Complex-analytic properties of certain Zariski open sets on algebraic varieties

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1. Introduction and statements of main results

One of the deepest and more mysterious procedures in complex analysis occurs by passing to the universal covering space $\tilde{U}$ of a complex manifold U. Of course, $\tilde{U}$ is again a complex manifold, but the global complex analytic properties of $\tilde{U}$ are generally quite difficult to fathom, especially when the fundamental group of U is infinite and non-abelian. For example, the classical uniformization theorem for Riemann surfaces does not seem to have ever been rigorously proved using only techniques from function theory; in fact, even the case of compact Riemann surfaces seems to require some discussion of potential theory. Another example of this is the theorem of Grauert-Oka that the universal covering of a domain of holomorphy is a Stein manifold (cf. [10, page 283]), whose proof relies on the construction of holomorphic functions from the data of a pseudo-convex exhaustion of the manifold in question. The answers to other easily stated questions involving universal coverings seem to be unknown (cf. § 8 (b) below).

In this paper we are primarily interested in the function-theoretic properties of the universal covering space $\tilde{U}$ of a smooth, quasi-projective algebraic variety U.[0] There are two central points we wish to make:

[0] By definition, U is a Zariski open subset of a projective algebraic variety U.

(1) These covering spaces $\tilde{U}$ are perhaps more interesting function-theoretically than they might seem at first glance. In particular, we are able to prove a sort of local, in the Zariski topology, uniformization theorem for arbitrary algebraic varieties.

(2) If we have a relatively simple complex manifold $\tilde{U}$, such as a bounded domain in $\mathbf{C}^n$, which has acting on it a properly discontinuous group Γ of holomorphic automorphisms whose quotient $U = \tilde{U}/\Gamma$ is a quasi-projective variety, then $\tilde{U}$ *should* have very strong function-theoretic properties (cf. §8 (d)).

We now state precisely our main results. Let V be an irreducible, smooth, quasi-projective algebraic variety over the complex numbers. Thus V is a complex manifold. Recall that a *Zariski open subset* U of V is an open set of the form $U = V - Z$ where Z is an algebraic subvariety of V. A basic principle in the study of the topological properties of algebraic varieties is the following:

Given a point $x \in V$, there is a Zariski neighborhood U of x in V such that the universal covering manifold $\tilde{U}$ of U is topologically a cell.(1)

It is always possible to choose U to be an affine algebraic variety, in which case both U and $\tilde{U}$ are Stein manifolds(2). Thus $\tilde{U}$ is again a Stein manifold which is homeomorphic to the unit ball in $\mathbf{C}^n$ ($n = \dim V$). We are interested in the function-theoretic properties of such a $\tilde{U}$, and our main results are:

THEOREM I. *Given a point $x \in V$, we may choose a Zariski neighborhood U of x such that the universal covering manifold $\tilde{U}$ of U is topologically a cell and is biholomorphically equivalent to a bounded domain of holomorphy in $\mathbf{C}^n$.*(3)

THEOREM II. *Given a point $x \in V$, we may choose a Zariski neighborhood U_1 of x such that there is a complete Kählerian metric $ds^2_{U_1}$ on U_1 with the properties that* (i) *the holomorphic sectional curvatures are all $\leqq -1$, and* (ii) *U_1 has finite volume. Furthermore, we may take U_1 to be the open neighborhood U of Theorem I, if we so desire.*

There are several drawbacks to Theorems I and II. The most serious is that we are unable to say that the universal covering $\tilde{U}$ belongs to a reason-

(1) Cf. Lefschetz [12, page 32] and M. Artin [3]. This statement will be verified in the course of proving Theorem I below.

(2) It is a theorem of K. Stein [15] that the universal covering of a Stein manifold is again a Stein manifold. This result will be discussed in §8 (b) below.

(3) This theorem is true if we do not assume that V is smooth, but only that x is a simple point on V. The qualitative character of the embedding $U \to C^n$ is given by the example (6.3) at the end of §6 below. Roughly speaking, the mapping functions are constructed as integrals of suitable differential equations on U.

able a priori given class of bounded domains in $\mathbf{C}^n$.[4] Secondly, we are unable to prove that the Bergman metric on U, which is in some ways the most interesting intrinsic metric, has the strong differential-geometric properties given in Theorem II. These and other related matters will be discussed in section 8 (a), (d) below.

We now give a few easily stated corollaries of Theorems I and II.

COROLLARY A (LOCAL PICARD THEOREM). *Suppose that V is complete and let U_1 be the Zariski neighborhood given by Theorem II. Suppose furthermore that A is an analytic space and B is an analytic subspace. Then any holomorphic mapping $f\colon A - B \to U_1$ extends to a meromorphic mapping $\bar{f}\colon A \to V$.*[5]

COROLLARY B (LOCAL UNIFORMIZATION THEOREM). *In Theorem I suppose that V is projectively embedded in $\mathbf{P}_N$ and that $\tilde{U}$ has been realized as a bounded domain in $\mathbf{C}^n$ with linear coordinates $z_1, \cdots, z_n$. Then the projection $\pi\colon \tilde{U} \to U \subset \mathbf{P}_N$ is given by N meromorphic functions $f_1(z), \cdots, f_N(z)$ which* (i) *are invariant under the group of covering transformations of $\tilde{U} \to U$, and* (ii) *satisfy the property that none of the $f_\alpha(z)$ can be analytically continued across any boundary point of $\tilde{U}$.*[6]

The proofs of Corollaries A and B will be given in sections 6 and 7 below. Section 8 contains a few more applications of Theorems I and II.

In order to illustrate Theorem I, let me give the

Example. Suppose that V_n is the smooth, projective surface given as a hypersurface in $\mathbf{P}_3$ by an equation

$$\xi_0^n + \xi_1^n + \xi_2^n + \xi_3^n = 0 .$$

Then V_n is simply connected. We let U_n be the Zariski open set on V_n obtained by removing the intersection of V_n with the planes

$$\xi_0 = \varepsilon^\mu \xi_1 \qquad (\varepsilon = e^{\pi i/n};\ \mu = 0, \cdots, n-1) .$$

Then, for $n \geqq 2$, the universal covering $\tilde{U}_n$ may be realized as a bounded domain in $\mathbf{C}^2$.

In concluding this introduction, it is my pleasure to acknowledge many

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(4) This is, of course, related to the complete breakdown of the theory of conformal mapping in more than one variable. There is some heuristic evidence that, in the problem at hand, we should be able to "rigidify" $\tilde{U}$ by posing a suitable extremal problem and using the existence of a "large" discontinuous group acting on U.

(5) We may take as definition of a *meromorphic mapping* a holomorphic mapping f as above such that, for any *rational* function φ on V, the composed function $f^*(\varphi)$ extends across B to give a meromorphic function on all of A.

(6) This will prove that U is *meromorphically* convex, and is therefore a domain of holomorphy.

valuable suggestions which were made by the referee. He pointed out several incomplete arguments in the original version of the paper and contributed in an essential way to our presentation of the material on quasi-Fuchsian groups. Also, I have had several helpful conversations with J. Carlson, and the discussion of differential equations on algebraic curves given in § 3 owes a great deal to a set of notes which he made on the subject. Finally, it was G. Washnitzer who first made me aware of trying to "do uniformization in the algebraic category" by locating suitable algebraic differential equations on algebraic varieties.

2. Algebro-geometric preliminaries

Let V be the quasi-projective variety appearing in the statements of Theorems I and II. We shall prove our results under the assumption that V is complete. This will be sufficient because (i) by resolution of singularities, any smooth, quasi-projective variety is a Zariski open set on a smooth, complete variety $\bar{V}$; and (ii) our proof will show that the desired Zariski neighborhood U of x in $\bar{V}$ may be chosen to lie in V.

The proof of Theorems I and II will be done by induction on the dimension n of V. The result for $n = 1$ is the classical uniformization theorem [2, chapter XVI], and we shall use this result in the strong form given by Bers [4] (cf. § 5 below) to make the induction step from $n - 1$ to n. For this, the following lemma is the essential algebro-geometric step:

LEMMA 2.1. *Let x be a point on the smooth, projective variety V, and let Z be a given algebraic subvariety of $V - x$ (Z may be empty). Then there exists a Zariski neighborhood U' of x in V with the following properties*: (i) *U' is contained in $V - Z$, and* (ii) *there is a smooth, quasi-projective variety S' of dimension $n - 1$ and a rational, holomorphic mapping*

$$\pi\colon U' \longrightarrow S'$$

which, when considered as a map of C^∞-manifolds, is differentially a locally trivial fibration.[7]

Proof. We first consider a projective embedding $V \to \mathbf{P}_{N'}$ given by a complete linear system $|D'|$ of divisors on V. Without loss of generality we may assume that one of these divisors, say D'_∞, contains the given subvariety Z. Now by replacing $|D'|$ by the linear series $|mD'|$ of divisors linearly equivalent to $m\cdot D'_\infty$, for m sufficiently large, we will obtain a complete linear

[7] Thus the fibres $C_{s'} = \pi'^{-1}(s')$ are all smooth algebraic curves of fixed genus g and constant topological Euler characteristic $2 - 2g - M$. The proof will show that we may assume that the $C_{s'}$ are irreducible, so that M is the number of "points at infinity" in $C_{s'}$.

series $|D|$ with the following properties: (i) $|D|$ gives a projective embedding $V \subset \mathbf{P}_N$; and (ii) if we take $n-1$ *generic* divisors $D_1, \cdots, D_{n-1}$ from $|D|$, then the intersection $D_1 \cdots D_{n-1}$ is a smooth, irreducible curve which meets the divisor $m \cdot D'_\infty = D_\infty$ in M distinct points. Letting D_∞ be the hyperplane at infinity, we may then choose affine coordinates $(x_1, \cdots, x_{n-1}; y_n, \cdots, y_N)$ for $\mathbf{C}^N \subset \mathbf{P}_N$ such that the hyper-planes $x_\alpha = 0$ $(\alpha = 1, \cdots, n-1)$ are all generic in the above sense.

Now the affine part of V, say V_a, will be a Zariski neighborhood of one given point which is contained in $V - Z$ and which is given in $\mathbf{C}^N$ by polynomial equations

$$P_\mu(x, y) = 0 \qquad (\mu = 1, \cdots, p) .$$

The projection $\mathbf{C}^N \to \mathbf{C}^{n-1}$ given by $(x, y) \to x$ induces a rational, holomorphic mapping π_a: $V_a \to \mathbf{C}^{n-1}$ such that, for generic $x \in \mathbf{C}^{n-1}$, the fibre $\pi_a^{-1}(x)$ is a smooth, irreducible, affine curve having exactly M points at infinity. Now we let $S' \subset \mathbf{C}^{n-1}$ be the Zariski open set of all x such that $\pi_a^{-1}(x)$ has these properties. Setting $U' = \pi_a^{-1}(S')$, we have the situation π': $U' \to S'$ required in the lemma. Q.E.D.

From the proof of this lemma and the induction assumption in Theorem I, we have

LEMMA 2.2. *Let x be a point of V and Z a proper subvariety of $V - x$. Then there exists a Zariski neighborhood U of x in V with the following properties*: (i) *U is contained in $V - Z$,* (ii) *there is a smooth, quasi-projective variety S of dimension $n - 1$ and a rational, holomorphic mapping*

$$\pi \colon U \longrightarrow S$$

which is a C^∞ locally trivial fibration; (iii) *the fibres $C_s = \pi^{-1}(s)$ of π are smooth curves of genus g having exactly M points at infinity and with $3g - 3 + M \geqq 0$*; *and* (iv) *the universal covering $\tilde{S}$ of S is topologically a cell and is biholomorphically equivalent to a bounded domain of holomorphy in $\mathbf{C}^{n-1}$.*

For the proof of Theorem II, we will use the

LEMMA 2.3. *Let x be a point on V and Z an algebraic subvariety of $V - x$. Then there exists a Zariski neighborhood U_1 of x which has a closed, rational, smooth embedding*

$$U_1 \subset S_1 \times \cdots \times S_N$$

where each S_j is a Zariski open subset of $\mathbf{P}_1$ of the form $\mathbf{P}_1 - \{z_1, \cdots, z_{N_j}\}$ $(N_j \geqq 3)$.

Proof. Referring to the proof of Lemma 2.1, we arrive at the situation

$$x \in V_a \subset \mathbf{C}^N .$$

Now we may remove from V_a hyperplane sections of the form $x_j = c_j$, $y_\alpha = c_\alpha$ to arrive at our lemma. Q.E.D.

3. Differential equations on algebraic curves

In this section we shall summarize some of the discussion in [2] pertaining to the use of certain algebraic differential equations to generate holomorphic mappings on the universal coverings of algebraic curves.

(a) Let C be a non-singular algebraic curve, which as always we may uniquely represent in the form

$$C = \bar{C} - \{z_1, \cdots, z_M\}$$

where $\bar{C}$ is a smooth, complete curve and the *punctures* $z_1, \cdots, z_M$ are distinct points on $\bar{C}$.(8) In order to explicitly represent the differential equations on C, we choose a holomorphic mapping $h: \bar{C} \to \mathbf{P}_2$ with the following properties: (i) h is a birational immersion whose image curve $h(\bar{C})$ has only ordinary double points as singularities, and none of these is a puncture z_α; (ii) upon identifying $\bar{C}$ with the image $h(\bar{C})$, we may choose a projection $\bar{C} \to \mathbf{P}_1$ such that all branch points are simple,(9) and such that none of the punctures or double points on $\bar{C}$ coincides with a branch point. We may then choose affine coordinates (x, y) on $\mathbf{P}_2$ so that $\bar{C}$ is given by an irreducible polynomial equation of degree n

$$p(x, y) = 0 , \tag{3.1}$$

and such that the projection $\bar{C} \to \mathbf{P}_1$ is induced by $(x, y) \to x$. We may furthermore assume that none of the points over $x = \infty$ is a double point, a branch point, or a puncture. Thus the punctures will all be finite points

$$z_\alpha = (x_\alpha, y_\alpha) ,$$

none of which will be a branch point or double point. Under all of these circumstances, we will say that the curve C given by the equation (3.1) is in *general position*.

Before discussing differential equations, we want to briefly discuss the concept of an *analytic family of algebraic curves*. For this we assume given a situation $\pi: U \to S$ where U, S are complex manifolds and π is a smooth mapping such that the fibres $C_s = \pi^{-1}(s)$ are of dimension one. We shall assume further that we may locally complete the fibres C_s to have closed Rie-

(8) These punctures were referred to as the points at infinity on C in § 2 above.

(9) A simple branch point is one which is locally of the form $s = \sqrt{t}$.

mann surfaces $\bar{C}_s$ in the following sense: Given $s_0 \in S$, there exists a neighborhood N of s_0 and a diagram

$$\begin{array}{ccc} U_N & \subset & \bar{U}_N \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ N & = & N \end{array} \qquad (U_N = \pi^{-1}(N))$$

where $\bar{U}_N$ is a complex manifold, $\bar{\pi}\colon \bar{U}_N \to N$ is a proper, smooth mapping, and where U_N is a Zariski open set in $\bar{U}_N$ such that

$$\bar{U}_N - U_N = Z_1 \cup \cdots \cup Z_N$$

is a *disjoint* union of smooth subvarieties Z_α which meet the fibres $\bar{C}_s = \bar{\pi}^{-1}(s)$ transversely in a point $z_\alpha(s)$. Thus we have that

$$C_s = \bar{C}_s - \{z_1(s), \cdots, z_N(s)\} .$$

The following lemma is easy (cf. Lemma 2.1):

LEMMA 3.2. *Let U, S be smooth algebraic varieties and $\pi\colon U \to S$ a smooth rational, holomorphic mapping such that the fibres $\pi^{-1}(s) = C_s$ are all Riemann surfaces of genus g and having M punctures. Then $\pi\colon U \to S$ gives an analytic family of algebraic curves according to the above definition.*

LEMMA 3.3. *Let $\pi\colon U \to S$ be an analytic family of algebraic curves. Then, given $s_0 \in S$ we may find a neighborhood N of s_0 such that the curves C_s $(s \in N)$ are given by an affine equation*

$$p(x, y; s) = 0 \tag{3.3}$$

where $p(x, y; s)$ is a polynomial in x, y whose coefficients are holomorphic functions of $s \in N$, and where the equation (3.3) *is in general position for all s.*

We return to the consideration of our fixed curve C with affine equation (3.1). On C we wish to consider 2nd order linear differential equations of the form

$$\frac{d^2\mu}{dx^2} + R(x, y)\mu = 0 \tag{E}$$

and which have the following properties: (i) the equation (E) is everywhere holomorphic on C, (ii) at a puncture $z_\alpha \in \bar{C} - C$, the D. E. has a *regular singular*

[10] To say that (E) has a regular singular point at z_α means that, upon using $t_\alpha = x - x_\alpha$ as a local uniformizing parameter around $z_\alpha = (x_\alpha, y_\alpha)$, the equation (E) has the form

$$\frac{d^2\mu}{dt_\alpha^2} + \frac{H(t_\alpha)}{t_\alpha^2}\mu = 0 \tag*{(E)$_\alpha$}$$

where $H(t_\alpha)$ is holomorphic for $|t_\alpha| < \varepsilon$. The roots of the indicial polynomial will be equal if $H(0) = 1/4$.

point; and (iii) at the regular singular point z_α, the roots of the *indicial polynomial* are equal.[10] We shall call such equations *admissible*. For an admissible equation, we may choose local solutions μ_1, μ_2 to the equation (E) in a punctured neighborhood Δ_α^* of z_α such that the ratio

$$\mu_1/\mu_2 = \frac{1}{2\pi\sqrt{-1}}\left(\log\left(x - x_\alpha\right)\right), \tag{3.4}$$

where we have used $x - x_\alpha$ as a local coordinate around $z_\alpha = (x_\alpha, y_\alpha)$.

It is easy to verify that the equation (E) satisfies the first two conditions just above if, and only if, the coefficient $R(x, y)$ is a rational function of total degree $2n - 4$ having a double pole at the punctures z_α, and which vanishes at the double points of C (*adjoint conditions*). From this it follows that all admissible equations (E) are obtained from a fixed one

$$\frac{d^2\mu}{dx^2} + R_0(x, y)\mu = 0 \tag{E_0}$$

by adding to $R_0(x, y)$ a rational function $Q(x, y)$ of degree $2n - 4$ and which has a simple pole at the punctures z_α. This leads to the

PROPOSITION 3.5. *The space $E(C)$ of admissible D.E.'s on C is an affine space whose associated vector space is the space of rational quadratic differentials*

$$\omega = Q(x, y)\left(\frac{dx}{\dfrac{\partial f}{\partial y}}\right)^2$$

which are everywhere holomorphic on C and have only simple poles at the punctures z_α. In particular, $\dim_{\mathbf{C}}\{E(C)\} = 3g - 3 + M$ *where g is the genus of C.*

For a complete curve C, this proposition appears in [9] under a slightly different guise. From (3.5) and our earlier discussion we have the

COROLLARY 3.6. *Let $\pi\colon U \to S$ be an analytic family of algebraic curves. Then there is a homomorphic affine bundle $\mathbf{E} \to S$ whose fibre $\mathbf{E}_s$ is in one-to-one correspondence with the space $E(C_s)$ of admissible D.E.'s on C_s, and whose associated vector bundle $V(\mathbf{E})$ is the bundle of quadratic differentials along the fibres of $\pi\colon U \to S$.*

(b) All of the preceding discussion is purely algebro-geometric. We shall now leave the "algebraic category" by utilizing *analytic continuation* to introduce the *monodromy group* $\Gamma(E)$ and corresponding étale mapping[11]

[11] In this paper, étale means *unramified*.

(3.7) $$\mu_E\colon \tilde{C} \longrightarrow \mathbf{P}_1$$

which is canonically associated to an admissible D.E. For this we choose a base point $z_0 \in C$ and consider a basis μ_1, μ_2 for the solutions to **E** which may be assumed to exist in a neighborhood of z_0. By the principle of analytic continuation, we may extend the domains of definition for μ_1 and μ_2 to obtain single-valued functions $\tilde{\mu}_1$, $\tilde{\mu}_2$ on the universal covering $\tilde{C}$ of C. Furthermore, if we let $\pi_1(S)$ operate as a group of covering transformations on $\tilde{C}$, then we will find a transformation rule

$$\begin{cases} \tilde{\mu}_1(\gamma\cdot\tilde{z}) = a_\gamma\tilde{\mu}_1(\tilde{z}) + b_\gamma\tilde{\mu}_2(\tilde{z}) \\ \tilde{\mu}_2(\gamma\cdot\tilde{z}) = c_\gamma\tilde{\mu}_1(\tilde{z}) + d_\gamma\tilde{\mu}_2(\tilde{z}) \,, \end{cases}$$

for $\gamma \in \pi_1(S)$ and all $\tilde{z} \in \tilde{C}$, and where the transformation matrix

$$M_\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in SL(2, \mathbf{C})$$

because the D.E. (E) has no term involving $d\mu/dx$ in it. In other words, letting $\mu_E = \tilde{\mu}_1/\tilde{\mu}_2$, what has been generated by the D.E. (E) is the *monodromy representation*

(3.8) $$\delta_E\colon \pi_1(C) \longrightarrow SL(2, \mathbf{C})$$

and corresponding *étale mapping* (3.7) which satisfies

(3.9) $$\mu_E(\gamma\tilde{z}) = \delta_E(\gamma)\mu_E(\tilde{z}) \qquad (\tilde{z} \in \tilde{C})\,.$$ [12]

Moreover, if $\Delta_\alpha^* = \{x\colon 0 < |x - x_\alpha| < \varepsilon\}$ is a neighborhood of the puncture z_α, and if we localize over Δ_α^* to have a diagram

$$\begin{array}{ccc} \tilde{\Delta}_\alpha^* & \longrightarrow & \tilde{C} \\ \downarrow & & \downarrow \\ \Delta_\alpha^* & \longrightarrow & C\,, \end{array}$$

then, after suitable conjugation within $SL(2, \mathbf{C})$ the restriction of μ_E to $\tilde{\Delta}_\alpha^*$ will be given by

(3.10) $$\mu_E(x) = \frac{1}{2\pi\sqrt{-1}}\{\log(x - x_\alpha)\}\,,$$

as is evident from (3.4).

Conversely, suppose we are given an étale map (3.7) having the properties (3.9) and (3.10). We may think of μ_E as being a meromorphic function on $\tilde{C}$, and we then consider the *Schwarzian derivative* [1, page 125],

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(12) The reason that μ_E is étale is that locally $\mu_E' = (\mu_1'\mu_2 - \mu_1\mu_2')/(\mu_2)^2 \neq 0$ because the Wronskian $W(\mu_1, \mu_2) = \mu_1'\mu - \mu_1\mu_2'$ is nowhere vanishing.

$$\{\mu_E, x\} = \frac{\mu_E'''}{\mu_E'} - \frac{3}{2}\left(\frac{\mu_E''}{\mu_E'}\right)^2 ,$$

the derivatives being taken with respect to x on the dense open subset of $\tilde{C}$ where x is a local holomorphic coordinate. Since

$$\left\{\frac{a\mu_E + b}{c\mu_E + d}, x\right\} = \{\mu_E, x\} ,$$

it follows from (3.9) that the Schwarzian derivative $\{\mu_E, x\} = R(x, y)$ is a single-valued function on C. From (3.10) we then deduce that the D.E.

$$\frac{d^2\mu}{dx^2} + R(x, y)\mu = 0$$

is an admissible D.E. whose associated étale map is exactly the one with which we began.

Let us agree to call admissible an étale mapping (3.7) with the properties (3.9) and (3.10). Then our conclusion is the

PROPOSITION 3.11. *There is a one-to-one correspondence between admissible differential equations* (E) *and admissible maps* μ_E.

To close this section we shall briefly discuss the following rigidity theorem from [2]:

PROPOSITION 3.12. *The monodromy group* $\Gamma(E)$ *uniquely determines the admissible D.E.* (E).[13]

If we recall that $\pi_1(S)$ has generators

$$\{\gamma_1, \cdots, \gamma_g; \delta_1, \cdots, \delta_g; \varepsilon_1, \cdots, \varepsilon_M\}$$

with the defining relation

$$\prod_{\mu=1}^{2g} (\gamma_\mu \delta_\mu \gamma_\mu^{-1} \delta_\mu^{-1}) \prod_{\alpha=1}^{M} \varepsilon_\alpha = 1 \tag{3.13}$$

then we see that $\Gamma(E)$ is represented by a point on the affine algebraic variety

$$R(g, M) \subset \underbrace{SL(2, \mathrm{C}) \times \cdots \times SL(2, \mathrm{C})}_{2g+M}$$

defined by the equations (cf. (3.13) and (3.4))

$$\begin{cases} \prod_{\mu=1}^{2g} (A_\mu B_\mu A_\mu^{-1} B_\mu^{-1}) \prod_{\alpha=1}^{M} C_\alpha = I , \\ \text{Trace } C_\alpha = 2 . \end{cases} \tag{3.14}$$

From (3.14) it follows that

$$\dim_{\mathrm{C}} [R(g, M)] = 6g + 2M - 3$$

(13) More precisely, if $\Gamma(E)$ is conjugate to $\Gamma(E')$ in $SL(2, \mathrm{C})$, then we may choose bases for the solutions of the respective differential equations such that $\mu_E \equiv \mu_{E'}$.

so that, speaking roughly, the space of conjugacy classes $R(g, M)/SL(2, \mathbf{C})$ has the dimension

$$\dim_{\mathbf{C}} [R(g, M)/SL(2, \mathbf{C})] = 2(3g - 3 + M) . \tag{3.15}$$

For a *fixed* curve C, we see from Propositions (3.5) and (3.12) that the monodromy group $\Gamma(E)$ depends on $3g - 3 + M$ parameters. On the other hand, it is well-known that the curve C has the same number $3g - 3 + M$ of moduli. Adding these up we may confirm the formula (3.15). The reason for the somewhat mysterious twofold appearance of the number $3g - 3 + M$ was finally explained by Bers [4], whose theorem we shall discuss in § 5 below.

4. Kleinian, quasi-Fuchsian, and Fuchsian groups

We shall discuss some important discrete subgroups of $SL(2, \mathbf{C})$; the references for this section are [2] and [5].

Let Γ be a discrete subgroup of $SL(2, \mathbf{C})$, which is the group of unimodular 2×2 matrices acting in the usual manner as linear fractional transformations on $\mathbf{P}_1$. A point $w = [w_0, w_1] \in \mathbf{P}_1$ is a *limit point* for Γ if there exists a sequence $\{\gamma_n\}$ of distinct elements of Γ and a point $w' \in \mathbf{P}_1$ such that

$$\lim_{n \to \infty} \gamma_n(w') = w .$$

We denote by $\Lambda(\Gamma)$ the set of all such limit points and $R(\Gamma) = \mathbf{P}_1 - \Lambda(\Gamma)$ the complement. If $R(\Gamma)$ is non-empty, it is open and dense on $\mathbf{P}_1$ and is called the *region of discontinuity* of Γ. When this happens, Γ is said to be a *Kleinian group*. We shall be exclusively interested in cases where Γ is finitely generated and contains no elliptic elements. Then it is a theorem of Ahlfors that

$$R(\Gamma)/\Gamma = S_1 \cup \cdots \cup S_M$$

is a union of finitely many smooth algebraic curves.[14]

A Kleinian group Γ is said to be *quasi-Fuchsian* with fixed curve Π if Π is an oriented Jordan curve on $\mathbf{P}_1$ which is transformed into itself by Γ. In this case we have the inclusion

$$\Lambda(\Gamma) \subset \Pi ,$$

and we shall say that Γ is of the *first kind* if $\Lambda(\Gamma) = \Pi$. For such groups the region of discontinuity is the disjoint union $R(\Gamma) = D_+ \cup D_-$ of two simply-connected regions. The transformations γ in Γ are elliptic, parabolic, or hyperbolic according to whether γ has one fixed point in D_+, one fixed point on Π,

[14] It is also the case that the S_j are smooth even when Γ contains elliptic elements. However, this is accidental to dimension one and should be regarded as being misleading.

or two distinct fixed points on Π. The quasi-Fuchsian groups we shall be interested in will be of the first kind and will consist entirely of parabolic and hyperbolic elements. In this case the quotients

$$\begin{cases} D_+/\Gamma = C_+ \\ D_-/\Gamma = C_- \end{cases}$$

are smooth algebraic curves. The group Γ will have generators $A_1, \cdots, A_g$, $B_1, \cdots B_g$, $C_1, \cdots, C_M$ with a single defining relation (3.14).

A *Fuchsian group* is a quasi-Fuchsian group Γ whose fixed curve Π is a circle on $\mathbf{P}_1$. In this case, by conjugation within $SL(2, \mathrm{C})$, we may assume that Γ is a discrete subgroup of $SL(2, \mathbf{R})$ and that Π is the positively oriented real axis. If Γ is of the first kind, then D_+ and D_- are respectively the upper and lower half planes given by

$$\begin{cases} H = \{\zeta = +\xi\sqrt{-1}\eta; \eta > 0\} \\ L = \{\zeta = +\xi\sqrt{-1}\eta; \eta < 0\} . \end{cases}$$

If Γ is a quasi-Fuchsian group, then by the Riemann mapping theorem there exists a holomorphic homeomorphism $v: D_+ \rightarrow H$ which transforms Γ into a Fuchsian group. Moreover, v extends continuously to map Π onto the real axis, and may then be extended to some homeomorphism $v: \mathbf{P}_1 \rightarrow \mathbf{P}_1$. It is a very subtle matter to see just how nearly conformal we may make v.

The Kleinian and Fuchsian groups are classical in origin; for instance, the terminology seems to have originated in the memoirs of Poincaré on the uniformization of algebraic curves [13]. The quasi-Fuchsian groups are of more recent origin and have come into importance through Bers' theorem on simultaneous uniformization [4]. We note that if Γ is a quasi-Fuchsian group which is *not* Fuchsian, then the fixed curve will nowhere have a tangent at any point of the fixed set $\Lambda(\Gamma) \subset \Pi$.

Let S be a connected complex manifold and $\{\Gamma_s\}_{s\in S}$ a family of discrete subgroups of $SL(2, \mathrm{C})$ parametrized by S. We will say that $\{\Gamma_s\}_{s\in S}$ is a *holomorphic family of discrete groups* if there is an abstract group G and a mapping

$$\delta: S \times G \longrightarrow SL(2, \mathrm{C})$$

with the properties: (i) for fixed $g \in G$, $\delta(s, g)$ is holomorphic in $s \in S$; and (ii) for fixed $s \in S$, $\delta(s, g): G \rightarrow SL(2, \mathrm{C})$ is an injective homomorphism with image Γ_s. If we suppose that G is generated by elements

$$\{\gamma_1, \cdots, \gamma_g; \delta_1, \cdots, \delta_g; \varepsilon_1, \cdots, \varepsilon_M\}$$

with the defining relation (3.13), then in all cases we shall consider $\{\Gamma_s\}_{s\in S}$ will be given by matrices

$$\{A_1(s), \cdots, A_g(s); B_1(s), \cdots, B_g(s); C_1(s), \cdots, C_M(s)\} \tag{4.1}$$

which vary holomorphically with $s \in S$ and which satisfy (3.14). Thus $\{\Gamma_s\}_{s \in S}$ is given by a holomorphic mapping $\delta: S \to R(g, M)$ where the latter space is defined in § 3.

In case the Γ_s are quasi-Fuchsian groups, we shall make the additional assumption that the boundary curve $\Pi(s)$ is given parametrically by

$$(\theta, s) \longrightarrow \pi(\theta, s) \qquad (0 \leqq \theta \leqq 2\pi,\ s \in S)$$

where $\pi(\theta, s)$ is continuous in both θ and s and is holomorphic in s.

LEMMA 4.2. *A holomorphic family of Fuchsian groups is constant, up to conjugation within* $SL(2, \mathbf{C})$.

Proof. This follows from the fact that the boundary circles $\Pi(s)$ may be holomorphically transformed to the real axis, so that we may assume the Γ_s give a holomorphically varying family of discrete subgroups of $SL(2, \mathbf{R})$. Such a family is obviously constant. Q.E.D.

Now we suppose that $\{\Gamma_s\}_{s \in S}$ is a holomorphically varying family of quasi-Fuchsian groups given by 2×2 matrices (4.1) which satisfy (3.14). Define $W \subset S \times \mathbf{P}_1$ to be the set of points $(s, w) \in S \times \mathbf{P}_1$ such that $w \in D_+(s)$.[15] The group G acts as holomorphic automorphisms of W by

$$g \cdot (s, w) = \big(s, \delta(s, g) \cdot w\big) ,$$

and this action is properly discontinuous and without fixed points.

LEMMA 4.3. *The quotient space* $U = W/G$ *is a holomorphic family of algebraic curves in the sense of* § 3.

Proof. This follows from standard methods involving fundamental domains of Fuchsian groups together with the observation that the "cusps" of Γ_s, i.e. the fixed points of the parabolic transformations $C_1(s), \cdots, C_M(s)$ will vary holomorphically with s.

From this Lemma together with (3.11) we conclude

PROPOSITION 4.4. *Let* $\{\Gamma_s\}_{s \in S}$ *be a holomorphic family of quasi-Fuchsian groups. Then, given* $s_0 \in S$ *there is a neighborhood* N *of* s_0 *in* S *such that:* (i) *for* $s \in N$, *the algebraic curve* $C_s = D_+(s)/\Gamma_s$ *is given by a polynomial equation* (3.3) *which is in general position*; (ii) *there is an admissible D.E.*

$$\frac{d^2\mu}{dx^2} + R(x, y; s)\mu = 0 \tag{E$_s$}$$

[15] $D_+(s)$ is the region "to the left" of the oriented Jordan curve $\Pi(s)$.

such that (E_s) *has monodromy group* Γ_s *and such that the étale map*

$$\mu_{E_s}\colon \tilde{C}_s \longrightarrow D_+(s)$$

is a conformal homeomorphism.

5. Bers theory of simultaneous uniformization

Let Γ_0 be a Fuchsian group which is of the first kind and which contains no elliptic elements. Then the quotient $L/\Gamma_0 = C_0$ of the lower-half-plane L by Γ_0 is a smooth algebraic curve C_0, which is also canonically a Zariski open set on a smooth, complete curve $\bar{C}_0$. We may assume that C_0 has a polynomial equation $p_0(x, y) = 0$, which is in general position as explained in § 3. There will be a *unique* admissible D.E.

$$\frac{d^2\mu}{dx^2} + R_0(x, y)\mu = 0 \tag{E_0}$$

on C_0 such that the associated étale mapping

$$\mu_{E_0}\colon \tilde{C}_0 \longrightarrow \mathbf{P}_1$$

gives the identification $\tilde{C}_0 = L$. The monodromy group $\Gamma(E_0)$ is Γ_0, and for the coefficient $R_0(x, y)$ we have the formula $R_0(x, y) = \{\zeta, x\}$ where $\zeta = \xi + \sqrt{-1}\eta$ is the coordinate on L.

The Poincaré metric $ds^2 = |d\zeta|^2/\eta^2$ on L is invariant under Γ_0 and induces a metric $ds^2_{\Gamma_0}$ on C_0 with constant Gaussian curvature -4. If $z_\alpha \in \bar{C}_0 - C_0$ is a puncture, then there is a parabolic transformation $\gamma_\alpha \in \Gamma_0$ which, under the isomorphism $\pi_1(C_0) \cong \Gamma_0$, corresponds to a path surrounding z_α. By conjugation within $SL(2, \mathbf{R})$, we may assume that γ_α is given by the standard translation $\gamma_\alpha(\zeta) = \zeta + 1$. When this is done, we may take

$$t_\alpha = e^{-2\pi\sqrt{-1}\zeta}$$

as a local holomorphic coordinate on $\bar{C}_0$ around the puncture z_α. Using this coordinate system, the Poincaré metric is given by

$$ds^2_{\Gamma_0} = \frac{|dt_\alpha|^2}{|t_\alpha|^2(\log|t_\alpha|)^2}\,. \tag{5.1}$$

Suppose that we now consider algebraic D.E.'s on C_0 of the form

$$\frac{d^2\mu}{dx^2} + [R_0(x, y) + Q(x, y)]\mu = 0 \tag{E_Q}$$

which are everywhere holomorphic on C_0 but which for the moment are not necessarily admissible. These D.E.'s are parametrized by rational, holomorphic quadratic differentials

$$Q = Q(x, y)\left(\frac{dx}{\frac{\partial p_0}{\partial y}}\right)^2$$

on C_0. Using the Poincaré metric we may define the sup norm

$$||Q|| = \max |Q(z_0)| , \qquad (z_0 \in C_0)$$

which may of course be infinite.

LEMMA 5.2. *The differential equation* (E_Q) *is admissible if, and only if, the sup norm* $||Q||$ *is finite.*

Proof. Around a puncture z_α, we write

$$Q = \frac{H(t_\alpha)(dt_\alpha)^2}{t_\alpha^N}$$

where $H(t_\alpha)$ is holomorphic and $H(0) \neq 0$. From (5.1) it follows that the norm of Q is bounded near z_α if, and only if, the order of the pole N is <2.

Q.E.D.

We let $E(\Gamma_0)$ denote the vector space of quadratic differentials on C_0 such that $||Q|| < \infty$.

Note that, by the Riemann-Roch theorem,

$$\dim_{\mathbf{C}} \{E(\Gamma_0)\} = 3g - 3 + M .$$

LEMMA 5.3. (*Nehari*): *For the differential equation* (E_Q) *we let* $\mu_Q: L \rightarrow \mathbf{P}_1$ *be the associated étale map. Then, if* μ_Q *is schlicht,*

$$||Q|| \leqq 3/2 .$$

Proof. By the usual Nehari theorem given in [1, page 126], we have the estimate

$$|\{\mu_Q, \zeta\}| \leqq \frac{3\eta^{-2}}{2} .$$

Our lemma follows from this together with the definition of the Poincaré metric on L and the relations (cf. [1, page 125]):

$$\begin{cases} \{\mu_Q, x\} = \{\mu_Q, \zeta\}\zeta'(x)^2 + \{\zeta, x\} \\ \{\mu_Q, x\} = Q(x, y) + R_0(x, y) \\ \{\zeta, x\} = R_0(x, y) . \end{cases}$$

Definition (*Bers*). The *Teichmüller space* $T(\Gamma_0)$ is the set of all $Q \in E(\Gamma_0)$ such that: (i) the differential equation (E_Q) is quasi-Fuchsian, and (ii) the resulting étale mapping

$$\mu_Q: L \longrightarrow \mathbf{P}_1$$

can be extended to a *quasi-conformal* homeomorphism of $\mathbf{P}_1$ to itself.

A central result concerning $T(\Gamma_0)$ is the

THEOREM 5.4. *$T(\Gamma_0)$ is a domain of holomorphy in $B(\Gamma_0)$ which is topologically a cell. Furthermore, if $B(\delta) = \{Q \in E(\Gamma_0): \|Q\| < \delta\}$ is the ball of radius δ in* $\mathrm{E}(\Gamma_0)$, *we have the strict inclusions*

$$B(1/2) \subset T(\Gamma_0) \subset B(3/2)\ .^{(16)}$$

Let Q be a point of $T(\Gamma_0)$ and $\mu_Q: L \to \mathbf{P}_1$ the associated étale mapping. This map is not unique, but following Bers [5], it may be normalized by requiring μ_Q to have the series expansion

$$\mu_Q(\zeta) = \frac{1}{\zeta + i} + \alpha_1(\zeta + i) + \cdots \tag{5.5}$$

around $\zeta = -i$.

LEMMA 5.6. *Let $Q \in T(\Gamma_0)$ and μ_Q be normalized by* (5.5). *Then, for a suitable constant r_0, the image $\mu_Q(L)$ contains the disc*

$$|\zeta| > r_0$$

on the Riemann sphere $\mathbf{P}_1$.

Proof. This is just a reformulation of Koebe's 1/4-theorem given in [11, page 350]. Q.E.D.

For $Q \in T(\Gamma_0)$, we normalize μ_Q by (5.5) and let Γ_Q be the transform of Γ_0 by μ_Q. Thus Γ_Q is the monodromy group of the algebraic differential equation (E_Q); by assumption it is a quasi-Fuchsian group. The family

$$\{\Gamma_Q\}_{Q \in T(\Gamma_0)} \tag{5.7}$$

gives a holomorphic family of quasi-Fuchsian groups according to the definition in § 4. If we write for the region of discontinuity

$$R(\Gamma_Q) = D_+(Q) \cup D_-(Q)$$

where $D_-(Q)$ is the transform of the lower half-plane L by μ_Q, then from Proposition 4.4 and Lemma 5.6 we have

PROPOSITION 5.8. *The curves $C_Q = D_+(Q)/\Gamma_Q$ form a holomorphic family $U(\Gamma_0) \to T(\Gamma_0)$ of algebraic curves as defined in* § 3. *On each curve C_Q we have a distinguished admissible differential equation*

$$\frac{d^2v}{dx^2} + R(x, y; Q)v = 0$$

such that the corresponding étale map

(16) Cf. [5] and the references given there.

$$v_Q: \tilde{C}_Q \longrightarrow D_+(Q)$$

gives an isomorphism of the universal covering $\tilde{C}_Q$ with a simply-connected domain $D_+(Q)$ contained in the disc $|\zeta| < r_0$ in the ζ-plane.

To conclude this section I shall relate the above definition of the Teichmüller space with the usual one as given, e.g., in [1]. The equivalence of definitions is essentially Bers' theorem on simultaneous uniformization as presented in [4].

Let $C_0^* = H/\Gamma_0$ be the conjugate algebraic curve to C_0, and consider all pairs (C, h) where $h: C_0^* \to C$ is a quasi-conformal homeomorphism from C_0^* to an algebraic curve C. Introduce the equivalence relation

$$(C_1, h_1) \sim (C_2, h_2)$$

whenever $h_2 \circ h_1^{-1}$ is homotopic to a conformal mapping of C_1 onto C_2. We denote by $T^\#(\Gamma_0)$ the set of all such equivalence classes of pairs (C, h), and this $T^\#(\Gamma_0)$ is the usual definition of the Teichmüller space. To get a map

$$\mu: T(\Gamma_0) \longrightarrow T^\#(\Gamma_0) \ , \tag{5.9}$$

we let $Q \in T(\Gamma_0)$ and choose a quasi-conformal extension U_Q of the schlicht mapping μ_Q. Then U_Q maps the upper half-plane H quasi-conformally onto $D_+(Q)$ in a manner such that the mapping diagram

$$\begin{array}{ccc} H & \xrightarrow{U_Q} & D_+(Q) \\ \downarrow & & \downarrow \\ C_0^* = H/\Gamma_0 & \xrightarrow{h_Q} & D_+(Q)/\Gamma_Q = C_Q \end{array}$$

is commutative and yields a point (C_Q, h_Q) in $T^\#(\Gamma_0)$. The map (5.9) is then given by

$$\mu(Q) = (C_Q, h_Q) \ ,$$

and it is a theorem [4] that this μ is an isomorphism of sets.

6. Proof of Theorem I

Let $\pi_S: U \to S$ be an analytic family of algebraic curves as defined in § 3. If T is a complex manifold and $g: T \to S$ a holomorphic mapping, then there is an induced analytic family of algebraic curves $\pi_T: U_T \to T$ where U_T is the *fibre product* $U \times_S T$.[17] Thus

$$C_t \cong C_{g(t)} \ ,$$

and there is a commutative diagram of holomorphic mappings

[17] Recall that $U \times_S T \subset U \times T$ is the set of all pairs (z, t) satisfying $\pi_S(z) = g(t)$.

$$\begin{array}{ccc} U_T & \longrightarrow & U \\ {\scriptstyle \pi_T}\big\downarrow & & {\scriptstyle \pi_S}\big\downarrow \\ T & \longrightarrow & S\,. \end{array}$$

In particular, the universal covering map $\pi\colon \tilde{S} \to S$ yields an analytic family of curves $U_{\tilde{S}} \to \tilde{S}$ with $C_{\tilde{s}} \cong C_{\pi(\tilde{s})}$ for all $\tilde{s} \in \tilde{S}$.

The main step in the proof of Theorem I is

LEMMA 6.1. *Choose a base point $\tilde{s}_0 \in \tilde{S}$. Then there is a quasi-Fuchsian group Γ_0 and a holomorphic mapping $Q\colon \tilde{S} \to T(\Gamma_0)$ such that the induced family $U(\Gamma_0)_{\tilde{S}}$ of algebraic curves is $U_{\tilde{S}}$.*[18]

Proof. We will define a mapping $P\colon \tilde{S} \to T^{\#}(\Gamma_0)$, and then Q will be the composition of P with the mapping μ given under (5.9). Fixing $\tilde{s}_0$, we choose a Fuchsian group Γ_0 such that

$$C_{\tilde{s}_0} \cong H/\Gamma_0 \qquad \text{(uniformization theorem)}\ .$$

For each point $\tilde{s} \in \tilde{S}$, there is a homeomorphism $h_s\colon C_{\tilde{s}_0} \to C_{\tilde{s}}$ and the homotopy class of $h_{\tilde{s}}$ is well-defined. Furthermore, as is evident from the definition of an analytic family of algebraic curves, we may choose $h_{\tilde{s}}$ to be quasi-conformal. The map P is then defined by:

$$P(\tilde{s}) = (C_{\tilde{s}}, h_{\tilde{s}}) \in T^{\#}(\Gamma_0)\ .$$

It is almost evident that $Q = \mu \circ P$ is continuous, and we must prove that it is holomorphic.

This is a local question and consequently may be reduced to the following: Let $\Delta \subset \mathbf{C}^m$ be a polycylinder with coordinates $t = (t_1, \cdots, t_m)$. Suppose that we are given a situation $\bar{f}\colon \bar{X} \to \Delta$, where $\bar{X}$ is a complex manifold and f is a connected, smooth, proper holomorphic mapping of fibre dimension one. Suppose furthermore that we are given smooth, disjoint analytic subvarieties $Z_1, \cdots, Z_M$ which meet each fibre $\bar{C}_t = f^{-1}(t)$ transversely in M points $\{z_1(t), \cdots, z_M(t)\}$. Letting $X = \bar{X} - \{Z_1 \cup \cdots \cup Z_M\}$, we arrive at an analytic family of algebraic curves $f\colon X \to \Delta$, and it will suffice to prove the analyticity of the mapping Q in this case.

Suppose first that $X = \bar{X}$ and choose a Fuchsian group $\bar{\Gamma}_0$ such that

$$\bar{C}_0 \cong H/\bar{\Gamma}_0\ .\text{[19]}$$

Then the same argument as above leads to a mapping

$$\bar{Q}\colon \Delta \longrightarrow T(\bar{\Gamma}_0)\ .$$

[18] Recall from §5 that $U(\Gamma_0) \to T(\Gamma_0)$ is the universal family of algebraic curves over the Teichmüller space given by Proposition 5.8.

[19] In this argument, the cases $g = 0, 1$ must be treated separately.

To see that $\bar{Q}$ is holomorphic, we observe that it follows from (3.3) that the (normalized) period matrix of $\bar{C}_t$ depends holomorphically on t.[20] Because of the theorem of Torelli-Rauch [14], these normalized periods of the abelian differentials, which depend only on the image point $\bar{Q}(t) \in T(\bar{\Gamma}_0)$, will essentially give local *holomorphic* coordinates on $T(\bar{\Gamma}_0)$. This implies that $\bar{Q}$ is holomorphic, and our lemma now follows in the case where the curves are complete.

In the general case, we choose a Fuchsian group Γ_0 such that

$$C_0 \cong H/\Gamma_0 .$$

Then an easy argument shows that there is a *holomorphic* fibration $T(\Gamma_0) \to T(\bar{\Gamma}_0)$ whose fibre over a curve $C \in T(\bar{\Gamma}_0)$ is the obvious Zariski open subset of the M-fold symmetric product of C. Moreover, we have a commutative diagram of holomorphic mappings

$$\begin{array}{ccc} \Delta & \xrightarrow{Q} & T(\Gamma_0) \\ \| & & \downarrow \\ \Delta & \xrightarrow[\bar{Q}]{} & T(\bar{\Gamma}_0) \end{array} ,$$

and it is clear that Q is obtained by composing the holomorphic mapping $\bar{Q}$ with a *holomorphic* cross-section of $T(\Gamma_0) \to T(\bar{\Gamma}_0)$ defined over the image $\bar{Q}(\Delta)$. From this we may deduce our lemma.

LEMMA 6.2. *Let π_S: $U \to S$ be an analytic family of algebraic curves and make the two assumptions*: (i) *for the fibres C_s we have*

$$3g - 3 + M \geqq 0 ,$$

and (ii) *the universal covering $\tilde{S}$ is biholomorphic to a bounded, contractible domain of holomorphy in $\mathbf{C}^{n-1}$. Then the universal covering $\tilde{U}$ of U is also biholomorphic to a bounded, contractible domain of holomorphy in $\mathbf{C}^n$.*

Proof. From the exact homotopy sequence of the fibration π_S: $U \to S$ and the fact that the higher homotopy groups $\pi_i(S) = 0$ $(i \geqq 2)$, we have

$$\begin{cases} 1 \longrightarrow \pi_1(C_{s_0}) \longrightarrow \pi_1(U) \longrightarrow \pi_1(S) \longrightarrow 1 \\ 1 \longrightarrow \pi_1(C_{\tilde{s}_0}) \longrightarrow \pi_1(U_{\tilde{s}}) \longrightarrow 1 . \end{cases}$$

[20] Referring to (3.3), we may choose polynomials $q_\alpha(x, y; t)$ $(\alpha = 1, \cdots, g)$, whose coefficients are holomorphic functions of t, such that the differentials

$$\varphi_\alpha(t) = \frac{q_\alpha(x, y; t)dx}{\frac{\partial p}{\partial y}} \qquad (\alpha = 1, \cdots, g)$$

give a basis for the holomorphic 1-forms on $\bar{C}_t$. The entries in the period matrix are then integrals $\int_\delta \varphi_\alpha(t)$ of these holomorphic differentials over closed paths on $\bar{C}_t$, and from this it is clear that the period matrix varies holomorphically with t.

From this we find a commutative diagram of holomorphic mappings

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\psi} & U_{\widetilde{S}} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\pi_{\widetilde{S}}} \\ \widetilde{S} & = & \widetilde{S} \end{array}$$

which has the property that, upon setting $\widetilde{C}_s = \varphi^{-1}(\widetilde{s})$,

$$\psi\colon \widetilde{C}_{\widetilde{s}} \longrightarrow C_{\widetilde{s}}$$

is the universal covering mapping of the algebraic curve $C_{\widetilde{s}} = \pi_s^{-1}(\widetilde{s})$. Using Lemma 6.1, we may replace $U_{\widetilde{S}}$ by $U(\Gamma_0)_{\widetilde{S}}$, so that it will suffice to prove the present lemma for the analytic family $U(\Gamma_0)_{\widetilde{S}}$ which is induced from the Teichmüller family.

We may consider $\widetilde{S}$ as already being embedded in $\mathbf{C}^{n-1}$. In $\mathbf{C}^n$ we consider the domain $\mathbf{D}$ of all points $(\widetilde{s}, \zeta) \in \mathbf{C}^{n-1} \times \mathbf{C}$ which satisfy

$$\zeta \in D_+(\Gamma_{Q(\widetilde{s})}) .$$

According to Proposition 5.8, $\mathbf{D}$ is a bounded, contractible domain in $\mathbf{C}^n$. It is essentially obvious that $\mathbf{D}$ is the universal covering of $U(\Gamma_0)_{\widetilde{S}}$, and $\mathbf{D}$ is therefore biholomorphic to $\widetilde{U}$.

To see that $\mathbf{D}$ is a domain of holomorphy, we recall that there is a continuous mapping

$$\Pi\colon \widetilde{S} \times S^1 \longrightarrow \mathbf{C} \qquad (S^1 = \text{circle})$$

given by $(\widetilde{s}, \theta) \rightarrow \Pi(\widetilde{s}, \theta)$ which has the properties of (i) being holomorphic in $\widetilde{s}$ for fixed θ, and (ii) mapping onto a simple, oriented Jordan curve $\Pi(\widetilde{s})$ in $\mathbf{C}$ for fixed $\widetilde{s}$. Now $\mathbf{D}$ consists of all pairs of points $(\widetilde{s}, \zeta) \in \mathbf{C}^{n-1} \times \mathbf{C}$ such that $\widetilde{s} \in \widetilde{S}$ and ζ is interior to the Jordan curve $\Pi(\widetilde{s})$. If K is a compact subset of such a domain $\mathbf{D}$, then an easy application of the maximum principle shows that the set

$$\hat{K} = \{z \in \mathbf{D}\colon |f(z)| \leqq ||f||_K \text{ for } f \in \mathcal{O}(\mathbf{D})\}$$

is compact. Thus $\mathbf{D}$ is holomorphically convex and, by a standard result [10], a domain of holomorphy. Q.E.D.

Proof of Theorem I. This now follows from Lemma 2.2, Lemma 3.2, and Lemma 6.2.

Proof of Corollary B. This follows from Lemma 2.2, Lemma 3.2, Lemma 6.1, and Theorem 4 in [4].

EXAMPLE 6.3. Suppose that the algebraic variety V is a smooth hypersurface in $\mathbf{P}_3$. We may choose affine coordinates (x, y, z) such that V has the affine equation

$$p(x, y, z) = 0 \tag{6.4}$$

for a polynomial $p(x, y, z)$ of degree n, and such that the axes are in general position relative to V. The Zariski open set U will be obtained from the affine surface (6.4) by removing the residual intersection with N tangent-planes

$$x = x_\alpha \qquad (\alpha = 1, \cdots, N)$$

where N is the *class* of V. We let $S = \mathbf{C}\{x\} - \{x_1, \cdots, x_N\}$ so that the plane curves $p(x, y, z) = 0$ (x = constant) are all homeomorphic for $x \in S$. Denote by $\tilde{S}$ the universal covering of $\tilde{S}$ and let $\tilde{x}$ be a global coordinate on $\tilde{S}$. Then the embedding

$$\tilde{U} \longrightarrow \mathbf{C}^2 \tag{6.5}$$

will be given by a pair of holomorphic functions μ_1, μ_2 on $\tilde{U}$ which are obtained as follows: We can find a pair of differential equations

$$\frac{d^2v}{dx^2} + R(x)v = 0 \tag{E_R}$$

$$\frac{d^2w}{dy^2} + T(\tilde{x}; y, z)w = 0 \ , \tag{E_T}$$

where $R(x)$ is a rational function of x and $T(\tilde{x}; y, z)$ is holomorphic in $\tilde{x}$ and rational in y and z, and such that μ_1 and μ_2 are given by

$$\begin{cases} \mu_1 = v_1/v_2 \\ \mu_2 = w_1/w_2 \ , \end{cases}$$

where v_1, v_2 are solutions to (E_R) and w_1, w_2 are solutions of (E_T). Thus, the embedding is given by integration of differential equations which are "partially algebraic". It is not known whether or not $T(\tilde{x}; y, z)$ is rational in x, as this is essentially the question of how the *Teichmüller modular group* acts on the Teichmüller space $T(\Gamma_0)$ relative to the embedding $T(\Gamma_0) \subset E(\Gamma_0)$.

7. Proof of Theorem II

We first need a few preliminary notions. Let M be a complex manifold with an Hermitian metric ds_M^2. Locally we may choose C^∞ (1, 0) forms on M so that $ds_M^2 = \sum_j \varphi_j \bar{\varphi}_j$. The *associated 2-form* is $\omega_M = \sqrt{-1}/2\,\{\sum_j \varphi_j \wedge \varphi_j\}$. There is an intrinsic Hermitian connection induced by the metric ds_M^2 [8, page 416], and we shall write $\Phi_{ij} = \sum_{k,l} R_{ij k\bar{l}} \varphi_k \wedge \bar{\varphi}_l$ for the curvature matrix. The *holomorphic sectional curvatures* $K_M(\xi)$ and *Ricci form* Ric_M are given respectively by

$$\begin{cases} K_M(\xi) = \sum_{i,j,k,l} R_{ij\bar{k}\bar{l}}\xi_j\bar{\xi}_j\xi_k\bar{\xi}_l \\ \mathrm{Ric}_M = \frac{\sqrt{-1}}{2}\{\sum_{i,k,l} R_{ii\bar{k}\bar{l}}\varphi_k \wedge \bar{\varphi}_l\} . \end{cases}$$

From [8, page 425], we have the

LEMMA 7.1. *Let M be a complex manifold with Hermitian metric ds^2_M, and let N be a complex submanifold with induced metric ds^2_M. Then the holomorphic sectional curvatures satisfy the inequalities*

$$K_N \leqq K_M .$$

From this lemma plus standard properties of the Poincaré metric on the unit disc, we have

LEMMA 7.2. *Let M be a complex manifold whose universal covering is the polycylinder in $\mathbf{C}^m$ and whose ds^2_M is the Poincaré metric. Let N be a complex submanifold of M with the induced metric ds^2_N. Then ds^2_N is a complete Kählerian metric whose holomorphic sectional curvatures and Ricci forms satisfy the inequalities*

$$\begin{cases} K_N \leqq -1 \\ \mathrm{Ric}_N \leqq -\omega_N . \end{cases}$$

Theorem II now follows from this lemma together with Lemma 2.3 and the following

PROPOSITION 7.3. *Let N be any smooth algebraic variety having an Hermitian metric whose Ricci form satisfies the inequality*

$$\mathrm{Ric}_N \leqq -\omega_N .$$

Then the total volume $\int_N (\omega_N)^n$ of N is finite.

Proof. The proof is based on the following two lemmas, the first of which is taken from [6], and the second of which is proved by an easy direct calculation. To state these, we first define a *punctured polycylinder* P^* to be given in $\mathbf{C}^n$ by

$$P^* = \{z = (u_1, \cdots, u_k, v_1, \cdots, v_{n-k}) \in \mathbf{C}^n\colon\ 0 < |u_\alpha| < 1, |v_\mu| < 1\} . \tag{7.4}$$

We let $P^*(\delta)$ be the "concentric" open subset of P^* in (7.4) which is defined by $|u_\alpha| < \delta$, $|v_\mu| < \delta$ ($\delta \leqq 1$). On P^* we take the Poincaré metric $ds^2_{P^*}$ induced from its universal covering.

LEMMA 7.5. *Let N be as in the statement of Proposition 7.3 and let $f\colon P^* \to N$ be a holomorphic mapping of a punctured polycylinder into N. Then*

f is volume-decreasing in the sense that we have the inequality

$$f^*(\omega_N)^n \leqq (\omega_{P^*})^n .$$

LEMMA 7.6. *For $\delta < 1$, the total volume*

$$\int_{P^*(\delta)} (\omega_{P^*})^n < \infty$$

of the concentric punctured polycylinder is finite.

To prove Proposition 7.3, we take a *smooth completion* $\bar{N}$ of N. Thus $\bar{N}$ is a smooth, complete algebraic variety such that

$$\bar{N} - N = D_1 \cup \cdots \cup D_k$$

is a union of non-singular divisors intersecting transversely. This implies that we may choose finitely many points $p_1, \cdots, p_l \in \bar{N} - N$ and neighborhoods U_j of p_j such that: (i) the intersections $U_j^* = U_j \cap N$ are biholomorphic to punctured polycylinders, and (ii) the complement $N - [\bigcup_{j=1}^{l} U_j^*]$ is a compact subset of N. Proposition 7.3 now follows from this together with Lemmas 7.5 and 7.6. Q.E.D.

Proof of Corollary A. This is evident from (2.3) and the usual Picard theorem.

In connection with Corollary A, we should like to pose the following question which is an analogue of Proposition 7.3.

Question 7.7. Let Δ be the disc $\{|z| < 1\}$ and Δ^* the punctured disc $\{0 < |z| < 1\}$. Let N be a smooth, quasi-projective algebraic variety which has a Kählerian metric ds_N^2 which is complete and has all holomorphic sectional curvatures $\leqq -1$. Then is any holomorphic mapping

$$f: \Delta^* \longrightarrow N$$

necessarily a meromorphic mapping?

8. Some applications and open questions

(a) *On rigidifying the universal coverings of Zariski open sets.*

Let D be a complex n-manifold which is topologically a cell and which has additionally the following two properties: (i) there is a subgroup $\Gamma \subset \mathrm{Aut}(D)$ which acts freely and properly discontinuously on D such that D/Γ is a quasi-projective algebraic variety;[21] (ii) there is at least one embedding

$$f: D \longrightarrow \mathbf{C}^n \tag{8.1}$$

[21] Thus D/Γ is a Zariski open subset of a smooth, projective variety.

of D into $\mathbf{C}^n$ as a bounded domain of holomorphy.

Problem A. Let $\mathcal{F}$ be the class of all embeddings (8.1) of D into $\mathbf{C}^n$ as a bounded domain. Then are there any distinguished embeddings $f \in \mathcal{F}$ which serve to single out the class of bounded domains in $\mathbf{C}^n$ which may be universal coverings of quasi-projective algebraic varieties?

Let me elaborate a little on this question. For $n = 1$ there is the *Riemann mapping theorem*, which gives a distinguished $f \in \mathcal{F}$ such that $f(D)$ is the unit disc in $\mathbf{C}$. For $n > 1$ there is certainly no Riemann mapping theorem and any bounded domain in $\mathbf{C}^n$ will, so to speak, depend on an ∞ of parameters. What I am asking is whether we can single out a class of embeddings $f \in \mathcal{F}$ such that the image $f(D)$ would appear as "nice" as possible and would depend on only *finitely* many continuous parameters. This is perhaps not unreasonable since the algebraic variety D/Γ depends on only finitely many parameters.

As a suggestion on how to select such distinguished embeddings $f \in \mathcal{F}$, let me recall from the one-variable case the following result [11, chapter 17]:

(8.2) Let D be a simply connected bounded domain in $\mathbf{C}$ and select a point $z_0 \in D$. Let $\mathcal{F}_0$ be the set of all univalent mappings $f\colon D \to \mathbf{C}$ such that $f(z_0) = 0$ and $f'(z_0) = 1$. For each such $f \in \mathcal{F}_0$, we set

$$V(f) = \int_D |f'(z)|^2 dxdy \,. \tag{8.3}$$ [22]

Then there is a unique $f_0 \in \mathcal{F}_0$ which minimizes $V(f)$, and the image $f_0(D)$ is a disc $\{|z| < r\}$ in $\mathbf{C}$.

In several variables, we may again normalize the embeddings and let $\mathcal{F}_0$ be the set of all embeddings (8.1) such that $f(z_0) = 0$ and such that the Jacobian matrix of f at z_0 is the identity matrix.[23] We may then define various functionals on $\mathcal{F}_0$ and look for mappings $f_0 \in \mathcal{F}_0$ which are extremal for these functionals. If we try to carry over to several variables the minimizing of the integral (8.2), then there is trouble because the group of all volume-preserving automorphisms of $\mathbf{C}^n$ which leave the origin fixed is an infinite dimensional group.[24] What might be tried is to minimize the *length-area ratio*

[22] $V(f)$ is the Euclidean area of the image $f(D)$ in $\mathbf{C}$.

[23] Recall that, since D has at least one embedding in $\mathbf{C}^n$ as a bounded domain, the automorphism group of D is a finite dimensional Lie group, and any automorphism of D which leaves z_0 fixed and which acts trivially in the tangent space at z_0 is necessarily the identity.

[24] For example, in $\mathbf{C}^2$ with coordinates (u, v), if $e(u, v)$ is any entire function with $e(0, 0) = 0$, then the transformation $(u, v) \to (u + e(u, v), v)$ will be a volume-preserving automorphism of $\mathbf{C}^2$ which leaves the origin fixed.

$$R(f) = \frac{\mathcal{H}_{2n-1}[\partial(f(D))]^{2n}}{\mathcal{H}_{2n}(f(D))^{2n-1}} \tag{8.4}$$

where $\mathcal{H}_k(S)$ is the *Hausdorff k-measure* of any subset $S \subset \mathbf{C}^n$. A first difficulty here is that, for the embeddings $f \colon \tilde{U} \to \mathbf{C}^n$ constructed in the proof of Theorem I, it is probably the case that $\mathcal{H}_{2n-1}[\partial(f(\tilde{U}))]$ is *infinite.* I would guess that this is not too serious, and being optimistic, it seems possible to me that we might always find an $f \in \mathcal{F}$ such that $f(D)$ is an *analytic polyhedron.*

(b) *Some general remarks on universal coverings of complex manifolds.*

The questions discussed in this paper may be thought of as special cases of the following general

Problem B. Let D be a complex manifold which is the universal covering of a quasi-projective algebraic variety. Then what function-theoretic properties does D possess?

We shall discuss a few examples related to this problem. For this we let $\Gamma \subset \operatorname{Aut}(D)$ be a group of covering transformations of D over a smooth complex-analytic variety D/Γ. We do *not* yet assume that D/Γ is algebraic.

(i) If $D \to D/\Gamma$ is a *finite covering* over a quasi-projective variety, then D is itself a quasi-projective variety (*Riemann existence theorem*).

(ii) If D/Γ has a complete Kählerian metric whose sectional curvatures are non-positive, then D is a *Stein manifold.* (This result is due to H. Wu, and proof goes as follows: Since D is simply-connected, the *exponential map*

$$\exp \colon T_{p_0}(D) \longrightarrow D$$

is a diffeomorphism. Thus we may use the geodesic distance on D to give a smooth exhaustion function

$$\varphi(p) = \operatorname{dist}_D(p_0, p) \,.$$

A computation then shows that the *E. E. Levi form* $dd^c\varphi$ is positive definite on the holomorphic tangent spaces to the level sets of φ. By Grauert's theorem it follows that D is a Stein manifold.) Wu has asked the following

Question 8.5. (*Wu*): If the sectional curvatures of D are negative and bounded away from zero, then is D a *bounded* domain of holomorphy in $\mathbf{C}^n$?

(iii) If D is a bounded domain in $\mathbf{C}^n$ and if Γ is a properly discontinuous group of automorphisms of D such that the quotient D/Γ is compact, then D/Γ is quasi-projective and D is a domain of holomorphy (C. L. Siegel). This result suggests the following two questions:

Question 8.6. If D is a bounded domain of holomorphy in $\mathbf{C}^n$ and $\Gamma \subset \mathrm{Aut}(D)$ is a properly discontinuous group of automorphisms such that the quotient D/Γ has *finite volume* with respect to the Bergman metric, then is D/Γ a quasi-projective algebraic variety?

Question 8.7. If D is a bounded domain in $\mathbf{C}^n$ and $\Gamma \subset \mathrm{Aut}(D)$ is a properly discontinuous group of automorphisms such that D/Γ is a quasi-projective variety, then (i) is D a domain of holomorphy, and (ii) does D/Γ have finite volume with respect to the Bergman metric?

We remark that perhaps the quickest proof of Siegel's theorem is by showing that a bounded domain in $\mathbf{C}^n$ whose Bergman metric is *complete* is necessarily a domain of holomorphy (Bremmerman). The existence of a Γ such that D/Γ is compact then guarantees the completeness of the Bergman metric. It is tempting to ask if the finite volume assumption for D/Γ also forces the Bergman metric of D to be complete?

(iv) To some extent, the basic question concerning the situation

$$D \longrightarrow D/\Gamma$$

is the following: Under what assumptions on D/Γ can we construct functions on D other than those which come from D/Γ? It is interesting to note that, at least in the nontrivial cases of which I am aware, the construction of functions on D is not done directly. For example, it is a theorem of K. Stein [15] that D is a Stein manifold if D/Γ is. Let me outline the proof of this result in order to illustrate how the necessary functions on D are proved to exist in this case.

The first step is to use the embedding theorem for Stein manifolds [10] to realize D/Γ as a complex submanifold in some $\mathbf{C}^N$. It is then easy to find an open tubular neighborhood X of D/Γ such that: (i) X is a domain of holomorphy, and (ii) X topologically retracts onto D/Γ. Passing to universal coverings, we have a situation

$$\begin{array}{ccc} D & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ D/\Gamma & \longrightarrow & X \end{array}$$

so that it will suffice to prove that $\tilde{X}$ is a Stein manifold.

Now $\tilde{X}$ is a so-called *Riemann domain*, and the coordinates $(z_1, \cdots, z_N)$ in $\mathbf{C}^N$ give an étale mapping

$$z\colon \tilde{X} \longrightarrow \mathbf{C}^N .$$

Given a point $\tilde{x} \in \tilde{X}$, we define the polycylinder $\Delta(\tilde{x}, r)$ to be the subset of $\tilde{X}$, defined by

$$|z_\alpha - z_\alpha(\tilde{x})| < r \qquad (\alpha = 1, \cdots, N) .$$

Then, following Oka, we may define

$$\delta(\tilde{x}) = \sup \{r \mid z: \Delta(\tilde{x}, r) \longrightarrow \Delta(z(\tilde{x}), r) \text{ is a homeomorphism}\} .$$

Similarly, for $x \in X$ we may let

$$\delta(x) = \sup \{r \mid \Delta(x, r) \text{ is contained in } X\} .$$

It is clear that

$$\delta(\tilde{x}) = \delta(z(\tilde{x})) .$$

On the other hand, it is a theorem of Grauert-Oka [10, page 283], that a general Riemann domain is a Stein manifold if, and only if, $-\log \delta$ is plurisubharmonic. Since $-\log [\delta(x)]$ is then plurisubharmonic, it follows that $-\log [\delta(\tilde{x})]$ is also plurisubharmonic. Q.E.D.

It should be noted that, in both this result and the theorem of Wu mentioned in (ii) above, the construction of holomorphic functions is based on Grauert's solution of the E. E. Levi problem [10].

Question 8.8. Let D be a complex manifold and $\Gamma \subset \mathrm{Aut}\,(D)$ a properly discontinuous group of automorphisms such that D/Γ is a quasi-projective algebraic variety. Then do the meromorphic functions separate points on D? Is D meromorphically convex?

(c) *On finding the Fuchsian equation.* Let C be an algebraic curve given by an affine equation (3.1). We consider the admissible differential equations (E) as described in § 3. Each such D. E. generates an étale mapping $\mu_E: \tilde{C} \rightarrow \mathbf{P}_1$, and the classical uniformization theorem is

THEOREM 8.9. *There exists a unique differential equation $E_f(C)$ (the Fuchsian D.E.) such that μ_E is univalent and the monodromy group $\Gamma(E)$ is a Fuchsian group.*

The original proof [13] of this theorem by Poincaré is more notable for its interesting discussion of the geometry of Fuchsian groups than for its mathematical rigor.[25] Subsequent complete proofs were given by Picard, Poincaré, and Koebe [2]; all of these were based on potential theory. In fact, all known complete proofs of (8.9) seem to be potential-theoretic and offer very little insight in just how to explicitly locate the Fuchsian D.E.

Now, rather than trying to locate $E_f(C)$ for a fixed curve C, we might do the following:

Let $\pi: U \rightarrow S$ be an analytic family of algebraic curves as defined in § 3.

[25] Cf. the criticism in [2, page 427].

Over S we construct the *affine* holomorphic vector bundle

$$\mathbf{E} \xrightarrow{\pi} S$$

whose fibre $\mathbf{E}_s$ is the space $E(C_s)$ of admissible D.E.'s on the curve C_s (cf. Corollary 3.6).[26] Now the Fuchsian equation $E_f(D_s)$ gives a cross-section

$$f\colon S \longrightarrow \mathbf{E} \tag{8.10}$$

of this affine bundle, and f is presumably real-analytic but, in any event it is never holomorphic unless of course the family $\pi\colon U \to S$ is trivial (cf. Lemma 4.2).

Problem C. Is it possible to characterize the *Fuchsian cross-section* (8.10)?

Here is a first guess as to how one might begin to characterize f. Given our affine vector bundle $\mathbf{E} \to S$, we let $V(\mathbf{E}) \to S$ be the corresponding ordinary vector bundle.[27] There is a well-defined operator

$$\bar{\partial}\colon C^\infty(\mathbf{E}) \longrightarrow A^{0,1}(V(\mathbf{E}))$$

from the C^∞ sections of $\mathbf{E}$ into the $C^\infty(0, 1)$ forms with values in $V(\mathbf{E})$. Given an hermitian metric h along the fibres $V(\mathbf{E}) \to S$, there is also defined a differential operator $\partial_h\colon A^{0,1}(V(\mathbf{E})) \to A^{1,1}(V(\mathbf{E}))$. Composing these we have a 2^{nd} order differential operator

$$\Delta_h\colon C^\infty(\mathbf{E}) \longrightarrow A^{1,1}(V(\mathbf{E})) ,$$

and we will say that a C^∞ section g of $\mathbf{E}$ is *harmonic* if $\Delta_h \cdot g = 0$.

Question 8.11. Does the Fuchsian cross-section f satisfy a differential equation in terms of $\bar{\partial}$ and ∂_h, where h is a suitable metric for the vector bundle $V(\mathbf{E}) \to S$ of quadratic differentials? For example, is f harmonic?

In concrete terms, if we locally give our family of curves $\pi\colon U \to S$ by polynomial equations (3.3), then for each s there will be a unique Fuchsian D.E.

$$\frac{d^2\mu}{dx^2} + R(x, y; s)\mu = 0 . \tag{$\mathrm{E}_f(C_s)$}$$

Now $R(x, y; s)$ will be rational in x, y but not in s. What I am asking for is a differential equation involving $\partial/\partial s_j$, $\partial/\partial \bar{s}_j$ which is satisfied by $R(x, y; s)$ as a function of s.

(d) *Some differential-geometric properties of Zariski open sets.*

(26) Relative to a suitable covering $\{W_\alpha\}$ of S, $\mathbf{E}$ will be obtained from product bundles $W_\alpha \times \mathbf{C}^N$ by transition relations $\zeta_\alpha = A_{\alpha\beta}(s)\zeta_\beta + B_{\alpha\beta}(s)$ $(s \in W_\alpha \cap W_\beta)$ taken from the group of affine transformations of $\mathbf{C}^N$.

(27) Referring to footnote 26, the transition functions of $V(\mathbf{E})$ are $\{A_{\alpha\beta}(s)\}$.

Referring to Theorem II, we see that sufficiently small Zariski open sets U on smooth quasi-projective algebraic varieties will have a Kählerian metric ds_U^2 which is complete and *negatively curved*.(28) However, ds_U^2 is not intrinsic and so, for example, it is not of much use in trying to determine how nice an embedding $\tilde{U} \subset \mathbf{C}^n$ we can have (cf. Problem A above).

Now the most important intrinsic differential geometric quantities associated to $\tilde{U}$ are (i) the distances and measures of Kobayashi-Eisenman [7], and (ii) the Bergman metric. We shall discuss the relevant definitions under (i).

Let M be a complex n-manifold. We shall consider holomorphic mappings $f\colon \Delta \to M$ of the unit disc Δ into M. Denote by $\delta(z, z')$ the Poincaré distance on Δ. Given $m, m' \in M$, a *chain* from m to m' will be given a sequence of pairs of points $(z_i, z_i') \in \Delta \times \Delta$ $(i = 1, \cdots, N)$ and holomorphic mappings $f_i\colon \Delta \to M$ such that: $f_1(z_1) = m$, $f_i(z_i') = f_{i+1}(z_{i+1})$, and $f_N(z_N') = m'$. We then define

$$d_M(m, m') = \inf \{\textstyle\sum_{i=1}^N \delta(z_i, z_i')\} ,$$

where the inf is taken over all chains. It is clear that d_M is a pseudo-distance which is intrinsically associated to M, and Kobayashi has defined M to be *hyperbolic* in case d_M is a true distance (cf. [7] and the references given there).

For a subset S of M, we shall now define the *hyperbolic k-measure* $\mu_M^k(S)$ of S [7]. To do this we consider holomorphic mappings $h\colon B_n \to M$ of the unit ball B_n in $\mathbf{C}^n$ into M. There is an intrinsic metric (the Bergman metric) on B_n, and we shall denote by $\sigma^k(T)$ the k-dimensional Hausdorff measure of a subset T of B_n. Given $S \subset M$, a *covering* of S will be given by subsets $\{T_i\}$ of B_n and holomorphic mappings $h_i\colon B_n \to M$ such that $S \subset \bigcup h_i(T_i)$. We now define

$$\mu_M^k(S) = \inf \{\textstyle\sum_{i=1}^\infty \sigma^k(T_i)\}$$

where inf is taken over all coverings of S. The complex manifold M is said to be *k-measure hyperbolic* if, for every real k-dimensional submanifold N and non-empty, relatively compact open subsets $S \subset N$, we have $0 < \mu_M^k(S) < \infty$.(29)

PROPOSITION 8.12. *Let V_n be a smooth quasi-projective algebraic variety. Then any point $x \in V$ has a Zariski neighborhood U which has the following properties*: (i) *U is complete hyperbolic*; (ii) *U is k-measure hyperbolic for*

(28) This means that the holomorphic sectional curvatures are all negative and bounded away from zero.

(29) Another intrinsic k-measure v_M^k is obtained by taking Hausdorff k-measure associated to d_M, the inequality $v_M^k \leq \mu_M^k$ is valid [7, page 55], and the advantage of μ_M^k is that it is more often non-trivial.

$0 \leqq k \leqq 2n$; *and* (iii) *the total volume* $\mu_U^{2n}(U)$ *of* U *is finite.*

Proof. This follows from Theorem II plus the methods used to prove Proposition (7.3).

Concerning the Bergman metric, we shall close with the following question whose answer, it seems to me, is very likely affirmative:

Question 8.13. Let D be a bounded domain in $\mathbf{C}^n$ which is topologically a cell and which has a properly discontinuous group $\Gamma \subset \mathrm{Aut}(D)$ such that D/Γ is a quasi-projective algebraic variety. Then (i) is the Bergman metric on D complete? And (ii) is the volume of D/Γ finite with respect to this metric?

The reason that we were able to prove Proposition 8.12 but are unable to answer (ii) in (8.13) is that the Kobayashi-Eisenman measures localize rather easily (once we know that they are true distances and measures), but this does not seem to be true of the Bergman metric. In this regard, let me mention that there are *finitely* many punctured polycylinders P_α^* $(\alpha = 1, \cdots, N)$ and commutative diagrams of holomorphic mappings

$$\begin{tikzcd} P_\alpha \arrow[r, "\tilde{f}_\alpha"] \arrow[d, "\pi_\alpha"'] & D \arrow[d, "\pi"] \\ P_\alpha^* \arrow[r, "f_\alpha"'] & D/\Gamma \end{tikzcd} ,$$

where P_α is an ordinary polycylinder, π_α and π are universal covering mappings, f_α is an injective holomorphic mapping, and such that

$$D/\Gamma - \textstyle\bigcup_{\alpha=1}^{N} f_\alpha(P_\alpha^*)$$

is a *compact* subset of D/Γ. Now using the facts that (i) $\tilde{f}_\alpha$ maps L^2-holomorphic n-forms on D into L^2-holomorphic n-forms on P_α, and (ii) P_α^* has finite volume with respect to the Bergman metric on P_α, it seems possible to me that question (8.13) can be localized and then perhaps resolved.

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