,B^4\,. \tag{E.6c}$$

As described in eq. (5.8), the vector $(\mathcal{I}_1, \ldots, \mathcal{I}_{16})^{\mathrm{T}}$ transforms under the modular symmetry [144, 115] with $R(\hat{\Sigma})$. These 16×16 matrices are generated by

$$R(\hat{C}_{\mathrm{S}}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{1}_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{1}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{1}_2 & 0 \end{pmatrix}, \quad R(\hat{C}_{\mathrm{T}}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{1}_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{1}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{E.7}$$

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order	1	2	2	2	2	2	3	3	4	4	4	4	6	6	6	6	12	12
size	1	1	6	6	9	9	4	4	6	6	18	18	4	4	12	12	12	12
name	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}
$\mathbf{1}_0$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_1$	1	1	−1	−1	1	1	1	1	1	1	−1	−1	1	1	−1	−1	1	1
$\mathbf{1}_2$	1	1	−1	−1	1	1	1	1	−1	−1	1	1	1	1	−1	−1	−1	−1
$\mathbf{1}_3$	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1
$\mathbf{1}_4$	1	−1	1	−1	1	−1	1	1	−i	i	−i	i	−1	−1	1	−1	−i	i
$\mathbf{1}_5$	1	−1	−1	1	1	−1	1	1	i	−i	−i	i	−1	−1	−1	1	i	−i
$\mathbf{1}_6$	1	−1	−1	1	1	−1	1	1	−i	i	i	−i	−1	−1	−1	1	−i	i
$\mathbf{1}_7$	1	−1	1	−1	1	−1	1	1	i	−i	i	−i	−1	−1	1	−1	i	−i
$\mathbf{2}_1$	2	2	0	0	−2	−2	2	2	0	0	0	0	2	2	0	0	0	0
$\mathbf{2}_2$	2	−2	0	0	−2	2	2	2	0	0	0	0	−2	−2	0	0	0	0
$\mathbf{4}_1$	4	−4	2	−2	0	0	1	−2	0	0	0	0	2	−1	−1	1	0	0
$\mathbf{4}_2$	4	−4	−2	2	0	0	1	−2	0	0	0	0	2	−1	1	−1	0	0
$\mathbf{4}_3$	4	4	0	0	0	0	−2	1	2	2	0	0	1	−2	0	0	−1	−1
$\mathbf{4}_4$	4	4	0	0	0	0	−2	1	−2	−2	0	0	1	−2	0	0	1	1
$\mathbf{4}_5$	4	4	2	2	0	0	1	−2	0	0	0	0	−2	1	−1	−1	0	0
$\mathbf{4}_6$	4	4	−2	−2	0	0	1	−2	0	0	0	0	−2	1	1	1	0	0
$\mathbf{4}_7$	4	−4	0	0	0	0	−2	1	2i	−2i	0	0	−1	2	0	0	−i	i
$\mathbf{4}_8$	4	−4	0	0	0	0	−2	1	−2i	2i	0	0	−1	2	0	0	i	−i

Table 4. Character table of $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144, 115]$, where 'size' means the number of elements in a conjugacy class and 'order' denotes the order of its elements.

In addition, $R(\hat{M})$ acts on each four-dimensional subspace defined by

$$\begin{pmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \mathcal{I}_3 \\ \mathcal{I}_4 \end{pmatrix}, \quad \begin{pmatrix} \mathcal{I}_5 \\ \mathcal{I}_8 \\ \mathcal{I}_{11} \\ \mathcal{I}_{13} \end{pmatrix}, \quad \begin{pmatrix} \mathcal{I}_6 \\ \mathcal{I}_9 \\ \mathcal{I}_{12} \\ \mathcal{I}_{14} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathcal{I}_7 \\ \mathcal{I}_{10} \\ \mathcal{I}_{15} \\ \mathcal{I}_{16} \end{pmatrix} \tag{E.8}$$

with the 4×4 matrix

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{E.9}$$

E.3 $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}}$ character table

A presentation for the group $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_4^{\hat{M}} \cong [144, 115]$ is given by

$$\Big\langle \hat{\mathsf{C}}_\mathrm{S},\ \hat{\mathsf{C}}_\mathrm{T},\ \hat{\mathsf{M}}\ \Big|\ \hat{\mathsf{C}}_\mathrm{S}^2 = \hat{\mathsf{C}}_\mathrm{T}^2 = \hat{\mathsf{M}}^4 = \left(\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}\right)^3 = \left(\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{M}}^2\right)^2 = \left(\hat{\mathsf{C}}_\mathrm{T}\hat{\mathsf{M}}^2\right)^2 = \mathbb{1} \ \text{ and} \tag{E.10a}$$

$$\hat{\mathsf{M}}\,\hat{\mathsf{C}}_i\,\hat{\mathsf{M}}\,\hat{\mathsf{C}}_j = \hat{\mathsf{C}}_j\,\hat{\mathsf{M}}\,\hat{\mathsf{C}}_i\,\hat{\mathsf{M}} \quad \text{where} \quad i, j \in \{\mathrm{S}, \mathrm{T}\}\Big\rangle\,. \tag{E.10b}$$

Furthermore, we define $\hat{\mathsf{K}}_\mathrm{S} = \hat{\mathsf{M}}^3\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{M}}$ and $\hat{\mathsf{K}}_\mathrm{T} = \hat{\mathsf{M}}^3\hat{\mathsf{C}}_\mathrm{T}\hat{\mathsf{M}}$, such that the names of the abstract generators $\hat{\mathsf{C}}_\mathrm{S}, \hat{\mathsf{C}}_\mathrm{T}, \hat{\mathsf{K}}_\mathrm{S}, \hat{\mathsf{K}}_\mathrm{T}$ and $\hat{\mathsf{M}}$ allow for an intuitive association with the modular

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order	1	3	3	2
size	1	4	4	3
name	C_1	C_2	C_3	C_4
$\mathbf{1}$	1	1	1	1
$\mathbf{1}'$	1	ω	ω^2	1
$\mathbf{1}''$	1	ω^2	ω	1
$\mathbf{3}$	3	0	0	-1

Table 5. Character table of A_4. We use the definition $\omega := \exp(2\pi\mathrm{i}/3)$.

transformations $\hat{C}_\mathrm{S}, \hat{C}_\mathrm{T}, \hat{K}_\mathrm{S}, \hat{K}_\mathrm{T}$ and $\hat{M}$. The group has 18 conjugacy classes

$$C_1 = [\mathbb{1}]\,, \qquad C_2 = [\hat{\mathsf{M}}^2]\,, \qquad C_3 = [\hat{\mathsf{C}}_\mathrm{S}]\,, \tag{E.11a}$$
$$C_4 = [\hat{\mathsf{M}}^2\,\hat{\mathsf{C}}_\mathrm{S}]\,, \qquad C_5 = [\hat{\mathsf{C}}_\mathrm{S}\,\hat{\mathsf{K}}_\mathrm{S}]\,, \qquad C_6 = [\hat{\mathsf{M}}^2\,\hat{\mathsf{C}}_\mathrm{S}\,\hat{\mathsf{K}}_\mathrm{S}]\,, \tag{E.11b}$$
$$C_7 = [\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}]\,, \qquad C_8 = [\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}\,\hat{\mathsf{K}}_\mathrm{S}\hat{\mathsf{K}}_\mathrm{T}]\,, \qquad C_9 = [\hat{\mathsf{M}}]\,, \tag{E.11c}$$
$$C_{10} = [\hat{\mathsf{M}}^3]\,, \qquad C_{11} = [\hat{\mathsf{M}}\,\hat{\mathsf{C}}_\mathrm{S}]\,, \qquad C_{12} = [\hat{\mathsf{M}}^3\,\hat{\mathsf{C}}_\mathrm{S}]\,, \tag{E.11d}$$
$$C_{13} = [\hat{\mathsf{M}}^2\,\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}\,\hat{\mathsf{K}}_\mathrm{S}\hat{\mathsf{K}}_\mathrm{T}]\,, \qquad C_{14} = [\hat{\mathsf{M}}^2\,\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}]\,, \qquad C_{15} = [\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}\,\hat{\mathsf{K}}_\mathrm{S}]\,, \tag{E.11e}$$
$$C_{16} = [\hat{\mathsf{M}}^2\,\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}\,\hat{\mathsf{K}}_\mathrm{S}]\,, \qquad C_{17} = [\hat{\mathsf{M}}\,\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}]\,, \qquad C_{18} = [\hat{\mathsf{M}}^3\,\hat{\mathsf{C}}_\mathrm{S}\hat{\mathsf{C}}_\mathrm{T}]\,. \tag{E.11f}$$

The character table is given in table 4. Note that out of the four-dimensional representations, the $\mathbf{4}_i$-plets with $i = 3, \ldots, 6$ are not faithful representations and span only the group $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_2^{\hat{M}} \cong [72, 40]$.

E.4 A_4 character table

We denote as a, b and c the abstract A_4 generators associated with h_1, h_2 and $(\hat{C}_\mathrm{T}\hat{C}_\mathrm{S})^2$, respectively, see section 4.3.1. In these terms, A_4 is defined by the presentation

$$A_4 = \left\langle \mathsf{a}, \mathsf{b}, \mathsf{c} \;\middle|\; \mathsf{a}^2 = \mathsf{b}^2 = \mathsf{c}^3 = (\mathsf{ab})^2 = \mathbb{1},\ \mathsf{bca} = \mathsf{bacb} = \mathsf{c} \right\rangle\,, \tag{E.12}$$

and has four conjugacy classes:

$$C_1 = [\mathbb{1}]\,, \quad C_2 = [\mathsf{c}^2]\,, \quad C_3 = [\mathsf{c}]\,, \quad C_4 = [\mathsf{a}]\,. \tag{E.13}$$

The character table of A_4 is shown in table 5, where we also present the order and number (size) of the elements in each conjugacy class.

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