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COMPLETIONS, TRANSLATIONAL HULLS AND IDEAL EXTENSIONS OF INVERSE SEMIGROUPS

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0. INTRODUCTION

Every inverse semigroup is canonically (naturally) ordered. In this paper we obtain some algebraic properties of inverse semigroups using essentially the order structure.

Every ordered set may be immersed into a complete lattice, some infima and suprema being preserved under such immersion. In Section 1 we consider completions of inverse semigroups. Every inverse semigroup may be immersed into another inverse semigroup in which all possible suprema of sets of elements exist. We construct a "universal" immersion of any inverse semigroup S into a complete inverse semigroup $C(S)$ with a distributive semilattice (in fact, an infinitely distributive complete lattice) of idempotents. Properties of $C(S)$ are studied.

Section 1 is of preparatory character to further sections which treat a more traditional material. Section 2 is devoted to translational hulls of inverse semigroups. The concept of a densely embedded ideal which was introduced by E. S. LJAPIN [16] to find an abstract characterization of symmetric inverse semigroups has led L. M. GLUSKIN [7, 8] to creating a theory of translational hulls of semigroups. This theory is connected with various branches of the theories of semigroups and rings. A survey of this theory can be found in [21]. It is well known that translational hulls are very useful in the theory of ideal extensions of semigroups [1, 2, 4, 9–13, 21, 23].

The central problem of the theory of translational hulls is the construction, for a given semigroup S, of its translational hull $\Omega(S)$. A general construction of $\Omega(S)$ was found by L. M. Gluskin [7, 8]. However, in case when S belongs to a special class of semigroups one can often simplify the general construction, make it more explicit (cf. [19–22]).

Inverse semigroups (called also "generalized groups", "pseudogroups", "groupids") constitute an important special class of semigroups. The main result of Section 2 is a description of $\Omega(S)$ in case when S is an inverse semigroup. However, we do not use Gluskin's construction of $\Omega(S)$. Instead, we construct an oversemigroup $\boldsymbol{T}(S)$ of S, this oversemigroup (which is the idealizer of S in $\boldsymbol{C}(S)$) turns out to be the largest essential (dense) ideal extension of S, hence, $\boldsymbol{T}(S)$ is isomorphic to $\Omega(S)$.

Every inverse semigroup is isomorphic to an inverse semigroup of univalent functions (=one-to-one partial transformations of a set). In case S is an inverse semigroup of univalent functions, we give an alternative construction for $\boldsymbol{T}(S)$.

We believe the structure of the elements of $\boldsymbol{T}(S)$ to be more transparent than that of bitranslations. This permits us to obtain numerous properties of translational hulls of general and special inverse semigroups.

It has been already mentioned that translational hulls are connected with ideal extensions of semigroups. In Section 3 we describe ideal extensions of inverse semigroups in general and of special classes of inverse semigroups. The construction of $\boldsymbol{T}(S)$ is of key role here: it permits to give a simple description of all dense ideal extensions of S.

Every inverse semigroup is isomorphic to a subdirect product of a family of subdirectly irreducible inverse semigroups. Section 4 of the paper is devoted to subdirectly irreducible inverse semigroups. Every such semigroup is a dense ideal extension of a [0-]simple subdirectly irreducible inverse semigroup. We give an explicit construction for all subdirectly irreducible inverse semigroups which possess a non-zero ideal satisfying the descending chain condition for principal one-sided ideals. In particular, all finite subdirectly irreducible inverse semigroups are constructed.

All results of the paper can be easily generalized and are applicable to generalized grouds (=generalized heaps) of V. V. Vagner.

The main results of the paper were reported by the author at the XXIII-d Herzen (Gercen) Readings (Leningrad, December 12, 1970), at a meeting of the seminar "Semigroups" in the Saratov State University (Saratov, December 24, 1970) and at a meeting of the algebraic seminar in the Kharkov Institute of Radioelectronics (Kharkov, May 13, 1971) [33].

1. COMPLETIONS OF INVERSE SEMIGROUPS

1.1. Definition. A semigroup S is called *inverse* if for every $s \in S$ there exists an inverse element s^{-1} (i.e. such an element that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$) and such inverse is unique for every $s \in S$.

Equivalent definitions of inverse semigroups can be found in [28, 30, 32].

1.2. Definition. For $s, t \in S$ define $s \leqq t$ in case $s = ss^{-1}t$. Then $\leqq$ is a (partial) order relation [37] which is called the *canonical order* of S. A subset H of an inverse semigroup S is called *minorantly saturated* if for every $s \in S$ and $t \in H$ $s \leqq t$ implies $s \in H$. $\boldsymbol{M}(S)$ denotes the set of all minorantly saturated subsets of S.

1.3. Definition. Elements s and t of an inverse semigroup S are called *compatible* if $st^{-1} \in E(S)$ and $s^{-1}t \in E(S)$ where $E(S)$ is the set of all idempotents of S. A subset $H \subset S$ is called *compatible* if the elements of H are pairwise compatible, i.e. if $HH^{-1} \subset E(S)$ and $H^{-1}H \subset E(S)$. Here $H^{-1} = \{h^{-1} : h \in H\}$ and $H_1H_2 = \{h_1h_2 : : h_1 \in H_1,\ h_2 \in H_2\}$.

Equivalent definitions of order and compatibility can be found in [37]. We use here these equivalent definitions without further notice.

1.4. Definition. Minorantly saturated and compatible subsets of an inverse semigroup S are called *permissible.* $\boldsymbol{C}(S)$ denotes the set of all permissible subsets of S.

1.5. Lemma. *A product of two minorantly saturated (permissible) subsets is minorantly saturated (permissible). If H is a minorantly saturated (permissible) subset, then the same holds for H^{-1}.*

Proof. A subset H is minorantly saturated if and only if $HE(S) = H$. For every subset H, $HE(S) = E(S)\,H$ [37]. If H_1 is a minorantly saturated and H_2 an arbitrary subset of S then $H_1H_2\,E(S) = H_1\,E(S)\,H_2 = H_1H_2$, i.e. H_1H_2 is minorantly saturated. Analogously, $H_2H_1 \in \boldsymbol{M}(S)$. If $s \leqq t$ then $s^{-1} \leqq t^{-1}$. It follows that $H \in \boldsymbol{M}(S) \to H^{-1} \in \boldsymbol{M}(S)$. Therefore, $\boldsymbol{M}(S)$ is an ideal of the involuted semigroup $\mathfrak{P}(S)$ of all subsets of S.

Two elements s and t are compatible if and only if s^{-1} and t^{-1} are. It follows that H is compatible if and only if H^{-1} is. Suppose $H_1, H_2 \in \boldsymbol{C}(S)$. Then, as we have just seen, $H_1H_2 \in \boldsymbol{M}(S)$. Now $(H_1H_2)(H_1H_2)^{-1} = H_1H_2H_2^{-1}H_1^{-1} \subset H_1\,E(S)\,H_1^{-1} = H_1H_1^{-1} \subset E(S)$. Analogously, $(H_1H_2)^{-1}(H_1H_2) \subset E(S)$. It follows that H_1H_2 is compatible and $H_1H_2 \in \boldsymbol{C}(S)$. Therefore, $\boldsymbol{C}(S)$ is an involuted subsemigroup of the involuted semigroup $\mathfrak{P}(S)$.

1.6. Definition. If $H \subset S$ then $\iota_1(H) = \{hh^{-1} : h \in H\}$ and $\iota_2(H) = \{h^{-1}h : h \in H\}$. We call $\iota_1(H)$ the *first projection* and $\iota_2(H)$ the *second projection* of H.

Clearly, $\iota_1(H) \cup \iota_2(H) \subset E(S) = \iota_1(S) = \iota_2(S)$ and $\iota_1(H^{-1}) = \iota_2(H)$.

1.7. Lemma. *$\iota_1(H) = HH^{-1}$ and $\iota_2(H) = H^{-1}H$ if and only if H is compatible and $HH^{-1}H = H$.*

Proof. Clearly, $\iota_1(H) \subset HH^{-1}$ and $\iota_2(H) \subset H^{-1}H$ for every $H \subset S$. If $\iota_1(H) = HH^{-1}$ and $\iota_2(H) = H^{-1}H$ then $HH^{-1} \subset E(S)$ and $H^{-1}H \subset E(S)$, hence H is compatible. Let $h_1, h_2, h_3 \in H$. Then $h_1h_2^{-1} = h_4h_4^{-1}$ and $h_4^{-1}h_3 = h_5^{-1}h_5$ for some elements $h_4, h_5 \in H$. Since $h_5^{-1}h_5 = (h_4^{-1}h_3)(h_4^{-1}h_3)^{-1} = h_4^{-1}h_3h_3^{-1}h_4 \leqq h_4^{-1}h_4$ and $h_4h_5^{-1} \in E(S)$, we obtain $h_1h_2^{-1}h_3 = h_4h_4^{-1}h_3 = h_4h_5^{-1}h_5 = h_5h_4^{-1}h_4 = h_5h_5^{-1}h_5h_4^{-1}h_4 = h_5h_5^{-1}h_5 = h_5 \in H$, i.e. $HH^{-1}H \subset H$. However, $H \subset HH^{-1}H$ is valid for every H, since $h = hh^{-1}h$.

Now let $HH^{-1}H = H$ for a compatible subset H. If $h_1, h_2 \in H$, then $h_2h_1^{-1}h_1 \in H$, therefore, $h_1h_2^{-1} = (h_1h_2^{-1}h_2)\, h_2^{-1} = (h_2h_1^{-1}h_1)\, h_2^{-1} = (h_2h_1^{-1}h_1)(h_2h_1^{-1}h_1)^{-1} \in \iota_1(H)$, i.e. $HH^{-1} \subset \iota_1(H)$ and $\iota_1(H) = HH^{-1}$. Analogously, $\iota_2(H) = H^{-1}H$.

1.8. Lemma. *If $H \in C(S)$, then $\iota_1(H) = HH^{-1}$ and $\iota_2(H) = H^{-1}H$.*

Proof. Let $h_1, h_2, h_3 \in H$. Then $h_1h_2^{-1} \in E(S)$, therefore, $h_1h_2^{-1}h_3 \leqq h_3 \in H$. Since H is minorantly saturated, $h_1h_2^{-1}h_3 \in H$, i.e. $HH^{-1}H \subset H$ and $HH^{-1}H = H$. It remains to apply Lemma 1.7.

1.9. Lemma. *$C(S)$ is an inverse semigroup.*

Proof. For every $H \in C(S)$, $HH^{-1}H = H$. Since the idempotents of an inverse semigroup commute, sets of idempotents commute as well. Using this fact and Lemma 1.8 we obtain $H_1H_1^{-1}H_2H_2^{-1} = H_2H_2^{-1}H_1H_1^{-1}$ for all $H_1, H_2 \in C(S)$. Therefore [28], $C(S)$ is an inverse semigroup.

1.10. For every $s \in S$ define $\tau(s) = \{t : t \leqq s\}$. Then $\tau(s) = s\,E(S) \in C(S)$, i.e. τ is a mapping of S into $C(S)$.

Lemma. *τ is an isomorphic embedding of S into $C(S)$.*

Proof of the Lemma consists of elementary computations and is omitted.

1.11. Let a subset H of an inverse semigroup S possess the least upper bound $\bigvee H$ in S. Since any two elements which have a common upper bound relative to $\leqq$ are compatible, H is compatible. Therefore, if a subset $H \subset S$ is not compatible, $\bigvee H$ can exist neither in S nor in any oversemigroup of S.

Definition. An inverse semigroup S is called *complete* whenever compatible subsets of S possess the l.u.b.'s in S.

S is called *(infinitely) distributive* if for finite (arbitrary) subsets $H \subset S$ for which $\bigvee H$ exists and for every $s \in S$ the l.u.b.'s $\bigvee Hs$ and $\bigvee sH$ exist and $(\bigvee H)\, s = \bigvee Hs$, $s(\bigvee H) = \bigvee sH$.

The two last identities are equivalent. Indeed, $(\bigvee H)^{-1} = \bigvee H^{-1}$ and therefore $s(\bigvee H) = ((\bigvee H)^{-1}\, s^{-1})^{-1} = ((\bigvee H^{-1})\, s^{-1})^{-1} = (\bigvee H^{-1}s^{-1})^{-1} = \bigvee sH$ which shows that the first identity implies the second one. Analogously, the second identity implies the first one.

1.12. Lemma. *Suppose $\bigvee H$ exists. Then $\bigvee \iota_1(H)$ and $\bigvee \iota_2(H)$ exist and $\bigvee \iota_1(H) = (\bigvee H)(\bigvee H)^{-1}$, $\bigvee \iota_2(H) = (\bigvee H)^{-1}(\bigvee H)$.*

Proof. Let $\bigvee H = h$. Then for every $h_1 \in H$ $h_1 \leqq h$, whence $h_1h_1^{-1} \leqq hh^{-1}$. Suppose $h_1h_1^{-1} \leqq s$ for some $s \in S$ and every $h_1 \in H$. Then $h_1 = h_1h_1^{-1}h \leqq sh$, therefore, $h = \bigvee H \leqq sh$. It follows that $hh^{-1} = shh^{-1} \leqq s$, i.e. $hh^{-1} = \bigvee \iota_1(H)$. The second identity for $\iota_2(H)$ may be proved in the same way.

1.13. Lemma. *For any inverse semigroup S the following properties are equivalent:*

1) *S is (infinitely) distributive;*

2) *the semilattice $E(S)$ is (infinitely) distributive;*

3) *if $\bigvee F$ and $\bigvee H$ exist for finite (or infinite) subsets $F, H \subset S$, then $\bigvee FH$ exists and $\bigvee FH = (\bigvee F)(\bigvee H)$.*

Proof. The implications 3) → 1) → 2) are obvious. Suppose 2) is valid and there exist $\bigvee F = f$ and $\bigvee H = h$. If $f_1 \in F$ and $h_1 \in H$ then $f_1 \leqq f$ and $h_1 \leqq h$, whence $f_1 h_1 \leqq fh$. Now let g be such an element of S that $f_1 h_1 \leqq g$ for all $f_1 \in F$ and $h_1 \in H$. Then $f_1^{-1} f_1 h_1 h_1^{-1} \leqq f^{-1} g h^{-1}$. Therefore, $f^{-1} f h_1 h_1^{-1} = (\bigvee F)^{-1} (\bigvee F)\, h_1 h_1^{-1} = (\bigvee \iota_2(F))\, h_1 h_1^{-1} = \bigvee(\iota_2(F)\, h_1 h_1^{-1}) \leqq f^{-1} g h$ for all $h_1 \in H$. Therefore, $f^{-1} f(\bigvee \iota_1(H)) = \bigvee (f^{-1} f \iota_1(H)) \leqq f^{-1} g h^{-1}$, i.e. $f^{-1} f h h^{-1} \leqq f^{-1} g h^{-1}$, whence $fh = f f^{-1} f h h^{-1} h \leqq f f^{-1} g h^{-1} h \leqq g$. Therefore, $fh = \bigvee FH$. Thus 2) implies 3).

1.14. Definition. A homomorphism φ of an inverse semigroup S into an inverse semigroup T is called *$\bigvee$-complete* (*$\bigwedge$-complete*) if for every subset $H \subset S$ for which $\bigvee H$ exists ($\bigwedge H$ exists) there exists $\bigvee \varphi(H)$ in T ($\bigwedge \varphi(H)$ in T) and $\varphi(\bigvee H) = \bigvee \varphi(H)$ ($\varphi(\bigwedge H) = \bigwedge \varphi(H)$).

A homomorphism which is $\bigvee$-complete and $\bigwedge$-complete is called *complete.*

Thus $\bigvee$-complete homomorphisms preserve all suprema existing, while $\bigwedge$-complete homomorphisms preserve all infima of subsets of S.

It should be noted that every homomorphism preserves all infima of finite subsets. Indeed, let $H \subset S$ be a finite subset, $H = \{h_1, \ldots, h_n\}$ and $h = \bigwedge H$ exist. For every $h_i \in H$ we have $h \leqq h_i$, whence $\varphi(h) \leqq \varphi(h_i)$. Now let $t \leqq \varphi(h_i)$ for all $h_i \in H$. Since $h = h_1 h_1^{-1} h_2 h_2^{-1} \ldots h_n h_n^{-1} h_n$, we obtain $t = tt^{-1} tt^{-1} \ldots t^{-1} t \leqq \varphi(h_1)\, \varphi(h_1^{-1}) \ldots \ldots \varphi(h_n)\, \varphi(h_n^{-1})\, \varphi(h_n) = \varphi(h)$, i.e. $\varphi(h) = \bigwedge \varphi(H)$.

1.15. Theorem. (i) *$C(S)$ is an infinitely distributive complete inverse semigroup;*

(ii) *if $F, H \in C(S)$ then $F \leqq H \leftrightarrow F \subset H$;*

(iii) *if $F, H \in C(S)$ then F is compatible with H in $C(S)$ if and only if $F \cup H \in C(S)$, i.e. if $F \cup H$ is a compatible subset of S;*

(iv) *if $H \in C(S)$ then H^{-1} is an inverse element for H in $C(S)$;*

(v) *if $\mathfrak{H} \subset C(S)$ then $\bigvee \mathfrak{H}$ exists if and only if $\bigcup \mathfrak{H} \in C(S)$; this being the case, $\bigvee \mathfrak{H} = \bigcup \mathfrak{H}$;*

(vi) *for every $\mathfrak{H} \subset C(S)$, $\mathfrak{H} \neq \emptyset$ the g.l.b. $\bigwedge \mathfrak{H}$ exists and $\bigwedge \mathfrak{H} = \bigcap \mathfrak{H}$;*

(vii) *for every homomorphism $S \to T$ of S into an infinitely distributive complete inverse semigroup T there exists a unique $\bigvee$-complete homomorphism $C(S) \to T$ such that the following diagram is commutative:*

$$\begin{array}{ccc} S & \xrightarrow{\tau} & C(S) \\ & \searrow & \downarrow \\ & & T \end{array}$$

Proof. By Lemma 1.9, $C(S)$ is an inverse semigroup. (iv) follows from Lemmas 1.7 and 1.8.

Lemma. *$F \in E(C(S))$ if and only if $F \subset E(S)$ and $F \in C(S)$, i.e., if F is an ideal of the semilattice $E(S)$.*

Proof. If $F \subset E(S)$, then F is compatible, therefore, $F \in C(S)$ is equivalent to $F \in M(S)$, i.e. F is a (possibly empty) ideal of $E(S)$. In this case $F \in E(C(S))$. Conversely, if $F \in E(C(S))$, then $F = FF^{-1} = \iota_1(F) \subset E(S)$.

(ii) If $F \leqq H$ then $F = FF^{-1}H = \iota_1(F)\, H \subset E(S)\, H = H$. Conversely, let $F \subset H$. Then $F = FF^{-1}F \subset HF^{-1}F \subset HH^{-1}F = \iota_1(H)\, F \subset E(S)\, F = F$, i.e. $F = HF^{-1}F$ and $F \leqq H$.

(iii) Since F and H are minorantly saturated, $F \cup H \in M(S)$. Therefore, $F \cup H$ is compatible in S if and only if $F \cup H \in C(S)$.

If $F \cup H \in C(S)$, then $F \subset F \cup H$ and $H \subset F \cup H$. By (ii), F and H possess a common upper bound in $C(S)$. Therefore, F is compatible with H. Conversely, let F, H be compatible in $C(S)$. Then $(F \cup H)(F \cup H)^{-1} = (F \cup H)(F^{-1} \cup H^{-1}) = FF^{-1} \cup FH^{-1} \cup HF^{-1} \cup HH^{-1} = \iota_1(F) \cup FH^{-1} \cup HF^{-1} \cup \iota_1(H) \subset E(S)$, since $HF^{-1}, FH^{-1} \in E(C(S))$ and, by Lemma 1.15, $HF^{-1}, FH^{-1} \subset E(S)$. Analogously, $(F \cup H)^{-1}(F \cup H) \subset E(S)$. Therefore, $F \cup H$ is a compatible subset of S.

(v) If $\mathfrak{H} \subset C(S)$ is a compatible subset, then $\bigcup \mathfrak{H} \in C(S)$. By (ii), $\bigcup \mathfrak{H}$ is the l.u.b. of $\mathfrak{H}$.

(i) follows from (v).

(vi) If $\mathfrak{H} \subset C(S)$ and $\mathfrak{H} \neq \emptyset$, then $\bigcap \mathfrak{H} \in C(S)$, therefore, by (iii), $\bigcap \mathfrak{H} = \bigwedge \mathfrak{H}$.

(vii) Suppose $\varphi : S \to T$ is a homomorphism of S into an infinitely distributive complete inverse semigroup T. If $H \in C(S)$ then H is compatible in S, hence $\varphi(H)$ is compatible in T and $\bigvee \varphi(H)$ exists in T. Let $\psi(H) = \bigvee \varphi(H)$. Then ψ is a mapping of $C(S)$ into T.

Let $F, H \in C(S)$. Then $\psi(FH) = \bigvee \varphi(FH) = \bigvee \varphi(F)\, \varphi(H) = (\bigvee \varphi(F))(\bigvee \varphi(H)) = \psi(F)\, \psi(H)$. We have used Lemma 1.13. Let $\mathfrak{H} \subset C(S)$ and $\bigcup \mathfrak{H} \in C(S)$. Then $\psi(\bigcup \mathfrak{H}) = \bigvee \varphi(\bigcup \mathfrak{H}) = \bigvee(\bigcup_{H_i \in \mathfrak{H}} \varphi(H_i)) = \bigvee_{H_i \in \mathfrak{H}}(\bigvee \varphi(H_i)) = \bigvee_{H_i \in \mathfrak{H}} \psi(H_i) = \bigvee \psi(\mathfrak{H})$. Therefore, ψ is a $\bigvee$-complete homomorphism of $C(S)$ into T.

If $s \in S$ then $\bigvee \varphi(\tau(s)) = \varphi(s)$, hence $\psi(\tau(s)) = \varphi(s)$. Therefore, the diagram (vii) is commutative.

If $\chi : C(S) \to T$ is another homomorphism making the diagram commutative, then $\chi(\tau(s)) = \varphi(s) = \varphi(\tau(s))$. Let $H \in C(S)$ and $h \in H$. Then $\tau(h) \leqq H$ in $C(S)$, whence $\chi(\tau(h)) \leqq \chi(H)$ in T, i.e. $\varphi(h) \leqq \chi(H)$ in T for every $h \in H$. Therefore, $\psi(H) = \bigvee \varphi(H) \leqq \chi(H)$. If χ is $\bigvee$-complete, then $H = \bigcup_{h \in H} \tau(h)$ implies $\chi(H) = \bigvee_{h \in H} \chi(\tau(h)) = \bigvee_{h \in H} \psi(\tau(h)) = \psi(H)$, i.e. $\chi = \psi$.

1.16. Remark. The isomorphism τ is obviously $\bigwedge$-complete. The l.u.b. $\bigvee H$ is called trivial if H contains the largest element. The isomorphism τ is not $\bigvee$-complete; moreover, τ does not preserve any nontrivial l.u.b.'s. In fact, let $h = \bigvee H$ exist and be nontrivial. Then $\tau(\bigvee H) = \tau(h) \neq H = \bigcup_{h_i \in H} \tau(h) = \bigvee_{h_i \in H} \tau(h_i)$.

1.17. An element $s \in S$ is called $\bigvee$-*indecomposable* if $s = \bigvee H$ implies $s \in H$ for any $H \subset S$. Clearly, the elements of the form $\tau(s)$ are precisely all the $\bigvee$-indecomposable elements of $C(S)$. Thus, $\bigvee$-indecomposable elements of $C(S)$ form an inverse subsemigroup isomorphic with S.

Corollary. *Let S and T be inverse semigroups. $C(S)$ and $C(T)$ are isomorphic if and only if S and T are isomorphic.*

1.18. Let S be a semilattice. By Lemma 1.15, $C(S)$ is the set of all ideals of S including the empty ideal. Let $I(S)$ be the set of all nonempty ideals of S. If $F, H \in I(S)$ then $FH = F \cap H$. Thus, $I(S)$ is a meet-semilattice. Clearly, $C(S) \cong I(S)^0$.

Corollary [6, 15]. *For two semilattices S and T, $I(S) \cong I(T)$ if and only if $S \cong T$.*

1.19. Every automorphism of $C(S)$ for every inverse semigroup S maps $\bigwedge$-indecomposable elements onto $\bigwedge$-indecomposable elements, i.e. it induces an automorphism of $\tau(S)$. On the other hand, every automorphism of $\tau(S)$ may be in an obvious and unique way extended to an isomorphism of $C(S)$. Thus, the automorphism group of $\tau(S)$ is a restriction to $\tau(S)$ of the automorphism group of $C(S)$. Moreover, every automorphism of $C(S)$ maps $\emptyset$ (which is a zero of $C(S)$) onto itself.

Corollary. *For every inverse semigroup S the groups of all automorphisms of S and of $C(S)$ are isomorphic. In particular, the automorphism groups of any semilattice S and of the semilattice $I(S)$ of all ideals of S are isomorphic.*

1.20. Corollary. *For every inverse semigroup S, $C(E(S)) = E(C(S))$.*

Proof. See Lemma 1.15.

1.21. Every homomorphism $S \to T$ of two inverse semigroups S and T may be in a unique way lifted up to a $\bigvee$-complete homomorphism of $C(S)$ into $C(T)$, i.e. τ is a functor from the category of inverse semigroups and their homomorphisms into the category of infinitely distributive complete inverse semigroups and $\bigvee$-complete homomorphisms. In particular, the former category can be isomorphically embedded into the latter one.

1.22. Not every inverse semigroup is distributive: e.g. nondistributive lattices can be considered as inverse semigroups with respect to one of their operations. However, in every inverse semigroup multiplication is distributive with respect to the operation of forming infima.

Proposition. *Let a subset H of an inverse semigroup S possess the g.l.b. $\bigwedge H$. Then for every $s \in S$ subsets Hs and sH possess g.l.b.'s and $\bigwedge Hs = (\bigwedge H)\, s$, $\bigwedge sH = s(\bigwedge H)$.*

Proof. Let $\bigwedge H = h$. Then for every $h_1 \in H$, $h \leqq h_1$, whence $sh \leqq sh_1$. Let $f \leqq sh_1$ for all $h_1 \in H$. Then $s^{-1}f \leqq s^{-1}sh_1 \leqq h_1$, hence $s^{-1}f \leqq h$. Therefore, $ss^{-1}f \leqq sh$. However, $ff^{-1} \leqq (sh_1)(sh_1)^{-1} \leqq ss^{-1}$. Thus, $ss^{-1}ff^{-1} = ff^{-1}$ and $f = ff^{-1}f = ss^{-1}ff^{-1}f = ss^{-1}f$. Therefore, $f \leqq sh$, i.e. $sh = \bigwedge sH$. The other equality may be proved analogously.

1.23. Every inverse semigroup is isomorphic with an inverse semigroup of univalent functions (i.e., of one-to-one partial transformations of a set). Every function φ on a set A is considered as a special binary relation on A: $\varphi = \{(a, \varphi(a)) : a \in pr_1\varphi\}$ where $pr_1\varphi$ is the domain of φ.

Let Φ be an inverse semigroup of univalent functions acting in a set A. It is known [37] that for $\varphi, \psi \in \Phi$, $\varphi \leqq \psi \leftrightarrow \varphi \subset \psi$, φ and ψ are compatible if and only if $\varphi \cup \psi$ is a univalent function. The inverse for $\varphi \in \Phi$ is the converse function φ^{-1}. A subset $H \subset \Phi$ is compatible if and only if $\bigcup H$ is a univalent function. If $\bigcup H \in \Phi$ then $\bigcup H$ is the l.u.b. of H in Φ, the converse being not true in general: the l.u.b. of H, if it exists, need not be equal to $\bigcup H$. A complete inverse semigroup need not be isomorphic to some inverse semigroup of univalent functions where all l.u.b.'s existing are set-theoretical unions of functions: infinite dictributivity is a necessary (but not sufficient) condition for the existence of such an isomorphism. Necessary and sufficient conditions for the existence of such an isomorphic representation are found in [31]. They are corollaries to results of [26].

Let $U(\Phi) = \{\bigcup H : H \in C(\Phi)\}$. If $H \in C(\Phi)$ then H is compatible and $\bigcup H$ is a univalent function. Thus, $U(\Phi)$ is a set of univalent functions. If H is a compatible subset of H and $\tau(H) = \{\varphi : \varphi \in \Phi$ and $\varphi \subset \psi$ for some $\psi \in H\}$ then $\tau(H) \in C(\Phi)$ and $\bigcup H = \bigcup\tau(H) \in U(\Phi)$, therefore $U(\Phi)$ consists of unions of arbitrary compatible subsets of Φ.

Proposition. *$U(\Phi)$ is an infinitely distributive complete inverse semigroup of univalent functions containing Φ as an inverse subsemigroup.*

Proof. If $\varphi \in \Phi$ then $\{\varphi\}$ is a compatible subset and $\varphi = \bigcup\{\varphi\} \in U(\Phi)$, i.e. $\Phi \subset U(\Phi)$. If H is a compatible subset of Φ then $H^{-1} = \{\varphi^{-1} : \varphi \in H\}$ is a compatible subset and $(\bigcup H)^{-1} = \bigcup H^{-1}$. Therefore, $\varphi \in U(\Phi) \leftrightarrow \varphi^{-1} \in U(\Phi)$. If H_1 and H_2 are compatible subsets of Φ then $(\bigcup H_2) \circ (\bigcup H_1) = \bigcup(H_2 \circ H_1)$ where $H_2 \circ H_1 = \{\varphi_2 \circ \varphi_1 : \varphi_1 \in H_1 \text{ and } \varphi_2 \in H_2\}$. Therefore, $H_2 \circ H_1$ is a compatible subset and $\varphi_1, \varphi_2 \in U(\Phi) \rightarrow \varphi_2 \circ \varphi_1 \in U(\Phi)$. Thus, $U(\Phi)$ is an inverse semigroup. Infinite distributivity and completeness of $U(\Phi)$ are obvious.

1.24. Corollary. *There exists a unique $\bigvee$-complete homomorphism $\xi : C(\Phi) \rightarrow U(\Phi)$ whose restriction to Φ coincides with τ^{-1}, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} \Phi & \longrightarrow & U(\Phi) \\ & \searrow & \uparrow \\ & & C(\Phi) \end{array}$$

Moreover, $\xi^{-1}(\Phi) = \tau(\Phi)$, for $H \in C(\Phi)$ $\xi(H) = \bigcup H$ and ξ is surjective, i.e. $U(\Phi)$ is a homomorphic image of $C(\Phi)$.

ξ need not be an isomorphism (consider such in inverse semigroup Φ that $\Phi = U(\Phi)$).

1.25. Let S be an inverse semigroup, $s \in S$. Define a binary relation $\tilde{\varrho}_s = \{(s_1, s_2) : s_1 s = s_2,\ s_2 s^{-1} = s_1\}$ on S. Then $\tilde{\varrho}_s$ is a univalent function which is called the *reduced right translation* of S defined by s. The mapping $\tilde{\varrho} : s \rightarrow \tilde{\varrho}_s$ is an isomorphic representation of S onto the inverse semigroup $\tilde{P}(S)$ of all reduced right translations of S [37].

Proposition. *$C(S)$ is isomorphic to $U(\tilde{P}(S))$.*

Proof. Let $H \in C(S)$. Then $\tilde{\varrho}(H)$ is a compatible subset of $\tilde{P}(S)$, therefore, $\bigcup\tilde{\varrho}(H)$ is a univalent function acting in S. Let $\beta(H) = \bigcup\tilde{\varrho}(H)$. Then β is a mapping of $C(S)$ into $U(\tilde{P}(S))$. Moreover, since $\tilde{\varrho}$ is an isomorphism of S onto $\tilde{P}(S)$, $\tilde{\varrho}$ may be naturally extended to an isomorphism $c(\tilde{\varrho}) : C(S) \rightarrow C(\tilde{P}(S))$. Then $\beta = \xi \circ c(\tilde{\varrho})$, i.e. β is a through homomorphism $C(S) \rightarrow C(\tilde{P}(S)) \rightarrow U(\tilde{P}(S))$ of $C(S)$ onto $U(\tilde{P}(S))$. Let $H_1, H_2 \in C(S)$ and $\beta(H_1) = \beta(H_2)$, i.e. $\bigcup_{s\in H_1} \tilde{\varrho}_s = \bigcup_{s\in H_2} \tilde{\varrho}_s$. Let $s \in H_1$. Then $(s^{-1}, s^{-1}s) \in \tilde{\varrho}_s$, whence $(s^{-1}, s^{-1}s) \in \tilde{\varrho}_{s_1}$ for some $s_1 \in H_2$. It follows that $s^{-1}s_1 = s^{-1}s$ and $ss^{-1}s_1 = ss^{-1}s = s$, i.e. $s \leqq s_1$. Since H_2 is minorantly saturated, $s \in H_2$, i.e. $H_1 \subset H_2$. Analogously, $H_2 \subset H_1$ and $H_1 = H_2$. Thus, β is one-to-one.

1.26. Definition. An ideal of a semigroup S is called a *retract ideal* if there exists an endomorphism of S onto the ideal, all the elements of the ideal being fixed points under the endomorphism.

Suppose T is an ideal of an inverse semigroup S. Then T is an inverse semigroup. If $H \in C(S)$ then $H \cap T \in C(T)$. Let $\varrho(H) = H \cap T$ for every $H \in C(S)$. Clearly, $\varrho(H_1)\,\varrho(H_2) \subset \varrho(H_1 H_2)$ for all $H_1, H_2 \in C(S)$. Conversely, let $h \in \varrho(H_1 H_2)$, i.e. $h \in T$ and $h = h_1 h_2$ for some $h_1 \in H_1$ and $h_2 \in H_2$. Then $h = (hh^{-1}h_1)(h_2 h^{-1}h)$ and $hh^{-1}h_1 \in \varrho(H_1)$, $h_2 h^{-1} h \in \varrho(H_2)$. Therefore, $\varrho(H_1 H_2) \subset \varrho(H_1)\,\varrho(H_2)$. Now for every $H \in C(T)$ we have $H \in C(S)$, i.e. $C(T) \subset C(S)$. Thus, $\varrho(H) = H$ for every $H \in C(T)$. Since T is an ideal of S, $C(T)$ is an ideal of $C(S)$. Thus, $C(T)$ is a retract ideal of $C(S)$, ϱ being the idempotent endomorphism of $C(S)$ onto $C(T)$.

Now let I be a retract ideal of $C(S)$, ϱ being an idempotent endomorphism of $C(S)$ onto I. Let $\varrho(E(S)) = E$. Then E is an ideal of $E(S)$ and $T = SES$ is an ideal of S. If $H \in C(T)$ then $H = HE \in I$, since one can easily verify that $E(T) = E$. Thus, $C(T) \subset I$. On the other hand, if $H \in I$ then $H = HI \subset T$. Therefore, $I = C(T)$. We have proved the following

Proposition. *Let T be an ideal of an inverse semigroup S. Then $C(T)$ is a retract ideal of $C(S)$, $H \to H \cap T$ being the corresponding endomorphism of $C(S)$ onto $C(T)$. Every retract ideal of $C(S)$ is of the form $C(T)$ where T is an ideal of S. In particular, the lattice $I(S)$ of all ideals of S is isomorphic to the lattice of all retract ideals of $C(S)$.*

1.27. Let $C_0(S) = C(S) \setminus \{\emptyset\}$. If $(S_i)_{i \in I}$ is a family of inverse semigroups, $H_i \in C(S_i)$ for each $i \in I$ then the Cartesian product $\mathsf{X}_{i \in I} H_i$ is an element of $C(\mathsf{X}_{i \in I} S_i)$ where $\mathsf{X}_{i \in I} S_i$ is the direct product of the family $(S_i)_{i \in I}$. Clearly, the correspondence $(H_i)_{i \in I} \to \mathsf{X}_{i \in I} H_i$ is a homomorphism of $\mathsf{X}_{i \in I} C(S_i)$ into $C(\mathsf{X}_{i \in I} S_i)$. If all H_i are nonempty, the correspondence is one-to-one, i.e. it is an isomorphism of $\mathsf{X}_{i \in I} C_0(S_i)$ into $C_0(\mathsf{X}_{i \in I} S_i)$.

1.28. Let φ be a surjective homomorphism of an inverse semigroup S onto an inverse semigroup T. If $H \in C(S)$, then $\varphi(H) \in C(T)$. Therefore, φ induces a homomorphism of $C(S)$ into $C(T)$. The latter homomorphism need not be surjective.

1.29. Let an inverse semigroup S be a subdirect product of an inverse semigroup T and a group G, i.e., $S \subset T \times G$ and the natural projection homomorphisms of S into T and G are surjective. Suppose $H \in C_0(S)$. If $(t_1, g_1), (t_2, g_2) \in H$ then, since H is compatible, t_1 is compatible with t_2 in T and $g_1 = g_2$. It follows that $H = H_T \times \{g\}$ for some compatible subset $H_T \subset T$ and an element $g \in G$. Since H is minorantly saturated, the same holds for H_T, i.e., $H_T \in C_0(T)$.

Conversely, if $H_1 \in C_0(T)$ and $g \in G$ then one can easily verify that $H_1 \times \{g\} \in C_0(S)$, provided $H_1 \times \{g\} \subset S$. Therefore, the mapping $H = H_T \times \{g\} \to (H_T, g)$

is an isomorphism of $C_0(S)$ into $C_0(T) \times G$. Clearly, the mapping $H_T \times \{g\} \to g$ is a homomorphism of $C_0(S)$ onto G. However, the mapping $H \to H_T$ is not surjective in general case. Therefore, $C_0(S)$ is isomorphic to a subdirect product of an inverse subsemigroup U of $C_0(T)$ and of the group G.

Clearly, $\tau(T) \subset U$. Suppose $H_1 \in E(C_0(T))$, i.e. H_1 is an ideal of the semilattice $E(T)$. For every $i \in H_1$ there exists $g \in G$ such that $(i, g) \in S$, whence $(i, 1) = (i, g) \cdot (i, g^{-1}) = (i, g)(i, g)^{-1} \in S$. Here 1 is the identity of G. Thus, $H_1 \times \{1\} \subset S$ and $H_1 \times \{1\} \in C(S)$. Therefore, $E(C_0(T)) \subset U$.

If T is a semilattice then $E(C_0(T)) = C_0(E(T)) = C_0(T)$ and we obtain the following

Proposition. *If an inverse semigroup S is isomorphic to a subdirect product of a semilattice T and a group G then $C_0(S)$ is isomorphic to a subdirect product of a lattice $I(T)$ of all ideals of T and of the group G.*

As we have proved above, $C_0(T \times G) \cong C_0(T) \times G$ for every inverse semigroup T and every group G.

1.30. Definition. Let $(S_i)_{i\in I}$ be a family of semigroups with zero and $S_i \cap S_j = \{0\}$ provided $i \neq j$. Let $S = \bigcup_{i\in I} S_i$. Define a multiplication in S in such a way that $S_iS_j = \{0\}$ for $i \neq j$, in every S_i the multiplication coincides with the operation of S_i. Then S is a semigroup which is called the *orthogonal sum* of the family $(S_i)_{i\in I}$ and is denoted by $\Sigma^0_{i\in I} S_i$.

Let $(S_i)_{i\in I}$ be a family of inverse semigroups with zero and $H \in C_0(\sum^0_{i\in I} S_i)$. Denote $H_i = H \cap S_i$. Then $H_i \in C_0(S_i)$ for every $i \in I$. Conversely, let $(H_i)_{i\in I} \in \times_{i\in I} C_0(S_i)$, $H = \bigcup_{i\in I} H_i$. Then $H \in C_0(\Sigma^0_{i\in I} S_i)$. Now let $F, H \in C_0(\Sigma^0_{i\in I} S_i)$. Then $FH = (\bigcup_{i\in I} F_i) \cdot (\bigcup_{i\in I} H_i) = \bigcup_{i,j\in I} F_iH_j = \bigcup_{i\in I} F_iH_i$, since $F_iH_j = \{0\}$ if $i \neq j$ and $0 \in H_i$ for all $i \in I$. Thus, $(FH)_i = F_iH_i$ and the following proposition is valid.

Proposition. *$C_0(\Sigma^0_{i\in I} S_i)$ is isomorphic to $\times_{i\in I} C_0(S_i)$, $H \to (H \cap S_i)_{i\in I}$ being the isomorphism.*

1.31. Definition. *A Croisot semigroup* (or a *primitive* inverse semigroup) is any inverse semigroup in which the product of any two distinct idempotents is zero. A [0-] bisimple Croisot semigroup is called a *Brandt semigroup.*

Motivation. Deleting zero from a Croisot semigroup one obtains "groupes partiels" of Croisot [5] (which are precisely the categories of isomorphisms). Therefore, Croisot semigroups are to partial groups of Croisot exactly as Brandt semigroups are to Brandt groupoids.

It is well known that any Croisot semigroup is an orthogonal sum of a family of Brandt semigroups.

Let G be a group, I a set, $0 \notin G \times I \times I$. Define a multiplication on the set $(G \times \times I \times I) \cup \{0\} : (g, i_1, j_1)(h, i_2, j_2) = (gh, i_1, j_2)$ if $i_2 = j_1$; all other products are 0. Then we obtain a Brandt semigroup $B(G, I)$ and every Brandt semigroup with zero may be constructed up to an isomorphism in such a fashion. Brandt semigroups without zero are groups.

1.32. Suppose $B(G, I)$ is a Brandt semigroup with zero and $H \in \boldsymbol{C}_0(B(G, I))$. Then $0 \in H$. Suppose $(g, i_1, j_1), (h, i_2, j_2) \in H$. Since H is compatible, either $(g, i_1, j_1) = (h, i_2, j_2)$ or $i_1 \neq i_2, j_1 \neq j_2$. Thus, $\varrho_H = \{(i, j) : (g, i, j) \in H \text{ for some } g \in G\}$ is a univalent function in I. Let $I_H = pr_1\varrho_H = \{i : (g, i, j) \in H$ for some $g \in G$ and $j \in I\}$. Define a mapping $f_H : I_H \to G$ in the following way: $f_H(i) = g$ for $i \in I_H$ if and only if $(g, i, j) \in H$ for some $j \in I$. Since H is compatible, j is completely determined by g and i and f_H is a one-valued function.

Conversely, let ϱ be a univalent function in I and f a mapping of $pr_1\varrho$ into G. Define $H = \{(g, i, j) : (i, j) \in \varrho \text{ and } f(i) = g\} \cup \{0\}$. Then $\varrho_H = \varrho$, $I_H = pr_1\varrho$, $f_H = f$. Thus, the correspondence $H \to (\varrho_H, f_H)$ is a bijection of $\boldsymbol{C}_0(B(G, I))$ onto the set of all pairs (ϱ, f) where ϱ is a univalent function in I and $f : pr_1\varrho \to G$.

Let $H_1, H_2 \in \boldsymbol{C}_0(B(G, I))$. It is easy to verify that $\varrho_{H_1H_2} = \varrho_{H_2} \circ \varrho_{H_1}$ and $f_{H_1H_2}(i) = f_{H_1}(i) f_{H_2}(\varrho_{H_1}(i))$. Therefore, $\boldsymbol{C}_0(B(G, I))$ is isomorphic to the wreath product of the symmetric inverse semigroup $\mathscr{I}_I$ of all univalent functions acting in I and the group G. For a definition of the wreath product see [41].

1.33. An inverse semigroup satisfying the identity $ss^{-1} = s^{-1}s$ is called a *Clifford inverse semigroup* [3]. Clifford inverse semigroups are precisely the semilattices of groups.

Let S be a Clifford inverse semigroup and H_i the $\mathscr{H}$-class of S containing an idempotent $i \in E(S)$. Let $i \leqq j$ for $i, j \in E(S)$. Define a mapping $\varphi_{j,i} : H_j \to H_i$ in the following way: $\varphi_{j,i}(s) = si$ for every $s \in H_j$. Then $\varphi_{j,i}$ is a homomorphism of H_j into H_i and $\{H_i, \varphi_{j,i}, i, j \in E(S)\}$ is a direct system of groups, $S = \bigcup_{i \in E(S)} H_i$ and if $s \in H_i$, $t \in H_j$ then $st = \varphi_{i,ij}(s)\, \varphi_{j,ij}(t)$. Conversely, if $\{H_i, \varphi_{j,i}, i, j \in I\}$ is a direct system of disjoint groups, one may define a multiplication on $\bigcup H_i$ in the above fashion, a Clifford inverse semigroup being obtained [3].

We are going to find $\boldsymbol{C}_0(S)$ for a Clifford inverse semigroup S. To this purpose suppose $H \in \boldsymbol{C}_0(S)$. Suppose $s, t \in H \cap H$. Since s and t are compatible and s, t belong to the same subgroup H_i of S, $s = t$. Therefore, H contains one element at most from H_i. Now let $s \in H_i$, $t \in H_j$ and $s \leqq t$. Then $i \leqq j$ and $s = ss^{-1}t = it = \varphi_{j,i}(t)$. Let $I_H = \{i : H \cap H_i \neq \emptyset\}$. Then I_H is minorantly saturated, i.e. I_H is an ideal of $E(S)$. Therefore, H is the set of elements h_i where $i \in I_H$, $h_i \in H_i$ and $\varphi_{j,i}(h_j) = h_i$ if $i \leqq j$. Thus H is an element of the inverse limit of the direct system $\{H_i, \varphi_{j,i}, i, j \in I_H\}$ of groups which is a subsystem of the initial direct system.

Conversely, let I be an ideal of $E(S)$ and let $H \in \operatorname{inv\,lim} H_i$ be an element of the inverse limit of the direct system $\{H_i\, \varphi_{j,i},\, i, j \in I\}$ of groups. One can easily verify that $H \in C_0(S)$. Thus, $C_0(S) = \bigcup_{I \in I(E(S))} \operatorname{inv}\lim H_i$.

Let $F, H \in C_0(S)$. Then $FH = F(I_F I_H)\, H = F(I_F \cap I_H)\, H$.

Thus, $C_0(S)$ is a Clifford inverse semigroup as well, $H_I = \operatorname{inv}\lim H_i$ being the maximal subgroups for all $I \in I(E(S))$, the order on $I(E(S))$ being the set-theoretical inclusion; if $I, J \in I(E(S))$, $I \subset J$ then the structure homomorphism $\varphi_{J,I} : H_J \to H_I$ is defined as follows: $\varphi_{J,I}(H) = H \cap S_I$ where $S_I = \bigcup_{i \in I} H_i = SIS$.

1.34. Now let S be an inverse semigroup with a linearly ordered semilattice $E(S)$. Let $H \in C_0(S)$. Suppose $s, t \in H$. Then either $ss^{-1} \leqq tt^{-1}$ or $tt^{-1} \leqq ss^{-1}$, since the canonical order on $E(S)$ is linear. If $ss^{-1} \leqq tt^{-1}$ then $s = ss^{-1}s \leqq tt^{-1}s \leqq t$, since $t^{-1}s \in E(S)$. If $tt^{-1} \leqq ss^{-1}$ we obtain $t \leqq s$. Therefore, H is a subchain of S. Conversely, if H is any minorantly saturated subset of S which is linearly ordered by the canonical order, then $H \in C(S)$. By Zorn's Lemma, every chain in S is included in a maximal chain. It is easy to verify that maximal chains are minorantly saturated, hence, they belong to $C_0(S)$.

Clearly, $\tau(S) \subset C_0(S)$ for any inverse semigroup S. Let $\tau(S) = C_0(S)$. Since $\tau(i) \cup \tau(j) \in C_0(S)$ for any $i, j \in E(S)$, there exists such $s \in S$ that $\tau(i) \cup \tau(j) = \tau(s)$. Then $s \in \tau(i) \cup \tau(j)$, hence, $s \leqq i$ or $s \leqq j$. In the first case, $s = i$ and $j \leqq i$, in the second case $s = j \leqq i$. Therefore, $E(S)$ is linearly ordered. Now let I be a nonempty subset of $E(S)$. Then $\tau(I) \in C_0(S)$, therefore, $\tau(I) = \tau(s)$. A simple argument shows that $s \in I$ and s is the largest element in I. Thus, $E(S)$ is dually well-ordered.

Conversely, let S be an inverse semigroup and let $E(S)$ be dually well-ordered. Suppose $H \in C_0(S)$. Then $HH^{-1} = \iota_1(H) \subset E(S)$, therefore $\iota_1(H)$ possesses the largest element, say, i. Then $i = hh^{-1}$ for some $h \in H$. If $h_1 \in H$ then $h_1 h_1^{-1} \leqq hh^{-1}$. Since h and h_1 are compatible, $h_1 \leqq h$. Therefore, $H = \tau(h)$ and $C_0(S) = \tau(S)$.

Proposition. *Let S be an inverse semigroup. Then $C_0(S) = \tau(S)$ if and only if $E(S)$ is dually well-ordered.*

Clifford inverse semigroups S with dually well-ordered $E(S)$ were considered in [38].

2. TRANSLATIONAL HULLS OF INVERSE SEMIGROUPS

2.1. Definition. An *ideal extension* of a semigroup S is any isomorphism $S \to T$ of S into a semigroup T such that S is mapped onto an ideal of T. Two ideal extensions $S \to T$ and $S \to U$ are called equivalent if there exists a bijective isomorphism $T \leftrightarrow U$ such that the following diagram is commutative:

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ T & \leftrightarrow & U \end{array}$$

An ideal I of a semigroup T is called *essential* or *dense* if every nontrivial (i.e. non-identical) congruence on T induces a nontrivial congruence on I. An ideal extension $\varphi : S \to T$ is called *essential* or *dense* if $\varphi(S)$ is a dense ideal of T. In other words, an ideal extension $S \to T$ is dense if and only if every homomorphism $T \to U$ of T into a semigroup U such that the through homomorphism $S \to T \to U$ is an ideal extension is an isomorphism.

A dense ideal extension $S \to T$ is called *maximal*, or *dense ideal embedding*, if every isomorphism $T \to U$ such that the through isomorphism $S \to T \to U$ is a dense ideal extension is bijective. An ideal I of a semigroup T is called *densely embedded* if the identical ideal extension $I \subset T$ is a dense ideal embedding.

2.2. To make the paper reasonably self-contained we give here some results of L. M. Gluskin [7, 8] which are necessary in the sequel. We use some terms, concepts and notation introduced by P. Dubreil, A. H. Clifford, G. Lallement and M. Petrich. A more detailed survey may be found in [21].

A binary relation ϱ on a semigroup S is called *left regular* if $(s, t) \in \varrho \to (us, ut) \in \varrho$ for every $s, t, u \in S$. *Right regular* binary relations are defined dually.

As we have already said, transformations of every set are considered to be special binary relations (one-valued and everywhere defined) on the set.

A left regular transformation of a semigroup S is called a *right translation* of S, right regular transformations are called *left translations*. Left (right) translations will be written as left (right) operators on S, i.e., ϱ is a left translation if $(\varrho s)\, t = \varrho(st)$ for all $s, t \in S$ and a right translation if $(st)\,\varrho = s(t\varrho)$ for all $s, t \in S$. A pair $\omega = (\lambda, \varrho)$ where ϱ is a right translation and λ a left translation is called a *bitranslation* if $(s\varrho)\, t = s(\lambda t)$ for all $s, t \in S$. We consider a bitranslation ω to be a two-sided operator on $S : s\omega = s\varrho$ and $\omega s = \lambda s$ for every $s \in S$. Products of bitranslations are defined in a natural way and are bitranslations, the set $\Omega(S)$ of all bitranslations of S is a semigroup, the *translational hull* of S.

To every $s \in S$ there corresponds an *inner* bitranslation $\pi_s \in \Omega(S) : \pi_s t = st$ and $t\pi_s = ts$ for all $t \in S$. The mapping $\pi : s \to \pi_s$ is a homomorphism of S into $\Omega(S)$, $\pi(S)$ being an ideal of $\Omega(S)$. Clearly, π is an isomorphism (and so an ideal extension) if and only if S is *weakly reductive*. i.e. if $su = tu$ and $us = ut$ for all $u \in S$ imply $s = t$.

If S is weakly reductive, then π is a dense ideal embedding and all dense ideal embeddings of S are equivalent. If S is not weakly reductive, then it possesses no maximal dense ideal extensions (of course, S has dense ideal extensions, say, the identical extension $S \subset S$). The first fact is due to L. M. Gluskin [8], the second is due to L. N. Shevrin [35]. In fact, for weakly reductive semigroups S, π is the largest dense ideal extension: for every dense ideal extension $S \to T$ there exists an isomorphism $T \to \Omega(S)$ such that the through isomorphism $S \to T \to \Omega(S)$ is π [8].

Inverse semigroups are weakly reductive, therefore, π is a dense ideal embedding for every inverse semigroup.

2.3. Definition. Let S be a subsemigroup of T. The *idealizer* of S in T is the subset $I_T(S) = \{t : tS \cup St \subset S\} \subset T$ which is the largest subsemigroup of T containing S as an ideal.

If S is an inverse semigroup, let $\boldsymbol{T}(S)$ denote the idealizer of $\tau(S)$ in $\boldsymbol{C}(S)$. Clearly, $\boldsymbol{T}(S) \subset \boldsymbol{C}_0(S)$ and $\tau : S \to \boldsymbol{T}(S)$ is an ideal extension of S. Of course, $\boldsymbol{T}(S)$ is an inverse subsemigroup of $\boldsymbol{C}(S)$.

2.4. Lemma. *Let S be an inverse semigroup, $s \in S$ and $H \in \boldsymbol{M}(S)$. Then $H\tau(s) = Hs$ and $\tau(s)\,H = sH$.*

Proof. Clearly, $Hs = (HE(S))\,s = H(E(S)\,s) = H\,\tau(s)$, since $\tau(s) = E(S)\,s = s\,E(S)$. The second equality may be proved analogously.

2.5. Lemma. *The following conditions are equivalent for any $H \in \boldsymbol{C}(S)$:*

1) $H \in \boldsymbol{T}(S)$;

2) $\iota_1(H) \in \boldsymbol{T}(S)$ *and* $\iota_2(H) \in \boldsymbol{T}(S)$.

Proof. 1) $\to$ 2) follows from Lemma 1.8. Now suppose $\iota_1(H) \in \boldsymbol{T}(S)$. Then $s\iota_1(H) = \tau(s)\,\iota_1(H) = \tau(t)$ for some $t \in S$. Since $\iota_1(H)\,H = H$, we obtain $sH = s\iota_1(H)\,H = \tau(t)\,H = tH$. Therefore, $t = shh^{-1}$ for some $h \in H$. Let $h_1 \in H$. Since h and h_1 are compatible, $h^{-1}h_1 \in E(S)$, hence $th_1 = shh^{-1}h_1 \leqq sh_1 \in sH$, i.e. $\tau(s)\,H \subset \tau(sh) \subset \tau(s)\,h \subset \tau(s)\,H$; thus, $\tau(s)\,H = \tau(sh)$. The second of conditions 2) implies that for every $s \in S$ there exists $u \in S$ such that $H\,\tau(s) = \tau(u)$. Therefore, $H \in \boldsymbol{T}(S)$.

2.6. Lemma. $E(\boldsymbol{T}(S)) = \boldsymbol{T}(E(S))$.

Proof. Let $H \in E(\boldsymbol{T}(S))$. Then $H \in E(\boldsymbol{C}(S)) = \boldsymbol{C}(E(S))$, by Corollary 1.20. For every $i \in E(S)$ $H\,\tau(i) = \tau(s)$ for some $s \in S$. However, $s \in H\,\tau(i) \subset E(S)$, therefore, $H \in \boldsymbol{T}(E(S))$.

Conversely, suppose $H \in \boldsymbol{T}(E(S))$, i.e. $H \in \boldsymbol{C}(E(S))$ and for every $i \in E(S)$ there exists $j \in E(S)$ such that $H\,\tau(i) = \tau(j)$. If $s \in S$ then $H\,\tau(s) = Hs = (Hss^{-1})\,s = \tau(j)\,s$ where $Hss^{-1} = \tau(j)$. Now $\tau(j)\,s = j\,E(S)\,s = js\,E(S) = \tau(js)$, therefore, $H\,\tau(s) = \tau(js)$. Analogously, $\tau(s)\,H = \tau(sk)$ for some $k \in E(S)$. Thus, $H \in E(\boldsymbol{T}(S))$.

2.7. Lemma. *Let $H \in \boldsymbol{C}(S)$ Then $H \in \boldsymbol{T}(S)$ if and only if $\iota_1(H) \in \boldsymbol{T}(E(S))$ and $\iota_2(H) \in \boldsymbol{T}(E(S))$.*

Lemma 2.7 follows from Lemmas 2.5 and 2.6 and shows that the problem of finding $\boldsymbol{T}(S)$ is very tightly connected with that of finding $\boldsymbol{T}(E(S))$.

2.8. Lemma. *An ideal I of a semilattice S is a retract ideal if and only if for every principal ideal (s) generated by an element $s \in S$ the intersection $I \cap (s)$ is* *a principal ideal of S. The set of all retract ideals under the operation of intersection is a semilattice coinciding with $\boldsymbol{T}(S)$.*

Proof. The first part of Lemma is due to M. Kolibiar [14]. If I is a retract ideal and f an idempotent endomorphism of S onto I, then $I \cap (s) = (f(s))$. Conversely, if $I \cap (s) = (t)$, then the mapping $s \to t$ is an idempotent endomorphism of S onto I, whence I is a retract ideal.

If I and J are ideals then $IJ = I \cap J$ and the second part of Lemma follows readily.

2.9. Theorem. (i) *$\boldsymbol{T}(S)$ is an inverse semigroup consisting of permissible subsets $H \subset S$ such that HH^{-1} and $H^{-1}H$ are retract ideals of the semilattice $E(S)$;*

(ii) $E(\boldsymbol{T}(S)) = \boldsymbol{T}(E(S))$;

(iii) *the isomorphism $\tau : S \to \boldsymbol{T}(S)$ is complete;*

(iv) *$\tau : S \to \boldsymbol{T}(S)$ is a dense ideal embedding; in other words, the ideal extensions $\tau : S \to \boldsymbol{T}(S)$ and $\pi : S \to \Omega(S)$ are equivalent, for every dense ideal extension $S \to T$ there exists such an isomorphism (which is necessarily unique) $T \to \boldsymbol{T}(S)$ that the through isomorphism $S \to T \to \boldsymbol{T}(S)$ is τ.*

Proof. (i) and (ii) are proved in Lemmas 2.6 and 2.7.

(iii) Let $H \subset S$ and $\bigvee H = h$. Then for every $h_1 \in H$ $h_1 \leqq h$, hence $\tau(h_1) \subset \tau(h)$. Let $F \in \boldsymbol{T}(S)$ and $\tau(h_1) \subset F$ for all $h_1 \in H$. Since F is minorantly saturated, $\tau(h_1) \subset F$ means that $h_1 \in F$. Therefore, $H \subset F$. Now $H\tau(h^{-1}h) = H$ since $h_1 h^{-1}h = h_1$ for all $h_1 \in H$. Therefore, $H \subset F\tau(h^{-1}h) = Fh^{-1}h \subset F$. Since $F \in \boldsymbol{T}(S)$, there exists $s \in S$ such that $Fh^{-1}h = \tau(s)$. Since $s \in Fh^{-1}h$, $s \in F$. If $h_1 \in H$ then $\tau(h_1) \subset F$ and $\tau(h_1) = \tau(h_1)\, h^{-1}h \subset Fh^{-1}h = \tau(s)$. Therefore, $h_1 \leqq s$ for all $h_1 \in H$, i.e. $h \leqq s$. Thus $h \in F$ and $\tau(h) \subset F$. It follows that $\tau(\bigvee H) = \tau(h) = \bigvee \tau(H)$, i.e. τ is $\bigvee$-complete.

If $H \subset S$ and $\bigwedge H = h$ then $\tau(\bigwedge H) = \bigcap_{f \in H} \tau(f) = \bigwedge \tau(H)$ and τ is $\bigwedge$-complete.

(iv) Let $\varphi : S \to T$ be a dense ideal extension of S. If $t \in T$, let a subset $H_t \subset S$ be defined as follows: $H_t = \{s : \varphi(s) = \varphi(ss^{-1})\, t\}$. We are going to show that the correspondence $t \to H_t$ is an isomorphism of T into $\boldsymbol{T}(S)$. This will be done in a series of statements.

(a) $H_t = \{s : \varphi(s) = t\, \varphi(s^{-1}s)\}$.

If $s \in H_t$ then $t\, \varphi(s^{-1}s) \in \varphi(S)$, since $\varphi(S)$ is an ideal of T. Therefore, $t\, \varphi(s^{-1}s) = \varphi(s_1)$ for some $s_1 \in S$. It follows that $\varphi(ss^{-1}s_1) = \varphi(ss^{-1})\, t\, \varphi(s^{-1}s) = \varphi(s) \,.\, \varphi(s^{-1}s) = \varphi(ss^{-1}s) = \varphi(s)$, whence $ss^{-1}s_1 = s$. Thus, $s_1 s^{-1} s = s$. On the other hand, $\varphi(s) = \varphi(s_1 s^{-1} s) = \varphi(s_1)\, \varphi(s^{-1}s) = t\varphi(s^{-1}s)\, \varphi(s^{-1}s) = t\, \varphi(s^{-1}ss^{-1}s) = t\, \varphi(s^{-1}s) = \varphi(s_1)$. Thus, $s = s_1$ and $\varphi(s) = t\, \varphi(s^{-1}s)$. Analogously, the latter equality implies $\varphi(s) = \varphi(ss^{-1})\, t$, i.e. $s \in H_t$.

(b) H_t is a compatible subset of S.

Let $s_1, s_2 \in H_t$. Then $\varphi(s_1 s_1^{-1} s_2) = \varphi(s_1 s_1^{-1}) \varphi(s_2) = \varphi(s_1 s_1^{-1}) \varphi(s_2 s_2^{-1}) t = \varphi(s_1 s_1^{-1} s_2 s_2^{-1}) t = \varphi(s_2 s_2^{-1} s_1 s_1^{-1}) t = \varphi(s_2 s_2^{-1}) \varphi(s_1 s_1^{-1}) t = \varphi(s_2 s_2^{-1}) \varphi(s_1) = \varphi(s_2 s_2^{-1} s_1)$. It follows that $s_1 s_1^{-1} s_2 = s_2 s_2^{-1} s_1$ and $s_1^{-1} s_2 = s_1^{-1} s_1 s_1^{-1} s_2 = (s_1^{-1} s_2) \cdot (s_1^{-1} s_2)^{-1} \in E(S)$.

Using (a) we obtain in an analogous fashion that $s_1 s_2^{-1} \in E(S)$.

(c) $H_t \in \boldsymbol{C}(S)$.

By (b), H_t is compatible. It remains to prove that $H_t \in \boldsymbol{M}(S)$.

Let $s_1 \leqq s$ and $s \in H_t$. Then $\varphi(s_1) = \varphi(s_1 s_1^{-1} s) = \varphi(s_1 s_1^{-1}) \varphi(s) = \varphi(s_1 s_1^{-1}) \cdot \varphi(s s^{-1}) t = \varphi(s_1 s_1^{-1} s s^{-1}) t = \varphi(s_1 s_1^{-1}) t$, whence $s_1 \in H_t$.

(d) $H_t \in \boldsymbol{T}(S)$.

Let $s \in S$. Since $\varphi(S)$ is an ideal of T, $t\, \varphi(s) \in \varphi(S)$ and the element $s_0 = \varphi^{-1}(t\, \varphi(s))$ is defined. If $s_1 \in H_t$ then $\varphi(s_1 s) = \varphi(s_1 s s^{-1} s_1^{-1}) \varphi(s_1) \varphi(s) = \varphi(s_1 s s^{-1} s_1^{-1}) \varphi(s_1 s_1^{-1}) \cdot t\, \varphi(s) = \varphi(s_1 s s^{-1} s_1^{-1} s_1 s_1^{-1}) t\, \varphi(s) = \varphi(s_1 s s^{-1} s_1^{-1}) t\, \varphi(s) = \varphi(s_1 s s^{-1} s_1^{-1}) \varphi(s_0) = \varphi(s_1 s s^{-1} s_1^{-1} s_0)$, whence $s_1 s = (s_1 s)(s_1 s)^{-1} s_0$ and $s_1 s \leqq s_0$. Therefore, $H_t s \subset \tau(s_0)$. Conversely, let $s_2 \in \tau(s_0)$, i.e. $s_2 \leqq s_0$ or $s_2 = s_2 s_2^{-1} s_0$. Then $\varphi(s_2) = \varphi(s_2 s_2^{-1}) \varphi(s_0) = \varphi(s_2 s_2^{-1}) t\, \varphi(s) = \varphi(s_3) \varphi(s) = \varphi(s_3 s)$, where $\varphi(s_3) = \varphi(s_2 s_2^{-1}) t$. Therefore, $s_2 = s_3 s$. Now $\varphi(s_3) = \varphi(s_2 s_2^{-1}) t = \varphi(s_2 s_2^{-1}) \varphi(s_2 s_2^{-1}) t = \varphi(s_2 s_2^{-1}) \cdot \varphi(s_3) = \varphi(s_2 s_2^{-1} s_3) = \varphi(s_2 s_2^{-1} s_3 s_3^{-1} s_3) = \varphi(s_3 s_3^{-1} s_2 s_2^{-1} s_3) = \varphi(s_3 s_3^{-1}) \varphi(s_2 s_2^{-1}) \cdot \varphi(s_3) = \varphi(s_3 s_3^{-1}) \varphi(s_2 s_2^{-1}) \varphi(s_2 s_2^{-1}) t = \varphi(s_3 s_3^{-1} s_2 s_2^{-1} s_2 s_2^{-1}) t = \varphi(s_3 s_3^{-1} s_2 s_2^{-1}) t = \varphi(s_2 s_2^{-1} s_3 s_3^{-1}) t = \varphi(s_2 s_2^{-1}) \varphi(s_3) \varphi(s_3^{-1}) t = \varphi(s_3) \varphi(s_3^{-1}) t = \varphi(s_3 s_3^{-1}) t$, since we have proved above that $\varphi(s_2 s_2^{-1}) \varphi(s_3) = \varphi(s_3)$. Thus, $s_3 \in H_t$. It follows that $\tau(s_0) \subset H_t s$ and $H_t\, \tau(s) = H_t s = \tau(s_0)$. Analogously, $\tau(s)\, H_t = \tau(s_4)$ for some $s_4 \in S$.

(e) The mapping $\psi : T \to \boldsymbol{T}(S)$ such that $\psi(t) = H_t$ for every $t \in T$ is an isomorphism of T into $\boldsymbol{T}(S)$ and $\psi \circ \varphi = \tau$.

Let $s_1 \in H_{t_1}$ and $s_2 \in H_{t_2}$. Then $\varphi(s_1 s_2 s_2^{-1} s_1^{-1}) t_1 t_2 = \varphi(s_1 s_2 s_2^{-1} s_1^{-1}) \varphi(s_1 s_1^{-1}) t_1 t_2 = \varphi(s_1 s_2 s_2^{-1} s_1^{-1}) \varphi(s_1) t_2 = \varphi(s_1) \varphi(s_2 s_2^{-1}) t_2 = \varphi(s_1) \varphi(s_2) = \varphi(s_1 s_2)$, i.e. $s_1 s_2 \in H_{t_1 t_2}$, hence $H_{t_1} H_{t_2} \subset H_{t_1 t_2}$. Now let $s \in H_{t_1 t_2}$. Denote $\varphi(s s^{-1}) t_1 = \varphi(s_1)$ and $t_2\, \varphi(s^{-1} s) = \varphi(s_2)$. Then $\varphi(s_1 s_2) = \varphi(s s^{-1})\, t_1 t_2\, \varphi(s^{-1} s) = \varphi(s)\, \varphi(s^{-1} s) = \varphi(s s^{-1} s) = \varphi(s)$, i.e. $s_1 s_2 = s$. We have seen above that $\varphi(s_3) = \varphi(s_2 s_2^{-1}) t$ implies $s_3 \in H_t$. Therefore, $s_1 \in H_{t_1}$. Analogously, $s_2 \in H_{t_2}$. Therefore, $s \in H_{t_1} H_{t_2}$ and $H_{t_1 t_2} \subset H_{t_1 t_2}$. Therefore, $H_{t_1 t_2} = H_{t_1} H_{t_2}$, i.e. $\psi(t_1 t_2) = \psi(t_1) \psi(t_2)$. If $s \in S$ then $H_{\varphi(s)} = \{s_1 : \varphi(s_1) = \varphi(s_1 s_1^{-1}) \varphi(s)\} = \{s_1 : \varphi(s_1) = \varphi(s_1 s_1^{-1} s)\} = \{s_1 : s_1 = s_1 s_1^{-1} s\} = \{s_1 : s_1 \leqq s\} = \tau(s)$. Therefore, $\psi(\varphi(s_1)) = \psi(\varphi(s_2)) \leftrightarrow \tau(s_1) = \tau(s_2) \leftrightarrow s_1 = s_2$, i.e., ψ induces an isomorphism on $\varphi(S)$ Since φ is dense, ψ is an isomorphism of T into $\boldsymbol{T}(S)$. Moreover, $\psi \circ \varphi(s) = \psi(\varphi(s)) = \tau(s)$ for all $s \in S$, i.e. $\psi \circ \varphi = \tau$.

(f) If $\chi : T \to \boldsymbol{T}(S)$ is a homomorphism such that $\chi \circ \varphi = \tau$, then $\chi = \psi$.

Suppose $\chi \circ \varphi = \tau$. Then $\chi \circ \varphi = \psi \circ \varphi$, i.e. χ and ψ coincide on $\varphi(S)$. Since φ is dense, χ is an isomorphism. Let $t \in T$ and $s \in \chi(t) \in \boldsymbol{T}(S)$. Then $\tau(s) \subset \chi(t)$ since $\chi(t)$ is a minorantly saturated subset of S. Therefore, $\chi(\varphi(s)) = \tau(s) \subset \chi(t)$. Thus, $s \in H_t$. It is easy to verify that $s \in H_t$ implies $s \in \chi(t)$, since our argument may be repeated backwards. Therefore, $\chi(t) = H_t = \psi(t)$ for all $t \in T$ and $\chi = \psi$.

(g) $\tau : S \to \boldsymbol{T}(S)$ is a dense ideal embedding.

By (e) it remains to prove that τ is a dense ideal extension. Let $\pi : \boldsymbol{T}(S) \to U$ be a homomorphism and let $\pi \circ \tau$ be an isomorphism of S into U. Let $H_1, H_2 \in \boldsymbol{T}(S)$ and $\pi(H_1) = \pi(H_2)$. If $h \in H_1$ then $H_1\, \tau(h^{-1}) = \tau(s)$ for some $s \in S$. Therefore, $hh^{-1} \in \tau(s)$, i.e. $hh^{-1} \leqq s$. On the other hand, $s \in H_1\, \tau(h^{-1}) = H_1 h^{-1}$, i.e. $s = h_1 h^{-1}$ for some $h_1 \in H_1$. It follows that $hh^{-1} = shh^{-1} = s$. Thus, $H_1\, \tau(h^{-1}) = \tau(hh^{-1})$ and $\pi(\tau(hh^{-1})) = \pi(H_1\, \tau(h^{-1})) = \pi(H_2\, \tau(h^{-1})) = \pi(\tau(s_1))$ where $\tau(s_1) = H_2\, \tau(h^{-1})$. Therefore, $\tau(hh^{-1}) = \tau(s_1)$ and $hh^{-1} = s_1$, i.e. $H_2\, \tau(h^1) = \tau(hh^{-1})$. It follows that $hh^{-1} \in H_2\, \tau(h^{-1}) = H_2 h^{-1}$ and $h = hh^{-1}h \in H_2 h^{-1} h \subset H_2$, i.e. $H_1 \subset H_2$. Analogously, $H_2 \subset H_1$, i.e. $H_1 = H_2$. Thus π is an isomorphism and τ is dense.

2.10. Remark. Since $T \to \boldsymbol{T}(S)$ is an isomorphism, T may be ordered in such a way that $T \to \boldsymbol{T}(S)$ is an order isomorphism. In this case $t_1 \leqq t_2 \leftrightarrow H_{t_1} \subset H_{t_2}$. Now $S \to T$ is an order isomorphism and $H_t = \{t_1 : t_1 \leqq t\} \cap \varphi(S)$.

2.11. Since $S \to \boldsymbol{T}(S)$ and $S \to \Omega(S)$ are equivalent ideal extensions, there exists a bijection α between $\boldsymbol{T}(S)$ and $\Omega(S)$, $\alpha \circ \tau = \pi$.

Corollary. *Let $\omega \in \Omega(S)$ and $H \in \boldsymbol{T}(S)$. In order that $\alpha(H) = \omega$ it is necessary and sufficient that for every $s \in S$, $sH = \tau(s\omega)$ and $Hs = \tau(\omega s)$ (i.e. H induces on $\tau(S)$ the same bitranslation as ω on S). $\alpha(H) = \omega$ if and only if $H = E(S)\,\omega = \omega\, E(S)$.*

Proof. Let $H \in \boldsymbol{T}(S)$. Then for every $s \in S$ there exist $s_1, s_2 \in S$ such that $Hs = \tau(s_1)$ and $sH = \tau(s_2)$. Define a two-sided operator ω_H on S as follows: $\omega_H s = s_1$, $s\omega_H = s_2$. It is a matter of straightforward computation to check that the correspondence $H \to \omega_H$ is a homomorphism of $\boldsymbol{T}(S)$ into $\Omega(S)$. If $H = \tau(s_0)$ then $Hs = \tau(s_0 s)$ and $sH = \tau(ss_0)$. It follows that $\omega_H = \pi_{s_0}$, i.e. the homomorphism of $\boldsymbol{T}(S)$ into $\Omega(S)$ induces an isomorphism on $\tau(S)$. Since τ is dense, the homomorphism of $\boldsymbol{T}(S)$ into $\Omega(S)$ is an isomorphism; since τ and π are dense ideal embeddings, this isomorphism coincides with α.

To prove the second part of our Corollary notice that $H = E(S)\, H = \bigcup_{i \in E(S)} iH = \bigcup_{i \in E(S)} \tau(i\omega) = \tau(E(S)\,\omega)$. Since $E(S)\,\omega = (E(S)\,E(S))\,\omega = E(S)\,(E(S)\,\omega)$, the subset $E(S)\,\omega$ is minorantly saturated, i.e. $\tau(E(S)\,\omega) = E(S)\,\omega$. Therefore, $H = E(S)\,\omega$. Analogously, $H = \omega\, E(S)$.

2.12. Corollary. *Let $\tau(S) \subset T_i \subset \boldsymbol{T}(S)$ for $i = 1, 2$, T_i being subsemigroups of $\boldsymbol{T}(S)$. If φ is an isomorphism of T_1 onto T_2 and all the elements of $\tau(S)$ are fixed points under φ, then $T_1 = T_2$ and φ is the identity.*

2.13. Corollary. *Let $\tau(S) \subset T \subset \boldsymbol{T}(S)$, T being a subsemigroup of $\boldsymbol{T}(S)$. Then $\tau : S \to T$ is a dense ideal extension.*

Proof may be given following the same lines as that of 2.9. (iv) (g).

Thus, the ideal extensions $S \to T$ such that $\tau(S) \subset T \subset T(S)$ are dense and every dense ideal extension of S is equivalent to the above ideal extension.

2.14. Corollary [24]. *If S is an inverse semigroup, then $\Omega(S)$ is an inverse semigroup.*

2.15. Corollary. *If S is an inverse semigroup then $E(\Omega(S))$ is isomorphic to $\Omega(E(S))$.*

This Corollary has been proved independently in [42].

2.16. Corollary. *$\boldsymbol{T}(S) = \boldsymbol{C}_0(S)$ if and only if every principal ideal of $E(S)$ is dually well-ordered (in other words, $E(S)$ is a tree semilattice satisfying the descending chain condition).*

Proof. $\boldsymbol{T}(S) = \boldsymbol{C}_0(S)$ if and only if every nonempty ideal of $E(S)$ is a retract ideal, by Theorem 2.9 (i), i.e., if $\boldsymbol{T}(E(S)) = \boldsymbol{I}(E(S))$. Let $i, j \in E(S)$. If i and j are not comparable relative to $\leqq$ and $i \leqq k$, $j \leqq k$, $k \in E(S)$, then $\tau(i) \cup \tau(j)$ is a non-principal ideal of $E(S)$ and $(\tau(i) \cup \tau(j)) \cap \tau(k) = \tau(i) \cup \tau(j)$, i.e. $\tau(i) \cup \tau(j)$ is not a retract ideal. Therefore, incomparable elements of $E(S)$ cannot possess a common upper bound, i.e. $E(S)$ is a tree semilattice.

Let $i \in E(S)$ and $\emptyset \neq H \subset \tau(i)$. Then $\tau(H) \in \boldsymbol{I}(E(S))$, therefore, $\tau(H) \cap \tau(i) = \tau(H)$ should be a principal ideal of $E(S)$ which means that H possesses the largest element. Therefore, $\tau(i)$ is dually well-ordered.

Conversely, let all principal ideals of $E(S)$ be dually well-ordered sets. Let $H \in \boldsymbol{I}(E(S))$ and $i \in E(S)$. Then $H \cap \tau(i)$ is not empty, therefore, $H \cap \tau(i) = \tau(j)$ where j is the largest element of the nonempty subset $H \cap \tau(i)$ of $\tau(i)$. Thus H is a retract ideal of $E(S)$.*)

2.17. Definition. An ideal I of an inverse semigroup S is called $\bigvee$-*basic* if every element of S is the l.u.b. of a subset from I.

A subset H of a semigroup S is called *left reductive* [36] if $hs = ht$ for all $h \in H$ implies $s = t$. Right reductive subsets are defined analogously. A subset which is both right and left reductive is called *reductive*. H is called *weakly reductive* if $hs = ht$ and $sh = th$ for all $h \in H$ imply $s = t$.

Added in the proof. Semilattices in which all ideals are retract are described in the paper: E. J. TULLY, Jr., *Semigroups in which each ideal is a retract*, J. Austral. Math. Soc. *9* (1969), 239—245.

Lemma. *For an ideal I of an inverse semigroup S the following properties are equivalent*:

1) *I is left reductive*;
2) *I is right reductive*;
3) *I is reductive*;
4) *I is weakly reductive*;
5) *the set $E(I)$ of all idempotents of I is left reductive*;
6) *$E(I)$ is right reductive*;
7) *$E(I)$ is reductive*;
8) *$E(I)$ is weakly reductive.*

Proof. Implications 5) → 1), 6) → 2), 7) → 3) → 1) → 4), 3) → 2) → 4), 7) → → 5) → 8) → 4) and 7) → 6) → 8) are obvious. It is sufficient to prove the implication 4) → 7).

Let I be a weakly reductive ideal of an inverse semigroup S. Suppose $is = it$ for all $i \in E(I)$. Then $hs = h(h^{-1}h)\,s = h(h^{-1}h)\,t = ht$ for every $h \in H$, since $h^{-1}h \in E(I)$. Now if $h \in I$ then $sh \in I$, since I is an ideal. Therefore, $sh = (sh)\,(sh)^{-1}sh = (sh)\,(sh)^{-1}\,th \leqq th$. Analogously, $th \leqq sh$, i.e. $sh = th$ for all $h \in I$. Since I is weakly reductive, $s = t$. Therefore, $E(I)$ is a left reductive subset of S. Right reductivity of $E(I)$ may be verified analogously.

2.18. Proposition. *For an ideal I of an inverse semigroup S the following properties are equivalent*:

1) *I is a dense ideal*;
2) *I is a $\bigvee$-basic ideal*;
3) *I is a reductive ideal.*

Proof. 1) → 2). Let I be a dense ideal of S. Then the isomorphism $\tau : I \to \boldsymbol{T}(I)$ can be extended to an isomorphism of S into $\boldsymbol{T}(I)$. Obviously, $\tau(I)$ is a $\bigvee$-basic ideal of $\boldsymbol{T}(I)$, therefore, $\tau(I)$ is a $\bigvee$-basic ideal of the image of S in $\boldsymbol{T}(I)$ and I is a $\bigvee$-basic ideal of S.

2) → 3). Let I be a $\bigvee$-basic ideal of S. Suppose $hs = ht$ for all $h \in I$ and some $s, t \in S$. Let $s = \bigvee H_1$, $t = \bigvee H_2$ for some $H_1, H_2 \subset I$. If $h \in H_1$, then $h \leqq s$ and $h = hh^{-1}s = hh^{-1}t \leqq t$. It follows that $s = \bigvee H_1 \leqq t$. Analogously, $t \leqq s$, i.e. $s = t$. Thus, I is left reductive. By Lemma 2.17, I is reductive.

3) → 1). Let I be a reductive ideal of S and ε a congruence on S which induces a trivial congruence on I. Suppose $s \equiv t(\varepsilon)$. Then $hs \equiv ht(\varepsilon)$ for all $h \in I$. Since ε induces identity on I and $hs, ht \in I$, $hs = ht$. By Lemma 2.17, I is left reductive. Therefore, $s = t$, i.e. ε is a trivial congruence.

2.19. Let Φ be an inverse semigroup of univalent functions acting in a set A. In Corollary 1.24 we have proved that there exists a unique $\bigwedge$-complete homomorphism

$\xi : C(\Phi) \to U(\Phi)$ such that $\xi \circ \tau$ is the natural embedding of Φ into $U(\Phi)$. Let $V(\Phi)$ be the idealizer of Φ in $U(\Phi)$. Then $\Phi \subset V(\Phi) \subset U(\Phi)$. The homomorphism ξ restricted to $T(\Phi)$ maps $T(\Phi)$ onto $V(\Phi)$. Since τ is dense, $\xi : T(\Phi) \to V(\Phi)$ is an isomorphism.

Lemma. $V(\Phi) = \{\varphi : \varphi \in U(\Phi)$ *and* $\varphi \circ \iota, \iota \circ \varphi \in \Phi$ *for all* $\iota \in E(\Phi)\}$.

Proof. If $\varphi \in V(\Phi)$, then $\varphi \circ \iota, \iota \circ \varphi \in \Phi$ for all $\iota \in E(\Phi)$. Now let $\varphi \in U(\Phi)$ and $\varphi \circ \iota, \iota \circ \varphi \in \Phi$ for all $\iota \in E(\Phi)$. Suppose $\psi \in \Phi$. Then $\psi^{-1} \circ \psi \in E(\Phi)$, whence $\psi^{-1} \circ \circ \psi \circ \varphi \in \Phi$. Therefore, $\psi \circ \varphi = \psi \circ (\psi^{-1} \circ \psi \circ \varphi) \in \Phi$. Analogously, $\varphi \circ \psi \in \Phi$, i.e. $\varphi \in V(\Phi)$.

We have proved

Proposition. *Let* Φ *be an inverse semigroup of univalent functions and* $V(\Phi) = \{\varphi : \varphi \in U(\Phi)$ *and* $\varphi \circ \iota, \iota \circ \varphi \in \Phi$ *for all* $\iota \in E(\Phi)\}$. *Then the natural embedding* $\Phi \to V(\Phi)$ *is a dense ideal embedding, i.e. it is equivalent to* $\Phi \to \Omega(\Phi)$.

Remark. $\varphi \in V(\Phi)$ if an only if φ is a set-theoretical union of a set of functions belonging to Φ and every restriction of φ and of φ^{-1} to the domain of a function from Φ belongs to Φ.

If inverse semigroups of univalent functions Φ and Ψ are isomorphic then $V(\Phi)$ and $V(\Psi)$ are isomorphic as well, though $U(\Phi)$ and $U(\Psi)$ need not be isomorphic.

2.20. Definition. An inverse semigroup S is called *fundamental* [18] if the Green equivalence $\mathscr{H}$ on S does not contain any nontrivial congruences.

Proposition. *If an inverse semigroup* S *is fundamental, then* $\Omega(S)$ *is fundamental.**)

Proof. Let ε be a congruence on $\Omega(S)$ which is included into the Green equivalence $\mathscr{H}_{\Omega(S)}$ on S. Then the restriction of ε to S is a congruence on S which is included into $\mathscr{H}_S$. Therefore, the restriction of ε to S is trivial. Since π is dense, ε is trivial as well, i.e. $\Omega(S)$ is fundamental.

Ideals of fundamental inverse semigroups are obviously fundamental, so if $\Omega(S)$ is fundamental, then S is fundamental as well.

2.21. Definition. Let S be a semigroup with identity 1. *Units* of S are the elements of the class modulo the Green equivalence $\mathscr{H}$ containing 1, i.e. the elements of the maximal subgroup of S containing 1.

*) Added in the proof: This Proposition has been recently (and independently) proved in the paper: N. R. Reilly, *The translational hull of an inverse semigroup*, to appear in the Canadian Journal of Mathematics. That paper has ofter minor intersections with the present one.

Let S be an inverse semigroup. Then $T(S)$ contains $E(S)$ as an identity. By Theorem 2.9 (i), $H \in T(S)$ is a unit if and only if $HH^{-1} = H^{-1}H = E(S)$. If H is a subset of S satisfying the latter equalities, then H is permissible by Lemma 1.7. Thus, the units of $T(S)$ are such compatible subsets $H \subset S$ that for every $i \in E(S)$ there exist $h_1, h_2 \in H$ such that $h_1h_1^{-1} = i = h_2^{-1}h_2$. In particular, $T(S)$ is $\tau(S)$ with units adjoined if and only if $T(E(S)) \cong E(S)^1$. If $E(S)$ is a chain, then $E(S)$ may contain only one retract ideal — $E(S)$ itself — which is not principal. So if $E(S)$ is a chain and S does not contain identity (i.e. $E(S)$ does not possess the largest element), then $T(S) \setminus \tau(S)$ is the group of units of $T(S)$. This result has been obtained independently in [42]. Any unit H of $T(S)$ must be linearly ordered by $\leqq$ if $E(S)$ is a chain (cf. 1.34).

2.22. By analogy with Proposition 1.26 we can prove that every retract ideal of $T(S)$ is of the form $T(U)$ where U is an ideal of S such that $E(U)$ is a retract ideal of $E(S)$. Conversely, if U is an ideal of S such that $E(U)$ is a retract ideal of $E(S)$, then $T(U)$ is a retract ideal of $T(S)$. In particular, the lattice of all retract ideals of $T(S)$ is isomorphic to the lattice of all ideals U of S such that $E(U)$ is a retract ideal of $E(S)$.

2.23. Proposition. *Let $(S_i)_{i\in I}$ be a family of inverse semigroups. The correspondence $(H_i)_{i\in I} \to \times_{i\in I} H_i$ where $H_i \in T(S_i)$ for all $i \in I$ is a bijective isomorphism between $\times_{i\in I} T(S_i)$ and $T(\times_{i\in I} S_i)$. In particular, $\times_{i\in I} \Omega(S_i)$ is isomorphic to $\Omega(\times_{i\in I} S_i)$.*

Proof. Let $H_i \in T(S_i)$ for every $i \in I$. Then, by 1.27, $\times_{i\in I} H_i \in C_0(\times_{i\in I} S_i)$. It is a matter of straightforward computation to verify that $\iota_1(\times_{i\in I} H_i) = \times_{i\in I} \iota_1(H_i)$ is a retract ideal of $\times_{i\in I} E(S_i) = E(\times_{i\in I} S_i)$. The same is true for $\iota_2(\times_{i\in I} H_i)$ so that $\times_{i\in I} H_i \in T(\times_{i\in I} S_i)$. It remains to prove that the isomorphism of $\times_{i\in I} T(S_i)$ into $T(\times_{i\in I} S_i)$ is surjective.

Let $H \in T(\times_{i\in I} S_i)$ and $pr_i H = H_i$ denote the i-th projection of H, i.e. the image of H under the canonical mapping of $\times_{i\in I} S_i$ onto S_i. Homomorphisms of inverse semigroups transform compatible subsets into compatible ones, surjective homomorphisms transform minorantly saturated subsets into minorantly saturated ones. Therefore, H_i is permissible for every $i \in I$. Let $(s_i)_{i\in I} \in \times_{i\in I} S_i$. Then $H(s_i)_{i\in I} = \tau((t_i)_{i\in I})$ where $\tau : \times_{i\in I} S_i \to T(\times_{i\in I} S_i)$. Therefore, $H_i s_i = pr_i(\tau((t_i)_{i\in I})) = \tau_i(t_i)$ where $\tau_i : S_i \to T(S_i)$. Analogously, $s_i H_i = \tau_i(u_i)$ for some $u_i \in S_i$. Therefore, $H_i \in T(S_i)$.

Clearly, $H \subset \times_{i\in I} H_i$. The converse inclusion also holds. Indeed, let $s_i \in H_i$ for every $i \in I$. This means that for every $i \in I$ there exist $s_{ij} \in S_j$ such that $s_{ii} = s_i$ and $(s_{ij})_{j\in I} \in H$. Consider the element $(s_i s_i^{-1})_{i\in I}$. Clearly, there exist $t_i \in S_i$ such that $\tau((t_i)_{i\in I}) = (s_i s_i^{-1})_{i\in I} H \subset H$, since H is minorantly saturated. It follows that $s_j s_j^{-1} s_{ij} \leqq t_j$ for all $i, j \in I$. If $i = j$ we obtain $s_i = s_i s_i^{-1} s_{ii} \leqq t_i$, i.e. $(s_i)_{i\in I} \leqq (t_i)_{i\in I} \in H$. Therefore, $(s_i)_{i\in I} \in H$.

Thus, $H = \times_{i\in I} H_i$, i.e. the isomorphism of $\times_{i\in I} T(S_i)$ into $T(\times_{i\in I} S_i)$ is surjective.

2.24. Let φ be a surjective homomorphism of an inverse semigroup S onto an inverse semigroup U and $H \in \boldsymbol{T}(S)$. It is easy to verify that φ transforms retract ideals of $E(S)$ into retract ideals of $E(\varphi(S)) = E(U)$. Together with 1.28 this shows that φ induces a homomorphism of $\boldsymbol{T}(S)$ into $\boldsymbol{T}(U)$, i.e. $\varphi(H) \in \boldsymbol{T}(U)$.

2.25. All the results obtained in 1.29 for $\boldsymbol{C}_0(S)$ can be obtained for $\boldsymbol{T}(S)$ in an analogous way. In particular, if an inverse semigroup S is a subdirect product of an inverse semigroup U and a group G then $\boldsymbol{T}(S)$ is isomorphic to a subdirect product of an inverse subsemigroup of $\boldsymbol{T}(U)$ containing $\tau(U)$ and $\boldsymbol{T}(E(U))$ and of G. As an analogue of Proposition 1.29 we obtain

Proposition. *If an inverse semigroup S is isomorphic to a subdirect product of a semilattice U and a group G then $\boldsymbol{T}(S)$ is isomorphic to a subdirect product of the lattice of all retract ideals of U and of the group G.*

This Proposition (with $\Omega(S)$ instead of $\boldsymbol{T}(S)$) has been obtained independently in [22].

2.26. Proposition. *Let $(S_i)_{i\in I}$ be a family of inverse semigroups with a common zero. Then $\boldsymbol{T}(\Sigma^0_{i\in I} S_i)$ is isomorphic to $\mathsf{X}_{i\in I}\, \boldsymbol{T}(S_i)$, $H \to (H \cap S_i)_{i\in I}$ being the isomorphism.*

Proof. Proceed as in the proof of Proposition 1.30. The only addition to the proof is the following argument: an ideal P of $E(\Sigma^0_{i\in I} S_i)$ is a retract ideal if and only if for every $i \in I$ $P_i = P \cap S_i$ is a retract ideal of $E(S_i)$.

This Proposition for arbitrary semigroups S_i such that $S_i^2 = S_i$ was proved in [1].

2.27. If S is a Brandt semigroup then, by Corollary 2.16, $\boldsymbol{T}(S) = \boldsymbol{C}_0(S)$. $\boldsymbol{C}_0(S)$ has been described in 1.32. Translational hulls of Brandt semigroups have been previously described in [19].

2.28. Now let S be a Clifford inverse semigroup, the notation of 1.33 being preserved. Evidently, Theorem 2.9. (i) and the results of 1.33 imply

Proposition. *Let S be a Clifford inverse semigroup with the maximal subgroups H_i, $i \in E(S)$. Then $\boldsymbol{T}(S)$ is a Clifford inverse semigroup with the maximal subgroups $\operatorname{inv\,lim} \{H_i\}_{i\in I}$ for all retract ideals I of $E(S)$. The structure homomorphisms $\varphi_{J,I}$ of $\boldsymbol{T}(S)$ are defined as follows: $I \subset J$ and $\varphi_{J,I}(H) = H \cap S_I$ for all $H \in$ $\in \operatorname{inv\,lim} \{H_i\}_{i\in J}$ and $S_I = \bigcup_{i\in I} H_i$.*

This result for $\Omega(S)$ has been obtained independently in [22].

3.1. Definition. An ideal extension $\varphi : S \to T$ is called an ideal extension *of S by Q* if the Rees quotient semigroup $T/\varphi(S)$ is isomorphic with Q.

We restrict ourselves to the case when both S and T are inverse semigroups. In Section 2 we have described all dense ideal extensions of inverse semigroups. However, not all ideal extensions are dense.

3.2. Theorem. *Let S be an inverse semigroup, U an inverse subsemigroup of* $\boldsymbol{T}(S)$, $\tau(S) \subset U$, *K an inverse semigroup with zero* 0 *and T an inverse semigroup which is a subdirect product of U and K such that* $\tau(S) \times \{0\} \subset T$. *Let* $\varphi(s) = (\tau(s), 0) \in T$ *for all* $s \in S$. *Then* $\varphi : S \to T$ *is an ideal extension of S and every ideal extension of S is equivalent to an ideal extension of the above form.*

Proof. Since $\tau(S) \times \{0\}$ is an ideal of $U \times K$, it is an ideal of T so that φ is indeed an ideal extension of S.

Now let $\psi : S \to T$ be an ideal extension of S. Let $T/\psi(S) = K$, then K is an inverse semigroup with zero. Define an equivalence relation ε_S on T as follows: $t_1 \equiv t_2(\varepsilon_S) \leftrightarrow \leftrightarrow t_1\, \psi(E(S)) = t_2\, \psi(E(S))$. Clearly, ε_S is left regular. Now $t\, \psi(E(S)) = t\, \psi(E(S))\,.\, \psi(E(S)) = \psi(E(S))\, t\, \psi(E(S)) = \psi(E(S))\, \psi(E(S))\, t = \psi(E(S))\, t$. It follows that $t_1 \equiv \equiv t_2(\varepsilon_S) \leftrightarrow \psi(E(S))\, t_1 = \psi(E(S))\, t_2$, whence ε_S is right regular. Therefore, ε_S is a congruence on T. Now $\psi(s_1) \equiv \psi(s_2)\, (\varepsilon_S) \leftrightarrow \psi(s_1)\, \psi(E(S)) = \psi(s_2)\, \psi(E(S)) \leftrightarrow \leftrightarrow \psi(s_1\, E(S)) = \psi(s_2\, E(S)) \leftrightarrow s_1\, E(S) = s_2\, E(S) \leftrightarrow s_1 = s_2$. It follows that ε_S induces the trivial congruence on $\psi(S)$. Therefore, the through homomorphism $S \to T \to \to T/\varepsilon_S$ is an ideal extension of S. Let α denote the canonical homomorphism of T onto T/ε_S. Suppose ε is a congruence on T/ε_S such that it induces the trivial congruence on $\alpha(\psi(S))$. If $\alpha(t_1) \equiv \alpha(t_2)\, (\varepsilon)$ then $\alpha(t_1)\, \alpha(\psi(i)) \equiv \alpha(t_2)\, \alpha(\psi(i))\, (\varepsilon)$ for every $i \in E(S)$. Since $\alpha(\psi(S))$ is an ideal of T/ε_S and ε is trivial on this ideal, $\alpha(t_1)\, \alpha(\psi(i)) = \alpha(t_2)\, \alpha(\psi(i))$ or $\alpha(t_1\, \psi(i)) = \alpha(t_2\, \psi(i))$, i.e. $t_1\, \psi(i) \equiv t_2\, \psi(i)\, (\varepsilon_S)$. In other words, $t_1\, \psi(i\, E(S)) = = t_1\, \psi(i)\, \psi(E(S)) = t_2\, \psi(i)\, \psi(E(S)) = t_2\, \psi(i\, E(S))$. Since this is true for every $i \in E(S)$, $t_1\, \psi(E(S)) = t_1\, \psi(E(S)\, E(S)) = t_2\, \psi(E(S)\, E(S)) = t_2\, \psi(E(S))$, i.e. $t_1 \equiv t_2(\varepsilon_S)$ and $\alpha(t_1) = \alpha(t_2)$. Thus, ε is trivial. It follows that $S \to T/\varepsilon_S$ is a dense ideal extension. Therefore it is equivalent to a dense ideal extension of the form $\tau : S \to U$ where $\tau(S) \subset U \subset \boldsymbol{T}(S)$.

Let β be a bijective isomorphism of T/ε_S onto U such that $\beta \circ \alpha \circ \psi = \tau$. If $\alpha(t_1) = = \alpha(t_2)$ and t_1, t_2 have the same image under the Rees homomorphism of T onto $T/\psi(S)$, then $t_1 = t_2$, i.e. T is isomorphic to a subdirect product of K and $T/\varepsilon_S \cong U$. Our Theorem readily follows.

Remark. It is clear from the proof of Theorem that we may suppose $\tau(S) = = \{u : (u, 0) \in T\}$.

Our Theorem follows from a known fact [23]: every ideal extension T of a semigroup S is isomorphic to a subdirect product of a dense ideal extension of S and another semigroup.

3.3. Let $\varphi : S \to T$ be an ideal extension constructed as in Theorem 3.2. Then T is an ideal extension of S by $Q = T/\varphi(S)$. Since T is a subdirect product of U and K, Q is naturally isomorphic to a subdirect product of $U/\tau(S)$ and K. In the proof of Theorem 3.2 we have seen that one may suppose $K = Q$.

3.4. Let Φ be an inverse semigroup of univalent functions acting in a set A, K an inverse semigroup of univalent functions acting in a set B, $\emptyset \in K$, $A \cap B = \emptyset$. Let U be an inverse subsemigroup of $V(\Phi)$, $\Phi \subset U$. Then, by 2.19 ,$\Phi \subset U$ is a dense ideal extension. Let Ψ be a subdirect product of U and K such that $\Phi \times \{\emptyset\} \subset \Psi$. Then we may represent each element $(\varphi, \varkappa)$ of Ψ as a univalent function $\varphi \cup \varkappa$ acting in the set $C = A \cup B$. We will consider the inverse semigroup Ψ to be a semigroup of univalent functions which is obtained under this representation. Then Φ is an ideal of Ψ, i.e. $\Phi \subset \Psi$ is an ideal extension of Φ. Conversely, every ideal extension of Φ is equivalent to the above ideal extension. In fact, every ideal extension $\Phi \to T$ is obviously equivalent to an ideal extension $\Phi \subset \Psi$ where Ψ is an inverse semigroup of univalent functions acting in a set C. Let $A = \bigcup_{\varphi \in \Phi} pr_1 \varphi$ and $B = C \setminus A$. Since Φ is an ideal of Ψ, the restriction of Ψ to A is an inverse semigroup U of univalent functions acting in A. For every $\varphi \in U$ and every $\iota \in E(\Phi)$, $\varphi \circ \iota \in \Phi$. It follows that $\varphi = \bigcup_{\iota \in E(\Phi)} \varphi \circ \iota$, therefore, Φ is a $\bigvee$-basic ideal of U. By Proposition 2.18, $\Phi \subset U$ is a dense extension. It is easy to verify that $U \subset V(\Phi)$. Let K be a restriction of Ψ to B, then K is an inverse semigroup of univalent functions acting in B and $\emptyset \in K$. For every $\psi \in \Psi$ the mapping $\psi \to (\psi|_A, \psi|_B)$ is a subdirect decomposition of Ψ in a subdirect product of U and K.

3.5. To illustrate how the preceding results work in concrete situations we consider several examples.

First of all we consider ideal extensions of Clifford inverse semigroups by Brandt semigroups.

Let S be a Clifford inverse semigroup and $\varphi : S \to T$ an ideal extension such that $T/\varphi(S)$ is a Brandt semigroup. Without loss of generality we may suppose that φ is constructed as in Theorem 3.2, i.e. T is a subdirect product of U and K, $S \times \{0\} \subset T$. Since $Q = T/\psi(S)$ is a Brandt semigroup and a subdirect product of $U/\tau(S)$ and K, both $U/\tau(S)$ and K are Brandt semigroups. Now U is a Clifford inverse semigroup so that $U/\tau(S)$ is both Brandt and Clifford. This is possible only if $U/\tau(S)$ is a group with zero adjoined. Therefore, $U = \tau(S) \cup G$ where G is a subgroup of a group inv lim H_i where $i \in I$ for some retract ideal I of $E(S)$. If I is a principal ideal then $U = \tau(S)$, whence T is isomorphic to a subdirect product of S and Q, $S \times \{0\}$ being included into the subdirect product.

Conversely, if Q is a Brandt semigroup with zero then any inverse subdirect product T of S and Q which contains $S \times \{0\}$ is an ideal extension of S by Q (the extension itself being $\varphi : S \to T$ where $\varphi(s) = (s, 0)$).

Now let I be a non-principal retract ideal of $E(S)$. Then $U/\tau(S) = G^0$ where G^0 is G with zero adjoined. We may suppose without loss of generality that $K = Q$. Then G^0 is a homomorphic image of Q. It follows that Q is a group with zero adjoined (we use here a well known and trivially verifiable fact that every congruence on a Brandt semigroup is either universal or idempotent-separating). Let $Q = H^0$ where H is a group. Then G is a homomorphic image of H and the subdirect product of G^0 and H^0 which is isomorphic to Q contains precisely two idempotents. Now $(0, 0)$ belongs to every subdirect product of G^0 and Q. If $Q = \{0\}$ then T is isomorphic to U; otherwise, precisely one of the idempotents $(0, 1), (1, 0), (1, 1)$ belongs to the subdirect product of G^0 and H^0. If this idempotent is $(0, 1)$ or $(1, 0)$ then $G = \emptyset$ or $H = \emptyset$ respectively and we arrive to a case already considered. Therefore, the idempotent is $(1, 1)$. It follows that the subdirect product of G^0 and H^0 is a subdirect product of G and H, say, $K \subset G \times H$, with $(0, 0)$ adjoined. Therefore, $T = (\tau(S) \times \{0\}) \cup K$. We have proved the following

Theorem. *Let $S \to T$ be an ideal extension of a Clifford inverse semigroup by a Brandt semigroup. Then one of the following cases holds:*

1) *T is a subdirect product of S and a Brandt semigroup with zero (possibly, a one-element Brandt semigroup) containing $S \times \{0\}$, S is mapped onto $S \times \{0\}$ under this extension;*

2) *$T = (\tau(S) \times \{0\}) \cup K$ where K is a subdirect product of a subgroup of a group* inv lim $\{H_i\}_{i\in I}$, *H_i being maximal subgroups of S and I a non-principal retract ideal of $E(S)$, and another group; in this case $S \to T$ is an ideal extension of S by a group with zero adjoined.*

3.6. Now let $\varphi : S \to T$ be an ideal extension of a Brandt semigroup S by a Clifford inverse semigroup $Q = T/\varphi(S)$. Then $T \subset U \times K$ and $Q = T/\varphi(S) \cong P \subset U/\tau(S) \times \times K$. Both $U/\tau(S)$ and K are Clifford inverse semigroups. If $U = \tau(S)$ then T is a subdirect product of S and Q containing $S \times \{0\}$, conversely, every such subdirect product provides an ideal extension of S by a Clifford inverse semigroup. Suppose now $U \neq \tau(S)$. Then $U/\tau(S)$ is a Clifford inverse semigroup if and only if for every $H \in U\backslash\tau(S)$, $HH^{-1} = H^{-1}H$. If $T(S)$ is represented as a wreath product of $\mathscr{I}_I$ and a group G as in 1.32 and 2.27, then U consists of such pairs (ϱ, f) where $pr_1\varrho = pr_2\varrho$ in case that $pr_1\varrho$ contains more than one element. Therefore, U is an inverse subsemigroup of a wreath product of an inverse semigroup Φ of univalent functions acting in the set I and of G, where Φ contains all the functions of the form $\{(i, j)\}$ for $i, j \in I$ and for every other function $\varrho \in \Phi$ $pr_1\varrho = pr_2\varrho$. Conversely, if U is an inverse subsemigroup of such a wreath product and $\tau(S) \subset U$ then any subdirect product of U

and a Clifford inverse semigroup with zero containing $\tau(S) \times \{0\}$ provides an ideal extension of S by a Clifford inverse semigroup.

3.7. Now consider ideal extensions of Brandt semigroups by Brandt semigroups. Such ideal extensions were described in [39] and, by another method, in [41, 2].

Let $\varphi : S \to T$ be an ideal extension of a Brandt semigroup S and let $Q = T/\varphi(S)$ be a Brandt semigroup with zero. Then $\tau(S) \times \{0\} \subset T \subset U \times Q$ where $\tau(S) \subset \subset U \subset T(S)$. We will use a description of $T(S)$ given in 2.27 and 1.32. Then $U/\tau(S)$ is a Brandt semigroup which is a homomorphic image of Q. If $U = \tau(S)$ then T is isomorphic to a subdirect product of S and Q and $S \times \{0\}$ is contained in this subdirect product. Conversely, any such subdirect product provides an ideal extension of S by Q.

Now let $U \neq \tau(S)$. Then $\tau : S \to U$ is an ideal extension of S by a Brandt semigroup $U/\tau(S)$. Elements of U are pairs (ϱ, f) where ϱ is a univalent function in I and $f : pr_1\varrho \to G$. It is easy to verify that (ϱ, f) is an idempotent if and only if $\varrho = \Delta_A$ for a subset $A \subset I$ and f maps A onto $\{1\}$ where 1 is the identity of G. Let (Δ_A, f_1) and (Δ_B, f_2) be idempotents of $U \backslash \tau(S)$, i.e. A and B contain more than one element each. If $A \neq B$ (i.e. the idempotents are different), then the product of these idempotents is $(\Delta_{A \cap B}, f_3)$ where f_3 maps $A \cap B$ onto $\{1\}$. If $U/\tau(S)$ is a Brandt semigroup, $(\Delta_{A \cap B}, f_3) \in \tau(S)$, i.e., $A \cap B$ contains one element at most: $|A \cap B| \leqq 1$ where $|A \cap B|$ is the cardinality of $A \cap B$.

Since $U/\tau(S)$ is 0-bisimple, there exists an element $(\varrho, f) \in U$ such that $(\varrho, f) \,.\, .\, (\varrho, f)^{-1} = (\Delta_A, f_1)$ and $(\varrho, f)^{-1} (\varrho, f) = (\Delta_B, f_2)$. It follows that $A = pr_1\varrho$ and $B = pr_2\varrho$, hence, $|A| = |B|$.

To fix a Brandt semigroup P one needs to give all idempotents of P, the structure group of P (which is isomorphic to any nontrivial maximal subgroup of P) and give mappings between different maximal subgroups of P. We are going to describe $U/\tau(S)$ and U following these lines.

Let $(A_j)_{j \in J}$ be the set of all such subsets $A_j \subset I$ that (Δ_{A_j}, f_j) is an idempotent of $U \backslash \tau(S)$ for $f_j : A_j \to \{1\}$. Then $|A_i \cap A_j| \leqq 1$ if $i \neq j$. Fix an element $0 \in J$. Let (ϱ, f) be an element of the maximal subgroup of U having the identity (Δ_{A_0}, f_0). Then $pr_1\varrho = pr_2\varrho = A_0$. It follows that this maximal subgroup of U is a subgroup of a wreath product of the symmetric group $\mathscr{S}_{A_0}$ of permutations of A_0 and G, the structure group of S. Denote this maximal subgroup of U by V. For every $j \in J$ fix an element $(\varrho_j, g_j) \in U$ such that $pr_1\varrho_j = A_j$, $pr_2\varrho_j = A_0$, g_j is an arbitrary mapping of A_j into G. Now U is determined in the unique way. In deed, let $(\varrho, f) \in U \backslash \tau(S)$. Suppose $pr_1\varrho = A_i$, $pr_2\varrho = A_j$. Then $(\varrho_i, g_i)^{-1} (\varrho, f) (\varrho_j, g_j) \in V$. It follows that $(\varrho, f) = (\varrho_i, g_i) (\bar{\varrho}, h) (\varrho_j, g_j)^{-1}$ for a uniquely determined element $(\bar{\varrho}, h) \in V$. Clearly, if U consists of $\tau(S)$ and the elements (ϱ, f) satisfying the above properties, then $U/\tau(S)$ is a Brandt semigroup. Thus, we have proved

Proposition. *Let $S \to T$ be a dense ideal extension of a Brandt semigroup S by a Brandt semigroup. Then it is equivalent to an ideal extension of the form $\tau : S \to U$ where $\tau(S) \subset U \subset T(S)$ and U has the following structure:*

a) *let $(A_j)_{j\in J}$ be a family of subsets of I (I is the index set of S) such that $|A_i \cap A_j| \leqq 1$ if $i \neq j$, $|A_i| = |A_j|$ for all $i, j \in J$;*

b) *let for every $j \in J$ ϱ_j be a bijection of A_j onto A_0 where 0 is a fixed element of J; let for every $j \in J$ f_j be a mapping of A_j into G where G is the structure group of S;*

c) *let V be a subgroup of the wreath product of $\mathcal{S}_{A_0}$, the symmetric group of permutations of A_0 and of G;*

Then U consists of $\tau(S)$ and all the elements of $T(S)$ of the form $(\varrho_i, f_i)(\varrho, f) \cdot (\varrho_j, f_j)^{-1}$ where $(\varrho, f) \in V$, i.e. $U = \tau(S) \cup RVR^{-1}$ (here $R = \{(\varrho_i, f_i)\}_{i\in J}$).

In particular, if $|A_j| \leqq 1$ for all $j \in J$ then $U = \tau(S)$.

To construct all ideal extensions of S by Q we should find subdirect products of $U/\tau(S)$ and Q which are Brandt semigroups. Suppose $u_1, u_2 \in U/\tau(S)$ and $q_1, q_2 \in Q$ are nonzero idempotents and $(u_1, q_1), (u_1, q_2), (u_2, q_1)$ belong to a subdirect product P of $U/\tau(S)$ and Q. Then $(u_1, q_1q_2), (u_1u_2, q_1) \in P$. If $u_1 \neq u_2$ then $(0, 0) < (u_1u_2, q_1) = (0, q_1) < (u_1, q_1)$; if $q_1 \neq q_2$ then $(0, 0) < (u_1, q_1q_2) = (u_1, 0) < (u_1, q_1)$. In neither case can P be a Brandt semigroup. If P is a Brandt semigroup then $u_1 = u_2$ and $q_1 = q_2$. Since P is a subdirect product of $U/\tau(S)$ and Q, $E(P)$ should be a bijection of $E(U/\tau(S))$ onto $E(Q)$. In particular, the index sets (i.e. the sets of nonzero idempotents) of $U/\tau(S)$ and Q should have the same cardinality. Since the index set of $U/\tau(S)$ is in a one-to-one correspondence with J, we may suppose $J = E(Q)\setminus\{0\}$.

Conversely, let P be a subdirect product of $U/\tau(S)$ and Q and $E(Q) = J \cup \{0\}$. Clearly, the product of two different idempotents of P is 0 in case of $E(P) = \{((\Delta_{A_j}, f_j), j)\}_{j\in J} \cup \{0\}$. For any $i, j \in J$ there exists $q \in Q$ such that $qq^{-1} = i$ and $q^{-1}q = j$. There exists $u \in U/\tau(S)$ such that $(u, q) \in P$. It follows that $(uu^{-1}, qq^{-1}) = (uu^{-1}, i) \in E(P)$ and $(u^{-1}u, q^{-1}q) = (u^{-1}u, j) \in E(P)$. Therefore, $uu^{-1} = (\Delta_{A_i}, f_i)$ and $u^{-1}u = (\Delta_{A_j}, f_j)$. Thus P is a Brandt semigroup.

We have given an outline of the proof of the following

Theorem. *Let S and Q be Brandt semigroups, let Q contain zero and let U be an inverse subsemigroup of $T(S)$ constructed as in Proposition 3.7. Let α be a bijection of J onto $E(Q)\setminus\{0\}$ and P an inverse semigroup which is a subdirect product of $U/\tau(S)$ and Q such that $E(P) = \{((\Delta_{A_j}, f_j), \alpha(j))\}_{j\in J} \cup \{(0, 0)\}$. Suppose $T = (\tau(S) \times \{0\}) \cup (P\setminus\{(0, 0)\})$. Then $\varphi(s) = (s, 0)$ is an ideal extension of S by P and P is a Brandt semigroup. Conversely, every ideal extension of S by a Brandt semigroup is equivalent to an extension constructed above or to an extension of the form $\varphi(s) = (s, 0)$, $\varphi : S \to T$ where T is a subdirect product of S and a Brandt semigroup with zero,*

$S \times \{0\} \subset T$. P is isomorphic to Q if and only if the structure group of P (which is a subdirect product of V, the structure group of $U/\tau(S)$, and of the structure group of Q) is isomorphic to a structure group of Q.

In particular, an ideal extension of S by Q which is not obtained via a subdirect product of S and Q (i.e. such that $U \neq \tau(S)$) exists if and only if $|E(Q)| + 1 \leqq |E(S)|$. We could easily obtain conditions for equivalence of two ideal extensions constructed as in the above Theorem; we omit them since the Theorem itself is of purely illustrative character: its aim is to show how results from Section 2 work. Theorem 3.7 is fairly analogous to the corresponding results from [2, 41].

4. SUBDIRECTLY IRREDUCIBLE INVERSE SEMIGROUPS

4.1. Every inverse semigroup is isomorphic to a subdirect product of subdirectly irreducible inverse semigroups. We consider here some properties of arbitrary subdirectly irreducible semigroups which will be used later in the inverse case. Necessary facts on the structure of subdirectly irreducible semigroups can be found in [27, 34].

Every subdirectly irreducible semigroup S contains a *core* K (i.e. the smallest non-null ideal).

Proposition [17]. *A core of a subdirectly irreducible semigroup is a dense ideal.*

Proof. Let K be a core of a subdirectly irreducible semigroup S and let ε be a congruence on S which induces a trivial congruence on K. Let $\varepsilon^K = (K \times K) \cup \Delta_S$ be the Rees congruence on S corresponding to K. Then $\varepsilon \cap \varepsilon^K = \Delta_S$. Since $\varepsilon^K \neq \Delta_S$ and S is subdirectly irreducible, $\varepsilon = \Delta_S$.

4.2. Proposition. *Let K be a core of a subdirectly irreducible semigroup S and let T be an oversemigroup of S containing K as a dense ideal. Then T is subdirectly irreducible.*

Proof. Let $(\varepsilon_i)_{i \in I}$ be a family of congruences on T, the intersection of the family being a trivial congruence. Let $\bar{\varepsilon}_i$ be a restriction of ε_i to S. Then the intersection of all $\bar{\varepsilon}_i$ is trivial, hence, $\bar{\varepsilon}_i = \Delta_S$ for some $i \in I$. It follows that ε_i induces a trivial congruence on K. Therefore, $\varepsilon_i = \Delta_T$ and T is subdirectly irreducible.

4.3. Proposition [17]. *If a semigroup S contains a subdirectly irreducible dense non-null ideal, then S is subdirectly irreducible.*

Proof. Let T be a subdirectly irreducible dense non-null ideal of S and let $(\varepsilon_i)_{i \in I}$ be a family of congruences on S having a trivial intersection. The intersection of restrictions of ε_i to T is trivial on T, therefore, there exists $i \in I$ such that ε_i induces a trivial congruence on T. Therefore, ε_i is trivial and S is subdirectly irreducible.

4.4. Definition. A congruence ε on a semigroup S is called *extendible to* T where $\varphi : S \to T$ is an ideal extension, if there exists a congruence η on T such that for every $s_1, s_2 \in S$ $s_1 \equiv s_2(\varepsilon) \leftrightarrow \varphi(s_1) \equiv \varphi(s_2)(\eta)$.

Proposition. *Let every congruence on a core K of a subdirectly irreducible semigroup S be extendible to S. Then K is a subdirectly irreducible semigroup.*

Proof. Let $(\varepsilon_i)_{i \in I}$ be a family of congruences on K having a trivial intersection. Let $\bar{\varepsilon}_i$ be an extension of ε_i to S and $\bar{\varepsilon} = \bigcap_{i \in I} \bar{\varepsilon}_i$. Then $\bar{\varepsilon}$ is a congruence on S which is trivial on K. Since, by Proposition 4.1, K is a dense ideal, $\bar{\varepsilon} = \Delta_S$. Since S is subdirectly irreducible, $\bar{\varepsilon}_i = \Delta_S$ for some $i \in I$. Therefore, $\varepsilon_i = \Delta_S$ and K is subdirectly irreducible.

4.5. Proposition. *Let a congruence ε on a weakly reductive semigroup S be extendible to every T such that $\varphi : S \to T$ is a dense ideal extension. Then for every $s_1, s_2 \in S$ and every $\omega \in \Omega(S)$*

$$(*) \qquad s_1 \equiv s_2(\varepsilon) \textit{ implies } s_1\omega \equiv s_2\omega(\varepsilon) \textit{ and } \omega s_1 \equiv \omega s_2(\varepsilon) .$$

If the quotient semigroup S/ε is weakly reductive, then the converse holds, i.e. (*) implies that ε is extendible for every dense extension of S.

Proof. Let ε be extendible. Then there exists a congruence $\bar{\varepsilon}$ on $\Omega(S)$ such that $s_1 \equiv s_2(\varepsilon) \leftrightarrow \pi(s_1) \equiv \pi(s_2)(\bar{\varepsilon})$. If $s_1 \equiv s_2(\varepsilon)$ then $\pi_{s_1} \equiv \pi_{s_2}(\bar{\varepsilon})$, whence $\pi_{s_1}\omega \equiv \pi_{s_2}\omega(\bar{\varepsilon})$ for every $\omega \in \Omega(S)$. Now $\pi_s\omega = \pi_{s\omega}$. It follows that $\pi_{s_1\omega} \equiv \pi_{s_2\omega}(\bar{\varepsilon})$. Therefore, $s_1\omega \equiv$ $\equiv s_2\omega(\varepsilon)$. Analogously, $\omega s_1 \equiv \omega s_2(\varepsilon)$.

Now let S/ε be weakly reductive and let (*) hold. Let $\varphi : S \to T$ be a dense ideal extension and $\alpha : T \to \Omega(S)$ an isomorphism such that $\alpha \circ \varphi = \pi$. Define a binary relation $\bar{\varepsilon}$ on $\Omega(S)$ as follows: $(\omega_1, \omega_2) \in \bar{\varepsilon}$ if and only if $s_1\omega_1 \equiv s_2\omega_2(\varepsilon)$ and $\omega_1 s_1 \equiv$ $\equiv \omega_2 s_2(\varepsilon)$ for every $s_1, s_2 \in S$, provided $s_1 \equiv s_2(\varepsilon)$. The reflexivity of $\bar{\varepsilon}$ follows from (*), the symmetricity of $\bar{\varepsilon}$ is obvious. Let $(\omega_1, \omega_2), (\omega_2, \omega_3) \in \bar{\varepsilon}$ and $s_1 \equiv s_2(\varepsilon)$. Then $s_1\omega_1 \equiv s_2\omega_2 \equiv s_2\omega_3(\varepsilon)$, analogously $\omega_1 s_1 \equiv \omega_3 s_2(\varepsilon)$, i.e. $(\omega_1, \omega_3) \in \bar{\varepsilon}$. Therefore $\bar{\varepsilon}$ is transitive, i.e. $\bar{\varepsilon}$ is an equivalence relation. Now let $\omega_1 \equiv \omega_2(\bar{\varepsilon})$ and $\omega_3 \equiv \omega_4(\bar{\varepsilon})$. If $s_1 \equiv s_2(\varepsilon)$ then $s_1(\omega_1\omega_3) = (s_1\omega_1)\,\omega_3 \equiv (s_2\omega_2)\,\omega_3 \equiv (s_2\omega_2)\,\omega_4 = s_2(\omega_2\omega_4)$, since $s_2\omega_2 \in S$. Analogously, $(\omega_1\omega_3)\,s_1 \equiv (\omega_2\omega_4)\,s_2(\varepsilon)$. Therefore, $\omega_1\omega_3 \equiv \omega_2\omega_4(\bar{\varepsilon})$, i.e. $\bar{\varepsilon}$ is a congruence.

Let $\pi_{s_1} \equiv \pi_{s_2}(\bar{\varepsilon})$, i.e. $s_3\pi_{s_1} \equiv s_4\pi_{s_2}(\varepsilon)$ and $\pi_{s_1}s_3 \equiv \pi_{s_2}s_4(\varepsilon)$ for every $s_3, s_4 \in S$ such that $s_3 \equiv s_4(\varepsilon)$. The latter formulae mean: $s_3s_1 \equiv s_4s_2(\varepsilon)$ and $s_1s_3 \equiv s_2s_4(\varepsilon)$. Let β be the canonical homomorphism of S onto S/ε and $x \in S/\varepsilon$. Then the latter formulae mean that $\beta(s_1)\,x = \beta(s_2)\,x$ and $x\,\beta(s_1) = x\,\beta(s_2)$. Since S/ε is weakly reductive, $\beta(s_1) = \beta(s_2)$, i.e. $s_1 \equiv s_2(\varepsilon)$. Therefore, $\bar{\varepsilon}$ is an extension of ε. Now define $t_1 \equiv$

$\equiv t_2(\bar{\bar{\varepsilon}}) \leftrightarrow \alpha(t_1) \equiv \alpha(t_2)(\bar{\varepsilon})$. Then $\bar{\bar{\varepsilon}}$ is a congruence on T and $s_1 \equiv s_2(\varepsilon) \leftrightarrow \varphi(s_1) \equiv \equiv \varphi(s_2)(\bar{\bar{\varepsilon}})$.

Corollary. *Let $S \to T$ be a dense ideal extension of an inverse semigroup. S Then every congruence of S is extendible to T.*

Proof. Inverse semigroups and their homomorphic images (which are inverse semigroups as well) are weakly reductive. It suffices to verify (*) for every congruence ε on an inverse semigroup S. Instead of $\Omega(S)$ we shall consider $\boldsymbol{T}(S)$ (this is possible by Theorem 2.9). Let $s_1 \equiv s_2(\varepsilon)$ and $H \in \boldsymbol{T}(S)$. Let $s_1H = \tau(s_3)$ and $s_2H = \tau(s_4)$. Thus $s_3 = s_1h_1$ and $s_4 = s_2h_2$ for some $h_1, h_2 \in H$. Therefore, $s_3 = s_1h_1 \equiv s_2h_1 \leqq \leqq s_4$ and $s_4 = s_2h_2 \equiv s_1h_2 \leqq s_3$. If γ is the canonical homomorphism of S onto S/ε then $\gamma(s_3) \leqq \gamma(s_4)$ and $\gamma(s_4) \leqq \gamma(s_3)$, i.e. $\gamma(s_3) = \gamma(s_4)$ and $s_3 \equiv s_4(\varepsilon)$. Analogously, if $Hs_1 = \tau(s_5)$ and $Hs_2 = \tau(s_6)$ then $s_5 \equiv s_6(\varepsilon)$. By Proposition 4.5, ε is extendible to T.

Remark. Corollary 4.5 is not true for arbitrary regular semigroups (consider, for example, left zero semigroups and their dense ideal extensions).

4.6. Theorem. (Cf. [40].) *An inverse semigroup is subdirectly irreducible if and only if it contains a dense ideal which is a subdirectly irreducible [0-] simple inverse semigroup.*

Proof. By Corollary 4.5, Propositions 4.1 – 4.4.

Notice that [0-] simple ideal of an inverse semigroup S is a core of S if S is subdirectly irreducible. Since ideals of ideals of inverse semigroups are ideals of the inverse semigroups themselves, a core of an inverse semigroup is a [0-] simple inverse semigroup.

Since all dense ideal extensions of inverse semigroups were described in Section 2, the problem of finding subdirectly irreducible inverse semigroups reduces to the same problem for [0-] simple inverse semigroups.

4.7. Precisely in the same way as Theorem 4.6 we could prove the following more general assertion. Let a semigroup S have a non-null ideal which is an inverse semigroup. Then S is subdirectly irreducible if and only if it has a dense subdirectly irreducible [0-] simple ideal.

Remark. Every congruence ε of an inverse semigroup S can be extended to $\boldsymbol{C}(S)$ and even to $\boldsymbol{M}(S)$. Indeed, for $H_1, H_2 \in \boldsymbol{M}(S)$ define $H_1 \equiv H_2(\bar{\varepsilon}) \leftrightarrow \varepsilon(H_1) = \varepsilon(H_2)$ Here $\varepsilon(H)$ is the union of all ε-classes having a nonempty intersection with H. Then $s_1 \equiv s_2(\varepsilon) \leftrightarrow \tau(s_1) \equiv \tau(s_2)(\bar{\varepsilon})$ and $\bar{\varepsilon}$ is the largest congruence on $\boldsymbol{M}(S)$ having this property. The same is true for $\boldsymbol{C}(S)$. The congruences $\bar{\varepsilon}$ constructed in 4.6 and 4.7 coincide on $\boldsymbol{T}(S)$.*)

4.8. Lemma. *An inverse semigroup is semisimple (i.e. all principal ideal factors of the semigroup are completely [0-] simple) if and only if it does not contain a bicyclic subsemigroup.*

Proof. If an inverse semigroup S contains a bicyclic subsemigroup $\mathscr{C}$ then all the elements of $\mathscr{C}$ are contained in the same $\mathscr{J}$-class, i.e. they generate the same principal ideal of S. Then S possesses a principal ideal factor containing $\mathscr{C}$ as a subsemigroup, i.e. this factor cannot be completely [0-] simple. Therefore, S cannot be semisimple.

Conversely, let S contain no bicyclic subsemigroups. If H is a principal ideal factor of S, it cannot contain two different comparable nonzero idempotents. Indeed, if $i, j \in H$ are nonzero idempotents and $i < j$ then $xix^{-1} = j$ and $x^{-1}jx = i$ for some $x \in S$. Then $(jxi)(jxi)^{-1} = jxiix^{-1}j = jxix^{-1}j = jjj = j$ and $(jxi)^{-1}(jxi) = ix^{-1}jjxi = ix^{-1}jxi = iii = i$, i.e. the element jxi generates a byciclic subsemigroup of S. Thus, H is a Croisot semigroup. Since H is [0-] simple, H is a Brandt semigroup. Thus, S is semisimple.

4.9. Lemma. *A Brandt semigroup is subdirectly irreducible if and only if its structure group is subdirectly irreducible, i.e. the structure group contains the smallest nontrivial invariant subgroup.*

Proof. This follows from a description of congruences on Brandt semigroups [25]. This Lemma was proved also in [40].

4.10. Lemma. *A [0-] simple inverse semigroup is a Brandt semigroup if and only if it satisfies the descending chain condition for principal right ideals.*

Proof. Every Brandt semigroup satisfies the descending chain condition for principal right ideals. Conversely, let S satisfy the descending chain condition. Then S cannot contain a bicyclic subsemigroup (since the idempotents of a bicyclic subsemigroup generate an infinite descending chain of principal right ideals). By Lemma 4.8, S is [0]- semisimple. Since S is [0-] simple, S is its own principal ideal factor, i.e. S is completely [0-] simple. Since S is an inverse semigroup, S is a Brandt semigroup.

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*) Added in the proof. After this paper had been submitted for publication there have been submitted and published two papers of N. R. Reilly: *Extensions of homomorphisms to dense extensions of semigroups,* Semigroup Forum *6* (1973), 153—170; *Extensions of congruences and homomorphisms,* Proc. Symposium on Inverse Semigroups and Their Generalizations, N. I. University, De Kalb, Ill., 1973, 140—166. In these papers Reilly has independently introduced the concept of compatible congruences similar to that of Definition 4.4 and proved Proposition and Corollary 4.5 and the maximality of $\bar{\varepsilon}$ on $\Omega(S)$.

4.11. Theorem. *An inverse semigroup S is a subdirectly irreducible inverse semigroup with a core satisfying the descending chain condition for principal right ideals if and only if S contains a $\bigvee$-basic ideal which is a Brandt semigroup with subdirectly irreducible structure group. This ideal is the smallest $\bigvee$-basic ideal of S.*

Proof. Theorem 4.11 follows from Theorem 4.6, Proposition 2.18 and Lemmas 4.9 and 4.10. Theorem 4.11 for finite S was proved in [40].

Example. Every inverse semigroup of univalent functions acting in a set A and containing all univalent functions of the form $\{(a_1, a_2)\}$ for $a_1, a_2 \in A$ is subdirectly irreducible [40].

4.12. Proposition [29]. *If $E(S)$ is a well-ordered linear semilattice than S is a Clifford inverse semigroup.*

Proof. If T is a homomorphic image of S then $E(T)$ is a homomorphic image of $E(S)$, hence $E(T)$ is well-ordered. Suppose T is subdirectly irreducible. Then T satisfies the descending chain condition for principal right ideals and the core of T is a Brandt semigroup, by Theorem 4.11. The idempotents of this Brandt semigroup are linearly ordered, therefore, the core is a group or a group with zero adjoined. It follows that S is a subdirect product of groups and groups with zeros adjoined, whence S is Clifford.

4.13. Proposition. *An inverse semigroup which possesses a dense ideal satisfying the descending chain condition for principal right ideals is isomorphic to a subdirect product of subdirectly irreducible inverse semigroups with Brandt cores.*

Proof. Let S have a dense ideal P satisfying the descending chain condition for principal right ideals. Let $(\varepsilon_i)_{i\in I}$ be a family of congruences on S such that $\bigcap_{i\in I} \varepsilon_i = \Delta_S$ and S/ε_i is subdirectly irreducible for every $i \in I$.

If ε^P is not included in ε_i, then the image of P in S/ε_i is a non-null ideal satisfying the descending chain condition for principal right ideals. Therefore, the core of S/ε_i satisfies this condition. By Theorem 4.11, S/ε_i has a Brandt core. Now let ε be the intersection of such ε_i that S/ε_i has a Brandt core. Since $\varepsilon^P \subset \varepsilon_i$ if S/ε_i has a non-Brandt core, ε induces a trivial congruence on P. Therefore, $\varepsilon = \Delta_S$, i.e. S is isomorphic to a subdirect product of subdirectly irreducible inverse semigroups with Brandt cores.

4.14. Proposition. *If an inverse semigroup has a dense ideal satisfying the identity $x^2x^{-2} = x^{-2}x^2$ then this semigroup is isomorphic to a subdirect product of subdirectly irreducible inverse semigroups with Brandt cores.*

Proof. Let S possess a dense ideal P satisfying the above identity and let $(\varepsilon_i)_{i\in I}$ be a family of congruences on S as in the proof of Proposition 4.13. We have already seen that without loss of generality we may suppose that ε^P is not included in ε_i for every $i \in I$. Therefore, S/ε_i possesses a non-null ideal satisfying the above identity. Therefore, the core of S/ε_i satisfies our identity. Such a core cannot contain a bicyclic subsemigroup. Therefore, the core is semisimple. Since it is [0-] simple, it is a Brandt semigroup.

Remark. Instead of $x^2x^{-2} = x^{-2}x^2$ we could consider any identity which bicyclic semigroups do not satisfy.

4.15. Let V be a variety of inverse semigroups defined by a system F of identities in two operations – multiplication and involution (in particular, F may consist of semigroup identities, i.e. identities where only the multiplication is used).

Suppose F contains an identity π which is not valid for a bicyclic semigroup $\mathscr{C}$. Since subdirectly irreducible inverse semigroups from V satisfy π, they cannot contain a bicyclic semigroup. Therefore, all subdirectly irreducible inverse semigroups from V possess Brandt cores. If $S \in V$ then any inverse subsemigroup of S belongs to V. Thus V contains Brandt semigroups. Every Brandt semigroup possesses a homomorphic image which is a Brandt semigroup with a trivial structure group. Therefore, V contains Brandt semigroups with trivial structure groups.

Let $\mathscr{B}$ denote a Brandt semigroup with trivial structure group and two-element index set ($\mathscr{B}$ contains precisely 5 elements: a zero, two nonzero idempotents and two non-idempotent elements). If V contains a Brandt semigroup S which is neither a group nor a group with zero, then $\mathscr{B}$ is isomorphic to a homomorphic image of a subsemigroup S, i.e. $\mathscr{B} \in V$.

Let σ be an identity which does not hold for $\mathscr{B}$. If $\sigma \in F$ then $\mathscr{B} \notin V$. It follows that all Brandt semigroups in V are either groups or groups with zero. This being the case, all subdirectly irreducible semigroups in V are groups or groups with zero (by Proposition 4.1), i.e. V is a variety of Clifford inverse semigroups. On the other hand, if $\sigma \in F$ but identites which do not hold for $\mathscr{C}$ do not follow from F, then each subdirectly irreducible semigroup in V is either a group or a group with zero, or it has a non-Brandt core. Non-Brandt cores are infinite.

We have proved the following

Proposition. *Let V be a variety of inverse semigroups. V is a variety of (not necessarily all) Clifford inverse semigroups if and only if $\mathscr{B}, \mathscr{C} \notin V$ (i.e. all inverse semigroups in V satisfy an identity which does not hold for $\mathscr{B}$ and an identity which does not hold for $\mathscr{C}$). If $\mathscr{B} \notin V$ then every finite semigroup belonging to V is a Clifford inverse semigroup. If $\mathscr{C} \notin V$ then all semigroups in V are isomorphic to subdirect products of subdirectly irreducible inverse semigroups with Brandt cores.*

Remark. The "only if" part of the first assertion follows from the fact that all Clifford inverse semigroups form a variety satisfying the law $xx^{-1} = x^{-1}x$.

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