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Complex Exponential Weighting Applied to Homomorphic Deconvolution

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Summary

The application of exponential weighting to homomorphic deconvolution has been treated extensively by Schafer. This paper considers complex exponential weighting, i.e. multiplication by $\alpha^n e^{i\phi n}$. At first glance it appears that the phase factor will have no significant effect (just a phase shift) on the complex cepstrum. However, it is shown that a slight modification of the weighting procedure yields a useful technique for the determination of delay times in the cepstrum.

1. Introduction

The real cepstrum (Bogart, Healy & Turkey 1963) and more recently the complex cepstrum (Oppenheim 1965; Schafer 1969) have been presented as a means of detecting echo delay times and deconvolving signals. The usefulness of such procedures in seismology is attested in several other papers (Ulrych 1971; Ulrych *et al.*, 1972; Bakun & Johnson 1973; Stoffa, Buhl & Bryan 1974). In order to avoid stringent restrictions on the signal (that it be minimum phase), Schafer (1969) introduced the concept of exponential weighting.

Essentially, the method is multiplication of the input signal $f(n)$ by a real number α^n, a transformation which does not destroy the echo. As well as insuring that the multipath operator be minimum phase, exponential weighting offers a tool for detecting delays in the cepstrum. It often drastically alters the cepstrum of the signal, while having only marginal effects on the periodic spikes representing the delay times.

A complex exponential weight, as proposed in this paper, represents an extension of the above. In place of α^n, a factor $\alpha^n e^{i\phi n}$ is introduced. Of course, merely a phase factor will have no significant effect (just a phase shift) on the complex cepstrum. However, we show that complex weighting followed by a symmetrization procedure yields a useful technique for the determination of delay times in the cepstrum.

As the basic theory of the complex cepstrum has been covered extensively in the literature, we shall limit ourselves to providing only the necessary definitions. For background material the reader is referred to the references.

2. Theory

For convenience of notation, we will assume that the absolute value of the complex

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exponential weight is unity; i.e. $\alpha = 1$. Let $s(n)$ be a time series and $S(z)$ be its z-transform. Consider the result of multiplying $s(n)$ by a time-dependent phase factor,

$$g(n) = s(n)\,e^{i\phi n}. \tag{1}$$

The z-transform of g becomes

$$\begin{aligned} G(z) &= \sum_{n=-\infty}^{\infty} g(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} s(n)\, e^{i\phi n} z^{-n} \\ &= S(e^{-i\phi} z). \end{aligned} \tag{2}$$

The complex cepstrum of $g(n)$ is then given by

$$\hat{g}(n) = \frac{1}{2\pi i} \oint_C [\log S(e^{-i\phi} z)] z^{n-1}\, dz \tag{3}$$

where C is the circle described by $z = e^{i\omega}$, $-\pi \leqslant \omega \leqslant \pi$; the logarithm is suitably defined (Schafer 1969), and $\hat{g}$ denotes the complex cepstrum. A change of variables, $z\,e^{-i\phi} \to z$, yields

$$\begin{aligned} \hat{g}(n) &= \frac{e^{i\phi n}}{2\pi i} \oint_C [\log S(z)] z^{n-1}\, dz \\ &= e^{i\phi n}\, \hat{s}(n). \end{aligned} \tag{4}$$

We note two things. Firstly, the cepstrum, $\hat{g}$, is a complex function. Secondly, its absolute value, $|\hat{g}|$, is the same as the absolute value of the original complex cepstrum, $|\hat{s}|$. Thus, a simple complex weighting such as this has not gained anything. In order to produce a useful result, we modify the above procedure as follows:

Assume that $s(n)$ is the sum of a wave and an attenuated echo (magnitude a)

$$s(n) = f(n) * (\delta(n) + a\delta(n-d)). \tag{5}$$

The logarithm of its z-transform is

$$\log S(z) = \log F(z) + \log(1 + az^{-d}). \tag{6}$$

Replacing z by its value, $e^{i\omega}$, on the unit circle, one obtains from equation (6)

$$\log S(e^{i\omega}) = \log F(e^{i\omega}) + \log(1 + a\,e^{-i\omega d}). \tag{7}$$

Since the z-transform of a real function is Hermitian, and since Schafer's definition of the complex logarithm insures that $\log \bar{A} = \overline{\log A}$, one has, for a *real* signal $s(n)$

$$\log S(e^{-i\omega}) = \overline{\log S(e^{i\omega})}. \tag{8}$$

(A complex function, $F(x)$ is considered Hermitian provided $F(-x) = \overline{F(x)}$, where the bar indicates complex conjugation.) On the other hand, note that the function $g(n)$ is not real and $G(e^{i\omega})$ is not Hermitian.

We now make $\log G$ Hermitian by reflecting about $\omega = 0$. Define

$$H(e^{i\omega}) = \left.\begin{matrix} \log G(e^{i\omega}) \\ \log G(e^{-i\omega}) \\ \mathrm{Real}(\log G(1)) \end{matrix}\right\} \quad \begin{matrix} 0 < \omega \leqslant \pi \\ -\pi < \omega < 0. \\ \omega = 0 \end{matrix} \tag{9}$$

It is easy to see that $H(e^{i\omega})$ is Hermitian for $|\omega| < \pi$. Finally, consider the transformed complex cepstrum defined by

$$\hat{g}^c(n) = \frac{1}{2\pi i} \oint_C H(z) z^{n-1} \, dz. \tag{10}$$

The superscript 'c' stands for complex exponential weight.

$\hat{g}^c(n)$ bears an important relationship to the original cepstrum $\hat{s}(n)$. Although, in passing from $\hat{s}$ to $\hat{g}^c$ the part of the cepstrum due to the source signal $f(n)$ has been completely transfigured, the delay, d, is still present. More specifically, from equations (2) and (7)

$$\begin{aligned} \log G(e^{i\omega}) &= \log S(z\, e^{-i\phi}) \\ &= \log F(z\, e^{-i\phi}) + \log\left(1 + a(z\, e^{-i\phi})^{-d}\right) \\ &= \log F(e^{i(\omega-\phi)}) + \log\left(1 + a\, e^{-i\omega d}\, e^{i\phi d}\right). \end{aligned} \tag{11}$$

The second term, representing the delay, will be Hermitian provided ϕd is an integral multiple of π; i.e.

$$\log\left(1 + a\, e^{-i\omega d}\, e^{i\phi d}\right) = \overline{\log\left(1 + a\, e^{i\omega d}\, e^{i\phi d}\right)} \tag{12}$$

when

$$\phi d = \pi l.$$

Under these conditions, equations (9), (10), (11) and (12) imply

$$\hat{g}^c(n) = \hat{f}^c(n) + \frac{1}{2\pi i} \oint_C \left[\log\left(1 + (-1)^l a\, e^{-i\omega d}\right)\right] z^{n-1} \, dz. \tag{13}$$

If we denote the operation of forming the complex cepstrum by D

$$\hat{s}(n) = Ds(n) \tag{14}$$

and define

$$f^c = D^{-1} \hat{f}^c \tag{15}$$

then $\hat{g}^c$ is merely the complex cepstrum of

$$f^c(n) * \left(\delta(n) + (-1)^l a \delta(n-d)\right). \tag{16}$$

(Since the cepstrum of a convolution is the sum of the cepstra.) In other words, $\hat{g}^c$ is still the cepstrum of a signal with an echo of delay d (with additional phase shift if l is odd), but the signal, f^c, is different.

The effect of the factor $e^{i\phi}$ on the spectrum of $s(n)$ is simply a frequency shift of $\phi/2\pi$ (rotation by ϕ) followed by truncation and reflection (the symmetrization detailed in equation (9)). In terms of an actual program implemented via the Fast Fourier Transform (FFT), the procedure is:

(1) Multiply the signal by a complex weight $\alpha\, e^i \phi$.

(2) Take the FFT of the result to obtain the spectrum.

(3) Retain the spectrum from 0 to the folding frequency, and determine the remainder of the spectrum by taking its complex conjugate and reflecting about the folding frequency.

(4) Take the complex logarithm and inverse FFT to obtain the cepstrum. (Note: the logarithm may be taken before (3) since it commutes with complex conjugation.)

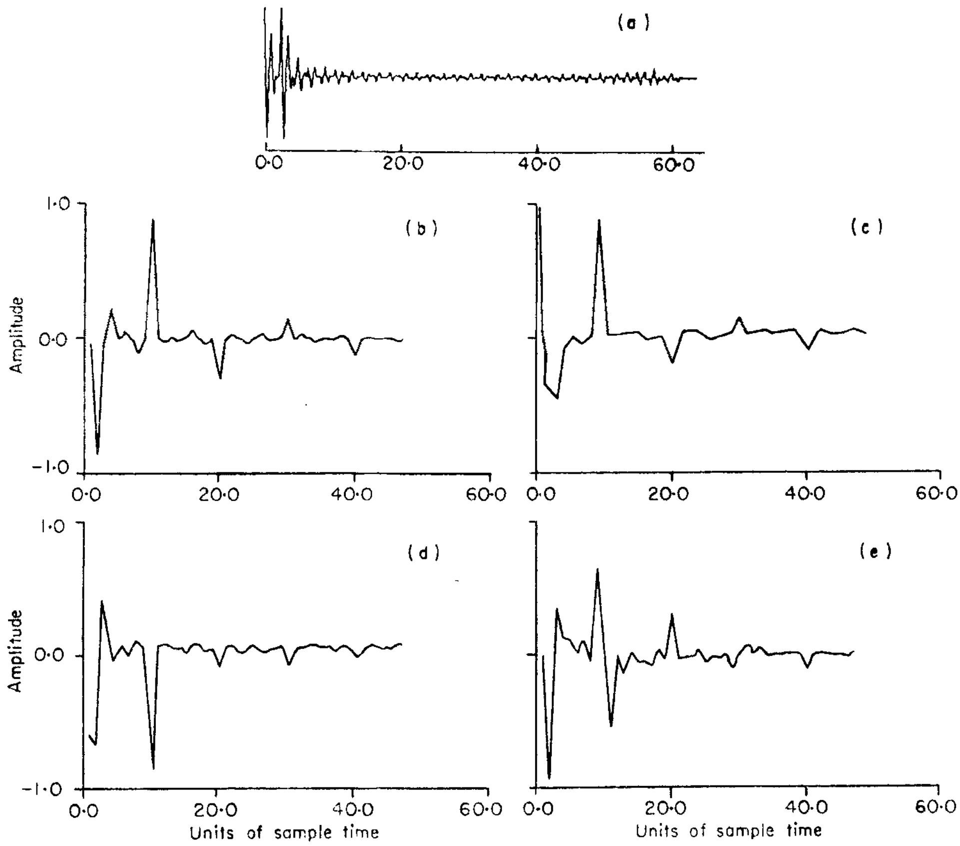

FIG. 1. Synthetic signal (a) containing an echo of delay d, and complex cepstra with complex exponential weights $\alpha e^{i\phi}$ where $\alpha = 1 \cdot 0$; and (b) no shift, $\phi = 0$; (c) $\phi d = -2\pi$; (d) $\phi d = \pi$; (e) $\phi d = \pi/2$.

3. Application

Examples of the treatment of a synthetic signal are given in Fig. 1. The delay is 10 samples and the echo is attenuated by 2 dB. From equation (13), it should be clear that, provided ϕd is an even multiple of π, the cepstrum of the multipath operator will remain unaltered (compare Fig. 1(b) and (c)). For ϕd an odd multiple of π, the echo will still appear, but with opposite phase (see Fig. 1(d)). As ϕd varies between these values, the cepstral peaks representing the delay (points $\omega = kd$) will vary between the two extremes (see appendix and Fig. 1(*e*)).

The usefulness of complex exponential weighting may be seen in the following: Firstly, the cepstrum of the transformed signal, as well as its phase relation to that of the multipath operator (delay peaks), is completely altered. In many cases this enhances the appearance of the delay in the cepstrum. Secondly, once a trial delay has been chosen, it can be tested by setting $\phi = \pi/d$ and checking to see if the first peak has changed sign. Finally, even if it is not possible to pick out a delay in the original cepstrum, a trial sweep of the complex weight over various angles ϕ may be attempted. Often the delay will now show up in the cepstrum of one of the complex exponentially weighted signals.

These techniques are illustrated in Fig. 2, which contains an analysis of a synthetic

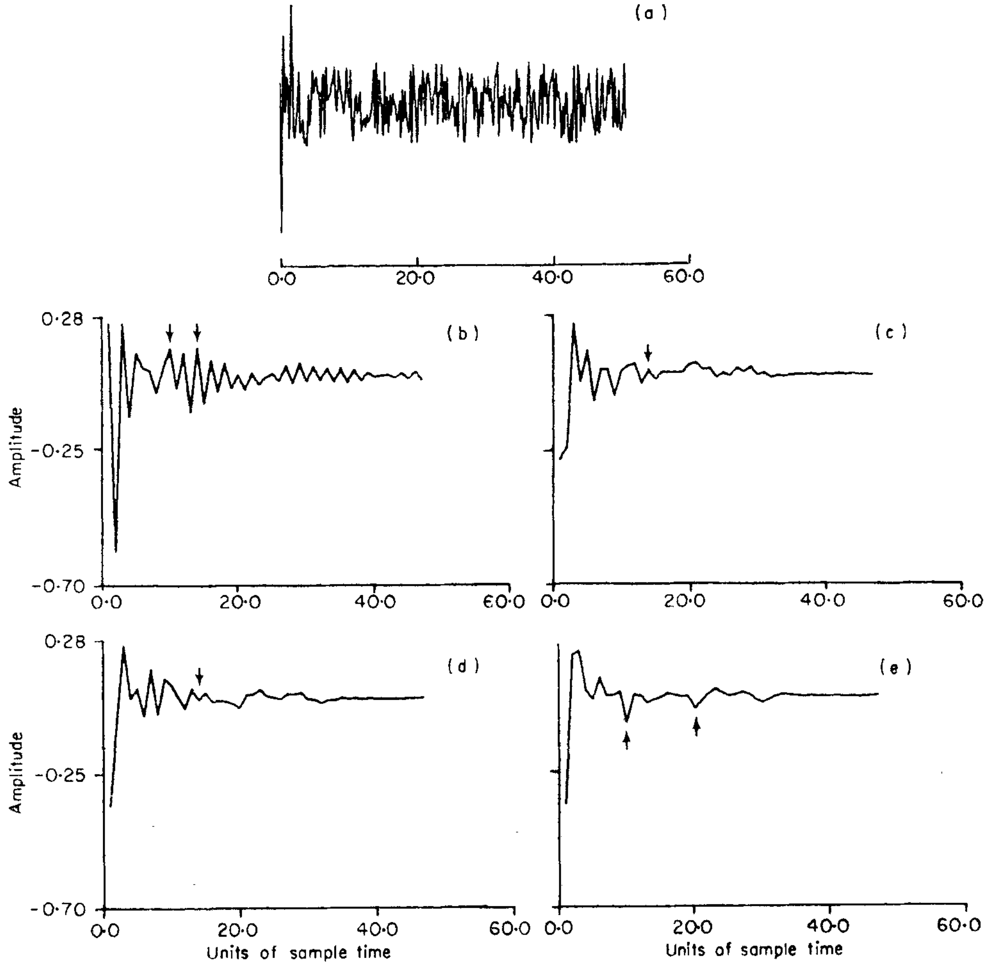

FIG. 2. Synthetic signal with noise (signal/noise ratio -6 dB) with complex exponential weights $\alpha\, e^{i\phi}$ where $\alpha = 0\cdot90$ and (b) no shift, $\phi = 0$; (c) $\phi d = 2\pi$, and $d = 14$ units; (d) $\phi d = 3\pi$, and $d = 14$ units; (e) $\phi = 3\pi$, and $d = 10$ units.

signal with noise (signal/noise ratio of $-6\cdot0$ dB and true delay of 10 sample units). The original cepstrum, Fig. 2(a), presents several possible choices for the peak; in particular, points 10 and 14. First assuming a true delay of 14 units, we introduce phase shifts of $\phi d = 2\pi$ and $\phi d = 3\pi$ (Figs 2(c) and (d)). There is no significant enhancement or reversal of the peak. On the other hand, in Fig. 2(e), where $d = 10$ units and $\phi d = 3\pi$, there is a clear phase reversal. In fact even a second peak at 20 units is visible. Thus, the procedure has clearly distinguished the true peak at 10 from the noise spike at 14.

We conclude with a real example. Let us preface the results by the following remarks: Complex exponential weighting is still valid if the complex logarithm is replaced by its real part. This will simply result in the cepstrum of Bogart *et al.* (1963) which for convenience we will refer to as the real cepstrum. It has been the experience of the author that the real cepstrum is superior to the complex cepstrum for spotting delays, since even with the removal of the linear phase component

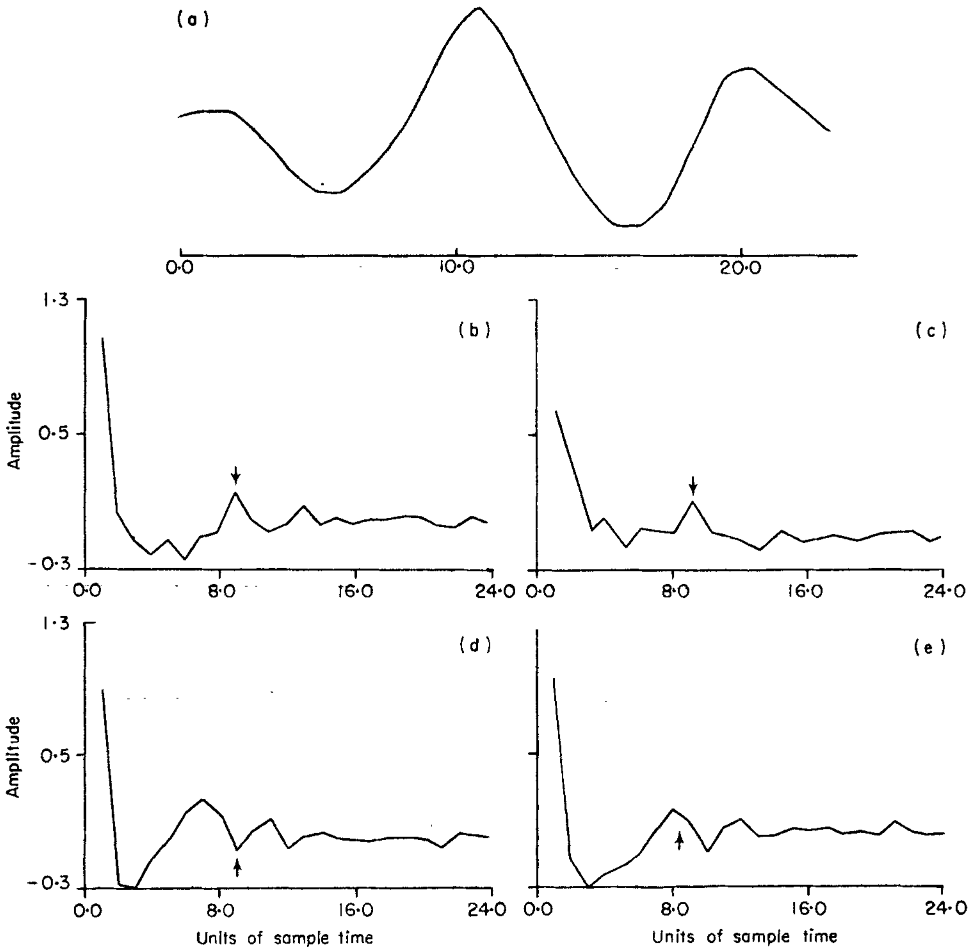

FIG. 3. Actual seismic event (a) and its real cepstra with complex exponential weights $\alpha\, e^{i\phi}$ where $\alpha = 0 \cdot 89$, and (b) no shift, $\phi = 0$; (c) $\phi d = -2\pi$; (d) $\phi d = \pi$; (e) $\phi d = \pi/2$.

(Schafer 1969) the contribution of the imaginary part of the logarithm tends to overshadow the delay.

Part of a seismic event (Fig. 3(a)) is analysed in Fig. 3. All the cepstra are 'real cepstra' and in addition to possible complex exponential weighting, include a real weight of 0·89 ($\alpha = 0 \cdot 89$). The delay time was initially estimated from Fig. 3(b) to be 9 units. The frequency shifts chosen for the complex exponential $e^{i\phi}$ are given by $\Delta f = \phi/2\pi$ where $\phi d = -2\pi$, π and $\pi/2$. They are thus the analogs of Fig. 1.

A comparison of Fig. 3(c) $(\Delta f = -(1/d) = -1 \cdot 1)$ and Fig. 3(b), the unshifted real cepstrum, exhibits no change in the multipath operator, but because of the transformed signal, the peak is enhanced. In Fig. 3(d), $\Delta f = 1/2d = 0 \cdot 56$, and the phase reversal is clearly visible. In Fig. 3(e), $\Delta f = 1/4d = 0 \cdot 28$, an intermediate phase shift ($\phi = \pi/2$). Here the peak has become obscured. These results all support the choice of 9 units for the delay. A further confirmation is evident in Fig. 4 which contains the output of a short pass filter (Schafer 1969) of the complex cepstrum (at point 9) and the difference between the signal and short pass output.

(It is very important to note that all filtering must be done on the real–exponen-

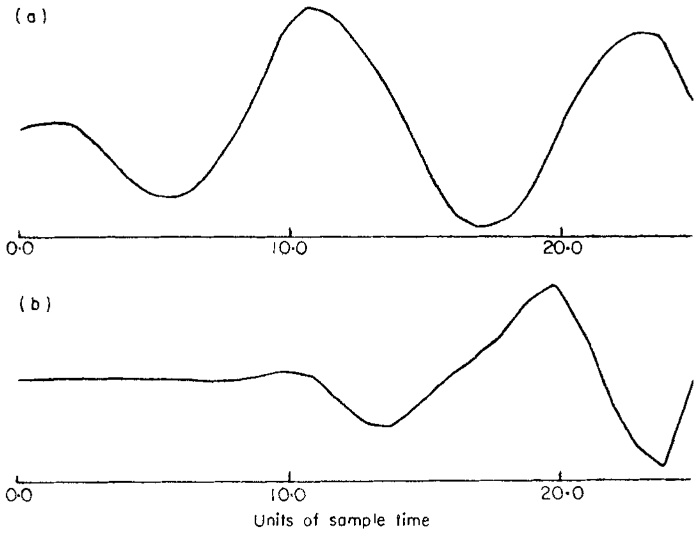

FIG. 4. (a) The output after short pass filtering the complex cepstrum; i.e. the presumed source signal. (b) The original signal minus the output pictured in Fig. 4(a); i.e. the presumed delayed source signal.

tially weighted signal since the complex exponentiation procedure destroys the signal—see equation (9).)

4. Discussion

An ability to distinguish the delay time in the cepstrum (as well as to deconvolve the signal) is important for two reasons:

(1) For a long signal it may not be reasonable to try deconvolution at a large number of points.

(2) Even when the filtering operation bears apparently fruitful results, it is not always easy to be certain that the filtered signal and the echo (input minus filtered signal) are sufficiently similar to justify the assumption of an echo. There is certainly room for more research in order to find suitable criteria for avoiding erroneous observations of delays.

Complex exponential weighting provides one possible means of dealing with the above problems. It can be used as both a hypothesis tester and as a tool for extracting a delay originally hidden by the cepstrum of the signal. Under the assumption of a particular delay, the complex exponential weight transforms the source signal in an arbitrary, non-linear manner, but has a predictable effect (a possible phase reversal) on the multipath operator, thus, it may confirm or deny the presumed value of the delay. In addition, as detailed in the previous section, a trial sweep over various phase shifts, $e^{i\phi}$, will often result in several cepstra with the delay enhanced.

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Appendix

Let

$$G(e^{i\omega}) = 1 + a\, e^{-i\omega d}\, e^{i\phi d} \tag{A1}$$

as found in equation (11). Then for $|a| < 1$

$$\log G(e^{i\omega}) = -\sum_{k=1}^{\infty} \frac{(-a)^k}{k}\, e^{i\phi kd}\, e^{-i\omega kd}. \tag{A2}$$

Consider the Hermitian function (*cf.* equation (9)) associated with $\log G(e^{i\omega})$

$$-\sum_{k=0}^{\infty} \frac{(-a)^k}{k}\, H_k(e^{i\omega}) \tag{A3}$$

where

$$H_k(e^{i\omega}) = \left.\begin{matrix} e^{i\phi kd}\, e^{-i\omega kd} \\ e^{-i\phi kd}\, e^{-i\omega kd} \\ \cos \phi kd \end{matrix}\right\} \begin{matrix} 0 < \omega < \pi \\ -\pi < \omega < 0 \\ \pi = 0. \end{matrix} \tag{A4}$$

Let

$$W_+(z) = \left.\begin{matrix} 1 \\ 0 \\ \tfrac{1}{2} \end{matrix}\right\} \begin{matrix} \operatorname{Im} z > 0 \\ \operatorname{Im} z < 0 \\ \operatorname{Im} z = 0 \end{matrix} \tag{A5}$$

and

$$W_-(z) = W_+(-z). \tag{A6}$$

Then

$$H_k(e^{i\omega}) = W_+(e^{i\omega})\, e^{i\phi kd}\, e^{-i\omega kd} + W_-(e^{i\omega})\, e^{-i\phi kd}\, e^{-i\omega kd}. \tag{A7}$$

The inverse z-transforms of W_+ and W_- on the unit circle are

$$w_+(n) = \left.\begin{matrix} \frac{1}{2} \\ \frac{-1}{\pi i n} \\ 0 \end{matrix}\right\} \begin{matrix} n = 0 \\ n \text{ odd} \\ n \text{ even and } n \neq 0 \end{matrix} \qquad \text{(A8)}$$

$$w_-(n) = \left.\begin{matrix} \frac{1}{2} \\ \frac{1}{\pi i n} \\ 0 \end{matrix}\right\} \begin{matrix} n = 0 \\ n \text{ odd} \\ n \text{ even and } n \neq 0. \end{matrix} \qquad \text{(A9)}$$

By the convolution theorem

$$\begin{aligned} h_k(n) &= w_+(n) * e^{i\phi kd}\,\delta(n-kd) + w_-(n) * e^{-i\phi kd}\,\delta(n-kd) \\ &= w_+(n-kd)\,e^{i\phi kd} + w_-(n-kd)\,e^{-i\phi kd} \\ &= \left.\begin{matrix} \cos(\phi kd) \\ \frac{-2\sin(\phi kd)}{\pi(n-kd)} \\ 0 \end{matrix}\right\} \begin{matrix} n = kd \\ n-kd \text{ odd} \\ n-kd \text{ even and } n \neq kd. \end{matrix} \end{aligned} \qquad \text{(A10)}$$

Note that the inverse transform of expression (A3),

$$-\sum_{k=d}^{\infty} \frac{(-a)^k}{k} h_k(n),$$

is the cepstrum of the complex exponentially weighted multipath operator. From (A10) it is seen that for ϕkd close to a multiple of π, $h_k(n)$ will behave as $\delta(n-kd)$. As ϕkd approaches $n\pi + (\pi/2)$, the peak will flatten out (with a dip in the centre), but the function will remain localized near $n = kd$ (because of the denominator $n-kd$).