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# Complex determinantal processes and $H^1$ noise

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**Abstract**

For the plane, sphere, and hyperbolic plane we consider the canonical invariant determinantal point processes $\mathcal{Z}_\rho$ with intensity $\rho d\nu$, where $\nu$ is the corresponding invariant measure. We show that as $\rho \to \infty$, after centering, these processes converge to invariant $H^1$ noise. More precisely, for all functions $f \in H^1(\nu) \cap L^1(\nu)$ the distribution of $\sum_{z \in \mathcal{Z}} f(z) - \frac{\rho}{\pi} \int f d\nu$ converges to Gaussian with mean zero and variance $\frac{1}{4\pi} \|f\|_{H^1}^2$
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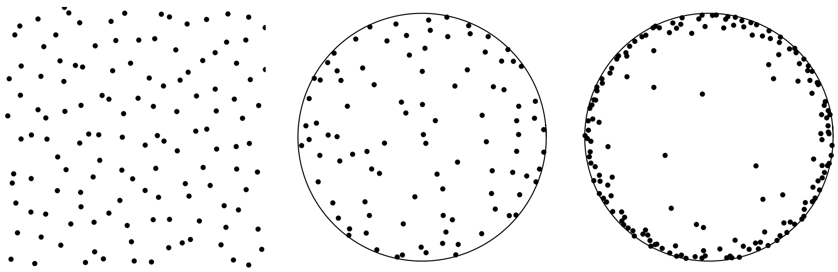

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## 1 Introduction

Determinantal processes are point processes with a built-in pairwise repulsion. They were first considered by Macchi (13) as a model for fermions in quantum mechanics, and have since been understood to arise naturally in a number of contexts, from eigenvalues of random matrices to random spanning trees and non-intersecting paths, see (1; 2; 4; 11; 12; 14; 17; 19).

A point process $\mathcal{Z}$ on $\mathbb{C}$ is determinantal if for disjoint sets $D_1, \ldots, D_k$ we have

$$\mathbf{E}\Big[\prod_{i=1}^{k} \#(D_i \cap \mathcal{Z})\Big] = \int_{\Lambda^k} \det\Big(K(x_i, x_j)\Big)_{1 \le i,j \le k} d\mu(x_1) \cdots d\mu(x_k), \tag{1}$$

for each $k \ge 1$. Here $K(x, y)$ is a symmetric, non-negative kernel and $\mu$ is a (also non-negative) Borel measure. The integrand is often called the joint intensity or correlation function of the point process.

Conversely, if a kernel $K$ defines a self-adjoint integral operator $\mathcal{K}$ on $L^2(\Lambda, \mu)$ which is locally trace class with all eigenvalues lying in $[0, 1]$, then there exists a point process satisfying (1). In this case we speak of the determinantal process $(K, \mu)$. In a weak sense, the points repel one another because the determinant vanishes on the diagonal.

The processes we consider are defined by two properties. First, they correspond to a projection $K$ to a subspace of analytic functions with respect to a radially symmetric reference measure. Second, their distribution is invariant under the symmetries of their underlying space $\Lambda$. The latter is either the complex plane $\mathbb{C}$, 2-sphere $\mathbb{S}$, or hyperbolic plane $\mathbb{U}$. We will think of them as a subsets of $\mathbb{C}$, or strictly speaking $\mathbb{C} \cup \{\infty\}$, though for the sphere $\mathbb{S}$ usually there is no harm in ignoring the point at infinity.

These properties are uniquely satisfied by a family of processes indexed by a single density parameter $\rho > 0$ on each space, see Krishnapur (10, Theorem 3.0.5).

**Planar model.** Here $\Lambda = \mathbb{C}$ and for any $\rho > 0$, consider the kernel

$$\check{K}_\rho(z, w) = e^{\rho z \bar{w}} \text{ with respect to } d\mu_\rho(z) = \frac{\rho}{\pi} e^{-\rho |z|^2} dz.$$

Here, as in the sequel, $dz$ stands for Lebesgue area measure on $\mathbb{C}$. We call the measure $\mu_\rho$ the *reference measure.*

Note that the kernel $\check{K}_\rho$ is the projection onto the span of the orthonormal set $\{\sqrt{\frac{\rho^k}{k!}} z^k\}_{k=0}^{\infty}$ in $L^2(\mathbb{C}, \mu_\rho)$, and so the above pair defines a determinantal process of infinitely many points in the complex plane.

**Spherical model.** The space $\Lambda$ is $\mathbb{S} = \mathbb{C} \cup \infty$, the two-sphere. Now $\rho$ is integer valued, $\rho = 1, 2, \ldots$ and the $(\check{K}, \mu)$ pair reads

$$\check{K}_\rho(z, w) = \sum_{k=0}^{\rho-1} \binom{\rho-1}{k} (z\bar{w})^k = (1 + z\bar{w})^{\rho-1}, \text{ with respect to } d\mu_\rho(z) = \frac{\rho}{\pi} (1 + |z|^2)^{-(\rho+1)} dz.$$

Note that the reference measure $\mu_\rho$ is typically *not* a constant multiple of the invariant measure $\nu$. $\check{K}_\rho$ is a projection kernel, onto the orthonormal polynomials $\sqrt{\binom{\rho-1}{k}} z^k$, $k = 0, \ldots, \rho - 1$ in $L^2(\mathbb{S}, \mu_\rho)$. In this case, $\rho$ is really the total number of particles.

**Hyperbolic model.** Take $\Lambda = \mathbb{U}$, the unit disk, which we identify with the hyperbolic plane. Let

$$\check{K}_\rho(z,\bar{w}) = \frac{1}{(1-z\bar{w})^{\rho+1}} \text{ with respect to } d\mu_\rho(z) = \frac{\rho}{\pi}(1-|z|^2)^{\rho-1}dz,$$

for any $\rho > 0$. As in the planar model, $(\check{K}_\rho, \mu_\rho)$ defines a determinantal process with infinitely many points; the orthonormal polynomials in $L^2(\mathbb{U}, \mu_\rho)$ being $\sqrt{\binom{\rho+k}{k}}z^k$ for $k = 1, 2, \ldots$.

The mean measure, or one-point function, in any determinantal processes is $K(z,z)d\mu(z)$. In the above models we find that $\check{K}_\rho(z,z)d\mu_\rho(z) = \frac{\rho}{\pi}d\nu(z)$, where $\nu$ is the *invariant measure* on $\Lambda$, unique up to constant multiples. Here we use

$$d\nu_{\mathbb{C}}(z) = dz,\ d\nu_{\mathbb{S}}(z) = \frac{dz}{(1+|z|^2)^2}. \quad \text{and } d\nu_{\mathbb{U}}(z) = \frac{dz}{(1-|z|^2)^2}; \tag{2}$$

The distribution of the above processes is invariant under symmetries of the respective $\Lambda$, i.e. linear fractional transformations of $\mathbb{C}$ preserving the measure $\nu_\Lambda$.

Our main theorem concerns the linear statistics for the point process. For $f : \Lambda \to \mathbb{R}$, $\nu_\Lambda$ the invariant measure, and the intrinsic gradient $\nabla_\iota$ we set

$$\|f\|^2_{H^1(\nu)} = \int_\Lambda |\nabla_\iota f|^2\, d\nu_\Lambda, \qquad \|f\|_{L^1(\nu)} = \int_\Lambda |f|\, d\nu_\Lambda,$$

and say that $f$ is in $H^1(\nu)$ or $L^1(\nu)$ if the corresponding norm is finite. With this definition $H^1(\nu)$ consists of equivalence classes of functions which differ by a constant.

**Theorem 1.** *For either the planar, spherical or hyperbolic model, let $f \in H^1(\nu) \cap L^1(\nu)$. Then, as $\rho \to \infty$, the distribution of*

$$\sum_{z\in\mathcal{Z}} f(z) - \frac{\rho}{\pi}\int_\Lambda f\, d\nu$$

*converges to a mean zero normal with variance $\frac{1}{4\pi}\|f\|^2_{H_1(\nu)}$.*

Note that for both the limiting variance and the shift to make sense it is necessary to have $f \in H^1 \cap L^1$, so the theorem holds for the most general test functions possible.

The fact that the variance is of order one manifests the advertised repulsion. The $H^1$-norm is conformally invariant, so one may replace the intrinsic gradient and intrinsic measure by the planar gradient and Lebesgue measure for the embedding.

This work is partially motivated by the recent results of Sodin and Tsilerson (18) on the three canonical *Gaussian analytic functions* (GAFs) with zero sets invariant under the symmetries of the plane, sphere, and hyperbolic plane. These processes are also indexed by a density parameter, and (18) establishes asymptotic normality for the corresponding linear statistics, with $f \in C_0^2$. What is striking is that for GAFs the variance actually decays as the density tends to infinity:

$$\mathrm{Var} \sum_{z\in\mathcal{Z}} f(z) \sim \text{const} \times \rho^{-1}\|\Delta_\iota \mathrm{f}\|^2_{\mathrm{L}^2(\nu)}.$$

Thus, the zeros of typical GAFs exhibit a higher level of repulsion than their determinantal counterparts.

The determinantal processes studied here, while attractive solely on the basis of their invariance, also arise as matrix models. The planar case is really just the infinite dimensional Ginibre ensemble. If $A$ is an $n \times n$ matrix of iid. standard complex Gaussians, then as $n \uparrow \infty$ the point process of $A$-eigenvalues converges to the planar model, and $\rho$ here corresponds to scaling. As for the spherical model, Krishnapur (10) has proved that this coincides with the eigenvalues of $A^{-1}B$, where $A$ and $B$ are independent $\rho \times \rho$ Ginibre matrices. Further, for integer $\rho$, Krishnapur provides strong evidence that the hyperbolic points have the same law as the singular points of $A_0 + zA_1 + z^2 A_2 + \cdots$ in $|z| < 1$ with again $A_0, A_1, \ldots$ independent $\rho \times \rho$ Ginibre matrices.

In all three cases, Krishnapur provides natural random analytic functions for which $\mathcal{Z}$ is the set of zeros. Using an integration by parts argument, Theorem 1 can be interpreted to say that the log characteristic polynomial of these matrix models converges to the Gaussian Free Field. See Section 3 in (15) for this relation in the Ginibre ensemble.

Theorem 1 also identifies the present as a companion paper to (15) which treats the limiting noise for the Ginibre eigenvalues. Those eigenvalues define a determinantal process in $\mathbb{C}$, see (6). In (15) it is shown that, along with an $H^1$-noise in the interior of $\mathbb{U}$ similar to above, there is an $H^{1/2}(\partial\mathbb{U})$ noise component in the corresponding $n \uparrow \infty$ Central Limit Theorem. (The $H^{1/2}$ boundary noise makes yet another connection to the, again determinantal, process of eigenvalues drawn from Haar measure on the Unitary group, see (5) or (9) for example.) The invariance and lack of boundary effects in the three models considered here makes for essentially different proofs that are shorter and rely less on combinatorial constructions.

The proof consists of three steps. In Section 2 we establish some general conditions under which (smooth) linear statistics are asymptotically normal, without computing the asymptotic variance. For this, the fact that the kernels are analytic projections, along with their specific decay properties, is crucial. In Section 3 we check that these properties are satisfied by our models. In Section 4 we determine the asymptotic variance and extend the convergence to general test functions.

## 2 General conditions for asymptotic normality

Taking a broader perspective, this section shows that under certain conditions satisfied by the models we are considering, linear functionals are asymptotically normal.

Let $B$ be a compact subset of $\mathbb{C}$. Consider the following set-up. $K_\rho : B^2 \to \mathbb{R}$ is a set of kernels indexed by $\rho$, which ranges in an unbounded subset of the positive reals. The kernels here are the Hermitian and are *with respect to Lebesgue measure*; more precisely, if $\check{K}_\rho$ denotes the kernels outlined above, then here and below

$$K_\rho(z, w) = \check{K}_\rho(z, w)\, (\kappa(z)\kappa(y))^{1/2} \tag{3}$$

where $\kappa = d\mu_\rho(z)/dz$ is the density of the reference measure.

### Kernel Properties

The eventual asymptotic normality rests on the following asymptotic properties of $K_\rho$ as $\rho \to \infty$; all limit statements and $o(\cdot)$ notations refer to this limit. Throughout, $c$ denotes a numerical constants which may change from line to line.

- **Uniform bound (UB).** It holds

$$\|K_\rho\|_\infty := \sup_{x,y\in B} |K_\rho(x,y)| \le c\rho. \tag{4}$$

- **$L^1$ bound ($L^1$B).** For $x, y \in B$ we have a bound

$$|K_\rho(x,y)| \le \varphi_\rho(x-y) \text{ with } \|\varphi_\rho\|_1 \le c. \tag{5}$$

- **Interaction decay (ID).** The above bounding function satisfies

$$\||y|^3\varphi_\rho\|_1 = o(\rho^{-1}). \tag{6}$$

- **Limited local analytic projection (LLAP) property.** Assume that $B \subset \mathbb{C}$. Fix $B_2$ compact so that $B_2 \subset B^0$. For $p = 0, 1, 2$ we have

$$\sup_{x,z\in B_2} \left| x^p K_\rho(x,z) - \int_B K_\rho(x,y)\, y^p\, K_\rho(y,z)\, dy \right| = o(1), \tag{7}$$

  Similarly,

$$\sup_{x,z\in B_2} \left| K_\rho(x,z)\bar{z}^p - \int_B K_\rho(x,y)\, \bar{y}^p\, K_\rho(y,z)\, dy \right| = o(1). \tag{8}$$

- **Covariance (CO)**. For any function $F$ with bounded third derivatives and compact support in the interior of $B$ we have $\mathrm{Cov}_\rho(\partial_{z,\bar{z}}F, z\bar{z}) = o(1)$.

Note of course that $\mathrm{Cov}_\rho(f,g)$ indicates the covariance of $\sum_{z\in\mathcal{Z}} f(z)$ and $\sum_{z\in\mathcal{Z}} g(z)$ in the $K_\rho$-process. The main proposition of this section (proved as Proposition 8) is:

**Proposition 2.** *Suppose that the above conditions are satisfied. Then, for the corresponding determinantal process any linear statistic $f$ with compact support in the interior $B^o$ and bounded third derivatives is asymptotically normal. We also have convergence of all moments.*

That these conditions are satisfied by the planar, spherical and hyperbolic models is delayed to the next section. Here we provide a lemma that sheds more light as to how condition LLAP arises.

**Lemma 3** (Analytic projections restrict to LLAP)**.** *Let $\hat{K}_\rho : S^2 \to \mathbb{R}$ be a kernel for the projection to the space of all analytic functions on the open set $S \subset \mathbb{C}$ with respect to measures $\mu_\rho$. For compact $B \subset S$, let $K_\rho$ denote the restriction of $\hat{K}_\rho(x,y)(\mu_\rho(x)\mu_\rho(y))^{1/2}$ to $B \times B$. If $K_\rho$ satisfies UB and ID, then $K_\rho$ satisfies the LLAP.*

*Proof.* Note that since for each $z$, the function $y \mapsto yK_\rho(y,z)$ is analytic, it follows from the analytic projection property that

$$\int_S K_\rho(x,y) y^p K_\rho(y,z)\, dy = x^p K_\rho(x,z).$$

Thus, for (7) it suffices to show that

$$\sup_{x,z\in B_2}\left|\int_{S\setminus B} K_\rho(x,y)\, y^p\, K_\rho(y,z)\, dy\right| = o(1), \tag{9}$$

where recall $B_2 \subset B^o$. Setting $s = y - z$, there is a polynomial $q$ of degree $p$ so that $|y^p| \le q(|s|)$ for all choices of $z \in B_2, y \in B^c$. Also, $q(|s|) \le c|s|^3$ as soon as $|s|$ bounded away from zero. So, for $z$ a positive distance from $S \setminus B$, we have that

$$|y^p\, K_\rho(y,z)| \le c|s|^3 \varphi_\rho(s),$$

and by UB, ID, the absolute value of the left hand side of (9) is bounded above by

$$c\rho \int |s|^3 \varphi_\rho(s) ds = c \times \rho \times o(\rho^{-1}) = o(1).$$

The proof of (8) is identical since $K_\rho$ is hermitian symmetric. □

## Cumulants

Recall that for any random variable $X$, the cumulants $\mathrm{Cum}_k(X)$, $k = 1, 2, \dots$, are the coefficients in the expansion of the logarithmic generating function,

$$\log \mathbf{E}[e^{itX}] = \sum_{k=1}^{\infty} \frac{(it)^k}{k!} \mathrm{Cum}_k(X),$$

and $X$ is Gaussian if and only of $\mathrm{Cum}_k(X) = 0$ for all $k \ge 3$. In any determinantal process $(K_\rho, \mu_\rho)$, the cumulants of the linear statistic $\sum f(z_k)$ have the explicit form,

$$\mathrm{Cum}_{k,\rho}(f) = \sum_{m=1}^{m} \frac{(-1)^{m-1}}{m!} \sum_{\substack{k_1+\cdots+k_m=k \\ k_1,\dots,k_m \ge 1}} \frac{k!}{k_1! \cdots k_m!} \int \Big[\prod_{i=1}^{m} f(x_i)^{k_i} K_\rho(x_i, x_{i+1})\Big] dx_1 \dots dx_m, \tag{10}$$

where $x_{m+1} = x_1$ is understood, the integral ranges over $m$ copies of the full space (here $B$), and again we are absorbing the reference measure $\mu_\rho$ into the $K_\rho$ kernel. The structure behind formula (10) has been employed several times in the past to establish asymptotic normality for various determinantal processes with various assumptions on the regularity of $f$. See in particular the pioneering work of Costin-Lebowitz (3) and the later papers of Soshnikov, (20) and (21). While going through cumulants, the method here is quite different.

We define the multiple integrals: for $f$ a function of $x_1, \dots x_k$,

$$\tilde{K}_\rho^{\circ k}(f) = \int \left[ f(x_1, \dots, x_k) \prod_{j=1}^{k} K_\rho(x_j, x_{j+1}) d\mu(x_j) \right] \tag{11}$$

(the indices are mod $k$), and, as another shorthand, if the $f_i$ are all functions of one variable, we set

$$\tilde{K}_\rho(f_1, \dots, f_k) = \tilde{K}^{\circ k}(f_1(x_1) \cdots f_k(x_k)). \tag{12}$$

Note that the cumulant (10) is just a weighted sum of terms of the form $\tilde{K}_\rho(g_1, \ldots, g_m)$, obtained by partitioning $\{1, 2, \ldots, k\}$ into $m$ parts $I_1, \ldots, I_m$ of sizes $k_1, \ldots k_m$ and setting $g_i = \prod_{j \in I_i} f_j$. Hence, we more generally seek conditions for the vanishing of $\mathrm{Cum}_\rho(f_1, \ldots, f_k)$, defined in the obvious way, for $k \geq 3$.

The first step is a collection of estimates on the integrals (12). The most fundamental of these are Lemmas 8 and 9 below. The former allows one to reduce the dimension in certain instances; the latter allows for the replacement of the test functions $f_i$ by their cubic approximations.

**Lemma 4.** *Assumptions $L^1B$ and UB imply that*

$$|\tilde{K}_\rho(f_1, \ldots, f_k)| \leq c\rho \|f_1\|_1 \prod_{\ell=2}^{\infty} \|f_\ell\|_\infty.$$

*Proof.* The integral is bounded above by

$$\int_{B^k} |f(x_1)| \, \varphi_\rho(x_2 - x_1) \cdots \varphi_\rho(x_k - x_{k-1}) \, dx_1 \cdots dx_k \times \|f_2\|_\infty \cdots \|f_\ell\|_\infty \times \sup_{x,y \in B} |K_\rho(x, y)|.$$

Changing variables, $y_i = x_i - x_{i-1}$ for $i \geq 2$ allows the remaining integral to be bounded above by

$$\int_{B \times (2B)^{k-1}} |f_1(x_1)| \varphi_\rho(y_2) \cdots \varphi_\rho(y_k) \, dx_1 dy_2 \cdots dy_k = \|f_1\|_1 \|\varphi_\rho\|_1^{k-1},$$

and the claim follows from assumptions ($L^1$B, UB). □

**Lemma 5.** *Assume $L^1B$, UB, ID and that $f_i$ are bounded for all $i$. If any $f_i$ and $f_j$ are supported on disjoint compact sets, then $\tilde{K}_\rho(f_1, \ldots, f_k) \to 0$.*

*Proof.* Now use the bound

$$\tilde{K}_\rho(f_1, \ldots, f_k) \leq c\|K\|_\infty \int \varphi_\rho(x_2 - x_1) \cdots \varphi_\rho(x_k - x_{k-1}) \, dx_1 \cdots dx_k$$

where the integral on the right is over the product of the supports of the $f_1$ to $f_k$. By adjusting $c$, we may insert $|x_i - x_j|^3$ (which is bounded below on the domain of integration) to produce

$$\begin{aligned}
\tilde{K}_\rho(f_1, \ldots, f_k) &\leq c\|K_\rho\|_\infty \int_{B \times (2B)^{k-1}} \varphi_\rho(y_2) \cdots \varphi_\rho(y_k) |y_{i+1} + \ldots + y_j|^3 \, dx_1 dy_2 \cdots dy_k \\
&\leq c\|K_\rho\|_\infty \sum_{\ell=i+1}^{j} \int_{B \times (2B)^{k-1}} \varphi_\rho(y_2) \cdots \varphi_\rho(y_k) |y_\ell|^3 \, dx_1 dy_2 \cdots dy_k \\
&\leq c \times \rho \times o(\rho^{-1}) = o(1),
\end{aligned}$$

after a change of variables ($y_i = x_i - x_{i-1}, i > 1$) in line one. □

Lemmas 4 and 5 were developed for the following purpose.

**Lemma 6.** *Assume $L^1B$, UB, ID, and LLAP. Let $f_1, \dots, f_k$ be bounded in $B$, with, for some $i$, $f_i$ supported in a compact $B_1 \subset B^o$, and, for some $j \neq i$, $f_j(z) = z^p$ for $z \in B$, with $p \in \{0, 1, 2\}$. Then*

$$\tilde{K}_\rho(f_1, \dots, f_k) = \tilde{K}_\rho(f_1, \dots, f_{j-1} \times f_j, f_{j+1} \dots, f_k) + o(1).$$

*Similarly, for $f_j(z) = \bar{z}^p$ we have*

$$\tilde{K}_\rho(f_1, \dots, f_k) = \tilde{K}_\rho(f_1, \dots, f_{j-1}, f_j \times f_{j+1} \dots, f_k) + o(1).$$

*Proof.* We prove the first claim with $f_j = z^p$; the proof of the second claim is identical. By the cyclic nature of $\tilde{K}_\rho$, we may assume $i = 1$. Fix a compact set $B_2$ such that $B_1 \subset B_2^o \subset B_2 \subset B^o$. By the disjoint union decomposition

$$(B \times B) \setminus (B_2 \times B_2) = ((B \setminus B_2) \times B\,) \cup (\,B_2 \times (B \setminus B_2))$$

and Lemma 5 we have the following (restrictions are placed only on functions with indices adjacent to $j$):

$$\begin{aligned}\Big|\tilde{K}_\rho(f_1, \dots, f_k) - \tilde{K}_\rho(f_1, \dots, f_{j-1}\mathbf{1}_{B_2}, f_j, f_{j+1}\mathbf{1}_{B_2}, \dots, f_k)\Big| \\ \leq \Big|\tilde{K}_\rho(f_1, \dots, f_{j-1}\mathbf{1}_{B\setminus B_2}, f_j, f_{j+1}, \dots, f_k)\Big| \\ + \Big|\tilde{K}_\rho(f_1, \dots, f_{j-1}\mathbf{1}_{B_2}, f_j, f_{j+1}\mathbf{1}_{B\setminus B_2}, \dots, f_k)\Big| = o(1).\end{aligned}$$

Also,

$$\begin{aligned}\tilde{K}_\rho(f_1, \dots, f_{j-1}\mathbf{1}_{B_2}, f_j, f_{j+1}\mathbf{1}_{B_2}, \dots, f_k) - \tilde{K}_\rho(f_1, \dots, f_{j-1}\mathbf{1}_{B_2} \times f_j, f_{j+1}\mathbf{1}_{B_2}, \dots, f_k) \\ = \int K_\rho(z_1, z_2) \cdots S(z_{j-1}, z_{j+1}) \cdots K_\rho(z_k, z_1) \prod_{\substack{i=1 \\ i \neq j}}^{k} f_i(z_i)\, dz_i,\end{aligned} \tag{13}$$

where

$$S(x, z) = \left(\int_B K_\rho(x, y) y^p K_\rho(y, z)\, dy - x^p K_\rho(x, z)\right) \mathbf{1}(x, z \in B_2).$$

But, by the LLAP assumption, we have that

$$\sup_{x,z \in B} |S(x, z)| = o(1),$$

and, after the familiar change of variables $y_i = x_i - x_{i-1}$ for $i \neq j + 1$, the argument used in Lemma 4 yields that the difference (13) converges to zero. An identical application of Lemma 5 gives

$$\tilde{K}_\rho(f_1, \dots, f_{j-1}\mathbf{1}_{B_2} \times f_j, f_{j+1}\mathbf{1}_{B_2}, \dots, f_k) - \tilde{K}_\rho(f_1, \dots, f_{j-1} \times f_j, f_{j+1}, \dots, f_k) = o(1),$$

which concludes the proof. □

We close this subsection by showing that the assumed conditions enable one to Taylor expand inside the $\tilde{K}_\rho$ integrals.

**Lemma 7.** *Assume that UB, $L^1B$ and ID hold. Let $f_j$, $1 \le j \le k$ have bounded third derivatives. Then we have*

$$\tilde{K}_\rho(f_1,\ldots,f_k) = \sum_{m=0}^{2} \sum_{m_2+\ldots+m_k=m} \tilde{K}_\rho^{\circ k}\left[f_1(x_1)\bigotimes_{i=2}^{k}\left(f_i^{(m_i)}(x_1)(x_i-x_1)^{\otimes m_i}\right)\right] + o(1),$$

*where we use the standard tensor notation for the full first and second derivatives.*

*Proof.* Starting at $\ell = 2$, we will step-by-step replace $f_1(x_1)f_2(x_2)\cdots f_\ell(x_\ell)$, $\ell \ge 2$ by an approximation $g_\ell$ of degree 2 at $x_1$:

$$g_\ell(x_1;x_2,\ldots,x_\ell) = f_1(x_1)\sum_{m=0}^{2}\sum_{m_2+\ldots+m_\ell=m}\left[\bigotimes_{i=2}^{\ell}\left(f_i^{(m_i)}(x_1)(x_i-x_1)^{\otimes m_i}\right)\right]$$

Note that all the $g_\ell$ are bounded on $B^\ell$. For the step-by-step replacement procedure we need to bound $d_\ell = g_{\ell-1}f_\ell - g_\ell$. Towards this end, let

$$f_\ell^* = f_\ell^*(x_1,x_\ell) = f_\ell(x_1) + f_\ell'(x_1)(x_\ell - x_1) + f_\ell''(x_1)(x_\ell-x_1)^{\otimes 2}.$$

Certainly,

$$|f_\ell^*(x_1,x_\ell) - f_\ell(x_\ell)| \le c|x_1 - x_\ell|^3,$$

for $C$ independent of $\ell, x_1, x_\ell$. Also,

$$|d_\ell| = |g_{\ell-1}f_\ell - g_\ell| \le |g_{\ell-1}||f_\ell - f_\ell^*| + |g_{\ell-1}f_\ell^* - g_\ell|.$$

Let $y_\ell = x_\ell - x_1$. Since $g_{\ell-1}$ is bounded, we have

$$|g_{\ell-1}||f_\ell - f_\ell^*| \le c\left(|y_2|^3 + \ldots + |y_\ell|^3\right) \tag{14}$$

for the range of $y_i$. As $g_\ell$ is produced from $g_{\ell-1}f_\ell^*$ by dropping all terms that are of degree 3 or 4 in the $y_i$, there is a constant $c$ such that

$$|g_{\ell+1} - g_\ell f_{\ell+1}^*| \le c\left(|y_2|^3 + \ldots + |y_\ell|^3\right) \tag{15}$$

on the range of the $y_i$. (Any monomial in $y_i$ of degree 3 or 4 and coefficient 1 is bounded above on the compact range by the right hand side for $c$ large enough).

Now we write the difference

$$\begin{aligned}\Big|\tilde{K}_\rho\left[g_{\ell-1}(x_1;x_2,\ldots,x_{\ell-1})f_\ell(x_\ell)\cdots f_k(x_k)\right] \\ - \tilde{K}_\rho\left[g_\ell(x_1;x_2,\ldots,x_\ell)f_{\ell+1}(x_{\ell+1})\cdots f_k(x_k)\right]\Big| \\ = \Big|\tilde{K}_\rho\left[d_\ell(x_1;\,x_2,\ldots,x_\ell)f_{\ell+1}(x_{\ell+1})\cdots f_k(x_k)\right]\Big|. \end{aligned}\tag{16}$$

By UB and $L^1$B (16) is bounded above by

$$\begin{aligned} c\|K_\rho\|_\infty \int_{B^k} \varphi_\rho(x_2-x_1)\cdots\varphi_\rho(x_k-x_{k-1})|d_\ell(x_1;\,x_2,\ldots,x_{\ell+1})|dx_1\cdots dx_k \\ \le \quad c\|K_\rho\|_\infty \int_{B\times(2B)^{k-1}} \varphi_\rho(y_2)\cdots\varphi_\rho(y_k)\Big(\sum_{i=2}^{\ell}|y_i|^3\Big)dx_1dy_2\cdots dy_k, \end{aligned}$$

with (14) and (15) used in the second line. Again by $L^1$B and ID, this in turn is upper bounded by

$$c|B|\|K_\rho\|_\infty \left\| \varphi_\rho \times |y|^3 \right\|_1 \leq c\rho \times o(\rho^{-1}) = o(1)$$

as required. □

## Proof of the proposition

The above bounds on $\tilde{K}_\rho$ made use of UB, $L^1$B , ID, and LLAP. If we add CO to the mix, the result is the following.

**Proposition 8.** *Assume that $K_\rho$ satisfies conditions UB, $L^1B$, and ID. For $k \geq 3$, let $f_1, \dots f_k$, be of compact support and have continuous third derivatives. If in addition CO holds, then* $\mathrm{Cum}_\rho(f_1, \dots, f_k) \to 0$.

To prove this, remember that $\mathrm{Cum}_\rho(f_1, \dots, f_k)$ is a weighted sum of terms in the form $\tilde{K}_\rho(g_1, \dots, g_m)$ each $g_j$ being a product of the underlying $f$'s (here $m \leq k$). Lemma 7 gives

$$\tilde{K}_\rho(g_1, \dots, g_m) = \sum_{\ell=0}^{2} \sum_{\ell_2+\dots+\ell_k=m} \tilde{K}_\rho^{\circ m} \left[ g_1(z_1) \bigotimes_{i=2}^{k} \left( g_i^{(z_i)}(\ell_1) s_i^{\otimes \ell_i} \right) \right] + o(1), \tag{17}$$

in which $s_i = z_i - z_1$, and we use complex coordinates $s_i, \bar{s}_i$. For example,

$$g_i^{(2)}(z_1) s_i^{\otimes 2} = \partial_z \partial_z g_i(z_1) s_i^2 + \partial_{\bar{z}} \partial_{\bar{z}} g_i(z_1) \bar{s}_i^2 + 2\partial_z \partial_{\bar{z}} g_i(z_1) s_i \bar{s}_i.$$

Further, the $k$-fold integrals on the right hand side of (17) are all of the form $\tilde{K}_\rho^{\circ k}(h(z_1)\sigma_i\sigma_j)$ where $h(z_1)$ is a $C^3$ compactly supported (in $B$) function of $z_1$ and $\sigma_\eta = s_\eta, \bar{s}_\eta$ or 1. Since functions of at most three of the $z_i$-s are present in this integrand, Lemma 6 with $p = 0$ allows us to reduce it to an at most threefold integral:

$$\tilde{K}_\rho^{\circ k}(h(z_1)\sigma_i\sigma_j) = \tilde{K}_\rho^{\circ (d+1)}(h(z_1)\sigma_i\sigma_j) + o(1),$$

where $d = 0, 1$ or 2 is the number of distinct variables in $\sigma_i\sigma_j$.

Next, two applications of Lemma 6 gives

$$\begin{aligned} \tilde{K}_\rho^{\circ 2}(h(z_1)s_2) &= \tilde{K}_\rho(h, z) - \tilde{K}_\rho(hz, 1) \\ &= \tilde{K}_\rho(h, z) - \tilde{K}_\rho(hz) + o(1) = o(1). \end{aligned}$$

Similarly, all except the $s_i\bar{s}_j$ terms vanish. Even among those, only half of the terms with $i \neq j$ survive, depending on the order of conjugation. Again by successive applications of Lemma 6

$$\tilde{K}_\rho^{\circ 3}(h(z_1)s_2\bar{s}_3) = \tilde{K}_\rho^{\circ 2}(\bar{s}_1 h(z_1)s_2) + o(1) = o(1),$$

while

$$\tilde{K}_\rho^{\circ 3}(h(z_1)\bar{s}_2 s_3) = \tilde{K}_\rho^{\circ 3}(h(z_1)\bar{s}_2 s_2) + o(1) = \tilde{K}_\rho(h, z\bar{z}) - \tilde{K}_\rho(hz\bar{z}) + o(1). \tag{18}$$

Apart from the error, the latter equals $-\mathrm{Cov}_\rho(h, z\bar{z})$.

Therefore, aside from a constant term, the only $O(1)$ contributions to the cumulant sum are of the form (18). The possible choices of $h$ are: with $F = f_1 f_2 \cdots f_k$,

$$\begin{aligned} G_i &= F \times (\partial_{z,\bar{z}} g_i)/g_i \\ G_{ij} &= F \times (\partial_z g_i)(\partial_{\bar{z}} g_j)/(g_i g_j), \end{aligned}$$

and our full $\tilde{K}_\rho$ formula reduces to $\tilde{K}_\rho(\cdot, z\bar{z}) - \tilde{K}_\rho(\cdot \times z\bar{z})$ applied to

$$\sum_{i=2}^{m} G_i + \sum_{2 \le i < j \le m} G_{ij}.$$

Reverting back to the original test functions $f_1, f_2, \ldots, f_k$, this is a weighted sum of the functions

$$\begin{aligned} F_u &= F \times (\partial_{z,\bar{z}} f_u)/f_u \\ F_{uv} &= F \times (\partial_z f_u)(\partial_{\bar{z}} f_v)/(f_u f_v). \end{aligned}$$

Since $\mathrm{Cum}_\rho(f_1, \ldots, f_k)$ is symmetric under permutations of indices of the $f_i$'s, it suffices to show that the total weight over the cumulant sum for each one of the two types of terms $F_u$ and $F_{uv}$ vanishes. Also, as each $g_i$ is a product of $k_i$ of the $f_i$, then $G_i$ is a sum of $k_i$ terms of type $F_u$ and $k_i(k_i - 1)$ terms of type $F_{uv}$. Similarly, when $i \ne j$, $G_{ij}$ is a sum of $k_i k_j$ terms of type $F_{uv}$. Thus, the total number of terms of each type is given by

$$\begin{aligned} \text{type } F_u &: \quad k_2 + \ldots + k_m \\ \text{type } F_{uv} &: \quad \sum_{i=2}^{m} k_i(k_i - 1) + \sum_{2 \le i < j \le m} k_i k_j, \end{aligned}$$

while each type appears in the cumulant sum with a different coefficient.

Finally we invoke property CO: $\tilde{K}_\rho(\partial_{z,\bar{z}} F, z\bar{z}) - \tilde{K}_\rho((\partial_{z,\bar{z}} F) \times z\bar{z}) \to 0$. Since

$$\partial_{z,\bar{z}} F = F \times \Big( \sum (\partial_{z,\bar{z}} f_u)/f_u + \sum \partial_z f_u(z) \partial_{\bar{z}} f_v/(f_u f_v) \Big),$$

after subtracting $\partial_{z,\bar{z}} F/k$ from each term of type $F_u$ it can be replaced by $k - 1$ terms of type $F_{uv}$ with the opposite sign. Thus, our final count is

$$\text{type } F_{uv} \quad : \quad \sum_{i=2}^{m} (k_i^2 - k k_i) + \sum_{2 \le i < j \le m} k_i k_j. \tag{19}$$

That is to say, each $m \le k$ term in the cumulant sum is the same constant multiple of (19). That this vanishes for $k \ge 3$ when summed over the full cumulant expansion is the content of the next lemma.

**Lemma 9.** *For each $m \ge 1$ let $\varphi(k, m, k_1, \ldots, k_m)$ be a real-valued function. With*

$$\Upsilon_k(\varphi) = \sum_{m=1}^{k} \frac{(-1)^{m-1}}{m} \sum_{\substack{k_1 + \ldots + k_m = k \\ k_1, \ldots, k_m \ge 1}} \frac{\varphi(k, m, k_1, \ldots, k_m)}{k_1! \cdots k_m!},$$

*it holds that*

$$\Upsilon_k \left( \sum_{i=2}^{k} (k_i^2 - k k_i) + \sum_{2 \le i < j \le m} k_i k_j \right) = 0. \tag{20}$$

*for all* $k \geq 3$.

*Proof.* First realize, if we denote $\varphi \equiv 1$ by 1, then

$$\begin{aligned} \log(e^x) &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left( \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)^m \\ &= \sum_{k=1}^{\infty} \Upsilon_k(1)\, x^k, \end{aligned}$$

which explains why the $k \geq 3$ cumulant of any constant is zero. Now set

$$f = f(x,y) = e^x + xye^x \quad = \quad 1 + \frac{x(1+y)}{1!} + \frac{x^2(1+2y)}{2!} + \frac{x^3(1+3y)}{3!} + \dots$$

so that the coefficient of the $y$ term in the $y$-power series expansion of $\log f$ is

$$\frac{d}{dy}(\log f)\Big|_{y=0} = \sum_{k=1}^{\infty} \Upsilon_k(k_1 + \ldots + k_m)\, x^k$$

and similarly,

$$\frac{1}{2}\frac{d^2}{d^2y}(\log f)\Big|_{y=0} = \sum_{k=1}^{\infty} \Upsilon_k \left( \textstyle\sum_{1 \le i < j \le m} k_i k_j \right) x^k.$$

In order to obtain the pure quadratic sums we set

$$g = g(x,y) = e^x + y(xe^x + x^2e^x) \quad = \quad 1 + \frac{x(1+y)}{1!} + \frac{x^2(1+2^2y)}{2!} + \frac{x^3(1+3^2y)}{3!} + \dots.$$

The coefficient of the $y$ term in this power series expansion reads

$$\frac{d}{dy}(\log g)\Big|_{y=0} = \sum_{k=1}^{\infty} \Upsilon_k(k_1^2 + \ldots + k_m^2)\, x^k.$$

These series produce the types of terms we are after up to the fact that our cumulant expressions do not have the first coefficient $k_1$. To omit this, the above may be modified as in

$$\begin{aligned} s_1 &= \frac{d}{dy}\frac{(\log f)(e^x - 1)}{f-1}\Big|_{y=0} = \frac{d}{dy}\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(e^x-1)(f-1)^{m-1}\Big|_{y=0} \\ &= \sum_{k=1}^{\infty} \Upsilon_k(k_2 + \ldots + k_m)\, x^k, \end{aligned}$$

and

$$s_2 = \frac{d}{dy} \frac{(\log g)(e^x - 1)}{g - 1}\bigg|_{y=0} = \sum_{k=1}^{\infty} \Upsilon_k (k_2^2 + \ldots + k_m^2)\, x^k,$$

and finally

$$s_{11} = \frac{1}{2} \frac{d^2}{d^2 y} \frac{(\log f)(e^x - 1)}{f - 1}\bigg|_{y=0} = \sum_{k=1}^{\infty} \Upsilon_k \left( \sum_{2 \le i < j \le m} k_i k_j \right) x^k.$$

Now we have an easy proof of the claim. Simply note that the right hand side of (20) equals the coefficient of the $x^k$ term in

$$-x\, \frac{d}{dx} s_1 + s_2 + s_{11} = \frac{x^2}{2};$$

the latter being a straightforward computation. □

## 3 Properties satisfied by our models

We now verify that the conditions for asymptotic normality of Section 2 hold for the three invariant models. First the simple bounds UB, $L^1B$ and ID are checked. Note that for the planar and hyperbolic models, property LLAP follows from Lemma 3: both kernels are projections onto the space of analytic functions on the plane or unit disk. For the spherical model, in which the kernel projects onto a set of finite degree, LLAP requires separate proof.

The first three properties follow from a similar pointwise bound for each of the kernels in question. Recall the definition (3) of $K_\rho(z, w)$.

**Planar model.** We have

$$K_\rho(z, w) = \frac{\rho}{\pi} e^{\rho(z\bar{w} - |z|^2/2 + |w|^2/2)},$$

and so

$$|K_\rho(z, w)| = \frac{\rho}{\pi} e^{\rho(\Re(z\bar{w}) - (|z|^2 + |w|^2)/2)} = \frac{\rho}{\pi} e^{-\rho|z-w|^2/2}. \tag{21}$$

That is, $\varphi_\rho(z) = \frac{\rho}{\phi} e^{-\rho|z|^2/2}$, and it is immediate that conditions UB, $L^1$B, and ID are satisfied for all $z, w \in \mathbb{C}$.

**Hyperbolic model.** Denote $t = (1 - |z|^2)(1 - |w|^2)$ and $s = (1 - z\bar{w})^2$. Then $|s| = t + |z - w|^2$, and we have

$$K_\rho(z, w) = \frac{\rho}{\pi s} (t/s)^{(\rho - 1)/2}.$$

If $|z| < R < 1$, then $|s| \in [(1 - R)^2, 4]$, and, assuming $\rho \ge 2$, we get

$$\begin{aligned} |K_\rho(z, w)| &= \frac{\rho}{\pi |s|} |t/s|^{\frac{\rho-1}{2}} = \frac{\rho}{\pi |s|} \left[ 1 - \frac{|z - w|^2}{|s|} \right]^{\frac{\rho-1}{2}} \\ &\le \frac{\rho}{(1 - R)^2} \left[ 1 - \frac{|z - w|^2}{4} \right]^{\frac{\rho-1}{2}} \le \frac{\rho}{(1 - R)^2}\, e^{-(\rho-1)|z-w|^2/8} =: \varphi_\rho(z - w). \end{aligned} \tag{22}$$

Thus, conditions UB, $L^1$B, and ID are satisfied for all $|z| \leq R$, $|w| < 1$.

**Spherical model.** Now put $t = (1+|z|^2)(1+|w|^2)$, $s = (1+z\bar{w})^2$. Then $t = |s| + |z-w|^2$, and the kernel reads

$$K_\rho(z,w) = \frac{\rho}{\pi t}(s/t)^{(\rho-1)/2}.$$

If, both $|z|$ and $|w| < R$, then $t \in [1,b]$ with $b = (1+R^2)^2$, and

$$\begin{aligned} |K_\rho(z,w)| &= \frac{\rho}{\pi t}|s/t|^{\frac{\rho-1}{2}} = \frac{\rho}{\pi t}\left[1 - \frac{|z-w|^2}{t}\right]^{\frac{\rho-1}{2}} \\ &\leq \frac{\rho}{\pi t}\, e^{-(\rho-1)|z-w|^2/(2t)} \leq \rho\, e^{-(\rho-1)|z-w|^2/(2b)} =: \varphi_\rho(z-w) \end{aligned} \tag{23}$$

as soon as $\rho \geq 2$. Just as before, we have UB, $L^1$B, and ID when $|z|, |w| \leq R$.

Rounding out the basic properties we have:

**Lemma 10.** *The spherical model restricted to a ball $\{z \leq |B|\}$ has the LLAP property.*

*Proof.* Fix $p > 0$, and assume that $\rho \geq 1+p$. We consider a truncated kernel, which is a projection to the space of polynomials of degree at most $\rho - 1 - p$ with respect to the same measure as $K_\rho$, that is $\mu_\rho$. That is, we introduce

$$\frac{\pi t^{\frac{\rho+1}{2}}}{\rho}\hat{K}_\rho(z,w) = \sum_{k=0}^{\rho-1-p}\binom{\rho-1}{k}(z\bar{w})^k,$$

with again $t = (1-|z|^2)(1-|w|^2)$. This truncated kernel is shown to have LLAP, and then the truncation is shown to make a negligible difference.

First note that

$$\int_{\mathbb{C}} K_\rho(x,y)y^p\hat{K}_\rho(y,z)dy = x^p\hat{K}_\rho(x,z), \tag{24}$$

since $K_\rho$ is a projection into the space of polynomials of degree at most $\rho-1$ and for $z$ fixed and $y^p\hat{K}_\rho(y,z)$ is $\mu_\rho(y)^{1/2}$ times such a polynomial. Also, since for $z$ fixed, $K_\rho(y,z) - \hat{K}_\rho(y,z)$ is orthogonal to all powers at least $\rho-1-p$ of $\bar{y}$ on any radially symmetric set, we have

$$\int_{|y|\leq B} K_\rho(x,y)y^p(K_\rho(y,z) - \hat{K}_\rho(y,z))dy = 0. \tag{25}$$

Next, using the bound $\binom{\rho-1}{k} \leq \binom{\rho-1-p}{k}\rho^p$ we find

$$\frac{\pi t^{\frac{\rho+1}{2}}}{\rho}|\hat{K}_\rho(z,w)| \leq \rho^p(1+|zw|)^{\rho-1-p}.$$

And, when $|z| < b < B < |w|$, we have $(1+|zw|)^2/t < a^2$ for some $a < 1$. Thus,

$$|\hat{K}_\rho(z,w)| \leq \frac{\rho^{p+1}}{t^{p/2+1}}\left[\frac{1+|zw|}{t^{1/2}}\right]^{\rho-1-p} \leq \frac{\rho^{p+1}}{t^{p/2+1}}a^{\rho-1-p}.$$

Similarly, for $|z| < b < |B| < |w|$, we have

$$|K_\rho(z,w)| \le \frac{\rho}{t}\left[\frac{1+|zw|}{t^{1/2}}\right]^{\rho-1} \le \frac{\rho}{t}a^{\rho-1}.$$

By Hermitian symmetry this implies that

$$\sup_{|x|,|z|<b}\left|\int_{|B|<|y|} K_\rho(x,y)y^p\hat{K}_\rho(y,z)dy\right| \le \rho^{p+2}a^{2\rho-2-p}\int_{|B|<|y|}\frac{|y|^p}{(1+|y|^2)^{p/2+2}}\,dy = o(1). \quad (26)$$

Finally, an easy estimate shows

$$\sup_{|x|,|z|<b}|x^pK_\rho(x,z) - x^p\hat{K}_\rho(x,z)| = o(1), \quad (27)$$

and (24),(26),(25),(27) together imply the first part of the LLAP property. The second part follows from Hermitian symmetry. □

# 4 Asymptotic variance and general test functions

Our goal is to prove asymptotic normality with explicit variances for any $f \in L^1 \cap H^1$. We do this by proving normality and determining the variance asymptotics for an $\|\cdot\|_{H^1}$-dense set of functions and then giving a uniform variance bound for all functions.

First note the general formula valid for all bounded $f, g$ of compact support:

$$\begin{aligned}\mathrm{Cov}_\rho(f,g) &= \int f(z)g(z)K_\rho(z,z)\,dz - \int\int f(z)g(w)K_\rho(z,w)K_\rho(w,z)\,dzdw \\ &= \int\int f(z)(g(z)-g(w))|K_\rho(z,w)|^2\,dzdw\end{aligned}$$

which, after symmetrization, reads

$$\mathrm{Cov}_\rho(f,g) = \frac{1}{2}\int\int (f(z)-f(w))(g(z)-g(w))|K_\rho(z,w)|^2\,dzdw \quad (28)$$

**Lemma 11.** *The subset of smooth functions with compact support not containing $\infty$ is $\|\cdot\|_{H^1}$-dense in $H^1(\nu)\cap L^1(\nu)$.*

Note that this subset is *not* dense in $H^1(\mathbb{U})$, only among $H^1 \cap L^1$ functions: harmonic functions $h$ in $\mathbb{U}$ are $H^1$-orthogonal to any compactly supported $f$. This can be seen via an integration by parts, moving the gradient from $f$ to produce a $\triangle h = 0$.

*Proof.* Replacing $f$ by $(f\wedge b)\vee(-b)$ and letting $b\to\infty$ shows that bounded functions are dense. Then convolving a bounded $f$ with a smooth probability density approaching $\delta_0$ shows that bounded $C^3$ functions are dense.

First consider the planar or hyperbolic case, and use the invariant gradient $\nabla_\iota$, measure $\nu$ and distance $\mathrm{dist}_\iota$. We may apply a sequence of smooth cutoff functions $g_r$ to $f$ which are equal to

1 on the ball of radius $r$ but are compactly supported and have $|\nabla_\iota g_r| \le 1$. Let $h_r = 1 - g_r$. We have

$$\nabla_\iota(f - fg_r) = h_r \nabla_\iota f + f \nabla_\iota h_r$$

and therefore

$$\begin{aligned} \|f - fg_r\|_{H^1}^2 &\le 2 \int |h_r \nabla_\iota f|^2 + |f \nabla_\iota h_r|^2 \, d\nu(z) \\ &\le 2 \int_{\mathrm{dist}_\iota(0,z)>r} |\nabla_\iota f(z)|^2 \, d\nu(z) + 2\|f\|_\infty \int_{\mathrm{dist}_\iota(0,z)>r} |f(z)| \, d\nu(z) \end{aligned}$$

these converge to 0 for bounded $f \in L^1 \cap H^1$ as $r \to \infty$.

For the sphere, we again consider smooth $f$ and now recall that adding a constant to $f$ does not change its $H^1$-norm. Nor does adding a constant change the fact that $f \in L^1$, as the invariant measure $\nu_{\mathbb{S}}$ is finite. Thus, we may assume that $f(\infty) = 0$, and by smoothness and compactness $f(z) \le c_f \, \mathrm{dist}_\iota(\infty, z)$. We now take $g_\varepsilon(z) = ((\mathrm{dist}_\iota(z, \infty)/\varepsilon - 1) \vee 0) \wedge 1$, which is supported on points at least $\varepsilon$ away from $\infty$. Also, $|\nabla_\iota g(z)| \le 1/\varepsilon$ and vanishes for $z$ more than $2\varepsilon$ away from $\infty$. As before, we have

$$\|f - fg_\varepsilon\|_{H^1}^2 \le 2 \int_{\mathrm{dist}_\iota(\infty,z)<2\varepsilon} |\nabla_\iota f(z)|^2 \, d\nu(z) + 2 \int_{\mathrm{dist}_\iota(\infty,z)<2\varepsilon} \frac{c_f^2 \, \mathrm{dist}_\iota(\infty, z)^2}{\varepsilon^2} \, d\nu(z)$$

Both terms converge to 0 when $\varepsilon \to 0$, as required. □

**Lemma 12** (Asymptotic variance for a dense set)**.** *Let $f$ and $g$ be $C^1$ and of compact support in $\Lambda$, where $\Lambda = \mathbb{C}$ for the plane or the sphere or $\mathbb{U}$ for the hyperbolic plane. Then*

$$\mathrm{Cov}_\rho(f, g) \to \frac{1}{4\pi} \langle f, g \rangle_{H_1}. \tag{29}$$

*Proof.* It suffices to compute $\lim_{\rho\to\infty} \mathrm{Var}_\rho(f)$ for $f \in C_0^1$ as the covariance may be identified by substituting $f + g$ for $f$. First, by Taylor's theorem with remainder there is a bounded non-negative function $\varepsilon(r)$ tending to 0 as $r \downarrow 0$ for which

$$\begin{aligned} &\Big|\mathrm{Var}_\rho(f) - \int_B \int_B (\nabla f(z) \cdot (w - z))^2 |K_\rho(z, w)|^2 \, dz dw\Big| \\ &\qquad \le \int_B \int_B \varepsilon(|z - w|)|z - w|^2 \phi_\rho^2(z - w) \, dz dw = o(1). \end{aligned}$$

Now examine the remaining integrand

$$I(z, w) := |\nabla f(z)|^2 |z - w|^2 \cos^2(\theta) |K_\rho(z, w)|^2,$$

where $\theta(z, w)$ is the angle between $f(z)$ and $w - z$, under the change of variables $w = z + \rho^{-1/2} w'$. Pointwise, in each of the three models, we have

$$\frac{1}{\rho^2} |K_\rho(z, z + \frac{w'}{\sqrt{\rho}})|^2 \to \frac{1}{\pi^2 \psi(z)^2} \exp\Big(-|w'|^2/\psi(z)\Big),$$

where $\psi(z) = 1$ (plane), $= 1 + |z|^2$ (sphere), $= 1 - |z|^2$ (hyperbolic plane). This would result in the limiting formula for the variance: with $\theta'$ denoting the limiting angle between $z$ and $w'$,

$$\begin{aligned}\mathrm{Var}_\rho(f) &\to \frac{1}{2\pi^2}\int_B |\nabla f(z)|^2 \int_{\mathbb{C}} \cos^2(\theta')\frac{|w'|^2}{\psi(z)^2} e^{-|w'|^2/\psi(z)}\, dw' dz \qquad (30)\\ &= \frac{1}{2\pi^2}\int_B |\nabla f(z)|^2 \int_0^\infty \int_0^{2\pi} \cos^2(\theta')|w'|^3 e^{-|w'|^2}\, d\theta' d|w'| dz = \frac{1}{4\pi}\int_B |\nabla f(z)|^2\, dz.\end{aligned}$$

On the other hand, $z$ and $w$ ranging in a bounded set and $||\nabla f||_{L^\infty} < \infty$, the right hand side of

$$|I(z, z + \frac{w'}{\sqrt{\rho}})| \le c|w'|e^{-|w'|^2/c}$$

is integrable on $B \times \mathbb{C}$ (this again uses (21), (22), and (23)). Therefore, dominated convergence validates (30) and completes the proof. □

**Corollary 13.** *The property CO holds for out models:* $\mathrm{Cov}_\rho(\partial z\bar{z}f, z\bar{z}\mathbf{1}_B(z)) \to 0$ *for* $f \in C^3$, $\mathrm{supp}(f) \subset B^o$ *and* $B$ *compact.*

*Proof.* Consider a smooth $g$ satisfying $\mathbf{1}_{B'} \le g \le \mathbf{1}_B$, where $B'$ is a neighborhood of $\mathrm{supp}(f)$. Lemma (12) with implies

$$4\pi\,\mathrm{Cov}_\rho(\partial_{z\bar{z}}f, g\,z\bar{z}) \to \langle\partial_{z\bar{z}}f, gz\bar{z}\rangle_{H_1} = -\langle f, \partial_{z\bar{z}}(gz\bar{z})\rangle_{H_1} = 0.$$

For the last equality, note that $\nabla f$ and $\nabla\partial_{z\bar{z}}(gz\bar{z})$ have disjoint support.

Now

$$\mathrm{Cov}(\partial_{z\bar{z}}f, (\mathbf{1}_B - g)) = \tilde{K}_\rho(\partial_{z\bar{z}}f, (\mathbf{1}_B - g)) - \tilde{K}_\rho(\partial_{z\bar{z}}f \times (\mathbf{1}_B - g)),$$

where the second term vanishes, and the first one converges to 0 by Lemma 5 and the fact that the arguments have disjoint support. □

**Lemma 14** (Variance bound)**.** *Let* $f \in H^1(\nu) \cap L^1(\nu)$. *There is a universal* $c > 0$ *so that for all* $\rho > 1$ *we have*

$$\mathrm{Var}_\rho(f) \le c\int_\Lambda |\nabla_\iota f(z)|^2\, d\nu(z). \qquad (31)$$

*Proof.* By considering the negative and positive parts of $f$ separately, we may assume $f \ge 0$. In each of the three models by (28) we have, for $f$ bounded and compact support

$$\mathrm{Var}_\rho(f) = \frac{1}{2}\int_\Lambda\int_\Lambda |f(z) - f(w)|^2|K^\iota(z,w)|^2 d\nu(z)\, d\nu(w), \qquad (32)$$

where $K^\iota(z,w) = K(z,w)(\eta(z)\eta(w))^{-1/2}$ and $\eta = d\nu(z)/dz$ is the density of the invariant measure.

Repeating the identities in (28) for the invariant $K^\iota$, and using that for $\rho$ fixed $K^\iota$ is bounded, we get that (32) extends to all $f \in L^2(\nu)$, in particular for bounded $f \in L^1(\nu)$. Now replace the nonnegative $f \in L^1(\nu)$ by $f_n = f \wedge n$. Let $\mathcal{V}(f)$ denote the right hand side of (32). Since $|f_n(z) - f_n(w)|$ is monotone increasing in $n$, the monotone convergence theorem gives $\mathcal{V}(f_n) \to \mathcal{V}(f)$. We also have $\mathrm{Var}_\rho(f_n) \to \mathrm{Var}_\rho(f)$: the mean converges to a finite limit as

$f \in L^1(\nu)$ and the second moment converges by the monotone convergence theorem. Thus (32) holds for all nonnegative $f \in L^1(\nu)$, although a priori both sides may be infinite.

For isometries $T$ of $\Lambda$ (i.e. linear fractional transformations preserving $\nu$) we have

$$|K^\iota(T(z), T(w))| = |K^\iota(z, w)| \tag{33}$$

A simple way to check (33) is to write $|K^\iota(z, w)|$ directly as a function of the single variable $|T_z(w)|$, where $T_z$ is the isometry taking $z$ to 0; such an expression is clearly invariant. It is also possible to get (33) from the invariance of the process and the covariance formula (28).

Let $\gamma_{zw}$ be a geodesic connecting $\omega$ and $z$ that proceeds at speed $s_{zw} = \mathrm{dist}_\iota(z, w)$ given by the invariant distance between $z$ and $w$. Then we have

$$\begin{aligned} |f(z) - f(w)|^2 &= \left| \int_0^1 \nabla_\iota f(\gamma_{zw}(t)) \cdot \gamma'_{zw}(t)\, dt \right|^2 \\ &\leq s_{zw}^2 \int_0^1 |\nabla_\iota f|^2(\gamma_{zw}(t))\, dt. \end{aligned}$$

Thus $\mathrm{Var}_\rho(f)$ is bounded above by $|\nabla_\iota f|^2$ integrated against the measure

$$d\vartheta(\zeta) = \frac{1}{2} \int_\Lambda \int_\Lambda \int_0^1 s_{zw}^2 \delta_{\gamma_{zw}(t)}(\zeta) |K^\iota(z, w)|^2\, dt\; d\nu(z)\, d\nu(w).$$

This is defined in an invariant way, so it must be a some $\alpha(\rho) \in (0, \infty]$ times the invariant measure. In fact, 'it is the invariant convolution on the symmetric space $\Lambda$ (see, for example (7) for background) of the standard invariant measure with the radially symmetric measure

$$d\vartheta'(\zeta) = \frac{1}{2} \int_\Lambda \int_0^1 s_{0w}^2 \delta_{\gamma_{0w}(t)}(\zeta) |K^\iota(0, w)|^2\, dt\; d\nu(w).$$

and therefore

$$\begin{aligned} \alpha(\rho) = \vartheta'(\Lambda) &= \frac{1}{2} \int_\Lambda s_{0w}^2 |K^\iota(0, w)|^2\, d\nu(w) \int_0^1 dt \\ &= \pi \int_0^{R_\Lambda} r s_{0r}^2 |K^\iota(0, r)|^2 \eta(r) dr = \frac{\pi}{\eta(0)} \int_0^{R_\Lambda} r s_{0r}^2 |K(0, r)|^2 dr \end{aligned}$$

Here $R_\Lambda = \infty$ or 1 is the radius of the planar model for $\Lambda$. A direct computation now shows that for all three models this quantity is bounded by an absolute constant, verifying our claim. $\square$

*Proof of Theorem 1.* Corollary (13) shows that condition CO holds, and the other conditions have been checked in Section 3. For $f \in C^3$ of compact support Proposition 2 gives asymptotic normality and Lemma 12 gives the limiting variance, so we have

$$Z_\rho(f) := \sum_{z \in \mathcal{Z}} f(z) - \frac{\rho}{\pi} \int_\Lambda |f| d\nu \;\Rightarrow\; \mathcal{N}\Big(0, \frac{1}{4\pi} \|f\|_{H_1(\nu)}^2\Big). \tag{34}$$

Lemma 14 allows us to extend the preliminary conclusion (34) to the advertised result. For any $f \in H^1(\Lambda)$ (and of appropriate support) there is a sequence of $f_\varepsilon \in C_0^3$ with $||f - f_\varepsilon||_{H^1} \to 0$ as $\varepsilon \to 0$. Moreover, Lemma 14 implies that the family $\{Z_\rho(f)\}$ is tight and also that

$$\Big| \mathbf{E}[e^{iZ_\rho(f)}] - \mathbf{E}[e^{iZ_\rho(f_\varepsilon)}] \Big|^2 \leq \mathbf{E} \Big| Z_\rho(f) - Z_\rho(f_\varepsilon) \Big|^2 \leq c \int_\Lambda |\nabla_\iota (f - f_\varepsilon)|^2.$$

The right hand side can be made small at will. Now, choosing a subsequence $\rho'$ over which $Z_\rho(f)$ has a limit in distribution, we find the Fourier transform of that limit is as close as we like to that of a mean zero Gaussian with variance $\frac{1}{4\pi}\int_\Lambda |\nabla_\iota f|^2$. (The full limit for $Z_\rho(f_\varepsilon)$ exists for any $\varepsilon > 0$). Since this appraisal is the same for any subsequence $\rho'$, we have pinned down the unique distributional limit of $Z_\rho(f)$. □

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