COLLOQUIUM MATHEMATICUM

VOL. XXXVIII 1978 FASC. 2

COMPLETION MEASURABLE LINEAR FUNCTIONALS ON A PROBABILITY SPACE

BY

MAREK KANTER (MONTREAL)

This paper was written in 1970. At that time there did not seem to be sufficient interest in the axiomatics of measures on linear spaces and the paper remained in a preprint form for several years. However, in recent years there has been an increase of interest in this field as summarized in the survey article by Urbanik [12]. In particular, Urbanik poses as a problem (P 949) a special case of Theorem 3.1 of our unpublished paper.

What follows is that paper. Although there is overlap with [12] we have chosen to keep our previous paper in its original form, partly because our treatment supplies a variety of details not present in [12]. One remark: we work with the sample space $S = R^{[0,b)}$, however, our arguments apply exactly as well to the sample space $S = L^2[0, b]$ which Urbanik deals with in [12]. Furthermore, although Urbanik allows random linear functionals F to be defined on linear subspaces D_F of $L^2[0, b]$ of total measure one, we note that a Hamel basis argument as in Lemma 2.1 allows F to be linearly extended to all of S in agreement with our own definitions.

1. Introduction. Let R stand for the real line. If $(X(t), t \in T)$ is a real-valued stochastic process defined on a parameter set T, then the sample space of X is the set S of all real-valued functions on T. S is a real linear space and it is possible to study, in a purely algebraic manner, the set of all real-valued linear functionals on S (the conjugate space of S). The probabilistic nature of this paper stems from the fact that we restrict ourselves to consider only the set S^* of real-valued linear functionals on S that can be defined as random variables in a natural way.

Let us go immediately into details. We usually set $T = [0, b)$, $b > 0$, or $T = [0, \infty)$. We define $(X(t), t \in T)$ as real-valued linear functionals on S by setting $X(t)(x) = x(t)$ for $x \in S$, $t \in T$. We define $\mathscr{A}$ as the smallest σ-field for which all the functions $(X(t), t \in T)$ are measurable.

If P is a probability measure on $(S, \mathscr{A})$, then $(X(t), t \in T)$ is a stochastic process. In our work we need to keep in mind that, by a fundamental theorem due to Kolmogorov, all stochastic processes X on T can be represented this way (see [10], 4.3A).

Our basic assumptions about X are the following:

(1) $X(0) = 0$ almost surely (a.s.).

(2) $X(t)$ is continuous in probability on T.

(3) $X(t)$ has independent increments.

We will in fact incorporate (1) into our definition of S by setting S to be the set of real-valued functions x on T such that $x(0) = 0$.

Let $\mathscr{A}^*$ stand for the completion of $\mathscr{A}$ with respect to P and let S^* denote the set of all real-valued linear functionals on S that are $\mathscr{A}^*$-measurable. We shall see that certain real-valued Borel measurable functions f defined on T are *X-integrable*, i.e., $\int_T f(t)\,dX(t)$ can be defined in a natural way as a random variable on $(S, \mathscr{A}^*, P)$ which is a.s. equal to an element F in S^*. Our main interest is to go in the reverse direction: that is, given F in S^* we wish to find some Borel measurable real-valued function f such that

$$F = \int_T f(t)\,dX(t) \quad \text{a.s.}$$

The nearest approach to our work is the paper of Cameron and Graves [1]. In this paper and in a subsequent one by Graves [5], the two authors prove, among other things, that if X is Brownian motion on $[0, 1]$ and F is an additive $\mathscr{A}$-measurable functional on S, then there is some Borel measurable real-valued function f such that

$$\int_{[0,1]} \big(f(t)^2\big)\,dt < \infty \quad \text{and} \quad F = \int_{[0,1]} f(t)\,dX(t) \ \text{a.s.}$$

Their proof consists of rather complicated calculations using a Fourier-Hermite development of F. A simpler proof of this theorem based on Lemma 3.10 and martingale methods can be found in the unpublished thesis of Kanter [6]. In fact, the same methods generalize the theorem so that it holds for any symmetric stable process of index $q \in (1, 2]$, where q is rational.

In this paper we usually assume that F is linear as well as additive, and we prove the representation theorem of Cameron and Graves for a rather large class of processes. The case of Brownian motion, however, is necessarily excluded by the methods of this paper.

The contents of this paper are arranged as follows. In Section 2 we present the concept of a *continuous countably additive stochastic measure*, i.e., a process X satisfying (1)-(3) and a countable additivity assumption. We shall then present the concept of stochastic integration; i.e., given a real-valued Borel measurable function f we shall specify conditions under which f is X-integrable. We will show that if f is X-integrable, then we can define $\int_T f(t)\,dX(t)$ as a random variable on $(S, \mathscr{A}^*, P)$ which is a.s. equal to an element of S^*.

In Section 3 we prove our main representation theorem, Theorem 3.1. For this theorem we need an extra hypothesis on X, namely:

(4) If a sequence $F_n \in S^*$ is such that, for some sequence $a_n \in R$, $F_n - a_n \to 0$ in probability, then, in fact, $a_n \to 0$ and $F_n \to 0$ in probability.

This hypothesis is satisfied by symmetric processes, strictly stable processes, and processes for which $P(\{\mathit{0}\}) > 0$ ($\mathit{0}$ stands for the zero element of S). Theorem 3.1 states that if X satisfies (1)-(4) and if it has no Gaussian part in its Lévy decomposition, then every $F \in S^*$ is a.s. equal to a stochastic integral.

In Section 4 we extend Theorem 3.1 to processes not necessarily satisfying (2), thus covering the case of a sequence of independent random variables X_n, $n \geqslant 1$. We then present some counterexamples to show the necessity of (4) and also the necessity of the assumption that there is no Gaussian part.

We end the introduction by presenting, in detail, the Lévy decomposition of a process $(X(t), t \in T)$ satisfying (1)-(3).

Let us call a function $x \in S$ *decomposable* if x has finite right-hand limits $x(t-)$,

$$x(t-) \equiv \lim_{t' \uparrow t} x(t'),$$

and finite left-hand limits $x(t+)$,

$$x(t+) \equiv \lim_{t' \downarrow t} x(t'),$$

existing for all $t \in T$. (At the endpoints of T only appropriate sided limits are assumed to exist; if $T = [0, \infty)$, then the limit of $x(t)$ as $t \uparrow \infty$ is not assumed to exist.) By [10], 37.2c and 37.3a, if X satisfies (1)-(3), then P assigns measure 1 to the set of decomposable functions. (Note that the set of decomposable functions is a linear subset of S.)

Let x be decomposable. If $x(0+) \neq 0$, then we shall say that x has a *jump at* 0 *of size* $h = x(0+)$. If for $t \in T$, t not a right-hand endpoint, $x(t+) - x(t-) \neq 0$, then we shall say that x has a *jump at* t *of size* $h = x(t+) - x(t-)$. By [9], p. 138, if x is decomposable, then the number $v_t(s)(x)$ of jumps of x in the interval $[0, t)$ with any size h such that $h/s > 1$ is finite for all $s \neq 0$ and $t \in T$.

Let us set $v_t(s)(x) = 0$ for all $s \neq 0$ and $t \in T$ if x is not decomposable. It follows from [10], 37.3A, that $v_t(s)(\cdot)$ is a random variable on $(S, \mathscr{A}^*, P)$ for $s \neq 0$ and $t \in T$. Let us denote this random variable by $V_t(s)$. By [10], 37.3c, if $s \neq 0$ is fixed, then the process $V_t(s)$ defined on T as t varies satisfies (1)-(3). On the other hand, if we fix $t \in T$, then for any $\gamma > 0$ the process $V_t(s)$, defined either on $(-\infty, -\gamma)$ or on (γ, ∞) as s varies, satisfies (3). Furthermore, $V_t(s)$ has the Poisson distribution and we can uniquely define a non-negative extended real-valued measure L on the Borel subsets of $T \times R$ by setting $L(T \times \{0\}) = 0$ as well as by setting

$$L\big([0, t) \times (\gamma, \infty)\big) = \mathrm{E}\big(V_t(\gamma)\big), \qquad L\big([0, t) \times (-\infty, -\gamma)\big) = \mathrm{E}\big(V_t(-\gamma)\big)$$

for all $t \in T$ and $\gamma > 0$, and then extending (see [2], p. 136, the Hahn extension theorem). L has the property that

$$\int_R \frac{s^2}{1+s^2} L\big([0, t) \times ds\big)$$

is finite for all $t \in T$.

Let $x \in S$ be decomposable and, for $t \in T$, let $h_1, \ldots, h_m$ be a list of the jumps of x of size h such that $|h| > \gamma > 0$, in the order of their occurrence in the interval $[0, t)$. Define $x^\gamma \in S$ by setting

$$x^\gamma(t) \equiv \sum_1^m h_i.$$

If x is not decomposable, then $x^\gamma \equiv 0$. It follows from [10], 37.3d, that $(\cdot)^\gamma(t)$ is a random variable on $(S, \mathscr{A}^*, P)$ for all $\gamma > 0$ and $t \in T$. Denote this random variable by $X^\gamma(t)$. Clearly, the process $\big(X^\gamma(t), t \in T\big)$ satisfies (1)-(3).

We are finally ready to present the Lévy decomposition of X. Namely, we can write

$$X(t) = a(t) + Y(t) + Z(t),$$

where a is a continuous real-valued function on T with $a(0) = 0$, and where Y and Z are independent stochastic processes defined on the probability triple $(S, \mathscr{A}^*, P)$ and both satisfy (1)-(3). Furthermore, $Y(t)$ has the Gaussian distribution of mean zero for all $t \in T$ while $Z(t)$ is the limit in probability of

$$X^\gamma(t) - \int_{|s|>\gamma} \frac{s}{1+s^2} L\big([0, t) \times ds\big) \quad \text{as } \gamma \downarrow 0.$$

For a proof of this decomposition, see [10], 37.3D.

2. Stochastic integration. In this section, we consider stochastic integration with respect to a process satisfying (1)-(3) and a countability assumption. We again work with $T = [0, b]$ or $[0, \infty)$, but we find it necessary to consider various families of subsets of T for which we need some nomenclature: $\mathscr{B}$ stands for the σ-field of all Borel subsets of T, $\mathscr{B}_0$ is the ring of all bounded Borel subsets of T, and ξ is the ring of subsets of T of the form

$$[t_0, t_1)\cup\ldots\cup[t_{2n}, t_{2n+1}), \qquad \text{where } t_0 < t_1 < \ldots < t_{2n} < t_{2n+1}.$$

We start by defining X to be a *finitely additive stochastic set function* on (T, ξ) if whenever $A_1, \ldots, A_n$ are disjoint elements of ξ, then

$$X(A_1\cup\ldots\cup A_n) = \sum_1^n X(A_m) \text{ a.s.}$$

If we set $X(t) = X\big([0, t)\big)$, then $\big(X(t), t \in T\big)$ satisfies (1). Conversely, starting from a process $\big(X(t), t \in T\big)$ that satisfies (1) we can define a finitely additive stochastic set function on (T, ξ) by setting

$$X\big([t_1, t_2)\big) = X(t_2) - X(t_1) \qquad \text{for } t_1 < t_2,$$

and then extending to all of ξ by additivity. From now on, X stands either for a stochastic process $X(t)$ or for a finitely additive stochastic set function on (T, ξ), the two interpretations being interrelated as above. The objects S, $\mathscr{A}$, P, $\mathscr{A}^*$, S^* remain exactly as defined in Section 1.

Definition 2.1. Let χ_A stand for the indicator function of $A \subset T$. Let M_0 stand for the set of all functions of the form $\sum_1^n c_i \chi_{A_i}$, where $A_i \in \xi$ and $c_i \in R$, $i = 1, \ldots, n$. For $f \in M_0$ and X, a finitely additive stochastic set function on (T, ξ) defines $\int_T f(t)dX(t)$ to be $\sum_1^n c_i X(A_i)$. Such an expression is called a *simple stochastic integral.*

It is trivially clear that every simple stochastic integral is in S^*.

In this section we define $\int_T f(t)dX(t)$ for a class of Borel functions f. We follow the treatment of Shale and Stinespring [11] and Urbanik and Woyczyński [13] on the subject of stochastic integration. However, we expand upon the treatment of these authors by bringing out the fact that all such expressions $\int_T f(t)dX(t)$ are a.s. equal to an element of S^*. For this we use the following lemma.

LEMMA 2.1. *Suppose that F_n is a sequence of elements in S^* such that $\lim_{n\to\infty} F_n(x)$ exists a.s. in S. Then there exists an F in S^* such that*

$$F(x) = \lim_{n\to\infty} F_n(x) \text{ a.s.}$$

Proof. Let

$$S_1 = \{x \in S\colon \lim_{n\to\infty} F_n(x) \text{ exists}\}.$$

Then S_1 is a linear subspace of S, $S_1 \in \mathscr{A}^*$, and $P(S_1) = 1$. Put

$$F(x) = \lim_{n\to\infty} F_n(x) \quad \text{for } x \in S_1$$

and extend F linearly to the rest of S by an argument using a well-ordered Hamel basis for S. Clearly, F has the required properties.

We now add a condition of countable additivity to X, in addition to (1)-(3).

Definition 2.2. Suppose we have a mapping $A \to X(A)$ from elements of $\mathscr{B}_0$ to random variables on some probability space such that if $X(t)$ is defined to be $X([0, t))$, then $(X(t), t \in T)$ satisfies (1)-(3). Suppose also that for all disjoint sequences $A_1, \ldots, A_n$ of sets in $\mathscr{B}_0$ we have

$$X(A_1 \cup \ldots \cup A_n) = \sum_1^n X(A_m) \text{ a.s.},$$

and if A_n, $n \geqslant 1$, is a decreasing sequence of sets in $\mathscr{B}_0$ such that $\lim_{n\to\infty} A_n$ is empty, then $X(A_n) \to 0$ in probability. We then say that X is a *continuous countably additive stochastic measure on* $(T, \mathscr{B}_0)$.

Let us start with a process $(X(t), t \in T)$ satisfying (1)-(3), and define for $A \in \xi$ the additive set functions $a(A)$, $Y(A)$, $Z(A)$ in terms of the functions $a(t)$, $Y(t)$, $Z(t)$ in exactly the same way as we defined $X(A)$ in terms of $X(t)$. We also put $\beta(A) = \mathrm{E}\big((Y(A))^2\big)$ for $A \in \xi$.

LEMMA 2.2. *The map $A \to Z(A) + Y(A)$ defined on (T, ξ) can be extended to be a continuous countably additive stochastic measure on $(T, \mathscr{B}_0)$.*

Proof. It suffices to prove this lemma in the case where T is bounded, i.e., $T = [0, b]$. Now β is a bounded non-negative additive set function on (T, ξ) and, furthermore, $\beta([0, t))$ is continuous. It follows again from the Hahn extension theorem that β has a unique countably additive extension to $(T, \mathscr{B}_0)$. Let δ_0 stand for the measure on R which assigns mass 1 to $\{0\}$ and no mass elsewhere. Let $\beta \times \delta_0$ stand for the product measure of β and δ_0, defined on the Borel subsets of $T \times R$. Define for any Borel subset B of $T \times R$ the measure

$$\psi(B) \equiv \int_B \frac{s^2}{1+s^2} L(dt \times ds) + (\beta \times \delta_0)(B).$$

ψ is a bounded non-negative measure on the Borel subsets of $T \times R$.

For any $A \in \xi$ we have

$$\mathrm{E}\big(\exp[iuX(A)]\big) = \exp\left[iu\,a(A) + \int_R \left(e^{ius} - 1 - \frac{ius}{1+s^2}\right)\frac{1+s^2}{s^2}\,\psi(A \times ds)\right],$$

where the integrand is defined as a continuous function on $T \times R$ by setting it equal to $-(1/2)u^2$ at points of the form $(u, 0)$.

We know that for any Borel subset A of T there is a sequence $A_n \in \xi$ such that

$$\psi\big(((A_n \sim A) \cup (A \sim A_n)) \times R\big) \to 0.$$

Letting $A_{m,n} = (A_n \sim A_m) \cup (A_m \sim A_n)$ we have

$$\psi(A_{m,n} \times R) \to 0 \quad \text{as } m, n \to \infty.$$

But

$$\log \mathrm{E}\big(\exp[iu\{Z(A_{m,n}) + Y(A_{m,n})\}]\big)$$
$$= \int_R \left(e^{ius} - 1 - \frac{ius}{1+s^2}\right)\frac{1+s^2}{s^2}\,\psi(A_{m,n} \times ds).$$

It follows that $Z(A_{m,n}) + Y(A_{m,n}) \to 0$ in probability as $n, m \to \infty$. Hence there is a random variable, which we denote by $Z(A) + Y(A)$, such that

$$Z(A_n) + Y(A_n) \to Z(A) + Y(A)$$

in probability. It is clear that $Z(A) + Y(A)$ is independent up to P-equivalence of the particular sequence A_n which we use to define it. It is also clear that

$$\log \mathrm{E}\big(\exp[iu(Z(A) + Y(A))]\big) = \int_R \left(e^{ius} - 1 - \frac{ius}{1+s^2}\right)\frac{1+s^2}{s^2}\,\psi(A \times ds).$$

Hence the map $A \to Z(A) + Y(A)$ is a countably additive stochastic measure on $(T, \mathscr{B}_0)$.

From Lemma 2.2 it is clear that if X satisfies (1)-(3), then X can be extended to be a countably additive stochastic measure on $(T, \mathscr{B}_0)$ if and only if the constant part a can be extended to be a measure on $(T, \mathscr{B}_0)$. Supposing now that this extension is possible we see that, for any A in $\mathscr{B}_0$, $X(A)$ is defined as the limit in probability of some sequence $X(A_n)$, $A_n \in \xi$. We can find some subsequence $\{n'\}$ such that $x(A_{n'})$ converges as $n' \to \infty$ a.s. The function

$$x \to \lim_{n' \to \infty} x(A_{n'})$$

is defined on a linear subset S_1 of S with $S_1 \in \mathscr{A}^*$ and $P(S_1) = 1$. Hence, as in Lemma 2.1, we may extend this function to all of S so that it is in S^*. We denote this extension by $\hat{I}_{\chi_A}$. If

$$f = \sum_0^n c_i \chi_{A_i}, \quad \text{where } A_i \in \mathscr{B}_0 \text{ and } c_i \in R,$$

we can similarly define $\hat{I}_f$ in S^* so that

$$\hat{I}_f = \sum_1^n c_i X(A_i) \text{ a.s.}$$

Definition 2.3. Suppose that X is a continuous countably additive stochastic measure on $(T, \mathscr{B}_0)$.

(a) If f is a real-valued simple Borel function, i.e.,

$$f = \sum_1^n c_i \chi_{A_i}, \quad \text{where } A_1, \ldots, A_n \in \mathscr{B}_0 \text{ and } c_1, \ldots, c_n \in R,$$

then we say that f is *X-integrable* and we define $\int_A f(t)\,dX(t)$ to be

$$\sum_1^n c_i X(A \cap A_i) \quad \text{for all } A \in \mathscr{B}.$$

(b) Let us suppose that f is a bounded real-valued function with $\{t\colon f(t) \neq 0\} \in \mathscr{B}_0$. (We call such functions *$\mathscr{B}_0$-bounded.*) We say that f is *X-integrable* and we define $\int_A f(t)\,dX(t)$ to be the limit in probability of

$$\int_A f_n(t)\,dX(t),$$

where f_n, $n \geqslant 1$, is any sequence of real-valued simple Borel functions such that $f_n \to f$ pointwise and $\sup_{n\geqslant 1} |f_n(t)|$ is $\mathscr{B}_0$-bounded. By [11] and [13] the above limit in probability is independent of the particular sequence f_n and exists for all $\mathscr{B}_0$-bounded f and for all $A \in \mathscr{B}$.

(c) Suppose now that f is any Borel measurable function on T. We say that f is *X-integrable* if for any $A \in \mathscr{B}$ we have

$$\lim_{n\to\infty} \int_A f\chi_{D_n}\,dX(t) \text{ a.s.}, \quad \text{where } D_n = \{t\colon |f(t)| \leqslant n\} \cap [0, n],$$

and we denote this limit by

$$\int_A f(t)\,dX(t).$$

By [13], for $G_n = \{t: |f(t)| \leqslant a_n\} \cap [0, b_n]$,

$$\lim_{n\to\infty} \int_A f\chi_{G_n} dX(t) = \int_A f(t) dX(t) \text{ a.s.}$$

for any sequences a_n and b_n such that $a_n \uparrow \infty$ and $b_n \uparrow \infty$, if f is X-integrable and $A \in \mathscr{B}$.

In [11] an analytical characterization of X-integrable functions is given. We will not need this characterization here, so we omit it.

Let us now show that, for any X-integrable function f, there is an element $\hat{I}_f$ of S^* such that

$$\hat{I}_f = \int_T f(t) dX(t).$$

We have already verified this for simple Borel functions f. If f is $\mathscr{B}_0$-bounded, then there exists a sequence f_n of simple Borel functions such that $f_n \to f$ uniformly. If we take a subsequence $\{n'\}$ such that $\hat{I}_{f_{n'}}$ converges a.s., then we can apply Lemma 2.1 as before to get the required result for f. If f is only assumed to be X-integrable, set $f_n = f\chi_{D_n}$. Now f_n is $\mathscr{B}_0$-bounded and $\hat{I}_{f_n}$ converges a.s. Apply Lemma 2.1 once more and we are done.

3. Completion of measurable linear functionals. We start by proving some facts about probability measures on Abelian groups.

LEMMA 3.1. *Suppose that S is an Abelian group. Suppose that T is some abstract index set and, for every $t \in T$, $X(t)$ is an additive real-valued functional on S, i.e., for all $x, y \in S$,*

$$X(t)(x+y) = X(t)(x) + X(t)(y).$$

Let $\mathscr{A}$ be the σ-field generated by the functionals $\{X(t), t \in T\}$. Define $\theta\colon S\times S \to S$ by $\theta(x, y) = x+y$ for $(x, y) \in S\times S$. We claim that $\theta^{-1}(A) \in \mathscr{A}\times\mathscr{A}$ for all $A \in \mathscr{A}$.

Proof. If A is of the form $\{Z: X(t)(z) < u\}$, where $t \in T$ and $u \in R$, then

$$\theta^{-1}(A) = \{(y, x): X(t)(x)+X(t)(y) < u\} = \bigcup_r A_r,$$

where the union is taken over all r rational and

$$A_r = \{(x, y): X(t)(x)+r < u\} \cap \{(x, y): X(t)(y) < r\}.$$

A_r is clearly in $\mathscr{A}\times\mathscr{A}$. Hence $\theta^{-1}(A)$ is in $\mathscr{A}\times\mathscr{A}$ for A of the above form. But sets A of the above form generate $\mathscr{A}$.

Definition 3.1. Suppose that $(S, \mathscr{A})$ satisfies the hypotheses of Lemma 3.1 and assume that P_1 and P_2 are probability measures on $(S, \mathscr{A})$. For any $B \in \mathscr{A}\times\mathscr{A}$ let us put

$$B_y = \{x: x \in S, (x, y) \in B\}.$$

By [10], 8.2a, $B_y \in \mathscr{A}$, and the functions $y \to P_1(B_y)$ and $y \to P_2(B_y)$ are $\mathscr{A}$-measurable. Also

$$\int_S P_1(B_y)P_2(dy) = \int_S P_2(B_y)P_1(dy),$$

and if we denote by $P_1 \times P_2(B)$ the common value of the last two integral expressions, then $P_1 \times P_2$ is a probability measure on $(S \times S, \mathscr{A} \times \mathscr{A})$.

If we set $B = \theta^{-1}(A)$ for $A \in \mathscr{A}$, we get $B_y = \{x: (x+y) \in A\} = A - y$. Let us define the probability measure $P_1 * P_2$ on $(S, \mathscr{A})$ by setting

$$P_1 * P_2(A) = \int_S P_1(A-y)P_2(dy) = \int_S P_2(A-y)P_1(dy).$$

If $(S, \mathscr{A})$ satisfies the hypotheses of Lemma 3.1, then for all $A \in \mathscr{A}$ and $y \in S$ we have just seen that $A - y \in \mathscr{A}$. If P is a probability measure on $(S, \mathscr{A})$, let us denote by P^y the probability measure on $(S, \mathscr{A})$ such that $P^y(A) \equiv P(A-y)$.

LEMMA 3.2. *Let $(S, \mathscr{A})$ be as in Lemma 3.1. Let P be a probability measure on $(S, \mathscr{A})$. Suppose that F is an additive real-valued functional on S which is measurable relatively to the completion of $\mathscr{A}$ with respect to any measure P^x, $x \in S$, x fixed. Then F is measurable relatively to the completion of $\mathscr{A}$ with respect to any other measure P^y, $y \in S$.*

Proof. It suffices to prove this in the case where $x = 0$. Let u be in R. We can find two sets in $\mathscr{A}$, denoted by A_u and B_u, such that

$$A_u \subset \{x: F(x) < u\} \subset B_u$$

and such that if $C_u = B_u \sim A_u$, then $P(C_u) = 0$. Now

$$A_u + y \subset \{x: F(x) < u\} + y \subset B_u + y$$

and

$$\{x: F(x) < u\} + y = \{x: F(x) < u + F(y)\}$$

by the additivity of F. Furthermore,

$$(B+y) \sim (A_u+y) = C_u + y \quad \text{and} \quad P^y(C_u+y) = P(C_u) = 0.$$

It follows that $\{x: F(x) < u + F(y)\}$ is in the completion of $\mathscr{A}$ with respect to P^y. But this argument works for all $u \in R$. It follows that F is measurable with respect to the P^y-completion of $\mathscr{A}$.

LEMMA 3.3. *Let $(S, \mathscr{A})$ be as in Lemma 3.1. Let P_1 and P_2 be two probability measures on $(S, \mathscr{A})$. Let $\mathscr{A}_i^*$ denote the completion of $\mathscr{A}_i$ with respect to P_i, $i = 1, 2$. Let $\mathscr{A}^*$ denote the completion of $\mathscr{A}$ with respect to P, where $P = P_1 * P_2$. Let F be a real-valued additive functional on S and suppose that F is measurable with respect to $\mathscr{A}^*$. Then F is measurable with respect to $\mathscr{A}_i^*$, $i = 1, 2$. Furthermore, if φ denotes the characteristic function of the random variable F on $(S, \mathscr{A}^*, P)$ while φ_i denotes the characteristic function of the random variable F on $(S, \mathscr{A}_i^*, P_i)$, $i = 1, 2$, then $\varphi = \varphi_1\varphi_2$.*

Proof. For r rational let $D_r = \{x: F(x) < r\}$. Let $A_r, B_r \in \mathscr{A}$ be such that

$$A_r \subset D_r \subset B_r \quad \text{and} \quad P(B_r \sim A_r) = \int_S P_1\big((B_r \sim A_r) - y\big) P_2(dy) = 0.$$

Let $C_r = \big\{y: P_1\big((B_r \sim A_r) - y\big) = 0\big\}$. Now $P_2(C_r) = 1$. Hence, if

$$C = \bigcap_r C_r,$$

then $P_2(C) = 1$. Now choose $x \in C$. Then $P_1\big((B_r \sim A_r) - x\big) = 0$ for all rational r, i.e., $P_1^x(B_r \sim A_r) = 0$ for all rational r. It follows that F is measurable with respect to the P_1^x-completion of $\mathscr{A}$. Now, since F is additive, we can use Lemma 3.2 to conclude that F is measurable with respect to $\mathscr{A}_1^*$. The functional F is measurable with respect to $\mathscr{A}_2^*$ by a similar argument.

As for the last statement, consider the map $y \to P_1(D_r - y)$. Since $P_2(C_r) = 1$, this function is $\mathscr{A}_2^*$-measurable and is P_2 a.s. equal to the function $y \to P_1(A_r - y)$. Now

$$P(A_r) = \int_S P_1(A_r - y) P_2(dy).$$

Also $P(A_r) = P(D_r)$. It follows that

$$P(D_r) = \int_S P_1(D_r - y) P_2(dy).$$

Hence the last statement follows.

Lemma 3.3 will play a crucial role in our attempt to represent elements of S^* by stochastic integrals. We now present some notation that we will use when we apply Lemma 3.3 to X as in Section 1 or 2.

Definition 3.2. Suppose that X is a continuous countably additive stochastic measure on $(T, \mathscr{B}_0)$. Let A be a fixed set in $\mathscr{B}$. We can define a *countably additive stochastic measure* on $(T, \mathscr{B}_0)$, denoted by X_A, by setting

$$X_A(B) = X(A \cap B) \quad \text{for all } B \in \mathscr{B}_0.$$

Let P_A be the measure induced on $(S, \mathscr{A})$ by the process $\big(X_A(t), t \in T\big)$, where $X_A(t) = X_A([0, t])$. Let A^c denote $T \sim A$. Then $P = P_A * P_{A^c}$ in the sense of Definition 3.1, and Lemma 3.3 can be applied. Denoting by $\mathscr{A}_A^*$ the completion of $\mathscr{A}$ with respect to P_A we see that if F is any additive functional on S which is $\mathscr{A}_T^*$-measurable, then F is also $\mathscr{A}_A^*$-measurable for any $A \in \mathscr{B}$. In particular, if $F \in S^*$, then F is $\mathscr{A}_A^*$-measurable for any $A \in \mathscr{B}$. (Remember that $\mathscr{A}^* = \mathscr{A}_T^*$.)

In connection with Definition 3.2, let us note that if f is a Borel measurable function and $A \in \mathscr{B}$, then $f\chi_A$ is X-integrable if and only if f is X_A-integrable. Furthermore, if that is the case, then, for all $B \in \mathscr{B}$, $\int_B f\chi_A dX(t)$ has the same distribution as $\int_B f dX_A(t)$. Also

$$\int_B f dX_A(t) = \int_B f\chi_A dX_A(t) \quad P_A \text{ a.s.}$$

Definition 3.3. Suppose that $(X(t), t \in T)$ satisfies (1)-(3). For any $x \in S$ and $\gamma > 0$ define x^γ as in Section 1 and put $x_\gamma \equiv x - x^\gamma - a$, where a is as defined in the Lévy decomposition of X. Let $E^\gamma(x) \equiv x^\gamma$ and let $E_\gamma(x) \equiv x_\gamma$. Now both E^γ and E_γ are measurable mappings from $(S, \mathscr{A}^*, P)$ to $(S, \mathscr{A})$. Let P^γ stand for the measure induced on $(S, \mathscr{A})$ by the mapping E^γ; define P_γ similarly in terms of E_γ. Then again $P = P^\gamma * P_\gamma$ in the sense of Definition 3.1; and if we denote by $\mathscr{A}^\gamma$ the completion of $\mathscr{A}$ with respect to P^γ, then any additive functional F on S which is $\mathscr{A}^*$-measurable is also $\mathscr{A}^\gamma$-measurable (and if we define $\mathscr{A}_\gamma$ to be the completion of $\mathscr{A}$ with respect to P_γ, then F is also $\mathscr{A}_\gamma$-measurable).

Let us remember the definition of X^γ from Section 1, and let us put

$$X_\gamma(t) \equiv X(t) - X^\gamma(t) - a(t).$$

Then X^γ (X_γ) induces the measure P^γ (P_γ) on $(S, \mathscr{A})$. Furthermore, both X^γ and X_γ satisfy (1)-(3).

Lastly, let us note that Definitions 3.2 and 3.3 can be applied in succession to a process X, yielding $(X_A)^\gamma$ or $(X^\gamma)_A$. However, it is easy to see that, for all $t \in T$,

$$\begin{aligned} \mathrm{E}\big(\exp[iu(X^\gamma)_A(t)]\big) &= \mathrm{E}\big(\exp[iu(X_A)^\gamma(t)]\big) \\ &= \exp\Big[\int_{|s|>\gamma} (e^{ius}-1) L\big(A \cap [0, t) \times ds\big)\Big]. \end{aligned}$$

We shall henceforth denote by X^γ_A the process which we have just defined and we shall denote by P^γ_A the probability measure which it induces on $(S, \mathscr{A})$. We shall denote by $\mathscr{A}^\gamma_A$ the completion of $\mathscr{A}$ with respect to P^γ_A.

We now prove a lemma whose nature is rather technical. It is used in the proof of Theorem 3.1. $\mathscr{B}$, as used below, stands for the Borel subsets of R.

LEMMA 3.4. *Suppose that $(X(t), t \in T)$ satisfies* (1)-(3). *Let G be a real-valued additive functional on S which is measurable with respect to $\mathscr{A}^*$. Suppose that, for some $\gamma > 0$, $G = 0$ P^γ a.s. Then G is measurable with respect to the P-completion of $(E_\gamma)^{-1}(\mathscr{A})$.*

Proof. Let $(\mathscr{A}\times\mathscr{A})(\gamma)$ stand for the completion of $\mathscr{A}\times\mathscr{A}$ with respect to $P_\gamma\times P^\gamma$. Let $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for $(x, y)\in S\times S$. Define $\theta\colon S\times S\to S$ as in Lemma 3.1. Now G is measurable with respect to $\mathscr{A}_\gamma$; hence $G\circ\pi_1$ is measurable with respect to $(\mathscr{A}\times\mathscr{A})(\gamma)$. Also G is measurable with respect to $\mathscr{A}^\gamma$; hence $G\circ\pi_2$ is measurable with respect to $(\mathscr{A}\times\mathscr{A})(\gamma)$. Hence $G\circ\theta = G\circ\pi_1+G\circ\pi_2$ is measurable with respect to $(\mathscr{A}\times\mathscr{A})(\gamma)$.

Now $G\circ\theta$ is measurable with respect to the $(P_\gamma\times P^\gamma)$-completion of $\pi_1^{-1}(\mathscr{A})$, since $G\circ\pi_2 = 0$ $P_\gamma\times P^\gamma$ a.s. by the hypothesis of the lemma. (Note that $G\circ\pi_2$ as a random variable on $\big(S\times S, (\mathscr{A}\times\mathscr{A})(\gamma), P_\gamma\times P^\gamma\big)$ has the same distribution as does G as a random variable on $(S, \mathscr{A}^\gamma, P^\gamma)$.)

Thus G is measurable with respect to the P-completion of $(E_\gamma)^{-1}(\mathscr{A})$, since the map $(G\circ\theta, \pi_1, \pi_2)$ defined on $\big(S\times S, (\mathscr{A}\times\mathscr{A})(\gamma), P_\gamma\times P^\gamma\big)$ has the same distribution on $(R\times S\times S, \mathscr{B}\times\mathscr{A}\times\mathscr{A})$ as does the map (G, E_γ, E^γ) defined on $(S, \mathscr{A}^*, P)$. (Note that $(P_\gamma\times P^\gamma)\theta^{-1} = P_\gamma * P^\gamma = P$.)

The following lemma shows how we use the hypothesis that X has trivial Gaussian part.

LEMMA 3.5. *Suppose that* $\big(X(t), t\in T\big)$ *satisfies* (1)-(3). *Then, if* $Y(t)$, *the Gaussian part of* $X(t)$ *in its Lévy decomposition, is identically zero, then* $\bigcap_{\gamma>0}(E_\gamma)^{-1}(\mathscr{A})$ *is P-trivial (i.e., P assumes only the values* 1 *and* 0 *on the σ-subfield.)*

Proof. $(E_\gamma)^{-1}(\mathscr{A})$ and $(E^\gamma)^{-1}(\mathscr{A})$ are P-independent for all $\gamma>0$. Hence $\bigcap_{\gamma>0}(E_\gamma)^{-1}(\mathscr{A})$ is independent of the σ-field $\bigvee_{\gamma>0}(E^\gamma)^{-1}(\mathscr{A})$ generated by $\bigcup_{\gamma>0}(E^\gamma)^{-1}(\mathscr{A})$.

Now, from the Lévy decomposition of X and the assumption that Y vanishes it follows that $\mathscr{A}$ is equal to the σ-field generated by the random variables $X^\gamma(t)$, $\gamma>0$, $t\in T$, up to P-equivalence. Furthermore, since the map

$$E^\gamma\colon (S, \mathscr{A}^*, P)\to(S, \mathscr{A})$$

is measurable and since $E^\gamma(x) = x^\gamma$ for all $x\in S$, $X^\gamma(t)$ is measurable with respect to the P-completion of $(E^\gamma)^{-1}(\mathscr{A})$. It follows that $\bigvee_{\gamma>0}(E^\gamma)^{-1}(\mathscr{A})$ contains $\mathscr{A}$ up to P-equivalence.

We now present some results about the possibility of representing stochastic integrals as linear functionals in a more direct manner than in Section 2.

Suppose that $\big(X(t), t\in T\big)$ satisfies (1)-(3) and let $L([0,t]\times R)<\infty$ for all $t\in T$. Then $\lim_{\gamma\to0}X^\gamma(t)$ exists as a limit in probability. We say then that X satisfies *Condition* A$'$ if

$$L([0,t]\times R)<\infty \quad\text{and}\quad X(t) = \lim_{\gamma\to0}X^\gamma(t) \text{ a.s.} \quad \text{for all } t\in T.$$

Note that if X satisfies Condition A$'$, then X can be extended to be a countably additive continuous stochastic measure on $(T, \mathscr{B}_0)$. Note also that if X satisfies (1)-(3), then X^γ satisfies Condition A$'$ for any $\gamma > 0$.

Definition 3.4. A function x: $T \to R$ is said to be *piecewise constant* if there exists a strictly increasing sequence of points $t_n \in T$, finite for T bounded, such that

$$\bigcup_n [0, t_n) = T$$

and x is constant on each of the intervals $[t_i, t_{i+1})$.

Let S' denote the set of all piecewise constant functions on T with $x(t) = x(t+)$ for $t \in T$, t not a right-hand boundary point. The following theorem can be found in [4], p. 274.

THEOREM A$'$. *Let $(X(t), t \in T)$ satisfy* (1)-(3). *Then P assigns measure one to S' if and only if X satisfies Condition* A$'$.

Definition 3.5. Let X satisfy (1)-(3) and Condition A$'$. For any real-valued function f on T with $\{t: f(t) \neq 0\}$ a bounded subset of T we put, for $x \in S'$,

$$I_f(x) \equiv \sum_1^\infty f(t_i) h_i,$$

where x has its jumps at the points t_i, and h_i are the corresponding sizes of the jumps. (Note that the sum has only a finite number of non-zero terms.) We extend I_f to be linear on all of S by a Hamel basis argument as in Lemma 2.1.

LEMMA 3.6. *Suppose that $(X(t), t \in T)$ satisfies* (1)-(3) *and Condition* A$'$. *Then, for any $A \in \mathscr{B}_0$ and any real-valued Borel measurable function f, $f\chi_A$ is X-integrable. Furthermore, if f_n, $n \geqslant 1$, is a sequence of Borel functions such that $f_n \to f$ pointwise, then*

$$\int_T f_n(t) \chi_A(t) dX(t)$$

converges to

$$\int_T f(t) \chi_A(t) dX(t)$$

in probability. Lastly, $I_{f\chi_A}$ is $\mathscr{A}^*$-measurable and

$$I_{f\chi_A} = \int_T f(t) \chi_A(t) dX(t) \quad a.s.$$

Proof. Let

$$f_n = f\chi_{C_n}, \quad \text{where } C_n = \{t: |f(t)| \leqslant n\}.$$

We need only to prove that, for any $C \in \mathscr{B}$, $\int_C f_n \chi_A dX(t)$ converges in probability. Consider

$$\log \mathrm{E}\left(\exp\left[iu\left(\int_C (f_n - f_m)\chi_A dX(t)\right)\right]\right)$$

$$= \int_{(C\cap A)\times R} \left(\exp\left[ius\left(f_n(t) - f_m(t)\right)\right] - 1\right) L(dt \times ds).$$

The last expression goes to zero as $n, m \to \infty$ by the Lebesgue bounded convergence theorem. It follows that $\int_C f_n \chi_A dX(t)$ is Cauchy in the topology of convergence in probability, and hence converges in this topology. This convergence is, in fact, a.s. since we can treat $\int_C f_n \chi_A dX(t)$ as a series of independent random variables.

The second statement in the lemma follows from the computation with characteristic functions that we have just accomplished.

Let us now prove the last statement. Let M_0 be as in Definition 2.1. Define M_n, $n \geqslant 1$, inductively in terms of M_{n-1} by setting M_n equal to the set of all real-valued functions that are pointwise limits of a sequence of elements of M_{n-1}. Clearly, $\bigcup_0^\infty M_n$ is equal to the set of all real-valued Borel measurable functions on T. Now for $f \in M_0$ and $A \in \xi$, clearly, $I_{f\chi_A}$ is $\mathscr{A}^*$-measurable and

$$I_{f\chi_A} = \int_T f\chi_A dX(t) \text{ a.s.}$$

If $f \in M_1$, then there is a sequence of functions $f_n \in M_0$ such that $f_n(t) \to f(t)$ for all $t \in T$. Hence

$$I_{f_n\chi_A}(x) \to I_{f\chi_A}(x) \quad \text{for all } x \in S'.$$

On the other hand, by the second statement of this lemma,

$$\int_T f_n \chi_A dX(t) \to \int_T f\chi_A dX(t)$$

in probability. It follows that $I_{f\chi_A}$ is $\mathscr{A}^*$-measurable and

$$I_{f\chi_A} = \int_T f\chi_A dX(t) \text{ a.s.} \quad \text{for all } f \in M_1 \text{ and } A \in \xi.$$

Proceeding inductively we infer that in the last equation f needs only to be assumed Borel measurable. Hence A may be assumed in $\mathscr{B}_0$ in the last equation, since $f\chi_A$ is then Borel measurable and there exists a set in ξ which contains A.

We now prove a basic lemma about the possibility of representing elements of S^* as stochastic integrals. This lemma and Lemma 3.3 are the two crucial points of our discussion.

LEMMA 3.7. *Suppose that* $(X(t), t \in T)$ *satisfies* (1)-(3) *and also Condition* A′. *Suppose that* F *is in* S^*. *Then there is some Borel measurable function* f *such that for all* $A \in \mathscr{B}_0$ *we have*

$$F = \int_T f(t)\,dX_A(t) \;\; P_A \text{ a.s.}$$

Proof. First note that if f is Borel measurable and $A \in \mathscr{B}_0$, then $f\chi_A$ is X-integrable by Lemma 3.6, and hence f is X_A-integrable. Note also that if $x \in S'$, then the map $B \to x(B)$ defined on ξ can be extended uniquely to be a countably additive measure on $(T, \mathscr{B}_0)$. For $A \in \mathscr{B}_0$ let us define the function x^A by setting

$$x^A(t) \equiv x\big(A \cap [0, t)\big).$$

If $A = [0, b]$, $b > 0$, let us write $x^b = x^{[0,b]}$. Note that if we define the map $E^A\colon S \to S$ by $E^A(x) \equiv x^A$ for $x \in S'$, and $E^A(x) = 0$ for $x \notin S'$, then $P(E^A)^{-1} = P_A$. Let us also note that if

$$A = [0, b] \quad \text{and} \quad H^b \equiv \{x \in S'\colon x(t) = x(b) \text{ for } t > b\},$$

then $H^b \in \mathscr{A}^*_{[0,b]}$ and $P_{[0,b]}(H^b) = 1$.

Let us now proceed to define the function f. First we define a real--valued function g on T. For $t \in T$ let x_t be an element of S' which is everywhere constant except for a jump at t of size $h(x_t)$. Put $g(t) \equiv F(x_t)/h(x_t)$. The function g is well defined since F and h are linear. Note that, for $x \in S'$ and $A \in \mathscr{B}_0$, $I_{g\chi_A}(x) = F(x^A)$.

We now show that we can change g on a Borel subset D of T such that $L(D \times R) = 0$, and we get a Borel measurable function f. To do this let $b > 0$ and define $B^b \subset S'$ as

$$B^b \equiv \{x \in S'\colon x \text{ has one and only one jump in } [0, b]\}.$$

Then $B^b \in \mathscr{A}^*_{[0,b]}$ and $P_{[0,b]}(B^b) > 0$. Let us now restrict ourselves to the set B^b. Let $\mathscr{A}(b)$ denote $\{C = B \cap B^b,\ B \in \mathscr{A}\}$. Let

$$Q_b(C) = P_{[0,b]}(C)/P_{[0,b]}(B^b) \qquad \text{for } C \in \mathscr{A}(b).$$

Then $\big(B^b, \mathscr{A}(b), Q_b\big)$ is a probability triple. Let $\mathscr{A}^*(b)$ denote the completion of $\mathscr{A}(b)$ with respect to Q_b. Now

$$\{x \in B^b\colon F(x) = F(x^b)\} \supset H^b \cap B^b.$$

It follows that the map $x \to F(x^b)$ defined on B^b is $\mathscr{A}^*(b)$-measurable. Let us define $\tau_b\colon B^b \to [0, b]$ by

$$\tau_b(x) = \sup_{t \in [0,b]} \{t\colon x(t) = 0\} \qquad \text{for } x \in B^b.$$

Write $g\big(\tau_b(x)\big) = F(x^b)/x(b)$ for $x \in B^b$. It follows that $g \circ \tau_b$ is measurable with respect to $\mathscr{A}^*(b)$. Hence $\tau_b^{-1}\big(\{t \in [0, b]\colon g(t) < a\}\big)$ is in

$\mathscr{A}^*(b)$ for all $a \in R$. It follows that there exist two sets $A_1, B_1 \in \mathscr{A}(b)$ such that

$$A_1 \subset \tau_b^{-1}\big(\{t \in [0, b]:\ g(t) < a\}\big) \subset B_1 \quad \text{and} \quad Q_b(B_1 \sim A_1) = 0.$$

Define V_b: $B^b \to [0, b] \times R$ by $V_b(x) \equiv \big(\tau_b(x), x(b)\big)$ for $x \in B^b$ and set

$$C_1 = \{(t, s):\ t \leqslant t_0,\ s < u\}, \quad C_2 = \{(t, s):\ t > t_0\} \quad \text{for } t_0 \in [0, b].$$

Clearly, the map V_b is measurable on $\big(B^b, \mathscr{A}(b)\big)$. Also

$$V_b^{-1}(C_1) = \{x \in B^b:\ x(t_0) < u\} \quad \text{for } u \leqslant 0$$

while

$$V_b^{-1}(C_1 \cup C_2) = \{x \in B^b:\ x(t_0) < u\} \quad \text{for } u > 0.$$

Hence V_b generates $\mathscr{A}(b)$, and so there exist two Borel sets A_2 and B_2 in $[0, b] \times R$ such that $V_b^{-1}(A_2) = A_1$ and $V_b^{-1}(B_2) = B_1$.

Now

$$V_b\Big(\tau_b^{-1}\big(\{t \in [0, b]:\ g(t) < a\}\big)\Big) = \{t \in [0, b]:\ g(t) < a\} \times R.$$

It follows that

$$A_2 \subset \{t \in [0, b]:\ g(t) < a\} \times R \subset B_2.$$

We also have $Q_b V_b^{-1}(B_2 \sim A_2) = 0$. Now define π: $T \times R \to T$ by

$$\pi\big((t, s)\big) = t \quad \text{for } (t, s) \in T \times R.$$

Let B_2^c stand for $([0, b] \times R) \sim B_2$. Then

$$A_2 \subset \big(\pi(A_2) \times R\big) \subset \{t \in [0, b]:\ g(t) < a\} \times R \subset \Big(\big([0, b] \sim \pi(B_2^c)\big) \times R\Big) \subset B_2.$$

Hence we have

$$Q_b V_b^{-1}\Big(\big([0, b] \sim \pi(B_2^c) \times R\big) \sim \big(\pi(A_2) \times R\big)\Big) = 0,$$

since all sets appearing in the last expression are analytic, and hence completion measurable with respect to any measure on $T \times R$ (see [7], p. 391). It follows likewise that

$$Q_b \tau_b^{-1}\Big(\big([0, b] \sim \pi(B_2^c)\big) \sim \pi(A_2)\Big) = 0.$$

But we have

$$\pi(A_2) \subset \{t \in [0, b]:\ g(t) < a\} \subset [0, b] \sim \pi(B_2^c).$$

It follows that if g is restricted to $[0, b]$, then it is measurable with respect to the completion of the Borel sets under $Q_b \tau_b^{-1}$.

To complete the proof of the lemma we need to examine the probability measure $Q_b \tau_b^{-1}$ defined on $([0, b], \mathscr{B})$ more closely. For $t \in [0, b]$ let

$$N\big([0, t)\big) = \lim_{\delta \to 0} V_t(\delta) + V_t(-\delta),$$

where the limit is taken in probability. (Note that $N([0, t))$ is the number of jumps in $[0, t)$.) Extend N to a finitely additive stochastic set function on $([0, b], \xi)$ in the usual way. Note that

$$\mathrm{E}\left(\exp[iuN(B)]\right) = \exp[(e^{iu}-1)L(B\times R)]$$

and

$$\mathrm{E}\left(N(B)\right) = L(B\times R) \quad \text{for } B\in\xi.$$

Hence N has a unique extension to $([0, b], \mathscr{B}_0)$ which is a continuous countably additive stochastic measure. Furthermore, $\mathrm{E}\left(N(B)\right) = L(B\times R)$ for $B\in\mathscr{B}_0$.

Now, for $B\in\mathscr{B}_0$ and $B\subset[0, b]$,

$$\begin{aligned} Q_b\tau_b^{-1}(B) &= \mathrm{P}\left[N(B) = 1 \mid N([0,b]) = 1\right] \\ &= \mathrm{P}\left[N(B) = 1,\ N([0,b]\sim B) = 0 \mid N([0,b]) = 1\right] \\ &= \frac{\exp[-L(B\times R)]L(B\times R)\exp\left[-L\left(([0,b]\sim B)\times R\right)\right]}{\exp\left[-L([0,b)\times R)\right]L([0,b]\times R)} \\ &= \frac{L(B\times R)}{L([0,b]\times R)}. \end{aligned}$$

Hence, we can change g on a Borel set D such that $L(D\times R) = 0$ to get a Borel measurable function f. Also $\mathrm{E}\left(N(D)\right) = 0$, whence, for almost all $x\in S'$, x has no jumps in D. From this it follows that $I_{f\chi_A}(x) = I_{g\chi_A}(x)$ a.s. for any $A\in\mathscr{B}_0$. Let us recall that, for $x\in S'$, $I_{g\chi_A}(x) = F(x^A)$. We conclude easily that $F(x^A) = I_{f\chi_A}(x^A)$ a.s., and then, by Lemma 3.6,

$$F = \int_T f(t)dX_A(t) \quad P_A \text{ a.s.}$$

Let us examine the proof of Lemma 3.7 to see under what extra conditions on X the statement of Lemma 3.7 would be true if F were only assumed to be additive and $\mathscr{A}^*$-measurable. We need only to define $g(t) = F(x_t)/x_t(b)$ for $x_t\in S'$. Now, suppose that there exists a countable subset $R'\subset R$ such that

1° $L([0, b]\times R) = L([0, b]\times R')$ for all $b > 0$, and

2° there is an element $d\in R'$ such that for all $a\in R'$ we have $a = rd$ for some rational number r.

Then almost all sample paths of X have jumps only of height rd for some rational r. If we restrict ourselves to this additive subspace of S, then we can carry out the proof of Lemma 3.7.

LEMMA 3.8. *Suppose that $\left(X(t), t\in T\right)$ satisfies* (1)-(3). *Let F be in S^*, $A\in\mathscr{B}_0$, and $\gamma > 0$. Then there exists some Borel measurable function f such that*

$$F = \int_T f(t)dX_A^\gamma(t) \quad P_A^\gamma \text{ a.s.}$$

Proof. By Lemma 3.3 we know that F is measurable with respect to $\mathscr{A}^{\gamma}$. Now P^{γ} satisfies Condition A'; so from Lemma 3.7 we conclude that for any $\gamma > 0$ there is a Borel function f (which may depend on γ) such that

$$F = \int_T f(t) dX_A(t) \quad P_A^{\gamma} \text{ a.s.}$$

Let us now eliminate the possible dependence of f on γ. Let $\gamma_n = 1/n$, $n = 1, 2, \ldots$, and let L_n denote the measure

$$B \to L(B \times \{s: |s| > \gamma_n\}) \qquad \text{for } B \in \mathscr{B}_0.$$

Define the non-negative measure L' by

$$L'(B) = \int_R \frac{s^2}{1+s^2} L(B \times ds) \qquad \text{for } B \in \mathscr{B}_0.$$

We infer that, for any Borel set B, if $L'(B) = 0$, then $L_n(B) = 0$ for all $n \geqslant 1$. On the other hand, if $L_n(B) = 0$ for all $n \geqslant 1$, then $L'(B) = 0$. From this it follows that g, as defined in Lemma 3.7, is measurable with respect to the completion of $\mathscr{B}$ under L'. So, if we assume that the Borel function f of Lemma 3.7 was constructed by changing g on a set of D of L' measure zero, then the proof is completed.

We now introduce our hypothesis (4) on X (see Section 1) and prove that if X satisfies (1)-(4), then X can be treated as a stochastic measure.

We note here the following equivalent version of (4):

If F_n, $n \geqslant 1$, is a sequence of elements in S^*, and c_n, $n \geqslant 1$, is a sequence of real numbers, then F_n converges in probability whenever $F_n - c_n$ converges in probability.

LEMMA 3.9. *If $(X(t), t \in T)$ satisfies* (1)-(4), *then the map $A \to X(A)$ defined on (T, ξ) can be extended to be a continuous countably additive stochastic measure on $(T, \mathscr{B}_0)$.*

Proof. Let $X(t) = a(t) + Y(t) + Z(t)$ be the Lévy decomposition of X. We now prove that a is a bounded variation on every finite interval $[0, b]$. Suppose not; then there is a sequence $A_n \subset [0, b]$, $A_n \in \xi$, such that $|a(A_n)| \to \infty$. Hence $X(A_n)/a(A_n) \to 1$ in probability. This contradicts (4). We conclude that a is of bounded variation. It has already been known that a is continuous. We conclude that a has a unique extension to a measure on $(T, \mathscr{B}_0)$. From Lemma 2.3 it follows that X itself can be extended, which completes the proof.

In the proof of Theorem 3.1 we will use the following lemma whose proof can be found in Loève [10], Lemma 37.Vb. Loève has a weaker statement but his proof works for our formulation.

LEMMA B. *Let $X_1, \ldots, X_n, \ldots$ be a sequence of independent random variables. Suppose that there exist a characteristic function φ and, for all $n \geqslant 1$, a characteristic function φ_n^c such that $\varphi = \varphi_n \varphi_n^c$, where φ_n is the characteristic function of*

$$S_n = \sum_1^n X_k.$$

Then there exists a sequence of constants a_n such that $S_n - a_n$ converges a.s.

THEOREM 3.1. *Suppose that $(X(t), t \in T)$ satisfies (1)-(4), and suppose also that $Y(t) = 0$ a.s., where Y is the Gaussian part of X in its Lévy decomposition. Then there exists a Borel function f such that f is X-integrable and*

$$F = \int_T f(t)\, dX(t) \ \ a.s.$$

Proof. Define f as in Lemma 3.8; we get

$$F = \int_T f(t)\, dX^{\gamma}_{[0,m]}(t) \ \ P^{\gamma}_{[0,m]} \ \text{a.s.}$$

for every $\gamma > 0$ and every positive integer $m \in T$. Write

$$f_n = f\chi_{D_n}, \quad \text{where } D_n = [0, n] \cap \{t\colon\ |f(t)| \leqslant n\}.$$

For every n let g_k^n be a sequence of simple Borel functions such that $g_k^n \to f_n$ uniformly and

$$\bigcup_k \{t\colon g_k^n(t) \neq 0\} \subset D_n.$$

Let $\gamma_j = \gamma/j$ for $j = 1, 2, \ldots$ Choose for each n a subsequence $\{k'\}$ of $\{k\}$ such that

$$\int_T g_{k'}^n(t)\, dX(t) \to \int_T f_n(t)\, dX(t) \ \text{a.s.}$$

and such that

$$\int_T g_{k'}^n(t)\, dX^{\gamma_j}(t) \to \int_T f_n(t)\, dX^{\gamma_j}(t) \ \ P^{\gamma_j} \ \text{a.s.} \quad \text{for all } j \geqslant 1.$$

(This can be done by a diagonalization argument.)

It follows that $\lim_{k' \to \infty} \hat{I}_{g_{k'}^n}$ can be extended to be a linear functional on all of S, called $\hat{I}_{f_n}$, such that

$$\hat{I}_{f_n} = \int_T F_n(t)\, dX(t) \ \text{a.s.}$$

and

$$\hat{I}_{f_n} = \int_T f_n(t)\,dX^{\gamma_j}(t) \quad P^{\gamma_j} \text{ a.s.} \qquad \text{for all } j \geqslant 1.$$

We now wish to prove that

$$F = \int_T f_n(t)\,dX_n(t) \quad P_n \text{ a.s.},$$

where X_n stands for X_{D_n} and P_n denotes the measure that X_n induces on $(S, \mathscr{A})$. Now

$$F = \int_T f_n(t)\,dX_n^{\gamma_j}(t) \quad P_n^{\gamma_j} \text{ a.s.} \qquad \text{for all } j \geqslant 1$$

by the choice of f and Lemma 3.8. Hence, if $G_n \equiv F - \hat{I}_{f_n}$, then $G_n = 0$ $P_n^{\gamma_j}$ a.s. for all $j \geqslant 1$. So, by Lemmas 3.4 and 3.5, G_n is constant P_n a.s. But X_n also satisfies (1)-(4); hence $G_n = 0$ P_n a.s.

Let φ stand for the characteristic function of F on $(S, \mathscr{A}^*, P)$. Let φ_n denote the characteristic function of F on $(S, \mathscr{A}_n^*, P_n)$ and let φ_n^c denote the characteristic function of F on $(S, \mathscr{A}_n^{*c}, P_n^c)$, where P_n^c stands for the measure induced on $(S, \mathscr{A})$ by $X_{T\sim D_n}$ and $\mathscr{A}_n^*$ $(\mathscr{A}_n^{*c})$ denotes the completion of $\mathscr{A}$ under P_n (P_n^c). By Lemma 3.3 we get $\varphi = \varphi_n \varphi_n^c$.

However, since $F = \int_T f_n(t)\,dX_n(t)$ P_n a.s. and since $\int_T f_n(t)\,dX_n(t)$ has the same distribution under P_n as does $\int_T f_n(t)\,dX(t)$ under P, we conclude that the characteristic function of $\int_T f_n(t)\,dX(t)$ is φ_n. By Lemma B this implies that there exists a sequence of constants c_n such that $\int_T f_n(t)\,dX(t) - c_n$ converges a.s. Applying (4) we conclude that $\int_T f_n(t)\,dX(t)$ converges a.s. (The fact that f is X-integrable follows, since we can prove that $\int_A f_n(t)\,dX(t)$ converges a.s. for any $A \in \mathscr{B}$, by considering $\int_T f_n(t)\,dX_A(t)$ and applying the arguments above to the measure P_A.)

Now, if we define $\hat{I}_f$ as a linear extension of $\lim_{n\to\infty} \hat{I}_{f_n}$, then it is clear that

$$\hat{I}_f = \int_T f(t)\,dX_{[0,m]}(t) \quad P_{[0,m]} \text{ a.s.} \qquad \text{for all } m.$$

However,

$$F = \int_T f(t)\,dX^{\gamma}_{[0,m]}(t) \qquad \text{for every } \gamma > 0$$

by our choice of f. Hence, if $G \equiv F - \hat{I}_f$, then $G = 0$ $P^{\gamma}_{[0,m]}$ a.s. for every $\gamma > 0$. So, from Lemmas 3.4 and 3.5 it follows that $G = 0$ $P_{[0,m]}$ a.s., using (4) as before. It T is bounded, we are done.

So assume that $T = [0, \infty)$. Define $E_m: S \to S$ by $E_m(x) = x_m$ for $x \in S$, where $x_m(t) = 0$ if $t \in [0, m]$ while $x_m(t) = x(t) - x(m)$ if $t > m$. Then $PE_m^{-1} = P_{(m,\infty)}$. Define $\mathscr{A}_m^*$ to be the completion of $E_m^{-1}(\mathscr{A})$ with respect to P. Now $G = 0\ P_{[0,m]}$ a.s., whence by a proof imitating Lemma 3.4 we know that G is measurable with respect to $\mathscr{A}_m^*$. It follows that G is measurable with respect to $\bigcap_{m \geqslant 1} \mathscr{A}_m^*$. Now for every $A \in \xi$ we infer that

$$X\big((A \sim [0, m]) \cup ([0, m] \sim A)\big)$$

converges to zero in probability as $m \to \infty$. From this it follows that $\bigcap_{m \geqslant 1} \mathscr{A}_m^*$ is P-trivial. Using (4) once again we conclude that $G = 0$ a.s. But $G = F - \hat{I}_f$ and we know that

$$\hat{I}_f = \int_T f(t)\, dX(t) \text{ a.s.}$$

We now give some applications of Theorem 3.1.

Definition 3.6. Suppose that X is a random variable such that if $X_1, \ldots, X_n$ are independent and have the same distribution as X, then $X_1 + \ldots + X_n$ is distributed like $(n)^{1/q} X$ for some $q \in (0, 2]$ which is supposed to be fixed while n is allowed to vary over all positive integers. If $q \neq 1$, then we shall say that X has a *strictly stable distribution*. For $q = 1$, we shall say that X has strictly stable distribution if X is also symmetrically distributed. The number q is called the *index of stability*.

Let $\big(X(t), t \in T\big)$ be a stochastic process that satisfies (1). Then we shall say that X is *strictly stable of index* q if, for all $t_1, \ldots, t_n \in T$ and $a_1, \ldots, a_n \in R$, $\sum_1^n a_i X(t_i)$ has a strictly stable distribution of index q.

LEMMA 3.10. *If $\big(X(t), t \in T\big)$ satisfies* (1) *and is strictly stable of index q, then any $F \in S$ is also strictly stable of index q.*

Proof. First note that since X satisfies (1), we can define the objects P, S, $\mathscr{A}$, $\mathscr{A}^*$, S^* as in Section 1. Now, for any positive integer n let S^n stand for $S \times \ldots \times S$, the set product of S with itself n times. Let $\mathscr{A}^n$ stand for $\mathscr{A} \times \ldots \times \mathscr{A}$, the n-product σ-field. Let $\hat{\mathscr{A}}^n$ stand for the completion of $\mathscr{A}^n$ with respect to P^n, the product measure $P \times \ldots \times P$.

If $(x_1, \ldots, x_n) \in S^n$, define $F_i(x_1, \ldots, x_n)$ to be equal to $F(x_i)$. Then $F_1, \ldots, F_n$ are independent random variables on $(S^n, \hat{\mathscr{A}}^n, P^n)$ and they all have the same distribution as F has on $(S, \mathscr{A}^*, P)$.

Let $\theta_n: S^n \to S$ be defined by

$$\theta_n(x_1, \ldots, x_n) = (1/n)^{1/q}(x_1 + \ldots + x_n).$$

Then $\theta_n^{-1}(\mathscr{A}) \subset \mathscr{A}^n$ by an easy extension of Lemma 3.1. Also, $P^n \theta_n^{-1} = P$ on $(S, \mathscr{A})$. We conclude that $\theta_n^{-1}(\mathscr{A}^*) \subset \hat{\mathscr{A}}^n$ and that the distribution of $F \circ \theta_n$ on $(S^n, \hat{\mathscr{A}}^n, P^n)$ is equal to the distribution of F on $(S, \mathscr{A}^*, P)$.

Now $F\circ\theta_n = n^{-1/q}(F_1+\ldots+F_n)$ by the additivity and homogeneity of F. Indeed,

$$F\left(\frac{x_1+\ldots+x_n}{n^{1/q}}\right) = F_1\big(n^{-1/q}(x_1,\ldots,x_n)\big)+\ldots+F_n\big(n^{-1/q}(x_1,\ldots,x_n)\big)$$

by the additivity of F, while

$$F_i\big(n^{-1/q}(x_1,\ldots,x_n)\big) = n^{-1/q}F_i(x_1,\ldots,x_n) \quad \text{for all } i\in\{1,\ldots,n\}$$

by the homogeneity of F.

It follows that if $q\neq 1$, then F is strictly stable of index q. If $q=1$, then we must also prove that the distribution of F is symmetric. However, if $q=1$, then the distribution of X is symmetric, i.e., the map $x\to -x$ leaves P-invariant. Also $-F(x) = F(-x)$ by the additivity of F. From this it follows that the distribution of F is symmetric.

Let us note that if q is rational, then $n^{1/q}$ will be rational for infinitely many n, in particular, for at least two relatively prime integers n. Hence the equation $F\circ\theta_n = n^{-1/q}(F_1+\ldots+F_n)$ is true for at least two relatively prime integers n. Moreover, from [3], p. 562, Problem 9, it follows that F is strictly stable of index q if such an equation holds for two relatively prime integers n. If we assume only that F is additive and $\mathscr{A}^*$-measurable, then Lemma 3.10 holds for such an F if q is rational.

LEMMA 3.11. *Let X_n, $n\geqslant 1$, be a sequence of strictly stable random variables of index q, and let a_n be a sequence of real numbers. Then $X_n+a_n\to 0$ in probability if and only if $X_n\to 0$ in probability and $a_n\to 0$.*

Proof. Suppose that $X_n+a_n\to 0$ in probability. For any positive integer k let us construct k sequences $X_{(n,1)},\ldots,X_{(n,k)}$ such that, for any n, $X_{(n,1)},\ldots,X_{(n,k)}$ are independent and have the same distribution as X_n. Now $\sum_1^k X_{(n,i)}+ka_n$ also converges to zero in probability as $n\to\infty$. Also $\sum_1^k X_{(n,i)}$ is distributed like $k^{1/q}X_n$. It follows that $k^{1/q}X_n+ka_n\to 0$ in probability. We conclude that $(k^{1/q}-k)a_n\to 0$. If $q\neq 1$, we are done. If $q=1$, then the distribution of the random variable X_n is symmetric for all n. But it is easy to see that the statement of this lemma holds for any sequence of symmetric random variables.

COROLLARY 3.1. *Suppose that $(X(t), t\in T)$ satisfies* (1)-(3) *and is strictly stable of index $q<2$. Then Theorem* 3.1 *is valid for X.*

Proof. In the Lévy decomposition of $X = a+Y+Z$, it can be seen that the Gaussian part Y vanishes. Furthermore, by Lemmas 3.10 and 3.11 it can be seen that X also satisfies (4).

COROLLARY 3.2. *Suppose that $(X(t), t\in T)$ satisfies* (1)-(3), *has the vanishing Gaussian part in its Lévy decomposition, and $X(t)$ is symmetrically distributed for all $t\in T$. Then Theorem* 3.1 *is valid for X.*

Proof. By the concluding remarks of the proof of Lemma 3.10 we see that if X is symmetric, then any F in S^* is symmetrically distributed. Now the statement of Lemma 3.11 is true for any sequence of symmetrically distributed random variables, hence X satisfies (4).

COROLLARY 3.3. *Suppose that* $\{X(t), t \in T\}$ *satisfies* (1)-(3) *and Condition* A′. *Suppose also that* $T = [0, b]$. *Then Theorem* 3.1 *is valid for* X.

Proof. P assigns positive measure to the trivial subspace $\{0\}$, hence it is easy to see that X satisfies (4). It is also clear that the Gaussian part of X in its Lévy decomposition vanishes.

Let us end this chapter by noticing that if X satisfies the conditions of Corollary 3.2 and the condition $L\{T \times (R \sim R')\} = 0$ mentioned after the proof of Lemma 3.7, then the statement of Theorem 3.1 is valid for any F which need only to be assumed additive and $\mathscr{A}^*$-measurable. This follows since in the chain of reasoning, that led up to Corollary 3.2, only Lemma 3.10 made use of the homogeneity of F, i.e., of the fact that $F(ax) = aF(x)$ for all $a \in R$ and $x \in S$.

4. Extensions and counterexamples. Theorem 3.1, as it stands, cannot be applied to a sequence X_n, $n \geqslant 1$, of independent random variables since hypothesis (2) is not satisfied. But to handle this case, simpler methods suffice. For the time being let $S = R \times R \times \ldots$, the linear space of all sequences of real numbers. Let $\mathscr{A}$ stand for the σ-field generated by the projections X_n. (If $x = (x_1, \ldots, x_n, \ldots) \in S$, then $X_n(x) = x_n$.) Let P denote the probability measure, induced on $(S, \mathscr{A})$, that is uniquely specified by assigning distributions on R^n, in a consistent way, to all random vectors $X_{n_1}, \ldots, X_{n_k}$, where $n_1, \ldots, n_k$ are positive integers. Let $\mathscr{A}^*$ denote the completion of $\mathscr{A}$ with respect to P. The set of real-valued linear functionals on S that are $\mathscr{A}^*$-measurable will be denoted by S^*.

THEOREM 4.1. *If* X_n, $n \geqslant 1$, *is a sequence of independent random variables, and if* (4) *is satisfied, then for every* F *in* S^* *there exists a sequence* c_n, $n \geqslant 1$, *of real numbers such that* $\sum_1^\infty c_n X_n$ *is unconditionally convergent* (*i.e., convergent under all reorderings*) *and*

$$F = \sum_1^\infty c_n X_n \quad a.s.$$

Proof. For $x = (x_1, \ldots, x_k, \ldots)$ in S, let

$$Q_k(x) = (0, \ldots, 0, x_k, 0, \ldots).$$

Then there exists a real number c_k such that $F(Q_k(x)) = c_k X_k(x)$ for all x in S. So, for all n,

$$F \circ \Big(\sum_1^n Q_k\Big) = \sum_1^n c_k X_k.$$

Let us write

$$Y_n(x) = F\Big(x - \sum_1^n q_k(x)\Big).$$

Then Y_n is independent of $X_1, \ldots, X_n$ and

$$Y_n + \sum_1^n c_k X_k = F \text{ a.s.}$$

From Lemma B it follows that there exists a sequence of constants d_n such that $\sum_1^\infty c_k X_k - d_k$ is convergent and, in fact, by [10], 37.1c, we can assume that we have chosen d_n in such a way that the series is unconditionally convergent. Using hypothesis (4) we conclude that $\sum_1^\infty c_k X_k$ is itself unconditionally convergent. Now $F - \sum_1^\infty c_k X_k$ is independent of X_n for any positive integer n; it follows from the well-known 0-1 law that $F - \sum_1^\infty c_k X_k$ is constant a.s. By hypothesis (4),

$$F = \sum_1^\infty c_k X_k \text{ a.s.}$$

We now use Theorem 4.1 to extend Theorem 3.1. Suppose that $(X(t), t \in T)$, S, $\mathscr{A}$, P are as in Section 1, but we assume that X satisfies only (1), (3), and (4). It follows that X is centered decomposable, i.e., for a constant c and for any $t \in T$ which is a limit point from the right (the left) the limit $X(t+)$ $(X(t-))$ exists a.s. and if $X(t+) - X(t) = c$ $(X(t) - X(t-) = c)$, then $c = 0$. Indeed, by [10], 37.3a, there is a real-valued function $c(t)$ with $c(0) = 0$ such that $X(t) - c(t)$ is centered decomposable. But from (4) it follows that $X(t)$ itself is centered decomposable.

The existence of limits in probability from the left and from the right implies that the set of fixed continuities of $X(t)$ is countable. Let us denote this set by $\{t_j: j = 1, 2, \ldots\}$. Let

$$U_j = X(t_j) - X(t_j -) \quad \text{and} \quad V_j = X(t_j +) - X(t_j).$$

By [10], p. 543, there exist constants c_j and d_j such that, for every interval $I \subset T$, $\sum_{t_j \in I} (U_j - c_j)$ and $\sum_{t_j \in I} (V_j - d_j)$ are unconditionally convergent. By (4), this implies that the series $\sum_{t_j \in I} U_j$ and $\sum_{t_j \in I} V_j$ are unconditionally convergent.

It follows that

$$X(t) = X^d(t) + X^c(t) \text{ a.s.,} \quad \text{where } X^d(t) = \sum_{0 \leqslant t_j \leqslant t} U_j + \sum_{0 \leqslant t_j < t} V_j$$

and where X^c satisfies (1)-(4) and is independent of X^d. (This is Loève's Decomposition Lemma, see [10], Lemma 37.2d.)

THEOREM 4.2. *Suppose that* $(X(t), t \in T)$ *satisfies* (1), (3), *and* (4). *Let* F *be in* S^*. *Suppose that the Gaussian part of* X^c *in its Lévy decomposition vanishes. Then there are a Borel measurable function* f *which is* X^c-*-integrable and sequences of real numbers* a_j $(j \geqslant 1)$ *and* b_j $(j \geqslant 1)$ *such that* $\sum_1^\infty a_j U_j$ *and* $\sum_1^\infty b_j V_j$ *are unconditionally convergent and, furthermore,*

$$F = \int_T f(t)\,dX^c(t) + \sum_1^\infty a_j U_j + \sum_1^\infty b_j V_j \quad a.s.$$

Proof. Let P^c denote the measure on $(S, \mathscr{A})$ corresponding to the process X^c, and let P^d denote the measure on $(S, \mathscr{A})$ corresponding to the process X^d. Let $\mathscr{A}^c$ denote the completion of $\mathscr{A}$ with respect to P^c, and $\mathscr{A}^d$ that of $\mathscr{A}$ with respect to P^d. By Lemma 3.3 we conclude that F is $\mathscr{A}^d$-measurable and is also $\mathscr{A}^c$-measurable. Now Theorem 3.1 implies that there is some Borel measurable function f such that

$$F = \int_T f(t)\,dX^c(t) \quad P^c \text{ a.s.},$$

where f is X-integrable. Also Theorem 4.1 implies that there are sequences $a_j, b_j \in R$ such that $\sum_1^\infty a_j U_j$ and $\sum_1^\infty b_j V_j$ are unconditionally convergent and such that

$$F = \sum_1^\infty a_j U_j + \sum_1^\infty b_j V_j \quad P^d \text{ a.s.}$$

(Define a_j to be $F(x_j)$, where $x_j(t) = 0$ for $t \leqslant t_j$, and $x_j(t) = 1$ for $t > t_j$. Define b_j to be $F(y_j)$, where $y_j(t) = 0$ for $t < t_j$ and $y_j(t) = 1$ for $t \geqslant t_j$.)

Now there exist linear measurable maps

$$\pi_i\colon (S, \mathscr{A}^*, P) \to (S, \mathscr{A}) \quad \text{for } i = 1, 2$$

such that $P\pi_1^{-1} = P^d$, $P\pi_2^{-1} = P^c$, and $\pi_1(x) + \pi_2(x) = x$ for all $x \in S$. (We skip the details of verifying this.) It follows that

$$F(x) = F \circ \pi_1(x) + F \circ \pi_2(x) \quad \text{for all } x \in S.$$

Now we have just seen that

$$F \circ \pi_1 = \sum_1^\infty (a_j U_j + b_j V_j) \text{ a.s.} \quad \text{and} \quad F \circ \pi_2 = \int_T f(t)\,dX^c(t) \text{ a.s.}$$

It is natural to ask just how much of a restriction is hypothesis (4). For instance, given X satisfying (1) and (3), can we find a real-valued function g such that $X-g$ satisfies (1), (3), and (4)? The answer is no. For instance, let X_n be a sequence of independent, identically distributed random variables such that

$$\mathrm{E}(\exp[iuX_n]) = \exp\left[\int_0^1 (e^{isu}-1-isu)\frac{1}{s^2}\,ds + \int_1^\infty (e^{isu}-1)\frac{1}{s^2}\,ds\right].$$

Then $X_1+\ldots+X_n$ is distributed like $nX_1+n\log n$ (see [8], p. 202). Now consider X_n-c_n for any sequence c_n of real numbers. If there exists some subsequence n' such that $|c_{n'}|\to\infty$, then $(X_{n'}-c_{n'})/c_{n'}\to -1$ in probability, contradicting (4). If, on the other hand, $|c_n|<M$ for all n, then

$$\frac{\sum_1^n X_k - c_k}{n\log n}\to 1$$

in probability, again contradicting (4). We see that for no sequence c_n the difference X_n-c_n satisfies (4).

Let us now consider the necessity of hypothesis (4) in Theorem 3.1. Suppose that $(X(t),\, t\in[0,\infty))$ is the standard Poisson motion, i.e., $X(t)$ satisfies (1)-(3) and

$$\mathrm{E}(\exp[iuX(t)]) = \exp[t(e^{iu}-1)].$$

Let F_n be a sequence of elements of S^* defined by $F_n(x) = n^{-1}(x(n))$ for $x\in S$. By the law of large numbers, $F_n\to 1$ in probability. By Lemma 2.1, there is an element F of S^* such that $F_n\to F$ in probability. It follows that $F=1$ a.s. Now it is clear that there is no X-integrable function f such that

$$1 = \int_T f(t)\,dX(t)\ \text{ a.s.}$$

We now consider the necessity of the condition that the process X has no Gaussian part Y. Suppose, for instance, that $Y(t)$ is the standard Brownian motion, i.e.,

$$\mathrm{E}(\exp[iuY(t)]) = \exp\left[-\frac{1}{2}tu^2\right] \quad \text{for } t\in[0,\infty).$$

Suppose that Z_1 and Z_2 are independent copies of the standard Poisson motion, also independent of Y. Write $X(t) = Y(t)+Z_1(t)-Z_2(t)$ for $t\in[0,\infty)$. Then $X(t)$ is a symmetric process and satisfies (1)-(4). Define F in S^* by setting $F(x)$ to be the sum of the jumps of x up to time 1. The distribution of F is the same as that of $Z_1(1)-Z_2(1)$, in particular, it is discrete. On the other hand, it is clear that, for any X-integrable function f, $\int_T f(t)dX(t)$ has no discrete distribution except for the trivial one with all mass at $\{0\}$. We conclude that the condition that X has no Gaussian part is necessary in Theorem 3.1.

Acknowledgement. I wish to express my appreciation to Professor Jacob Feldman for the guidance he gave me during the course of my doctoral research, in which time I proved earlier versions of the theorems in this paper.

REFERENCES

[1] R. H. Cameron and R. E. Graves, *Additive functionals on a space of continuous functions I*, Transactions of the American Mathematical Society 70 (1951), p. 160-176.
[2] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, New York 1964.
[3] W. Feller, *An introduction to probability theory and its applications*, Vol. II, New York 1966.
[4] I. I. Gikhman and A. V. Skorokhod, *Introduction to the theory of random processes*, London 1969.
[5] R. E. Graves, *Additive functionals on a space of continuous functions II*, Annals of Mathematics 54 (1951), p. 275-285.
[6] M. Kanter, *Measures on linear spaces*, Doctoral dissertation, University of California, Berkeley 1969.
[7] C. Kuratowski, *Topologie I*, 2-nd edition, Warszawa 1948.
[8] P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris 1937.
[9] — *Processus stochastiques et mouvement Brownian*, Paris 1948.
[10] M. Loève, *Probability theory*, 3-rd edition, Princeton, New Jersey, 1963.
[11] D. Shale and W. F. Stinespring, *Wiener processes*, Journal of Functional Analysis 2 (1968), p. 378-394.
[12] K. Urbanik, *Random linear functionals and random integrals*, Colloquium Mathematicum 33 (1975), p. 255-263.
[13] — and W. A. Woyczyński, *A random integral and Orlicz spaces*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 15 (1967), p. 161-169.

Reçu par la Rédaction le 17. 3. 1976