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# COMPLEX ANALYSIS IN ONE AND SEVERAL VARIABLES

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**Abstract.** This is an expository article concerning complex analysis, in particular, several complex variables. Several subjects are discussed here to demonstrate the development and the diversity of several complex variables. Hopefully, the brief introduction to complex analysis in several variables would motivate the reader's interests to this subject.

The purpose of this article is to give a brief expository introduction to complex analysis, in particular, to several complex variables. Complex analysis differs dramatically between one and several variables. Many fundamental features change when space dimension jumps from one to greater than one. For instance, any domain $D$ on the complex plane is a domain of holomorphy, that is, there is a holomorphic function $f$ on $D$ which cannot be extended holomorphically across any boundary point of $D$. In $\mathbb{C}^n$, $n \geq 2$, this is not the case. It is easy to construct a domain $D \subset \mathbb{C}^n$, $n \geq 2$, such that any holomorphic function $f$ on $D$ extends holomorphically to a fixed strictly larger domain $D_1$ containing $D$. Another effect is that the set of singularities of a meromorphic function $g$ in $\mathbb{C}$ is always discrete, e.g., $g(z) = 1/z$ has a pole at zero and is holomorphic otherwise. Such an effect cannot happen in several variables due to the Hartogs extension theorem. As a matter of fact, every function $f$ holomorphic on $D \setminus K$, where $K$ is a compact subset of $D$ such that $D \setminus K$ is connected, extends holomorphically to $D$. Also, there does not exist an analog of the famous Riemann mapping theorem for one variable in higher-dimensional spaces. Even the open unit ball and polydisc in $\mathbb{C}^n$, $n \geq 2$, are not biholomorphically equivalent. This fundamental discovery is due to H.

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Poincaré. It indicates that the classification of domains in several variables is extremely difficult. Thus, new ideas and methods are needed for investigating these problems.

Obviously, it is not possible to survey every aspect of complex analysis in such a short article. Instead, we shall address only to several main features that set up the courses of complex analysis in one and several variables. We hope that this would motivate the reader's interests for exploring several complex variables.

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## 1. Complex Structure

Let us recall, when we first learned one complex variable, how we defined the holomorphic functions. A $C^1$-function $f(z) = u(z) + iv(z)$ is called holomorphic on the domain $D \subset \mathbb{C}$ if the derivative

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \tag{1.1}$$

exists for all $z \in D$, where $z = x + iy$. The existence of (1.1) is equivalent to the so-called Cauchy-Riemann equations:

$$u_x(z) = v_y(z) \quad \text{and} \quad u_y(z) = -v_x(z). \tag{1.2}$$

In my own opinion, it seems that this definition cannot faithfully reflect the complex structure on $\mathbb{C}$. Thus, we would like to look at this problem from a slightly different point of view. We first give the definition for a complex structure here.

**Definition 1.3.** Let $V$ be a real vector space of real dimension $2m$. A complex structure $J$ on $V$ is an endomorphism from $V$ onto $V$ such that $J^2 = -1$.

From the definition a complex structure $J$ on an even-dimensional real vector space $V$ is an isomorphism of $V$ with $J^2 = -1$. With such a structure, $V$ can be naturally converted into a complex vector space of complex dimension $m$ as follows. For $\alpha = a + ib \in \mathbb{C}$ and $x \in V$, define

$$\alpha x = (a + ib)x = ax + bJx. \tag{1.4}$$

A routine calculation shows that $(V, J)$ is indeed a vector space over $\mathbb{C}$ with multiplication defined by (1.4). Let $x_1, \cdots, x_m \in V$ be $m$ linearly independent vectors over $\mathbb{R}$. Then we claim that $x_1, Jx_1, \cdots, x_m, Jx_m$ form a basis for $V$ over $\mathbb{R}$. For if

$$a_1x_1 + b_1Jx_1 + \cdots + a_mx_m + b_mJx_m = 0, \tag{1.5}$$

where $a_1, b_1, \cdots, a_m, b_m \in \mathbb{R}$, after applying $J$ to (1.5), we would get

$$a_1Jx_1 - b_1x_1 + \cdots + a_mJx_m - b_mx_m = 0. \tag{1.6}$$

This implies

$$(a_1^2 + b_1^2)x_1 + \cdots + (a_m^2 + b_m^2)x_m = 0,$$

and hence

$$a_1 = b_1 = \cdots = a_m = b_m = 0.$$

This proves the claim. It follows immediately from the above arguments that $x_1, \cdots, x_m$ form a basis for $(V, J)$ over $\mathbb{C}$. Thus, $\dim_{\mathbb{C}}(V, J) = m$.

On the other hand, one may complexify $V$ by letting

$$\mathbb{C}V = V \otimes_{\mathbb{R}} \mathbb{C}. \tag{1.7}$$

Then $\mathbb{C}V$ is a complex vector space of complex dimension $2m$ with multiplication given by

$$\beta(x \otimes \alpha) = x \otimes (\beta\alpha),$$

where $x \in V$ and $\alpha, \beta \in \mathbb{C}$. Now the complex structure $J$ on $V$ can be extended to $\mathbb{C}V$ by

$$J(x \otimes \alpha) = (Jx) \otimes \alpha.$$

Still we have $J^2 = -1$. This means that the eigenvalues of $J$, acting on $\mathbb{C}V$, are $i$ and $-i$. Let $V^{1,0}$ and $V^{0,1}$ be the eigenspaces of $J$ corresponding to the eigenvalues $i$ and $-i$, respectively. Then $V^{1,0}$ and $V^{0,1}$ are conjugate $\mathbb{C}$-linear isomorphic with $\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = m$ and we have

$$\mathbb{C}V = V \otimes_{\mathbb{R}} \mathbb{C} = V^{1,0} \oplus V^{0,1}. \tag{1.8}$$

It is then important to observe the following:

**Lemma 1.9** *$(V, J)$ is $\mathbb{C}$-linear isomorphic to $V^{1,0}$ as complex vector spaces.*

*Proof.* We observe that if $x \otimes 1 \in V^{1,0}$, then

$$(Jx) \otimes 1 = J(x \otimes 1) = i(x \otimes 1).$$

Hence

$$x \otimes 1 = \frac{1}{2}(x \otimes 1 - (Jx) \otimes i).$$

On the other hand, for any $x \in V$, if we set

$$w = x \otimes 1 - (Jx) \otimes i,$$

then $w \in V^{1,0}$ since

$$Jw = (Jx) \otimes 1 + i(x \otimes 1) = iw.$$

Thus, a direct calculation shows that the mapping

$$\begin{aligned} \varphi : (V, J) &\to V^{1,0} \\ x \quad &\mapsto x \otimes 1 - (Jx) \otimes i \end{aligned}$$

is a $\mathbb{C}$-linear isomorphism. This proves the lemma. ■

Lemma 1.9 shows that the piece $V^{1,0}$ carries naturally the structure induced by $J$. Now we return to the analysis on $\mathbb{C}$. First, we may identify $\mathbb{C}$ with $\mathbb{R}^2$ via sending $z = x + iy$ to $(x, y)$. The tangent space $T_p(\mathbb{C})$ at the point $p$, viewed as a vector space over $\mathbb{R}$, is spanned by

$$T_p(\mathbb{C}) = \left\{ \left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p \right\}.$$

Define a complex structure $J$ on $T_p(\mathbb{C})$ by

$$J\left(\left(\frac{\partial}{\partial x}\right)_p\right) = \left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad J\left(\left(\frac{\partial}{\partial y}\right)_p\right) = -\left(\frac{\partial}{\partial x}\right)_p.$$

Then the complexified tangent space $\mathbb{C}T_p(\mathbb{C})$ can be written as

$$\mathbb{C}T_p(\mathbb{C}) = T_p^{1,0}(\mathbb{C}) \oplus T_p^{0,1}(\mathbb{C}).$$

Any vector $v_p \in T_p^{1,0}(\mathbb{C})$ is called a vector of type (1,0), and $\overline{v}_p \in T_p^{0,1}(\mathbb{C})$ is called a vector of type (0,1). It is also easily seen that $T_p^{1,0}(\mathbb{C})$ is spanned by

$$\left(\frac{\partial}{\partial z}\right)_p = \frac{1}{2}\left(\frac{\partial}{\partial x} - iJ\left(\frac{\partial}{\partial x}\right)\right)_p = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)_p.$$

By duality, $J$ also induces a complex structure $J^*$ on the cotangent space $T_p^*(\mathbb{C})$ by

$$J^*((dx)_p) = -(dy)_p \quad \text{and} \quad J^*((dy)_p) = (dx)_p.$$

As before, the complexified cotangent space $\mathbb{C}T_p^*(\mathbb{C})$ can be decomposed as

$$\mathbb{C}T_p^*(\mathbb{C}) = {T^*}_p^{1,0}(\mathbb{C}) \oplus {T^*}_p^{0,1}(\mathbb{C}).$$

Any 1-form $\omega \in {T^*}_p^{1,0}(\mathbb{C})$ is called a (1,0)-form, and $\overline{\omega} \in {T^*}_p^{0,1}(\mathbb{C})$ is called a (0,1)-form. We see easily that ${T^*}_p^{1,0}(\mathbb{C})$ $({T^*}_p^{0,1}(\mathbb{C}))$ is generated by

$$dz = dx + i dy \quad (d\overline{z} = dx - i dy).$$

The importance of ${T^*}_p^{1,0}(\mathbb{C})$ is that it reflects the structure induced from $J$. Thus, we make the following definition.

**Definition 1.10.** A $C^1$-function $f$ defined on $\mathbb{C}$ is called *holomorphic* if $df$ reflects such a structure, that is, if $df$ is sitting in ${T^*}^{1,0}(\mathbb{C})$ completely.

For any $C^1$-function $f$, we have

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \\ &= \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z}. \end{aligned}$$

Hence, $f$ is holomorphic if and only if

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{1.11}$$

If $f = u + iv$, then condition (1.11) is equivalent to the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.12}$$

From here the theory of one complex variable follows.

For higher-dimensional complex Euclidean spaces, a similar procedure can be carried out. We identify $\mathbb{C}^n$, $n \geq 2$, with $\mathbb{R}^{2n}$ via sending $z = (z_1, \cdots, z_n)$ to $(x_1, y_1, \cdots, x_n, y_n)$, where $z_j = x_j + i y_j$ for $1 \leq j \leq n$. The real tangent space $T_p(\mathbb{C}^n)$ at the point $p$ is spanned by

$$T_p(\mathbb{C}^n) = \left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_n}\right)_p \right\}.$$

Define a complex structure $J$ on $T_p(\mathbb{C}^n)$ by

$$J\left(\left(\frac{\partial}{\partial x_j}\right)_p\right) = \left(\frac{\partial}{\partial y_j}\right)_p \quad \text{and} \quad J\left(\left(\frac{\partial}{\partial y_j}\right)_p\right) = -\left(\frac{\partial}{\partial x_j}\right)_p,$$

for $1 \le j \le n$. Then the complexified tangent space $\mathbb{C}T_p(\mathbb{C}^n)$ can be decomposed as

$$\mathbb{C}T_p(\mathbb{C}^n) = T_p^{1,0}(\mathbb{C}^n) \oplus T_p^{0,1}(\mathbb{C}^n).$$

Any vector $v_p \in T_p^{1,0}(\mathbb{C}^n)$ is called a vector of type (1,0), and $\overline{v}_p \in T_p^{0,1}(\mathbb{C}^n)$ is called a vector of type (0,1). Thus, $T_p^{1,0}(\mathbb{C}^n)$ $(T_p^{0,1}(\mathbb{C}^n))$ is spanned by

$$\left(\frac{\partial}{\partial z_j}\right)_p = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)_p \quad \left(\left(\frac{\partial}{\partial \overline{z}_j}\right)_p = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)_p\right)$$

for $1 \le j \le n$. By duality, $J$ also induces a complex structure $J^*$ on the cotangent space $T_p^*(\mathbb{C}^n)$ by

$$J^*((dx_j)_p) = -(dy_j)_p \quad \text{and} \quad J^*((dy_j)_p) = (dx_j)_p$$

for $1 \le j \le n$. As before, the complexified cotangent space $\mathbb{C}T_p^*(\mathbb{C}^n)$ can be decomposed as

$$\mathbb{C}T_p^*(\mathbb{C}^n) = {T^*}_p^{1,0}(\mathbb{C}^n) \oplus {T^*}_p^{0,1}(\mathbb{C}^n).$$

Any 1-form $\omega \in {T^*}_p^{1,0}(\mathbb{C}^n)$ is called a (1,0)-form, and $\overline{\omega} \in {T^*}_p^{0,1}(\mathbb{C}^n)$ is called a (0,1)-form. It follows that ${T^*}_p^{1,0}(\mathbb{C}^n)$ $({T^*}_p^{0,1}(\mathbb{C}^n))$ is generated by

$$dz_j = dx_j + i dy_j \quad (d\overline{z}_j = dx_j - i dy_j)$$

for $1 \le j \le n$.

As before, we shall call a $C^1$-function $f(z)$ defined on a domain $D$ in $\mathbb{C}^n$ *holomorphic* if $df \in {T^*}^{1,0}(D)$. Since for any $C^1$-function $f$ we have

$$df = \frac{\partial f}{\partial z_1} dz_1 + \cdots + \frac{\partial f}{\partial z_n} dz_n + \frac{\partial f}{\partial \overline{z}_1} d\overline{z}_1 + \cdots + \frac{\partial f}{\partial \overline{z}_n} d\overline{z}_n,$$

thus $f$ is holomorphic if and only if

$$\frac{\partial f}{\partial \overline{z}_j} = 0 \quad \text{for} \quad 1 \le j \le n. \tag{1.13}$$

In other words, $f$ is holomorphic if and only if $f$ is holomorphic in each variable. The set of holomorphic functions on $D$ will be denoted by $\mathcal{O}(D)$.

## 2. Domains of Holomorphy

In this section we shall discuss the domains of holomorphy. Let $D$ be a domain in $\mathbb{C}^n$, $n \ge 1$. We first make the following definition.

**Definition 2.1.** $D$ is called a domain of holomorphy if there is a holomorphic function $f$ defined on $D$ that is singular at every boundary point, that is, $f$ cannot be extended holomorphically across any boundary point.

In one complex variable, every domain is a domain of holomorphy. One way to see this fact is as follows. Let $D$ be a domain on the complex plane, and let $p \in bD$ be a boundary point of $D$. Then $g(z) = 1/(z-p)$ is a holomorphic function on $D$, but singular at $p$. To construct a holomorphic function $f$ on $D$ that is singular at every boundary point, we use the following fact.

**Theorem 2.2.** *Let $K$ be a compact subset of $D$ in $\mathbb{C}$ and let $z_0 \in D \setminus K$. If the component of $D \setminus K$ that contains $z_0$ is not relatively compact in $D$, then there is a holomorphic function $h(z)$ on $D$ such that $|h(z_0)| > \sup_{z\in K}|h(z)|$.*

The proof of Theorem 2.2 uses the Runge approximation theorem. Actually, we can make $|h(z_0)|$ arbitrarily large and $\sup_{z\in K}|h(z)|$ arbitrarily small. For instance, if we set $m = (|h(z_0)| + \sup_{z\in K}|h(z)|)/2$, then $(h/m)^l$, for large $l \in \mathbb{N}$, will do the job.

Now, let $\mathcal{P}$ be the set containing all points in $D$ with rational coordinates. Clearly, $\mathcal{P}$ is countable and dense in $D$. Let $\{\zeta_i\}_{i=1}^{\infty}$ be a sequence of points in $D$ such that every point belonging to $\mathcal{P}$ appears infinitely many times in the sequence. Next, we exhaust $D$ by a sequence of increasingly compact subsets $\{K_j\}_{j=1}^{\infty}$ of $D$ defined as follows:

$$K_j = \{z \in D \mid \operatorname{dist}(z, D^c) \geq j^{-1}\} \cap \overline{B(0;j)}, \quad j \in \mathbb{N},$$

where $\operatorname{dist}(z, D^c)$ denotes the distance from $z$ to the complement of $D$ and $B(0;j) = \{z \in \mathbb{C} |\ |z| < j\}$. Hence, we have $K_j \subset \overset{\circ}{K}_{j+1}$, where $\overset{\circ}{K}_{j+1}$ is the interior of $K_{j+1}$. For each $i$, denote by $B_{\zeta_i}$ the largest disc centered at $\zeta_i$ and contained in $D$. Then, inductively choose a subseqence $\{K_{n_j}\}$ of $\{K_j\}$ such that for each $j$, pick a $z_j \in (B_{\zeta_j} \setminus K_{n_j}) \cap \overset{\circ}{K}_{n_{j+1}}$, and a $f_j(z) \in \mathcal{O}(D)$ satisfying

$$|f_j(z)| < \frac{1}{2^j}, \quad z \in K_{n_j},$$

and

$$|f_j(z_j)| \geq \sum_{i=1}^{j-1} |f_i(z_j)| + j + 1.$$

It follows that

$$h(z) = \sum_{j=1}^{\infty} f_j(z)$$

defines a holomorphic function on $D$ and that

$$|h(z_j)| \geq |f_j(z_j)| - \sum_{i=1}^{j-1} |f_i(z_j)| - \sum_{i=j+1}^{\infty} |f_i(z_j)| \geq j,$$

which implies $h(z)$ is singular at every boundary point of $D$. For if $h(z)$ extends holomorphically across some boundary point, then $h(z)$ would be bounded on $\overline{B}_{\zeta_i}$ for some $\zeta_i$. Obviously, it contradicts the construction of $h$.

We now move to higher-dimensional Euclidean spaces. We see that the phenomenon is totally different. Let us start with the following simple example in $\mathbb{C}^2$. We consider the domain $D$ defined by

$$D = \left\{(z_1, z_2) \in \mathbb{C}^2 \,\middle|\, \frac{1}{2} < |z_2| < 1 \text{ and } |z_1| < 1\right\} \cup \left\{(z_1, z_2) \in \mathbb{C}^2 \,\middle|\, |z_2| \leq \frac{1}{2} \text{ and } |z_1| < \frac{1}{2}\right\}.$$

For any $f \in \mathcal{O}(D)$, set

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z_1, w)}{w - z_2}\, dw,$$

where $\Gamma = \{w \in \mathbb{C} |\ |w| = 3/4\}$. Then it is not hard to see that $F|_D = f$ on $D$ and $F \in \mathcal{O}(D_1)$, where $D_1 = \{(z_1, z_2) |\ |z_1| < 1 \text{ and } |z_2| < 1\}$. This implies that $D$ is not a domain of holomorphy since any holomorphic function $f$ on $D$ can be holomorphically extended to a strictly larger set $D_1$ via applying Cauchy integral formula in $z_2$. It also indicates that the analog of Theorem 2.2 in higher-dimensional case in general need not hold.

Thus, it becomes fundamental to determine whether a given domain $D$ is a domain of holomorphy or not. When $n = 1$, the key in the above construction of a holomorphic function singular at every boundary point is Theorem 2.2. We observe that one of the crucial hypotheses stated in Theorem 2.2 is to exclude the points sitting in the components, which are relatively compact in $D$, of the complement of $K$ due to the maximum modulus principle. Technically, to avoid such a difficulty one may enlarge $K$ by considering the holomorphically convex hull of $K$. Define

(2.4) $$\widehat{K}_D = \{z \in D |\ |f(z)| \leq \sup{}_K |f| \text{ for all } f \in \mathcal{O}(D)\}.$$

From Definition 2.4, it is clear that if $z \in D \setminus \widehat{K}_D$, then there is $f \in \mathcal{O}(D)$ such that $|f(z)| > \sup_{\widehat{K}_D} |f|$. This is exactly the property required in Theorem 2.2. If $K = \widehat{K}_D$, we shall call $K$ holomorphically convex.

For $n = 1$, $\widehat{K}_D = K \cup_j K_j$, where $K_j$ is a component, relatively compact in $D$, of the complement of $K$ in $\mathbb{C}$. Obviously, if $K$ is a compact subset of $D$, so is $\widehat{K}_D$. When $n > 1$, $\widehat{K}_D$ is still a closed subset of $D$ if $K$ is a compact subset of $D$. But $\widehat{K}_D$ in general need not be compact in $D$. Again, we use the domain $D$ defined by (2.3) to illustrate this effect. Let $K = \{(3/4, (3/4)e^{i\theta}); \theta \in [0, 2\pi]\}$ be a compact subset of $D$. Since any function holomorphic in $D$ can be extended holomorphically to $D_1$, it is easily seen by the maximum modulus principle that

$$\widehat{K}_D = \left\{ \left( \frac{3}{4}, z_2 \right) \middle| \ \frac{1}{2} < |z_2| \leq \frac{3}{4} \right\},$$

which is clearly not relatively compact in $D$. Hence, we make the following definition.

**Definition 2.5.** Let $D$ be a domain in $\mathbb{C}^n$. $D$ is called holomorphically convex if $\widehat{K}_D$ is relatively compact in $D$ for any compact subset $K$ of $D$.

Since holomorphically convex hull of a compact set is always contained in the geometrically convex hull of this set, it follows that we have

**Lemma 2.6.** *Any convex domain is holomorphically convex.*

It turns out that holomorphic convexity is the right condition for characterizing domains of holomorphy in $\mathbb{C}^n$.

**Theorem 2.7.** *Let $D$ be a domain in $\mathbb{C}^n$, $n \geq 1$. Then the following statements are equivalent*:

(1) *$D$ is a domain of holomorphy.*

(2) $\operatorname{dist}(K, D^c) = \operatorname{dist}(\widehat{K}_D, D^c)$ *for every compact subset $K$ in $D$, where* $\operatorname{dist}(K, D^c)$ *denotes the distance between $K$ and $D^c = \mathbb{C}^n \setminus D$.*

(3) *$D$ is holomorphically convex.*

Theorem 2.7 is due to H. Cartan and P. Thullen [14].

Thus, any convex domain is a domain of holomorphy according to Theorem 2.7. Before proving Theorem 2.7 we first introduce some terminologies. Denote by $P(a;r) = \prod_{j=1}^{n} B(a_j;r_j)$ a polydisc in $\mathbb{C}^n$, $n \geq 2$, centered at $a = (a_1, \cdots, a_n)$ with multiradii $r = (r_1, \cdots, r_n)$, $r_j > 0$. Then we have the following Cauchy integral formula on polydiscs.

**Theorem 2.8.** *If $f \in \mathcal{O}(P(a;r)) \cap C(\overline{P(a;r)})$, then*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma_n} \cdots \int_{\Gamma_1} \frac{f(\zeta_1, \cdots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n$$

*for* $z \in P(a;r)$, *where* $\Gamma_j = \{\zeta_j \in \mathbb{C} |\, |\zeta_j - a_j| = r_j\}$, $1 \leq j \leq n$.

The proof is a direct application of the Cauchy integral formula in one variable. Theorem 2.8 also indicates that power series representation and the identity theorem for holomorphic functions in several variables are valid. Now we return to the proof of Theorem 2.7.

*Proof of Theorem* 2.7. (2) $\Rightarrow$ (3) is obvious. The proof of (3) $\Rightarrow$ (1) is similar to the one that we adopted for $n = 1$ with compact subsets and discs being replaced by holomorphically convex compact subsets and polydiscs respectively. This is possible since by hypothesis the domain is holomorphically convex. Thus, we need to show (1) $\Rightarrow$ (2).

Let $P(0;r)$ be a polydisc centered at zero with multiradii $r = (r_1, \cdots, r_n)$. For each $z \in D$, we set

$$\delta_r(z) = \sup\{\lambda > 0 |\, \{z\} + \lambda P(0;r) \subset D\}.$$

To prove (1) $\Rightarrow$ (2), we first show:

**Lemma 2.9.** *Let $K$ be a compact subset of $D$, and let $f \in \mathcal{O}(D)$. Suppose that*

$$|f(z)| \leq \delta_r(z) \quad \text{for } z \in K.$$

*Let $\zeta$ be a fixed point in $\widehat{K}_D$. Then any $g \in \mathcal{O}(D)$ extends holomorphically to $D \cup (\{\zeta\} + |f(\zeta)|P(0;r))$.*

*Proof.* For each $0 < t < 1$, the union of the polydiscs centered at $z \in K$,

$$K_t = \bigcup_{z \in K} (\{z\} + t|f(z)|\overline{P(0;r)}),$$

is a compact subset of $D$. Hence, there exists $M_t > 0$ such that $|g(z)| \leq M_t$ on $K_t$. Using Cauchy's estimates of $g$, we obtain

$$\frac{|\frac{\partial^\alpha g}{\partial z^\alpha}(z)| t^{|\alpha|} |f(z)|^{|\alpha|} r^\alpha}{\alpha!} \leq M_t \tag{2.10}$$

for $z \in K$ and all multiindices $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $\alpha_j \in \{0\} \cup \mathbb{N}$. Here, by definition, $\partial^\alpha g / \partial z^\alpha = \partial^{|\alpha|} g / \partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}$ and $r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}$. Since $(\partial^\alpha g / \partial z^\alpha)(z) f(z)^{|\alpha|}$ is holomorphic on $D$, by definition, (2.10) also holds for $z \in \widehat{K}_D$. Letting $t$ tend to one, we see that $g(z)$ extends holomorphically to $D \cup (\{\zeta\} + |f(\zeta)|P(0;r))$. This proves the lemma. ∎

Now, we write

$$\begin{aligned}\operatorname{dist}(z, D^c) &= \sup\{r > 0 \mid \ z + aw \in D \text{ for all } w \in \mathbb{C}^n,\ |w| \le 1 \text{ and}\\ &\qquad a \in \mathbb{C},\ |a| < r\}\\ &= \inf_{|w|\le 1} d_w(z),\end{aligned}$$

where

$$d_w(z) = \sup\{r > 0 \mid \ z + aw \in D \text{ for all } \ a \in \mathbb{C},\ |a| < r\}.$$

Fix a $w$, and we may assume that $w = (1, 0, \cdots, 0)$. Denote by $P_j = P(0; r(j))$ the polydisc with multiradii $r(j) = (1, 1/j, \cdots, 1/j)$ for $j \in \mathbb{N}$. Then it is easily seen that

$$\lim_{j\to\infty} \delta_{r(j)}(z) = d_w(z).$$

Thus, given $\epsilon > 0$, if $j$ is sufficiently large, we have

$$\operatorname{dist}(K, D^c) \le (1+\epsilon)\delta_{r(j)}(z), \quad z \in K. \tag{2.11}$$

We let $f(z) = \operatorname{dist}(K, D^c)/(1+\epsilon)$ be the constant function. Since $D$ is a domain of holomorphy, using estimate (2.11), Lemma 2.9 shows that

$$\operatorname{dist}(K, D^c) \le (1+\epsilon)\delta_{r(j)}(\zeta) \le (1+\epsilon)d_w(\zeta) \quad \text{for all } \zeta \in \widehat{K}_D.$$

Letting $\epsilon$ tend to zero, we get

$$\begin{aligned}\operatorname{dist}(K, D^c) &\le \inf_{\zeta\in\widehat{K}_D} \big(\inf_{|w|\le 1} \ d_w(\zeta)\big)\\ &= \inf_{\zeta\in\widehat{K}_D} \ \operatorname{dist}(\zeta, D^c)\\ &= \operatorname{dist}(\widehat{K}_D, D^c).\end{aligned}$$

Since $\operatorname{dist}(\widehat{K}_D, D^c) \le \operatorname{dist}(K, D^c)$ is obvious, this proves (1) $\Rightarrow$ (2) and hence Theorem 2.7. ∎

It is now clear that, in order to be a domain of holomorphy, $D$ must be holomorphically convex. However, such a condition is difficult to verify. Thus, we need to develop other equivalent conditions that can be actually computed. When the domain $D \subset \mathbb{C}^n$, $n \ge 2$, has smooth boundary, we also make the following definition.

**Definition 2.12.** Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a smooth bounded domain, and let $r$ be a smooth defining function for $D$. $D$ is said to be (Levi) pseudoconvex (or strongly Levi pseudoconvex) if

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(z) a_j \overline{a}_k \geq 0 \quad (\text{or } > 0) \tag{2.13}$$

for any $z \in bD$ and $a = (a_1, \cdots, a_n) \in \mathbb{C}^n$ with $\sum_{j=1}^{n} a_j (\partial r / \partial z_j)(z) = 0$.

By a smooth defining function for $D$ we mean $r : U \to \mathbb{R}$ a smooth function on some open neighborhood $U$ of $\overline{D}$ such that $D = \{z \in U |\ r(z) < 0\}$ and $bD = \{z \in U |\ r(z) = 0\}$, the boundary of $D$, and that $|dr| \neq 0$ on $bD$. The Hermitian form (2.13) is called the Levi form of $D$. Thus, the domain $D$ is pseudoconvex (or strongly pseudoconvex) if the Levi form is positive semi-definite (or positive definite) at every boundary point when applied to tangential type (1,0) vector fields on the boundary. It can also be easily verified that the semi-definiteness or definiteness of the Levi form is independent of the choice of the defining function for $D$.

The concept of pseudoconvexity is fundamental in several complex variables. It can be shown that if the domain $D$ has smooth boundary, then $D$ is a domain of holomorphy if and only if $D$ is pseudoconvex. Condition (2.13) is computable. The Levi form can be computed easily to determine if a domain is a domain of holomorphy. However, we shall not get into the details of the proof here.

For materials of this section the reader is referred to the books [23], [37], [46] and [48].

## 3. Hartogs Extension Theorem

Another peculiar phenomenon that occurs in several complex variables is the so-called Hartogs extension theorem. In one variable there are meromorphic functions that have singularities at certain discrete set of a domain. For instance, $g(z) = 1/z$ has a singularity at zero and is holomorphic otherwise. Such a phenomenon does not occur in several variables. As a matter of fact, if $D$ is a domain in $\mathbb{C}^n$, $n > 1$, and $K$ is a compact subset of $D$ such that $D \setminus K$ is connected, then every $f \in \mathcal{O}(D \setminus K)$ extends holomorphically to $D$. This also implies that $D \setminus K$ cannot be a domain of holomorphy.

To exploit this phenomenon, we start with the Cauchy integral formula in $\mathbb{C}$.

**Lemma 3.1.** *Let $D$ be a bounded domain in $\mathbb{C}$ with $C^1$ boundary. If $f \in C^1(\overline{D})$, we have*

$$f(z) = \frac{1}{2\pi i}\left(\int_{bD} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \iint_D \frac{\frac{\partial f}{\partial \overline{\zeta}}}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}\right) \tag{3.2}$$

*for any $z \in D$.*

The proof is an easy consequence of Stokes' theorem. In what follows we shall assume that $D$ is a bounded domain in $\mathbb{C}$ with smooth boundary such that $\mathbb{C} \setminus D$ is connected. For any $f \in C^k(\overline{D})$, $k \in \mathbb{N}$, set

$$u(z) = \frac{1}{2\pi i}\iint_D \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}. \tag{3.3}$$

Then, we have the following solvability theorem for the Cauchy-Riemann operator in $D$.

**Theorem 3.4.** *Let $D$ be as above and $f$, $u$ be given as in* (3.3). *Then we have*

(1) *$u \in C^k(D)$ and $\partial u/\partial \overline{z} = f$ in $D$, and*
(2) *$u$ is supported in $\overline{D}$ if and only if*

$$\iint_D f(\zeta)\zeta^m\, d\zeta \wedge d\overline{\zeta} = 0 \quad \text{for } m \in \{0\} \cup \mathbb{N}. \tag{3.5}$$

*Proof.* (1) follows immediately from Lemma 3.1. To prove (2), we first observe that $u$ is continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \overline{D}$. Now assume that (3.5) holds. If $z$ satisfies $|z| > |\zeta|$ for all $\zeta \in \overline{D}$, we have

$$\begin{aligned} u(z) &= \frac{1}{2\pi i}\iint_D \frac{f(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta} \\ &= \frac{-1}{2\pi i}\sum_{m=0}^{\infty}\left(\iint_D f(\zeta)\zeta^m\, d\zeta \wedge d\overline{\zeta}\right) z^{-m-1} \\ &= 0. \end{aligned}$$

Since $u(z)$ is holomorphic on $\mathbb{C} \setminus \overline{D}$, the identity theorem shows that $u(z) \equiv 0$ for all $z \in \mathbb{C} \setminus D$.

Conversely, if $u$ is supported on $\overline{D}$, by reversing the above arguments for $z$ outside a large disc centered at the origin, it is easy to see that (3.5) must hold. This proves the theorem. ∎

Thus, it is in general not possible to get a solution of compact support to the Cauchy-Riemann equation in one variable even when $f$ has compact support. However, this is not the case in several variables as shown in the next theorem. There always exists a solution with compact support to the Cauchy-Riemann equation for a given form with compact support.

We consider the inhomogeneous Cauchy-Riemann equations in $\mathbb{C}^n$, $n \geq 2$,

$$\overline{\partial} u = \sum_{j=1}^{n} \frac{\partial u}{\partial \overline{z}_j} d\overline{z}_j = f, \tag{3.6}$$

where $f$ is a (0,1)-form of class $C^k$ with $k \geq 1$. Write $f$ as $f = \sum_{j=1}^{n} f_j d\overline{z}_j$. Since $\overline{\partial}^2 = \overline{\partial} \circ \overline{\partial} = 0$, a necessary condition for solving the $\overline{\partial}$-equation is

$$\overline{\partial} f = \sum_{j=1}^{n} \overline{\partial} f_j \wedge d\overline{z}_j = 0. \tag{3.7}$$

More explicitly, the equation (3.6) is overdetermined. In order to solve (3.6) for some function $u$, it is necessary from (3.7) that the $f_i$'s satisfy the following compatibility conditions:

$$\frac{\partial f_j}{\partial \overline{z}_k} = \frac{\partial f_k}{\partial \overline{z}_j}, \tag{3.8}$$

for all $1 \leq j < k \leq n$.

Then we have the following theorem:

**Theorem 3.9.** *Let $f_j \in C_0^k(\mathbb{C}^n)$, $k \in \mathbb{N}$, $n \geq 2$ and $1 \leq j \leq n$, such that (3.8) is satisfied. Then there is a function $u \in C_0^k(\mathbb{C}^n)$ satisfying (3.6). In particular, $u$ vanishes on the unbounded component of $\mathbb{C}^n \setminus (\cup_j \operatorname{supp} f_j)$.*

*Proof.* Set

$$\begin{aligned} u(z) &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f_1(\zeta, z_2, \cdots, z_n)}{\zeta - z_1} \, d\zeta \wedge d\overline{\zeta} \\ &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{f_1(\zeta + z_1, z_2, \cdots, z_n)}{\zeta} \, d\zeta \wedge d\overline{\zeta}. \end{aligned}$$

It is easily seen that $u \in C^k(\mathbb{C}^n)$, $k \in \mathbb{N}$, from differentiation under the integral sign. We also have $u(z) = 0$ when $|z_2| + \cdots + |z_n|$ is sufficiently large, since $f$ vanishes on the set.

By Theorem 3.4, we have

$$\frac{\partial u}{\partial \overline{z}_1} = f_1(z).$$

For $j > 1$, using the compatibility condition (3.8), we obtain

$$\begin{aligned}\frac{\partial u}{\partial \bar{z}_j} &= \frac{1}{2\pi i}\iint_{\mathbb{C}} \frac{\frac{\partial f_j}{\partial \bar{\zeta}}(\zeta, z_2, \cdots, z_n)}{\zeta - z_1}\, d\zeta \wedge d\bar{\zeta} \\ &= f_j(z).\end{aligned}$$

Hence, $u(z)$ is a solution to the $\bar{\partial}$-equation (3.6). In particular, $u$ is holomorphic on the unbounded component of the complement of the support of $f$. Since $u(z) = 0$ when $|z_2| + \cdots + |z_n|$ is sufficiently large, we see from the identity theorem for holomorphic functions that $u$ must be zero on the unbounded component of the complement of the support of $f$. This completes the proof of the theorem. ∎

Using Theorem 3.9, we can easily prove the Hartogs extension theorem.

**Theorem 3.10 (Hartogs).** *Let $D$ be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let $K$ be a compact subset of $D$ so that $D \setminus K$ is connected. Then any holomorphic function $f$ defined on $D \setminus K$ can be extended holomorphically to $D$.*

*Proof.* Choose a cut-off function $\zeta \in C_0^\infty(D)$ such that $\zeta = 1$ in some open neighborhood of $K$. Then $-f(\bar{\partial}\zeta) \in C^\infty_{(0,1)}(\mathbb{C}^n)$ satisfies the compatibility conditions (3.8), and it has compact support. By Theorem 3.9, there is a $u \in C_0^\infty(\mathbb{C}^n)$ such that

$$\bar{\partial} u = -f\bar{\partial}\zeta$$

and $u = 0$ in some open neighborhood of $\mathbb{C}^n \setminus D$. Then, it is easily seen that

$$F = (1 - \zeta)f - u$$

is the desired holomorphic extension of $f$. ∎

Theorem 3.10 implies that the set of singularities of a meromorphic function cannot be relatively compact in the domain in several complex variables. Theorem 3.10 was proved by F. Hartogs [34]. The proof of Theorem 3.10 we present here is essentially based on Theorem 3.9, the existence of compactly supported solutions to the Cauchy-Riemann equations; see [30].

For further study of the materials of this section, the reader is referred to the books [23], [37], [46] and [48].

## 4. Classification of Domains

In complex analysis, classification of domains is always a fundamental problem. When $n = 1$, the problem is completely understood when the domain is simply connected due to the following famous Riemann mapping theorem.

**Theorem 4.1 (Riemann Mapping Theorem).** *Let $D$ be a proper subdomain of $\mathbb{C}$. If $D$ is simply-connected, then $D$ is biholomorphically equivalent to the unit disc $\triangle = \{z \in \mathbb{C} \mid |z| < 1\}$.*

It is remarkable to note that the classification problem of simply connected domains in one variable can be reduced to a topological condition of the domains. Thus, we are curious about what will happen in several variables. In 1907, H. Poincaré discovered that the unit ball $B_n$ and $\triangle^n$ cannot be biholomorphically equivalent to each other in $\mathbb{C}^n$, $n \geq 2$. Here $B_n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\}$ is the open unit ball and $\triangle^n$ is the Cartesian product of $n$ copies of $\triangle$ in $\mathbb{C}^n$. This discovery reveals that the classification problem in several variables is quite complicated. In fact, at present it is still far from being understood. So we first present a proof to this nonequivalence theorem.

A holomorphic mapping $f$ from $D_1$ into $D_2$ in $\mathbb{C}^n$ is called a biholomorphism if $f$ is one-to-one and onto. In this case, $f^{-1}$ is also a one-to-one holomorphic mapping from $D_2$ onto $D_1$.

**Theorem 4.2 (Poincaré).** *There exists no biholomorphic map*

$$f : \triangle^n \to B_n \quad for\ n \geq 2.$$

*Proof.* We shall assume that $n = 2$. The proof is the same for $n > 2$. Suppose that $f = (f_1, f_2) : \triangle^2 \to B_2$ is a biholomorphism. Let $(z, w)$ be the coordinates in $\mathbb{C}^2$. For any point sequence $\{z_j\}$ in $\triangle$ with $|z_j| \to 1$ as $j \to \infty$, the sequence $g_j(w) = f(z_j, w) : \triangle \to B_2$ is uniformly bounded. Hence, by Montel's theorem, there is a subsequence, still denoted by $g_j(w)$, that converges uniformly on compact subsets of $\triangle$ to a holomorphic map $G(w) = (G_1(w), G_2(w)) : \triangle \to \overline{B}_2$. Since $f$ is a biholomorphism, we must have $|G(w)| = 1$ for all $w \in \triangle$. Hence, $|G'(w)| = 0$ for all $w \in \triangle$, which implies $G'(w) \equiv 0$ on $\triangle$. It follows that

$$\lim_{j \to \infty} f_w(z_j, w) = G'(w) \equiv 0. \tag{4.3}$$

Equation (4.3) implies that for each fixed $w \in \triangle$, $f_w(z, w)$, when viewed as a function of $z$ alone, is continuous up to the boundary with boundary value identically equal to zero. Therefore, by the maximum modulus principle we get

$$f_w(z, w) \equiv 0 \quad \text{for all } (z, w) \in \triangle^2.$$

This implies that $f$ is independent of $w$, contradicting the fact that $f$ is a biholomorphic map. This completes the proof of the theorem. ■

The proof we presented here is based on the idea of R. Remmert and K. Stein [49], not the original one found by H. Poincaré. See also the books by R. Narasimhan [47] and R. M. Range [48]. Thus, the main task of this section is to discuss how one approaches the classification problem in several variables during the last few decades.

Note first that the boundary of a smooth bounded domain $D$ in $\mathbb{C}$ is a finite union of smooth closed curves. A curve in general does not carry any information induced from the ambient complex structure. Thus, it is of less use to study the boundary geometry in treating the classification problem when $n = 1$. In several variables the situation is completely different as we see now.

Let $M$ be a smooth hypersurface in $\mathbb{C}^n$, $n \geq 2$. Then $M$ is a smooth submanifold of real dimension $2n - 1$ greater than or equal to three. Let $\mathbb{C}T(M)$ be the complexified tangent bundle on $M$. In $\mathbb{C}T(M)$, there is a natural subbundle $T^{1,0}(M)$ induced from the complex structure of the ambient space, that is, $T^{1,0}(M) = \mathbb{C}T(M) \cap T^{1,0}(\mathbb{C}^n)$. Hence, $\dim_{\mathbb{C}} T^{1,0}(M) = n - 1$ and

$$\mathbb{C}T(M) = T^{1,0}(M) \oplus T^{0,1}(M) \oplus E,$$

where $T^{0,1}(M) = \overline{T^{1,0}(M)}$ and $E$ is a line bundle.

**Definition 4.4.** The subbundle $T^{1,0}(M)$ is called the Cauchy-Riemann structure on $M$, and we call $(M, T^{1,0}(M))$ a $CR$ manifold. Any smooth section of $T^{1,0}(M)$ ($T^{0,1}(M)$) is called a type $(1,0)$ ($(0,1)$) vector field on $M$.

See also [13] and [23]. With this $CR$ structure on $M$ one can study the geometry and calculate the invariants resulting from it. For instance, the number of positive eigenvalues of the Levi form associated with the boundaries of pseudoconvex domains is a $CR$ invariant. The geometry on a $CR$ manifold is one of the main research subjects in several complex variables which has been intensively studied by many authors. The paper by S. S. Chern and J. Moser [24] is an excellent reference for dealing with the $CR$ invariants, a subject that is beyond the scope of this article.

We make the following definitions.

**Definition 4.5.** Let $M$ and $N$ be two smooth hypersurfaces in $\mathbb{C}^n$, $n \geq 2$, and let $\varphi : M \to N$ be a smooth mapping. $\varphi$ is said to be a $CR$ mapping if $\varphi_* T^{1,0}(M) \subset T^{1,0}(N)$. If $\varphi$ is a diffeomorphic $CR$ mapping, we say that $M$ is $CR$ diffeomorphic to $N$.

**Definition 4.6.** A $C^1$ function $f$ on $M$ is called a $CR$ function if $\overline{L}f = 0$ for any type $(0,1)$ vector field $\overline{L}$ on $M$.

Thus, if $f \in C^1(\overline{D}) \cap \mathcal{O}(D)$, where $D$ is a domain in $\mathbb{C}^n$, $n \geq 2$, with smooth boundary, the restriction of $f$ to the boundary is automatically a $CR$ function on $bD$. We also have the following immediate consequence.

**Corollary 4.7.** *Let $\varphi : M \to N$ be a smooth mapping between two smooth hypersurfaces in $\mathbb{C}^n$, $n \geq 2$. Then, $\varphi$ is a $CR$ mapping if and only if $\varphi_j$, $1 \leq j \leq n$, is a $CR$ function on $M$, where $\varphi = (\varphi_1, \cdots, \varphi_n)$.*

So far we have done a very brief introduction to the $CR$ manifolds. From here a natural question arises. Since the boundaries carry rich information of the $CR$ geometry, why not use the various $CR$ invariants naturally attached to the boundaries to classify domains? If this is the case, then the investigation of the $CR$ invariants attached to the boundaries will give us a method to tackle the classification problem. It sounds reasonable to do so, provided that one can make clear first what the connection between $CR$ diffeomorphisms of the boundaries and biholomorphisms of the domains is. The remainder of this section is thus devoted to explain this scheme.

From now on, we shall always confine ourselves to smooth bounded pseudoconvex domains $D$ in $\mathbb{C}^n$ with $n \geq 2$ unless the contrary is explicitly stated. The extension from a $CR$ diffeomorphism between two boundaries $bD_i$, $i = 1, 2$, to a biholomorphism between $D_1$ and $D_2$ is considerably much easier than the other direction. This can be done via the aid of the generalized version of the Hartogs extension theorem.

**Theorem 4.8.** *Let $D$ be a smooth bounded domain in $\mathbb{C}^n$ with connected boundary, and let $f$ be a smooth $CR$ function defined on $bD$. Then, $f$ extends holomorphically to a function $F \in C^\infty(\overline{D}) \cap \mathcal{O}(D)$ such that $F|_{bD} = f$.*

For a proof of Theorem 4.8, the reader is referred to [35] or [48]. Using Theorem 4.8, one can easily prove the following extension theorem from the boundaries to the domains.

**Theorem 4.9.** *Let $f : bD_1 \to bD_2$ be a $CR$ diffeomorphism between two connected boundaries. Then $f$ extends smoothly to a $CR$ diffeomorphism between $\overline{D}_1$ and $\overline{D}_2$, that is, $f$ (and $f^{-1}$) extends smoothly to $\overline{D}_1$ ($\overline{D}_2$) such that $f : D_1 \to D_2$ is a biholomorphism.*

*Proof.* According to Theorem 4.8, $f$ ($f^{-1}$) extends smoothly to $\overline{D}_1$ ( $\overline{D}_2$) such that $f$ ($f^{-1}$) is holomorphic in $D_1$ ($D_2$). Using the fact that $f$ is a

$CR$ diffeomorphism between $bD_1$ and $bD_2$, we deduce that $\det(f')(z) \neq 0$ for $z \in D_1$, where $(f')$ is the complex Jacobian of $f$. Thus, $f$ is locally a biholomorphism. Similar arguments also hold for $f^{-1}$. It follows that $f$ $(f^{-1})$ must map $D_1$ $(D_2)$ into $D_2$ $(D_1)$. Since $f$ is one-to-one on the boundary $bD_1$, $f$ must be a biholomorphism (see also [22]). This proves the theorem. ∎

Thus, it remains to see how to extend smoothly a biholomorphism from the domains to their closures. Before we proceed any further, let us digress a little to introduce the Bergman projection and Bergman kernel function.

For any domain $D$ in $\mathbb{C}^n$, let $\mathcal{H}(D)$ be the space of square integrable holomorphic functions on $D$. Obviously, $\mathcal{H}(D)$ is a closed subspace of $L^2(D)$, and hence is itself a separable Hilbert space. If $D = \mathbb{C}^n$, then $\mathcal{H}(\mathbb{C}^n) = \{0\}$. Thus, we are interested in the case when $\mathcal{H}(D)$ is nontrivial, in particular, when $D$ is bounded. Since $\mathcal{H}(D)$ is a closed subspace of $L^2(D)$, we may consider the orthogonal projection from $L^2(D)$ onto $\mathcal{H}(D)$.

**Definition 4.10.** The orthogonal projection $P$ from $L^2(D)$ onto $\mathcal{H}(D)$ is called the Bergman projection on $D$.

On the other hand, one may construct a reproducing kernel for $\mathcal{H}(D)$ as follows. For any $w \in D$, let $\Lambda_w$ be the point evaluation map

$$\begin{aligned} \Lambda_w : \mathcal{H}(D) &\to \mathbb{C} \\ f \quad &\mapsto f(w). \end{aligned}$$

It is easily verified by Cauchy's estimate that $\Lambda_w$ satisfies

$$|\Lambda_w(f)| = |f(w)| \leq c d(w)^{-n} \parallel f \parallel_{L^2(D)}, \tag{4.11}$$

where $d(w)$ is the distance from $w$ to the complement of $D$, and the constant $c$ depends only on the space dimension $n$. Hence, by the Riesz representation theorem, there is a unique element, denoted by $K_D(\cdot, w)$, in $\mathcal{H}(D)$ such that

$$f(w) = \Lambda_w(f) = (f, K_D(\cdot, w)) = \int_D f(z)\overline{K_D(z,w)}dV_z$$

for all $f \in \mathcal{H}(D)$. It is easy to check that $K(z,w) = \overline{K(w,z)}$ and

$$\|K_D(\cdot, w)\|_{L^2(D)} \leq c d(w)^{-n} \quad \text{for any } w \in D.$$

The function $K_D(z,w)$ thus defined is called the Bergman kernel function for $D$.

For any $f \in L^2(D)$, write $f = f_1 + f_2$, where $f_1 \in \mathcal{H}(D)$ and $f_2 \in \mathcal{H}(D)^{\perp}$. We have

$$Pf = f_1 = \int_D K_D(z,w) f_1(w) dV_w = \int_D K(z,w) f(w) dV_w.$$

Thus, we have the following representation of the Bergman projection.

**Theorem 4.12.** *The Bergman projection $P_D : L^2(D) \to \mathcal{H}(D)$ is represented by*

$$P_D f(z) = \int_D K(z,w) f(w) dV_w$$

*for all $f \in L^2(D)$ and $z \in D$.*

Now, we return to the problem of how to extend a biholomorphism of domains smoothly up to the boundaries. This extension problem is proved case by case. The first significant achievement in this aspect is due to C. Fefferman [31], who proved the following fundamental result.

**Theorem 4.13 (Fefferman).** *Let $f$ be a biholomorphism between two smooth bounded strongly pseudoconvex domains $D_1$ and $D_2$. Then $f$ extends smoothly to a $CR$ diffeomorphism between $\overline{D}_1$ and $\overline{D}_2$.*

It follows from Theorem 4.13 that if two strongly pseudoconvex domains have different $CR$ invariants attached to the boundaries, then there is no biholomorphism between them. The proof of Theorem 4.13 relies heavily on the detailed analysis of the Bergman kernel function near the strongly pseudoconvex boundary. In this case, the boundary behavior of the Bergman kernel function has the following expression:

$$K_D(z,z) = \frac{\phi(z)}{(-r(z))^{n+1}} + \tilde{\phi}(z) \log(-r(z)), \tag{4.14}$$

where $r$ is a strictly plurisubharmonic defining function for $D$, $\phi, \tilde{\phi} \in C^\infty(\overline{D})$ and $\phi \neq 0$ near $bD$. By a smooth (strictly) plurisubharmonic function we mean a smooth real-valued function such that the complex Hessian is positive (definite) semi-definite at every point of the domain.

However, it seems not easy to extend Fefferman's approach to weakly pseudoconvex domains. Around 1980, S. Bell and E. Ligocka [7] proposed a new approach to this extension problem. We describe it here.

First, we introduce a condition concerning the regularity of the Bergman projection operator which is useful in proving the regularity of a biholomorphic mapping near the boundary.

**Definition 4.15.** A smooth bounded domain $D$ in $\mathbb{C}^n$ is said to satisfy condition $R$ if the Bergman projection $P$ associated with $D$ maps $C^\infty(\overline{D})$ continuously into $C^\infty(\overline{D}) \cap \mathcal{O}(D)$.

Various equivalent statements of conditin $R$ are given in the next theorem.

**Theorem 4.16.** *Let $D$ be a smooth bounded domain in $\mathbb{C}^n$ with Bergman projection $P$ and Bergman kernel function $K(z,w)$. The following conditions are equivalent*:

(1) *$D$ satisfies condition $R$.*

(2) *For each positive integer $s$, there is a nonnegative integer $m = m_s$ such that $P$ is bounded from $W_0^{s+m}(D)$ to $\mathcal{H}^s(D)$.*

(3) For each multiindex $\alpha$, there are constants $c = c_\alpha$ and $m = m_\alpha$ such that

$$\sup_{z \in D} \left| \frac{\partial^\alpha}{\partial z^\alpha} K(z,w) \right| \leq c d(w)^{-m},$$

where $d(w)$ is the distance from the point $w$ to the boundary $bD$.

For a proof of Theorem 4.16, see [6] and [23]. Here $W_0^s(D)$ is the completion of $C_0^\infty(D)$ in $W^s(D)$, the Sobolev space of order $s$ on $D$, and $\mathcal{H}^s(D) = W^s(D) \cap \mathcal{O}(D)$.

Condition $R$ in general does not hold on smooth bounded domain. A smooth bounded non-pseudoconvex domain in $\mathbb{C}^2$ on which condition $R$ fails was constructed by D. Barrett [2]. However, the powerfulness of condition $R$ still can be seen from the following important result proved by S. Bell and E. Ligocka [7].

**Theorem 4.17 (Bell-Ligocka).** *Let $D_1$ and $D_2$ be two smooth bounded domains* (*not necessarily pseudoconvex*) *in $\mathbb{C}^n$, $n \geq 2$, and let $f$ be a biholomorphic mapping from $D_1$ onto $D_2$. Suppose that condition $R$ holds on both $D_1$ and $D_2$, then $f$ extends smoothly to the boundary.*

Note that if both $D_1$ and $D_2$ are assumed to be pseudoconvex, then the assertion of Theorem 4.17 is still valid if condition $R$ holds only on one of the domains $D_1$ and $D_2$; see [4]. Roughly speaking, to prove Theorem 4.17 we first construct, under condition $R$, local holomorphic coordinates smooth up to the boundary near a boundary point. Then we show that the biholomorphic map becomes linear in this coordinate system. Hence, it extends smoothly up to the boundary.

Theorem 4.17 indicates that condition $R$ is sufficient for the extension problem of biholomorphic map. Thus, it becomes extremely important to verify the validity of condition $R$ on a given domain, in particular, on a smooth bounded pseudoconvex domain in view of the non-pseudoconvex counterexample discovered by D. Barrett. Obviously, from Theorem 4.16, the regularity of the Bergman projection is closely related to that of the Bergman kernel function. Unfortunately, the Bergman kernel function in general cannot be obtained explicitly. Therefore, it is not easy to get the regularity of the Bergman projection directly from the Bergman kernel function unless on certain very special domains, say, balls or Reinhardt domains (domains that are invariant under rotations in each variable).

Usually, condition $R$ on pseudoconvex domains is verified through the so-called $\overline{\partial}$-Neumann problem which had been intensively studied long before condition $R$ was proposed. The $\overline{\partial}$-Neumann problem can be formulated as follows. For any smooth bounded pseudoconvex domain $D \subset \mathbb{C}^n$, $n \geq 2$, denote by $L^2_{(p,q)}(D)$ the space of all $(p,q)$-forms with square integrable coefficients. Hence, any $f \in L^2_{(p,q)}(D)$ can be written as

$$f = \sum_{|I|=p,|J|=q}{}' f_{I,J} dz^I \wedge d\overline{z}^J, \tag{4.18}$$

where $I = (i_1, \cdots, i_p)$, $J = (j_1, \cdots, j_q)$ are increasing multi-indices with $i_s, j_t \in \mathbb{N}$, $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\overline{z}^J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$, and the prime means that the summation ranges only over those increasing multi-indices. The coefficients $f_{I,J}$ are defined to be anti-symmetric in $I$ and $J$. The inner product on $L^2_{(p,q)}(D)$ is defined by

$$(f, g) = \sum{}' \int_D f_{I,J} \overline{g_{I,J}}\, dV$$

for $f, g \in L^2_{(p,q)}(D)$.

The $\overline{\partial}$ operator acts on $L^2_{(p,q)}(D)$ in the distribution sense. Hence, $f \in L^2_{(p,q)}(D)$ is in the domain of $\overline{\partial}$ if $\overline{\partial} f \in L^2_{(p,q+1)}(D)$, where

$$\overline{\partial} f = \sum{}' (\overline{\partial} f_{I,J}) \wedge dz^I \wedge d\overline{z}^J. \tag{4.19}$$

It is easily seen that $\overline{\partial}$ is a closed, linear, densely defined operator, and $\overline{\partial}$ forms a complex, i.e., $\overline{\partial}^2 = 0$. The Hilbert space adjoint

$$\overline{\partial}^* : L^2_{(p,q)}(D) \to L^2_{(p,q-1)}(D) \tag{4.20}$$

of $\overline{\partial}$ is also a closed, linear, densely defined operator. Hence, $g \in \mathrm{Dom}(\overline{\partial}^*)$ if there is a $h \in L^2_{(p,q-1)}(D)$ such that for any $\phi \in \mathrm{Dom}(\overline{\partial}) \cap L^2_{(p,q-1)}(D)$, we have

$$(g, \overline{\partial}\phi) = (h, \phi).$$

We then define $\overline{\partial}^* g = h$. Clearly, $\overline{\partial}^*$ also forms a complex, and we have

$$L^2_{(p,q-1)}(D) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(p,q)}(D) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(p,q+1)}(D).$$

Then we form the complex Laplacian

$$\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : L^2_{(p,q)}(D) \to L^2_{(p,q)}(D) \tag{4.21}$$

on $\mathrm{Dom}(\Box) = \{f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \mid \overline{\partial} f \in \mathrm{Dom}(\overline{\partial}^*) \text{ and } \overline{\partial}^* f \in \mathrm{Dom}(\overline{\partial})\}$. $\Box$ is a closed, linear, self-adjoint, densely defined operator.

The $\overline{\partial}$-Neumann problem is, given $f \in L^2_{(p,q)}(D)$, to prove the existence and regularity of the solution $u$ to the equation (4.21), i.e.,

$$\Box u = f. \tag{4.22}$$

Using Hilbert space techniques, the existence of the solution $u$ in the $L^2$ sense can be obtained. In fact, one can prove

**Theorem 4.23 (Hörmander [36]).** *Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. For each $0 \leq p \leq n$, $1 \leq q \leq n$, there exists a bounded operator $N_{(p,q)} : L^2_{(p,q)}(D) \to L^2_{(p,q)}(D)$ such that:*

(1) $\mathcal{R}(N_{(p,q)}) \subset \mathrm{Dom}(\Box_{(p,q)})$, $N_{(p,q)}\Box_{(p,q)} = \Box_{(p,q)}N_{(p,q)} = I$ on $\mathrm{Dom}(\Box_{(p,q)})$.

(2) *For any* $f \in L^2_{(p,q)}(D)$, $f = \bar{\partial}\bar{\partial}^* N_{(p,q)} f \oplus \bar{\partial}^*\bar{\partial} N_{(p,q)} f$.

(3) $\bar{\partial} N_{(p,q)} = N_{(p,q+1)}\bar{\partial}$ *on* $\mathrm{Dom}(\bar{\partial})$, $1 \leq q \leq n-1$.

(4) $\bar{\partial}^* N_{(p,q)} = N_{(p,q-1)}\bar{\partial}^*$ *on* $\mathrm{Dom}(\bar{\partial}^*)$, $2 \leq q \leq n$.

(5) *Let $\delta$ be the diameter of $D$. The following estimates hold for any $f \in L^2_{(p,q)}(D)$:*

$$\|N_{(p,q)} f\| \leq \frac{e\delta^2}{q}\|f\|,$$

$$\|\overline{\partial} N_{(p,q)} f\| \leq \sqrt{\frac{e\delta^2}{q}}\|f\|,$$

$$\|\overline{\partial}^* N_{(p,q)} f\| \leq \sqrt{\frac{e\delta^2}{q}}\|f\|.$$

The operator $N = N_{(p,q)}$ is called the $\overline{\partial}$-Neumann operator. Now, it is clear how to relate the $\overline{\partial}$-Neumann operator with the Bergman projection. If $f \in \mathrm{Dom}(\overline{\partial})$ and $g \in \mathcal{H}(D)$, then

$$(g, \overline{\partial}^* N_{(0,1)} \overline{\partial} f) = (\overline{\partial} g, N_{(0,1)} \overline{\partial} f) = 0.$$

This shows $\overline{\partial}^* N_{(0,1)} \overline{\partial} f \perp \mathcal{H}(D)$. On the other hand, using (1) and (3) of Theorem 4.23, we have

$$\begin{aligned} \overline{\partial}(f - \overline{\partial}^* N_{(0,1)} \overline{\partial} f) &= \overline{\partial} f - \overline{\partial}\overline{\partial}^* N_{(0,1)} \overline{\partial} f \\ &= \overline{\partial} f - (\overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}) N_{(0,1)} \overline{\partial} f \\ &= \overline{\partial} f - \overline{\partial} f \\ &= 0. \end{aligned}$$

Hence,

$$Pf = f - \overline{\partial}^* N_{(0,1)} \overline{\partial} f \tag{4.24}$$

for $f \in \mathrm{Dom}(\overline{\partial})$. However, (4.24) also holds for any $f \in L^2(D)$ if $\overline{\partial}^* N_{(0,1)} \overline{\partial}$ is first viewed as a bounded operator on the dense subspace $\mathrm{Dom}(\overline{\partial})$ and then extended by continuity to the whole space $L^2(D)$. Thus, using (4.24), the regularity of the Bergman projection $P$ will follow immediately from that of the $\overline{\partial}$-Neumann operator $N$.

Vast efforts had been devoted to proving the regularity of the $\overline{\partial}$-Neumann problem in the last few decades. The program was first initiated by J. J. Kohn [39] when $D$ is a strongly pseudoconvex domain. In this case, a subelliptic 1/2-estimate for $(p,q)$-forms was established. To be more precise, we make the following definition.

**Definition 4.25.** Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, and let $z_0 \in bD$. The $\overline{\partial}$-Neumann problem on $D$ is said to satisfy a subelliptic $\epsilon$-estimate for $(p,q)$-forms at $z_0$, $0 < \epsilon < 1$, if there exists an open neighborhood $U$ of $z_0$ such that

$$\| f \|_\epsilon^2 \leq C(\| \overline{\partial} f \|^2 + \| \overline{\partial}^* f \|^2 + \| f \|^2) \tag{4.26}$$

for $f \in \mathcal{D}_{(p,q)}(D) = C^\infty_{(p,q)}(\overline{D}) \cap \mathrm{Dom}(\overline{\partial}^*)$ with $\mathrm{supp}(f)$ contained in $U$, where $\| \cdot \|_\epsilon$ denotes the Sobolev $\epsilon$-norm and $C > 0$ is independent of $f$.

Then, we have

**Theorem 4.27 (Kohn).** *Let $D$ be a smooth bounded strongly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. Then subelliptic 1/2-estimate for $(p,q)$-forms holds on $D$, $0 \leq p \leq n$ and $1 \leq q \leq n-1$.*

A complete characterization of subelliptic 1/2-estimate for $(p,q)$-forms at $z_0$ was obtained by Hörmander [36]. That is, (4.26) holds with $\epsilon = 1/2$ for $(p,q)$-forms at $z_0$ if and only if the Levi form at $z_0$ has either at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues. This is usually called condition $Z(q)$. See also [23] and [32].

In [45], it was shown through a general scheme that if (4.26) holds, then the $\overline{\partial}$-Neumann operator $N$ will gain $2\epsilon$ derivatives in Sobolev scale, that is,

$$\| \zeta N f \|_{s+2\epsilon} \leq C_s(\| \zeta' f \|_s + \| f \|), \tag{4.28}$$

where $\zeta, \zeta' \in C_0^\infty(U)$ and $\zeta' = 1$ on the support of $\zeta$. Thus, we have the following theorem.

**Theorem 4.29.** *The Bergman projection preserves Sobolev spaces $W^s(D)$, $s \geq 0$, on any smooth bounded strongly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. In particular, condition $R$ holds on any smooth bounded strongly pseudoconvex domain in $\mathbb{C}^n$.*

See also [44] and [23]. Using Theorems 4.17 and 4.29, we reestablish Fefferman's mapping theorem (Theorem 4.13).

Apart from strong pseudoconvexity, we begin to encounter the problem caused by the vanishing of the Levi form. In [40], Kohn launched another program to study the regularity of the $\overline{\partial}$-Neumann problem on weakly pseudoconvex domains. When $D$ is a smooth bounded pseudoconvex domain in $\mathbb{C}^2$, the Levi form $\lambda$ is just a nonnegative function on the boundary. Hence, $D$ is strongly pseudoconvex at a boundary point $z_0$ if and only if $\lambda(z_0) > 0$. Let $r$ be a smooth defining function for $D$, and let

$$L = \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1}$$

be the generator of the tangential type (1,0) vector fields on $bD$. Define the subspaces $\mathcal{L}_k$'s inductively by

$$\mathcal{L}_1 = \langle L, \overline{L} \rangle$$

and

$$\mathcal{L}_k = \langle \mathcal{L}_{k-1}, [X,Y] \rangle \quad \text{for} \quad X, Y \in \mathcal{L}_{k-1},\ k \geq 2.$$

Then we make the following definition of finite type.

**Definition 4.30.** Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$. $D$ is said to be of finite type at $z_0 \in bD$ if $\lambda(z_0) \neq 0$ or $X\lambda(z_0) \neq 0$

for some $X \in \mathcal{L}_k$. The order of finite type at $z_0 \in bD$ is defined to be 2 if $\lambda(z_0) \neq 0$, and be $2+k$ if $\lambda(z_0) = 0$ and $k$ is the smallest positive integer such that $X\lambda(z_0) \neq 0$ for some $X \in \mathcal{L}_k$. Otherwise, $D$ is said of infinite type at $z_0$.

Obviously, finite type is an open condition on the boundary. According to Definition 4.30, strictly pseudoconvex boundary points are of type 2. Also, due to pseudoconvexity of the domain, the type is always an even integer. With this setup, Kohn was able to establish the subelliptic estimate at the finite type points.

**Theorem 4.31.** *Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^2$. Suppose that $D$ is of finite type $m$ at $z_0 \in bD$. Then a subelliptic $1/m$-estimate for the $\overline{\partial}$-Neumann problem holds at $z_0$.*

The necessity of a subelliptic $1/m$-estimate at $z_0$ is due to P. Greiner [33]. The type condition in $\mathbb{C}^2$ is equivalent to the largest order of contact that a one-dimensional complex manifold can have with the boundary at $z_0$. See also [8]. If we move now to higher-dimensional spaces, the situation is becoming more complicated.

In [43], Kohn introduced the concept of finite ideal type for the subellipticity of the $\overline{\partial}$-Neumann problem. Define the ideals $I_k^q$ of germs of $C^\infty$-functions at a boundary point $z_0 \in bD$ inductively as follows:

$$I_1^q(z_0) = \sqrt[\mathbb{R}]{(r, \text{coeff}\{\partial r \wedge \overline{\partial} r \wedge (\partial\overline{\partial} r)^{n-q}\})},$$

where $r$ is a smooth defining function for $D$ and coeff$\{\cdot\}$ stands for the coefficients of the forms with respect to some holomorphic coordinate system, and, for $k \geq 2$,

$$I_k^q(z_0) = \sqrt[\mathbb{R}]{(I_{k-1}^q(z_0), \text{coeff}\{\partial f_1 \wedge \cdots \wedge \partial f_j \wedge \partial r \wedge \overline{\partial} r \wedge (\partial\overline{\partial} r)^{n-q-j}\})},$$

where $f_1, \cdots, f_j \in I_{k-1}^q$. Here $\sqrt[\mathbb{R}]{I}$ means the radical ideal of $I$, that is,

$$\sqrt[\mathbb{R}]{I} = \{f \mid \text{there exists } g \in I \text{ and } m \text{ such that } |f|^m \leq |g|\}.$$

Then the following theorem was proved in [43].

**Theorem 4.32.** *Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, and let $z_0$ be a boundary point. If $1 \in I_k^q(z_0)$ for some $k$, then a subelliptic $\epsilon$-estimate for the $\overline{\partial}$-Neumann problem for $(p,q)$-forms holds at $z_0$, where $0 < \epsilon \leq 1/2$.*

Conversely, if a subelliptic $\epsilon$-estimate for $(p,q)$-forms holds at $z_0$, is it necessary that $1 \in I_k^q(z_0)$ for some $k$? The necessity of the finite ideal type so far is still open. When the defining function $r$ is real analytic near $z_0$, with the aid of a theorem due to K. Diederich and J. E. Fornaess [29], we have a complete characterization of the subellipticity of the $\overline{\partial}$-Neumann problem at $z_0$.

**Theorem 4.33.** *Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, and let $z_0$ be a boundary point. If there exists a defining function $r$ which is real analytic near $z_0$, then a subelliptic $\epsilon$-estimate for the $\overline{\partial}$-Neumann problem for $(p,q)$-forms holds at $z_0$ if and only if there does not exist germs of complex variety of dimension $q$ at $z_0$ in the boundary.*

In particular, we have

**Theorem 4.34.** *If $D$ is a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with real analytic boundary, then a subelliptic $\epsilon$-estimate for the $\overline{\partial}$-Neumann problem for $(p,q)$-forms, $0 \leq p \leq n$, $1 \leq q \leq n-1$, holds on $D$.*

According to the theorem of Diederich and Fornaess [29], a smooth bounded pseudoconvex domain with real analytic boundary does not contain any nontrivial complex variety in the boundary. Hence, one may apply Theorem 4.33 to obtain Theorem 4.34.

On the other hand, using the order of contact of complex varieties with the boundary at $z_0$, D'Angelo [26] also introduced a notion of finite type which we now present. Denote by $\nu(f)$ the order of vanishing at the origin in $\mathbb{C}$ of a smooth vector-valued function $f$ defined in an open neighborhood of the origin in $\mathbb{C}$.

**Definition 4.35.** Let $M$ be a smooth hypersurface of $\mathbb{C}^n$ containing $z_0$. Let $r$ be a local defining function for $M$ near $z_0$. Then $z_0$ is called a point of finite type if

$$\tau^*(r) = \sup_g \frac{\nu(r \circ g)}{\nu(g)} \leq \tau < \infty, \tag{4.36}$$

where the supremum is taken over all germs of nonconstant holomorphic maps $g : \mathbb{C} \to \mathbb{C}^n$ with $g(0) = z_0$. The least such number $\tau$ is defined to be the type of the point $z_0$, denoted by $\triangle(M, z_0)$. A smooth bounded domain $D$ is of finite type if every boundary point of $D$ is of finite type.

See also the book by J. D'Angelo [27]. Thus, the type $\tau$ of a point $z_0 \in bD$ is equal to the largest order of contact of a one-dimensional complex variety can have with the boundary $bD$ at $z_0$. Then, the following theorem is proved in [26].

**Theorem 4.37.** *The type given by* (4.36) *is an open condition on the boundary.*

D. Catlin also proposed a notion of type condition closely related to the one in the sense of Definition 4.35. Then, he proved the following result.

**Theorem 4.38.** *Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, and let $z_0$ be a boundary point. Then a subelliptic $\epsilon$-estimate for the $\overline{\partial}$-Neumann problem holds at $z_0$ if and only if $D$ is of finite type at $z_0$.*

Detailed proofs of Theorem 4.38 and related results can be found in [15, 16, 18]. So far, the quantitative estimate of $\epsilon$ is not very precise. However, the story of seeking a subelliptic $\epsilon$-estimate for the $\overline{\partial}$-Neumann problem almost ended here.

Next, we need to treat the regularity of the $\overline{\partial}$-Neumann problem on pseudoconvex domains with infinite type boundary points. For this situation, in general, there is no subelliptic $\epsilon$-estimate. Thus, we are seeking for global regularity of the $\overline{\partial}$-Neumann problem.

In [17], a condition named property (P) was introduced by D. Catlin for global regularity of the $\overline{\partial}$-Neumann problem. Domains with property (P) may be viewed as a generalization of finite type domains. Here is the definition.

**Definition 4.39.** Let $D$ be a smooth bounded domain in $\mathbb{C}^n$, $n \geq 2$. The boundary of $D$ is said to satisfy property (P) if for every positive number $M > 0$, there is a plurisubharmonic function $\lambda \in C^\infty(\overline{D})$ with $0 \leq \lambda \leq 1$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \lambda}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k \geq M|t|^2 \quad \text{for all} \quad z \in bD, \tag{4.40}$$

where $t = (t_1, \cdots, t_n) \in \mathbb{C}^n$.

See also the paper concerning $B$-regular set by N. Sibony [50]. Obviously, property (P) implies the absence of any nontrivial complex variety on the boundary $bD$.

Using property (P), Catlin was able to prove the following compactness estimate for the $\overline{\partial}$-Neumann problem on $D$.

**Theorem 4.41.** *Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. Suppose that $bD$ satisfies property $(P)$. Then, for any $\epsilon > 0$, there is a $C(\epsilon) > 0$ such that*

$$\|f\|^2 \leq \epsilon(\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2) + C(\epsilon)\|f\|_{-1}^2 \tag{4.42}$$

*for $f \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*)$, where $\|\cdot\|_{-1}$ denotes the Sobolev norm of order $-1$ on $D$.*

Now, combining Theorem 4.41 with a theorem of Kohn and Nirenberg [45] gives the following consequence of property (P).

**Theorem 4.43.** *Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$. Suppose that $bD$ satisfies property $(P)$. Then the $\overline{\partial}$-Neumann problem is globally regular on $D$.*

Typical examples of pseudoconvex domains with property (P) are those whose boundaries can be stratified by submanifolds $M$ of holomorphic dimension zero. $M$ is said of holomorphic dimension zero if the restriction of the Levi form to type (1,0) vector fields tangent to $M$ is positive. Thus, property (P) provides us with a tool for proving the global regularity of the $\overline{\partial}$-Neumann problem when the domain does not contain any complex variety on the boundary. However, it was pointed out by Sibony [50] that there exists a smooth bounded pseudoconvex domain $D$ in $\mathbb{C}^2$ with no analytic disc contained in the boundary, yet property (P) fails on this domain. Whether the absence of analytic disc on the boundary implies the global regularity of the $\overline{\partial}$-Neumann problem remains to be unknown.

On the other hand, a vector field technique is developed to prove the global regularity of the $\overline{\partial}$-Neumann problem for general weakly pseudoconvex domains. Let $D \subseteq \mathbb{C}^n, n \geq 2$, be a smooth bounded pseudoconvex domain, and let $r$ be a smooth defining function for $D$. Set

$$L_n = \frac{4}{|\nabla r|^2} \sum_{j=1}^{n} \frac{\partial r}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} \tag{4.44}$$

if $|\nabla r| \neq 0$, and

$$L_{jk} = \frac{\partial r}{\partial z_j} \frac{\partial}{\partial z_k} - \frac{\partial r}{\partial z_k} \frac{\partial}{\partial z_j} \quad \text{for } 1 \leq j < k \leq n. \tag{4.45}$$

We have $L_n r = 1$ in a neighborhood of the boundary and the $L_{jk}$'s are tangent to the level sets of $r$. Also, the $L_{jk}$'s span the space of tangential type (1,0) vector fields at every boundary point of $D$.

The main idea of this method is to construct a real tangential vector field $T$ on some open neighborhood of the boundary such that the commutators of $T$ with type (1,0) and (0,1) vector fields have small modulus in $L_n$ direction on the boundary. We formulate the required properties of $T$ in the following condition:

**Condition (T).** *For any given $\epsilon > 0$, there exists a smooth real vector field $T = T_\epsilon$, depending on $\epsilon$, defined in some open neighborhood of $\overline{D}$ and tangent to the boundary with the following properties*:

(1) *On the boundary, $T$ can be expressed as*

$$T = a_\epsilon(z)(L_n - \overline{L}_n), \quad \text{mod } (T^{1,0}(bD) \oplus T^{0,1}(bD)),$$

*for some smooth function $a_\epsilon(z)$ with $|a_\epsilon(z)| \geq \delta > 0$ for all $z \in bD$, where $\delta$ is a positive constant independent of $\epsilon$.*

(2) *If $\mathcal{V}$ is any one of the vector fields $L_n$, $\overline{L}_n$, $L_{jk}$ and $\overline{L}_{jk}$, $1 \leq j < k \leq n$, then*

$$[T, \mathcal{V}]|_{bD} = A_{\mathcal{V}}(z)L_n, \quad \text{mod } (T^{1,0}(bD) \oplus T^{0,1}(bD), \overline{L}_n),$$

*for some smooth function $A_{\mathcal{V}}(z)$ with $\sup_{bD} |A_{\mathcal{V}}(z)| < \epsilon$.*

Using Condition (T), we can prove the following theorem.

**Theorem 4.46.** *Let $D$ be a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, with a smooth defining function $r$. Suppose that condition (T) holds on $D$. Then the $\overline{\partial}$-Neumann operator $N$ maps $W^s_{(p,q)}(D)$, $0 \leq p \leq n$, $1 \leq q \leq n$, boundedly into itself for each nonnegative real $s$.*

A related version of Theorem 4.46 was obtained by H. P. Boas and E. Straube [11]. They first proved the regularity for the Bergman projections under a condition similar to condition (T). Then, using the equivalence of regularity for the $\overline{\partial}$-Neumann operators and the Bergman projections proved earlier [10], they deduced the regularity for the $\overline{\partial}$-Neumann operators. See also [20]. Theorem 4.46 gives a direct treatment for the regularity of the $\overline{\partial}$-Neumann operators. A detailed proof of Theorem 4.46 can be found in [23].

Roughly speaking, to prove Theorem 4.46, using elliptic regularization method (see the paper by J. J. Kohn and L. Nirenberg [45]), it suffices to prove an *a priori* estimate for the solution $u = Nf$ to the $\overline{\partial}$-Neumann problem. Hence, we assume $u$ is smooth up to the boundary and start to estimate

$$\|\overline{\partial}T^k u\|^2 + \|\overline{\partial}^* T^k u\|^2 \tag{4.47}$$

for $k \in \{0\} \cup \mathbb{N}$. Commuting $T^k$ with $\overline{\partial}$ and $\overline{\partial}^*$, we see that it suffices to estimate

$$\| [\overline{\partial}, T]T^{k-1}u\|^2 + \| [\overline{\partial}^*, T]T^{k-1}u\|^2, \tag{4.48}$$

modulo some lower order terms that can be controlled by the induction hypotheses. The most difficult terms to estimate in (4.48) occur when $[\overline{\partial}, T]$ or $[\overline{\partial}^*, T]$ gives nontrivial component in $L_n$-direction. The assumption, condition (T), is used here to guarantee that such terms can be absorbed by the left-hand side, namely, (4.47). This proves Theorem 4.46.

So far, the vector field method can be well applied to two large classes of pseudoconvex domains to obtain the regularity of the $\overline{\partial}$-Neumann operators. Unlike pseudoconvex domains of finite type, both cases allow the boundaries of the domains to contain a large piece of complex variety. One is the case when the domain has a plurisubharmonic defining function. The other case is when the domain enjoys a circular transverse symmetry. We now describe them below.

**Definition 4.49.** Let $D$ be a smooth bounded domain in $\mathbb{C}^n$, $n \geq 2$. $r$ is called a smooth plurisubharmonic defining function for $D$ if $r$ is a defining function for $D$ and plurisubharmonic on the boundary $bD$.

Thus, if $D$ has a plurisubharmonic defining function, then $D$ is automatically pseudoconvex. It is crucial to observe that if $r$ is a plurisubharmonic defining function, then, for each $k$, the derivatives of $\partial r/\partial z_k$ of type (0,1) in the directions that lie in the null space of the Levi form must vanish. This observation enables us to construct the required real tangential vector fields $T$ such that the commutators of $T$ and type (1,0) (or (0,1)) vector fields have small modulus in the $L_n$-direction. Using Theorem 4.46, we have the following theorem.

**Theorem 4.50.** *Let $D \subseteq \mathbb{C}^n, n \geq 2$, be a smooth bounded pseudoconvex domain admitting a plurisubharmonic defining function $r(z)$. Then the $\overline{\partial}$-Neumann operator $N$ is exactly regular on $W^s_{(p,q)}(D)$ for $0 \leq p \leq n$, $1 \leq q \leq n$ and all real $s \geq 0$.*

By "exactly regular" we mean that the $\overline{\partial}$-Neumann operator $N$ preserves $W^s_{(p,q)}(D)$ for all $s \geq 0$. For a proof of Theorem 4.50, the reader is refered to [11] and [23].

Theorem 4.50 can be applied to the class of convex domains. Let $D$ be a smooth bounded convex domain in $\mathbb{R}^N$, $N \geq 2$, containing the origin. For any

$x \in \mathbb{R}^N$, the Minkowski functional $\mu(x)$ is defined by

$$\mu(x) = \inf\{\lambda > 0 \mid x \in \lambda D\}, \tag{4.51}$$

where $\lambda D = \{\lambda t \mid t \in D\}$. It is not hard to verify that the Minkowski functional $\mu$ is a plurisubharmonic defining function for $D$. Thus, we have

**Theorem 4.52.** *Let $D$ be a smooth bounded convex domain in $\mathbb{C}^n$, $n \geq 2$. Then the $\overline{\partial}$-Neumann operator $N$ is exactly regular on $W^s_{(p,q)}(D)$ for $0 \leq p \leq n$, $1 \leq q \leq n$ and all real $s \geq 0$.*

For a proof of Theorem 4.52, the reader is refered to [11], [21] and [23].

The other important class of pseudoconvex domains that satisfy the hypotheses of condition (T) are circular domains with transverse symmetries.

**Definition 4.53.** A domain $D$ in $\mathbb{C}^n$ is called circular if $e^{i\theta} \cdot z = (e^{i\theta} z_1, \cdots, e^{i\theta} z_n) \in D$ for any $z \in D$ and $\theta \in \mathbb{R}$. $D$ is called Reinhardt if $(e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) \in D$ for any $z \in D$ and $\theta_1, \cdots, \theta_n \in \mathbb{R}$, and $D$ is called complete Reinhardt if $z = (z_1, \cdots, z_n) \in D$ implies $(w_1, \cdots, w_n) \in D$ for all $|w_j| \leq |z_j|, 1 \leq j \leq n$.

Thus, a Reinhardt domain is automatically circular. Let $D$ be a smooth bounded circular domain in $\mathbb{C}^n, n \geq 2$. Define $r(z)$ by

$$r(z) = \begin{cases} d(z, bD) & \text{for } z \notin D, \\ -d(z, bD) & \text{for } z \in D, \end{cases} \tag{4.54}$$

where $d(z, bD)$ denotes the distance from $z$ to the boundary $bD$. It is easy to see that $r$ is a smooth defining function for $D$ such that $r(z) = r(e^{i\theta} \cdot z)$ and that $|\nabla r| = 1$ on the boundary. Denote by $\Lambda$ the map of the $S^1$-action on $D$ from $S^1 \times D$ to $D$ defined by

$$\begin{aligned} \Lambda : S^1 \times D &\to D \\ (e^{i\theta}, z) &\mapsto e^{i\theta} \cdot z = (e^{i\theta} z_1, \cdots, e^{i\theta} z_n). \end{aligned}$$

For each fixed $\theta$, $\Lambda$ is an automorphism of $D$ and $\Lambda$ can be extended smoothly to a map from $S^1 \times \overline{D}$ to $\overline{D}$. Hence, for each fixed $z \in \overline{D}$, we consider the orbit of $z$, namely, the map

$$\begin{aligned} \pi_z : S^1 &\to \overline{D} \\ e^{i\theta} &\mapsto e^{i\theta} \cdot z. \end{aligned}$$

Then, $\pi_z$ induces a vector field $T$ on $\overline{D}$, in fact on $\mathbb{C}^n$, by

$$T_z = \pi_{z,*}\left(\frac{\partial}{\partial \theta}\bigg|_{\theta=0}\right) = i \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} - i \sum_{j=1}^n \overline{z}_j \frac{\partial}{\partial \overline{z}_j}, \tag{4.55}$$

where $\pi_{z,*}$ is the differential map induced by $\pi_z$. Note that $T$ is tangent to the level sets of $r$. In particular, $T$ is tangent to the boundary of $D$.

**Definition 4.56.** Let $K$ be a compact subset of the boundary of a smooth bounded circular domain $D$. $D$ is said to have transverse circular symmetry on $K$ if for each point $z \in K$ the vector field $T$ defined in (4.55) is not contained in $T_z^{1,0}(bD) \oplus T_z^{0,1}(bD)$.

Transverse symmetry was first introduced by D. Barrett in [1]. It is obvious from (4.55) and Definition 4.56 that $D$ has transverse circular symmetry on the whole boundary if and only if $\sum_{j=1}^n z_j(\partial r/\partial z_j)(z) \neq 0$ on $bD$. Then, we prove

**Theorem 4.57.** *Let $D \subseteq \mathbb{C}^n, n \geq 2$, be a smooth bounded circular pseudoconvex domain with a smooth defining function $r$ defined by* (4.54)*. Suppose that $\sum_{j=1}^n z_j(\partial r/\partial z_j)(z) \neq 0$ on the boundary. Then the $\overline{\partial}$-Neumann problem is exactly regular on $D$.*

*Proof.* Let $T$ be the vector field defined in (4.55). By assumption, $T$ is transversal to $T^{1,0}(bD) \oplus T^{0,1}(bD)$ everywhere on the boundary. The key of the proof is to observe that $Tr \equiv 0$, $[T, \partial/\partial\overline{z}_j] = i\partial/\partial\overline{z}_j$ and $[T, \partial/\partial z_j] = -i\partial/\partial z_j$. Hence, we have

$$[T, L_{jk}] = -2iL_{jk}$$

for all $1 \leq j < k \leq n$, and

$$[T, L_n]|_{bD} = [T, \overline{L}_n]|_{bD} = 0,$$

where $L_n$ and $L_{jk}$ are defined in (4.44) and (4.45). It follows that condition (T) holds on $D$. By Theorem 4.46, this proves Theorem 4.57. ∎

Next a geometric argument shows that a complete Reinhardt domain always enjoys transverse circular symmetry.

**Theorem 4.58.** *Let $D \subseteq \mathbb{C}^n, n \geq 2$, be a smooth bounded complete Reinhardt pseudoconvex domain with a smooth defining function $r(z) = r(e^{i\theta_1}z_1, \cdots, e^{i\theta_n}z_n)$ for all $\theta_1, \cdots, \theta_n \in \mathbb{R}$. Then, we have $\sum_{j=1}^n z_j(\partial r/\partial z_j) \neq 0$ on $bD$. In particular, the $\overline{\partial}$-Neumann operator $N$ is exactly regular on $W^s_{(p,q)}(D)$ for $0 \leq p \leq n$, $1 \leq q \leq n$ and all real $s \geq 0$.*

Theorems 4.57 and 4.58 can be found in [19] or [23].

Now, it is natural to ask whether the vector field method can be applied to any weakly pseudoconvex domain or not. Unfortunately, the answer is negative. An important counterexample known as "worm domain" was contructed by K. Diederich and J. E. Fornaess in [28]. Let us present it here (see Kiselman [38]).

Let $\beta > \pi/2$. Fix a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ with the following properties:

(1) $\eta(x) \geq 0$, $\eta$ is even and convex;

(2) $\eta^{-1}(0) = I_{\beta-\pi/2}$, where $I_{\beta-\pi/2} = [-\beta+\pi/2, \beta-\pi/2]$;

(3) there exists an $a > 0$ such that $\eta(x) > 1$ if $x < -a$ or $x > a$;

(4) $\eta'(x) \neq 0$ if $\eta(x) = 1$.

Note that (4) follows from (1) and (2). The existence of such a function is obvious. Define

$$D_\beta = \{(z_1, z_2) \in \mathbb{C}^2 \mid r(z) < 0\}, \tag{4.59}$$

where $r(z) = |z_1 + e^{i\log|z_2|^2}|^2 + \eta(\log|z_2|^2) - 1$ is the defining function for $D_\beta$.

It is not hard to see that $D_\beta$ has smooth boundary and is defined locally by a plurisubharmonic function. Thus, we have:

**Proposition 4.60.** *For each fixed $\beta > \pi/2$, $D_\beta$ is a smooth bounded pseudoconvex domain in $\mathbb{C}^2$.*

$D_\beta$ is strictly pseudoconvex everywhere except on the closure of the annulus $A$, where

$$A = \{(0, z_2) \in \mathbb{C}^2 |\ |\log|z_2|^2| < \beta - \pi/2\}.$$

The following result shows that for each fixed $\beta > \pi/2$, there is no $C^2$ *global* defining function which is plurisubharmonic on the boundary of $D_\beta$.

**Theorem 4.61.** *For any $\beta > \pi/2$, there is no $C^2$ defining function $\rho(z)$ for $D_\beta$ such that $\rho(z)$ is plurisubharmonic on the boundary of $D_\beta$.*

*Proof.* Let $\rho(z)$ be such a $C^2$ defining function for $D_\beta$ that is plurisubharmonic on the boundary $bD_\beta$. Then there is a $C^1$ positive function $h$ defined in some neighborhood of $bD_\beta$ such that $\rho(z) = hr$. A direct calculation shows that the complex Hessian of $\rho(z)$ acting on any $(\alpha, \beta) \in \mathbb{C}^2$ for any point $p \in A \subset bD_\beta$ is given by

$$\begin{aligned} \mathcal{L}_{\rho(z)}(p; (\alpha, \beta)) = {} & 2\mathrm{Re}\left[\overline{\alpha}\beta\left(\frac{ih}{z_2} + \frac{\partial h}{\partial z_2}\right)e^{i\log|z_2|^2}\right] \\ & + \left[h + 2\mathrm{Re}\left(\frac{\partial h}{\partial z_1}e^{i\log|z_2|^2}\right)\right]|\alpha|^2. \end{aligned} \tag{4.62}$$

Since, by assumption, (4.62) is always nonnegative, we must have

$$\left(\frac{ih}{z_2} + \frac{\partial h}{\partial z_2}\right) e^{i \log |z_2|^2} \equiv 0$$

on $A$, or, equivalently,

$$\frac{\partial}{\partial \overline{z}_2}(h e^{-i \log |z_2|^2}) \equiv 0$$

on $A$. Consequently,

$$g(z_2) = h(0, z_2) e^{-i \log |z_2|^2}$$

is a holomorphic function on the annulus $A$. It follows that

$$g(z_2) e^{i \log z_2^2} = h(0, z_2) e^{-2\arg z_2} = c$$

is also locally a holomorphic function on $A$, and hence it must be a constant $c$, since the right-hand side is real. This implies that

$$h(0, z_2) = c e^{2\arg z_2}$$

is a well-defined, $C^1$ positive function on $A$, which is impossible. This proves Theorem 4.61. ■

In particular, Theorem 4.50 is not applicable to worm domains. Actually, D. Barrett [3] was able to show the following:

**Theorem 4.63.** *For any $\beta > \pi/2$, the Bergman projection on $D_\beta$ does not map $W^k(D_\beta)$ into $W^k(D_\beta)$ when $k \geq \pi/(2\beta - \pi)$.*

Basically, the idea of the proof is that we first dilate the $z_1$-component of $D_\beta$ to an unbounded worm domain $D'_\beta$ for $z_2$ belonging to $\{z_2 \in \mathbb{C} \mid |\log |z_2|^2| < \beta - \pi/2\}$. The Bergman kernel function on $D'_\beta$ can be related to the weighted Bergman kernel function on a horizontal strip $I_\beta$ of the complex plane. Now, using the explicit asymptotic expansion of the weighted Bergman kernel function on $I_\beta$, we deduce the assertion of the theorem.

With Barrett's result at hand, it was finally proved by M. Christ [25] that the Bergman projection is not globally regular on worm domains. Namely, we have

**Theorem 4.64.** *Condition R fails on worm domain $D_\beta$ for any $\beta > \pi/2$.*

To prove Theorem 4.64, he showed that for each $\beta > \pi/2$, there is a sequence of positive numbers $\{s_j = s(\beta)_j\}_{j=1}^{\infty}$ tending to infinity such that if $f \in C^\infty_{(p,1)}(\overline{D}_\beta)$ and $Nf = N_{(p,1)}f \in C^\infty_{(p,1)}(\overline{D}_\beta)$ (this would be the case if condition $R$ holds on $D_\beta$; see [9]), then, for each $j$, the following estimate holds:

$$\| Nf \|_{W^{s_j}(D_\beta)} \leq C_j \| f \|_{W^{s_j}(D_\beta)} \tag{4.65}$$

for some $C_j > 0$. Since $C^\infty(\overline{D}_\beta)$ is dense in every $W^k(D_\beta)$, $k \geq 0$, (4.65) will violate Barrett's result. This proves Theorem 4.64.

It might be interesting if one can reprove Theorem 4.64 via a direct treatment of the Bergman kernel function on $D_\beta$. This will shed more insights into the behavior of the Bergman projection near the weakly pseudoconvex boundary points on worm domains.

Although condition $R$ fails on weakly pseudoconvex domains in general, yet the classification problem still remains. A complete understanding of the classification problem of domains in $\mathbb{C}^n$, $n \geq 2$, is still far from reach. It is possible that any biholomorphic map between two smooth bounded domains extends automatically smoothly up to the boundary without any additional assumption. New ideas and techniques are needed for further investigations. Hopefully, one can get a clearer picture of it in the near future. See also the survey papers by S. Bell [5] and H. P. Boas and E. Straube [12]. Finally, we should also note that if the metric is changed, then it is always possible to obtain Sobolev estimates for the weighted $\overline{\partial}$-Neumann problem. See [41, 42].

I think it is about time to stop at this stage. If, after reading this article, the reader wishes to learn more about several complex variables, I highly recommend the books by S.-C. Chen and M.-C. Shaw [23], G. B. Folland and J. J. Kohn [32], L. Hörmander [37], S. Krantz [46], R. Narasimhan [47] and R. M. Range [48].

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