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Complex Extension of Quantum Mechanics

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This talk examines Hamiltonians H that are not Hermitian but do exhibit space-time reflection ($\mathcal{PT}$) symmetry. If the ($\mathcal{PT}$) symmetry of H is not spontaneously broken, then the spectrum of H is entirely real and positive. Examples of $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians are $H = p^2 + ix^3$ and $H = p^2 - x^4$. The apparent shortcoming of quantum theories arising from $\mathcal{PT}$-symmetric Hamiltonians is that the $\mathcal{PT}$ norm is not positive definite. This suggests that it may be difficult to develop a quantum theory based on such Hamiltonians. In this talk it is shown that these difficulties can be overcome by introducing a previously unnoticed underlying physical symmetry $\mathcal{C}$ of Hamiltonians having an unbroken $\mathcal{PT}$ symmetry. Using $\mathcal{C}$, it is shown how to construct an inner product whose associated norm *is* positive definite. The result is a new class of fully consistent complex quantum theories. Observables in these theories exhibit $\mathcal{CPT}$ symmetry, probabilities are positive, and the dynamics is governed by unitary time evolution.

1 Introduction

In this talk we discuss an alternative to a standard axiom of quantum mechanics; namely, that the Hamiltonian H, which incorporates the symmetries and specifies the dynamics of a quantum theory, must be Hermitian: $H = H^\dagger$. It is commonly believed that H must be Hermitian in order to ensure that the energy spectrum (the eigenvalues of H) is real and that the time evolution is unitary (probability is conserved in time). Although this axiom is sufficient to guarantee these desired properties, it is not necessary. We believe that the condition of Hermiticity is a mathematical requirement whose physical basis is obscure. We demonstrate here that there is a more physical alternative axiom, which we refer to as space-time reflection symmetry ($\mathcal{PT}$ symmetry): $H = H^{\mathcal{PT}}$. This symmetry allows for the possibility of non-Hermitian and complex Hamiltonians but still leads to a consistent theory of quantum mechanics.

We also show that because $\mathcal{PT}$ symmetry is an alternative condition to Hermiticity, it is now possible to construct infinitely many new Hamiltonians that would have been rejected in the past because they are not Hermitian. An example of such a Hamiltonian is $H = p^2 + ix^3$. It should be emphasized that we do not regard the condition of Hermiticity as wrong. Rather, the condition of $\mathcal{PT}$ symmetry offers the possibility of studying new quantum theories.

Let us recall the properties of the space reflection (parity) operator $\mathcal{P}$ and the time-reflection operator $\mathcal{T}$: $\mathcal{P}$ is *linear* and has the effect $p \to -p$ and $x \to -x$; $\mathcal{T}$ is antilinear and has the effect $p \to -p$, $x \to x$, and $i \to -i$. Note that $\mathcal{T}$ changes the sign of i because, like $\mathcal{P}$, it preserves the fundamental commutation relation of quantum mechanics, $[x, p] = i$.

It is easy to construct Hamiltonians that are not Hermitian but do possess $\mathcal{PT}$ symmetry. For example, consider the one-parameter family of Hamiltonians

$$H = p^2 + x^2(ix)^\epsilon \qquad (\epsilon \text{ real}). \tag{1}$$

While H in (1) is not symmetric under $\mathcal{P}$ or $\mathcal{T}$ separately, it is invariant under their combined operation. We say that such Hamiltonians possess space-time reflection symmetry. Other complex Hamiltonians having $\mathcal{PT}$ symmetry are $H = p^2 + x^4(ix)^\epsilon$ and $H = p^2 + x^6(ix)^\epsilon$ [1][1].

[1]These classes of Hamiltonians are all *different*. For example, the Hamiltonian obtained by continuing H in (1) along the path $\epsilon : 0 \to 8$ has a different spectrum from the Hamiltonian that is obtained by continuing

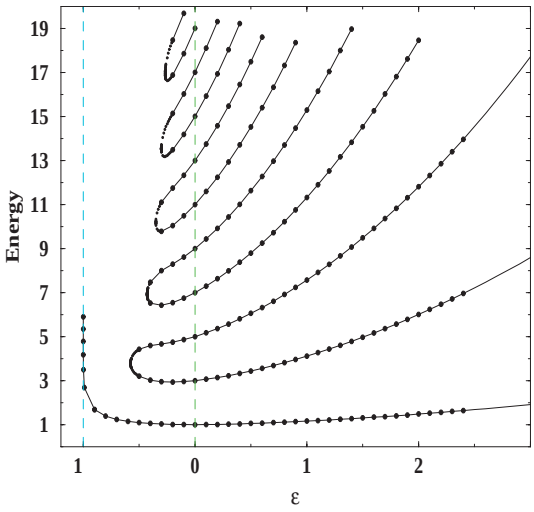

Figure 1. Energy levels of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon$ as a function of the parameter ϵ. There are three regions: When $\epsilon \geq 0$, the spectrum is real and positive and the energy levels rise with increasing ϵ. The lower bound of this region, $\epsilon = 0$, corresponds to the harmonic oscillator, whose energy levels are $E_n = 2n + 1$. When $-1 < \epsilon < 0$, there is a finite number of real positive eigenvalues and an infinite number of complex conjugate pairs of eigenvalues. As ϵ decreases from 0 to -1, the number of real eigenvalues decreases; when $\epsilon \leq -0.57793$, the only real eigenvalue is the ground-state energy. As ϵ approaches -1^+, the ground-state energy becomes infinite. When $\epsilon \leq -1$ there are no real eigenvalues.

The class of $\mathcal{PT}$-symmetric Hamiltonians is larger than and includes real symmetric Hermitians because any real symmetric Hamiltonian is automatically $\mathcal{PT}$-symmetric. For example, consider the real symmetric Hamiltonian $H = p^2 + x^2 + 2x$. This Hamiltonian is time-reversal symmetric, but according to the usual definition of space reflection for which $x \to -x$, this Hamiltonian appears not to have $\mathcal{PT}$ symmetry. However, recall that the parity operator is defined only up to unitary equivalence. In this example, if we express the Hamiltonian in the form $H = p^2 + (x+1)^2 - 1$, then it is evident that H is $\mathcal{PT}$ symmetric, provided that the parity operator performs a space reflection about the point $x = -1$ rather than $x = 0$. See Ref. [2] for the construction of the relevant parity operator.

With properly defined boundary conditions the spectrum of the Hamiltonian H in (1) is *real and positive* when $\epsilon \geq 0$ [3] and the spectrum is partly real and partly complex when $\epsilon < 0$. The eigenvalues have been computed numerically to very high precision, and the real eigenvalues are plotted as functions of ϵ in Fig. 1.

We say that the $\mathcal{PT}$ symmetry of a Hamiltonian H is *unbroken* if all of the eigenfunctions of H are simultaneously eigenfunctions of $\mathcal{PT}$[2]. Here is a proof that if the $\mathcal{PT}$ symmetry of a Hamiltonian H is unbroken, then the spectrum of H is real: Assume that H possesses $\mathcal{PT}$ symmetry (that is, that H commutes with the $\mathcal{PT}$ operator), and that if ϕ is an eigenstate of H with eigenvalue E, then it is simultaneously an eigenstate of $\mathcal{PT}$ with eigenvalue λ:

$$H\phi = E\phi \qquad \text{and} \qquad \mathcal{PT}\phi = \lambda\phi. \tag{2}$$

$H = p^2 + x^6(ix)^\epsilon$ along the path $\epsilon : 0 \to 4$. This is because the boundary conditions on the eigenfunctions are different.

[2]If a system is defined by an equation that possesses a discrete symmetry, the solution to this equation need not exhibit that symmetry. For example, the differential equation $\ddot{y}(t) = y(t)$ is symmetric under time reversal $t \to -t$. The solutions $y(t) = e^t$ and $y(t) = e^{-t}$ do not exhibit time-reversal symmetry while the solution $y(t) = \cosh(t)$ is time-reversal symmetric. The same is true with a system whose Hamiltonian is $\mathcal{PT}$ symmetric. Even if the Schrödinger equation and the corresponding boundary conditions are $\mathcal{PT}$ symmetric, the wave function that solves the Schrödinger equation boundary value problem may not be symmetric under space-time reflection. When the solution exhibits $\mathcal{PT}$ symmetry, we say that the $\mathcal{PT}$ symmetry is unbroken. Conversely, if the solution does not possess $\mathcal{PT}$ symmetry, we say that the $\mathcal{PT}$ symmetry is broken.

We begin by showing that the eigenvalue λ is a pure phase. Multiply $\mathcal{PT}\phi = \lambda\phi$ on the left by $\mathcal{PT}$ and use the fact that $\mathcal{P}$ and $\mathcal{T}$ commute and that $\mathcal{P}^2 = \mathcal{T}^2 = 1$ to conclude that $\phi = \lambda^*\lambda\phi$ and thus $\lambda = e^{i\alpha}$ for some real α. Next, introduce a convention that we use throughout this talk. Without loss of generality we replace the eigenstate ϕ by $e^{-i\alpha/2}\phi$ so that its eigenvalue under the operator $\mathcal{PT}$ is unity: $\mathcal{PT}\phi = \phi$. Next, multiply the eigenvalue equation $H\phi = E\phi$ on the left by $\mathcal{PT}$ and use $[\mathcal{PT}, H] = 0$ to obtain $E\phi = E^*\phi$. Hence, $E = E^*$ and E is real.

The crucial assumption in this argument is that ϕ is simultaneously an eigenstate of H and $\mathcal{PT}$. In quantum mechanics if a linear operator X commutes with H, then the eigenstates of H are also eigenstates of X. However, the operator $\mathcal{PT}$ is not linear (it is antilinear) and thus we must make the extra assumption that the $\mathcal{PT}$ symmetry of H is unbroken; that is, that ϕ is simultaneously an eigenstate of H and $\mathcal{PT}$. This extra assumption is nontrivial because it is hard to determine *a priori* whether the $\mathcal{PT}$ symmetry of a given H is broken or unbroken. For the Hamiltonian H in (1) the $\mathcal{PT}$ symmetry is unbroken when $\epsilon \geq 0$ and broken when $\epsilon < 0$. The conventional Hermitian Hamiltonian for the quantum mechanical harmonic oscillator lies at the boundary of the unbroken and the broken regimes. Recently, Dorey *et al.* proved rigorously that the spectrum of H in (1) is real and positive [4] in the region $\epsilon \geq 0$. Many other $\mathcal{PT}$-symmetric Hamiltonians for which space-time reflection symmetry is not broken have been investigated, and the spectra of these Hamiltonians have also been shown to be real and positive [5].

While it is useful to show that a given non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian operator has a positive real spectrum, the urgent question is whether such a Hamiltonian defines a physical theory of quantum mechanics. By a *physical theory* we mean that there is a Hilbert space of state vectors and that this Hilbert space has an inner product with a positive norm. In quantum mechanics we interpret the norm of a state as a probability and this probability must be positive. Furthermore, we must show that the time evolution of the theory is unitary. This means that as a state vector evolves in time the probability does not leak away.

It is not obvious whether a Hamiltonian such as H in (1) gives a consistent quantum theory. Indeed, past investigations of this Hamiltonian have shown that while the spectrum is entirely real and positive when $\epsilon \geq 0$, one inevitably encountered the severe problem of Hilbert spaces endowed with indefinite metrics [6]. We will identify a new symmetry that all $\mathcal{PT}$-symmetric Hamiltonians having an unbroken $\mathcal{PT}$-symmetry possess. We denote the operator representing this symmetry by $\mathcal{C}$ because the properties of this operator resemble those of the charge conjugation operator in particle physics. This will allow us to introduce an inner product structure associated with $\mathcal{CPT}$ conjugation for which the norms of quantum states are positive definite. We will see that $\mathcal{CPT}$ symmetry is an alternative to Hermiticity; it introduces the new concept of a *dynamically determined* inner product (one that is defined by the Hamiltonian itself). As a consequence, we will extend the Hamiltonian and its eigenstates into the complex domain so that the associated eigenvalues are real and the underlying dynamics is unitary.

2 Construction of the $\mathcal{C}$ operator

We begin by summarizing the mathematical properties of the solution to the Sturm–Liouville differential equation eigenvalue problem

$$-\phi_n''(x) + x^2(ix)^\epsilon\phi_n(x) = E_n\phi_n(x) \tag{3}$$

associated with the Hamiltonian H in (1). This differential equation must be imposed on an infinite contour in the complex-x plane. For large $|x|$ this contour lies in wedges placed symmetrically with respect to the imaginary-x axis [3]. The boundary conditions on the eigenfunctions are that $\phi(x) \to 0$ exponentially rapidly as $|x| \to \infty$ on the contour. For $0 \leq \epsilon < 2$, the contour may lie on the real axis.

When $\epsilon \geq 0$, the Hamiltonian has an unbroken $\mathcal{PT}$ symmetry. Thus, the eigenfunctions $\phi_n(x)$ are simultaneously eigenstates of the $\mathcal{PT}$ operator: $\mathcal{PT}\phi_n(x) = \lambda_n\phi_n(x)$. As we argued above, λ_n is a pure phase and, without loss of generality, for each n this phase can be absorbed into $\phi_n(x)$ by a multiplicative rescaling so that the new eigenvalue is unity:

$$\mathcal{PT}\phi_n(x) = \phi_n^*(-x) = \phi_n(x). \tag{4}$$

There is strong evidence that, when properly normalized, the eigenfunctions $\phi_n(x)$ are complete. The coordinate-space statement of completeness reads

$$\sum_n (-1)^n \phi_n(x)\phi_n(y) = \delta(x-y) \qquad (x, y \text{ real}). \tag{5}$$

This nontrivial result has been verified numerically to extremely high accuracy (twenty decimal places) [7, 8]. There is a factor of $(-1)^n$ in the sum. This unusual factor does not appear in conventional quantum mechanics. The presence of this factor is explained in the following discussion of orthonormality [see (7)].

Here is where we encounter the problem associated with non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians. The obvious choice for the inner product of two functions $f(x)$ and $g(x)$ is

$$(f, g) \equiv \int dx\, [\mathcal{PT}f(x)]g(x), \tag{6}$$

where $\mathcal{PT}f(x) = [f(-x)]^*$ and the integral path is the above contour in the complex-x plane. The apparent advantage of this inner product is that the associated norm (f, f) is independent of the overall phase of $f(x)$ and is conserved in time. Phase independence is desired because in quantum mechanics the objective is to construct a space of rays to represent quantum mechanical states. With respect to this inner product the eigenfunctions $\phi_m(x)$ and $\phi_n(x)$ of H in (1) are orthogonal for $n \neq m$. However, when $m = n$ the norm is evidently *not positive*:

$$(\phi_m, \phi_n) = (-1)^n \delta_{mn}. \tag{7}$$

This result is apparently true for all values of ϵ in (3) and it has been verified numerically to extremely high precision. Because the norms of the eigenfunctions alternate in sign, the metric associated with the $\mathcal{PT}$ inner product $(\cdot, \cdot)$ is indefinite. This split signature (sign alternation) is a *generic* feature of the $\mathcal{PT}$ inner product. Extensive numerical calculations verify that the formula in (7) holds for all $\epsilon \geq 0$.

Despite the lack of positivity of the inner product, we proceed with the usual analysis that one would perform for any Sturm–Liouville problem of the form $H\phi_n = E_n\phi_n$. First, we use the inner product formula (7) to verify that (5) is the representation of the unity operator. That is, we verify that $\int dy\, \delta(x-y)\delta(y-z) = \delta(x-z)$.

Second, we reconstruct the parity operator $\mathcal{P}$ in terms of the eigenstates. The parity operator in position space is $\mathcal{P}(x, y) = \delta(x+y)$, so from (5) we get

$$\mathcal{P}(x, y) = \sum_n (-1)^n \phi_n(x)\phi_n(-y). \tag{8}$$

By virtue of (7) the square of the parity operator is unity: $\mathcal{P}^2 = 1$.

Third, we construct H in coordinate space: $H(x, y) = \sum_n (-1)^n E_n \phi_n(x)\phi_n(y)$. Using (5)–(7) we can see that this Hamiltonian satisfies $H\phi_n(x) = E_n\phi_n(x)$.

Fourth, we construct the Green's function $G(x, y) = \sum_n (-1)^n \frac{1}{E_n}\phi_n(x)\phi_n(y)$ in coordinate space. The Green's function is the functional inverse of H; that is, G satisfies

$$\int dy\, H(x, y)G(y, z) = \left[-\frac{d^2}{dx^2} + x^2(ix)^\epsilon\right] G(x, z) = \delta(x-z). \tag{9}$$

While the time-independent Schrödinger equation (3) cannot be solved analytically, the differential equation for $G(x,z)$ in (9) *can* be solved in closed form [8]. The technique is to consider the case $0<\epsilon<2$ so that we may treat x as real and then to decompose the x axis into two regions, $x>z$ and $x<z$. We can solve the differential equation in each region in terms of Bessel functions. From this representation of the Green's function we construct an exact closed-form expression for the *spectral zeta function* (sum of the inverses of the energy eigenvalues). To do so we set $y=x$ in $G(x,y)$ and use (7) to integrate over x. For all $\epsilon>0$ we obtain [8]

$$\sum_n \frac{1}{E_n} = \left[1+\frac{\cos\left(\frac{3\epsilon\pi}{2\epsilon+8}\right)\sin\left(\frac{\pi}{4+\epsilon}\right)}{\cos\left(\frac{\epsilon\pi}{4+2\epsilon}\right)\sin\left(\frac{3\pi}{4+\epsilon}\right)}\right]\frac{\Gamma\left(\frac{1}{4+\epsilon}\right)\Gamma\left(\frac{2}{4+\epsilon}\right)\Gamma\left(\frac{\epsilon}{4+\epsilon}\right)}{(4+\epsilon)^{\frac{4+2\epsilon}{4+\epsilon}}\Gamma\left(\frac{1+\epsilon}{4+\epsilon}\right)\Gamma\left(\frac{2+\epsilon}{4+\epsilon}\right)}. \tag{10}$$

Having presented these general Sturm–Liouville constructions, we now address the crucial question of whether a $\mathcal{PT}$-symmetric Hamiltonian defines a physically viable quantum mechanics or whether it merely provides an intriguing Sturm–Liouville eigenvalue problem. The apparent difficulty with formulating a quantum theory is that the vector space of quantum states is spanned by energy eigenstates, of which half have norm $+1$ and half have norm -1. Because the norm of the states carries a probabilistic interpretation in standard quantum theory, the existence of an indefinite metric in (7) seems to be a serious obstacle.

The situation here in which half of the energy eigenstates have positive norm and half have negative norm is analogous to the problem that Dirac encountered in formulating the spinor wave equation in relativistic quantum theory [9]. Following Dirac's approach, we attack the problem of an indefinite norm by finding a physical interpretation for the negative norm states. We claim that in *any* theory having an unbroken $\mathcal{PT}$ symmetry there exists a symmetry of the Hamiltonian connected with the fact that there are equal numbers of positive-norm and negative-norm states. To describe this symmetry we construct a linear operator denoted by $\mathcal{C}$ and represented in position space as a sum over the energy eigenstates of the Hamiltonian [10]:

$$\mathcal{C}(x,y)=\sum_n \phi_n(x)\phi_n(y). \tag{11}$$

As stated earlier, the properties of this new operator $\mathcal{C}$ resemble those of the charge conjugation operator in quantum field theory. For example, we can use (5)–(7) to verify that the square of $\mathcal{C}$ is unity ($\mathcal{C}^2=1$): $\int dy\,\mathcal{C}(x,y)\mathcal{C}(y,z)=\delta(x-z)$. Thus, the eigenvalues of $\mathcal{C}$ are ±1. Also, $\mathcal{C}$ commutes with the Hamiltonian H. Therefore, since $\mathcal{C}$ is linear, the eigenstates of H have definite values of $\mathcal{C}$. Specifically, if the energy eigenstates satisfy (7), then we have $\mathcal{C}\phi_n=(-1)^n\phi_n$ because

$$\mathcal{C}\phi_n(x)=\int dy\,\mathcal{C}(x,y)\phi_n(y)=\sum_m \phi_m(x)\int dy\,\phi_m(y)\phi_n(y).$$

We then use $\int dy\,\phi_m(y)\phi_n(y)=(\phi_m,\phi_n)$ according to our convention. Thus, $\mathcal{C}$ is the operator observable that represents the measurement of the signature of the $\mathcal{PT}$ norm of a state[3].

The operators $\mathcal{P}$ and $\mathcal{C}$ are distinct square roots of the unity operator $\delta(x-y)$. That is, $\mathcal{P}^2=\mathcal{C}^2=1$, but $\mathcal{P}\neq\mathcal{C}$. Indeed, $\mathcal{P}$ is real, while $\mathcal{C}$ is complex[4]. Furthermore, these two operators do not commute; in the position representation

$$(\mathcal{CP})(x,y)=\sum_n \phi_n(x)\phi_n(-y) \qquad \text{but} \qquad (\mathcal{PC})(x,y)=\sum_n \phi_n(-x)\phi_n(y), \tag{12}$$

which shows that $\mathcal{CP}=(\mathcal{PC})^*$. However, $\mathcal{C}$ *does* commute with $\mathcal{PT}$.

[3]The $\mathcal{PT}$ norm of a state determines its parity type. We can regard $\mathcal{C}$ as representing the operator that determines the $\mathcal{C}$ charge of the state. Quantum states having opposite $\mathcal{C}$ charge possess opposite parity type.

[4]The parity operator in coordinate space is explicitly real $\mathcal{P}(x,y)=\delta(x+y)$; the operator $\mathcal{C}(x,y)$ is complex because it is a sum of products of complex functions, as we see in (11). The complexity of the $\mathcal{C}$ operator can be seen explicitly in perturbative calculations of $\mathcal{C}(x,y)$ [11].

Finally, having obtained the operator $\mathcal{C}$ we define a new inner product structure having *positive definite* signature by

$$\langle f|g\rangle \equiv \int_{\mathrm{C}} dx\,[\mathcal{CPT}f(x)]g(x). \tag{13}$$

Like the $\mathcal{PT}$ inner product (6), this inner product is phase independent and conserved in time. This is because the time evolution operator, just as in ordinary quantum mechanics, is e^{iHt}. The fact that H commutes with the $\mathcal{PT}$ and the $\mathcal{CPT}$ operators implies that both inner products, (6) and (13), remain time independent as the states evolve in time. However, unlike (6), the inner product (13) is positive definite because $\mathcal{C}$ contributes -1 when it acts on states with negative $\mathcal{PT}$ norm. In terms of the $\mathcal{CPT}$ conjugate, the completeness condition (5) reads

$$\sum_n \phi_n(x)[\mathcal{CPT}\phi_n(y)] = \delta(x-y). \tag{14}$$

Unlike the inner product of conventional quantum mechanics, the $\mathcal{CPT}$ inner product (14) is *dynamically determined*; it depends implicitly on the Hamiltonian.

The operator $\mathcal{C}$ does not exist as a distinct entity in conventional quantum mechanics. Indeed, if we allow the parameter ϵ in (1) to tend to zero, the operator $\mathcal{C}$ in this limit becomes identical to $\mathcal{P}$. Thus, in this limit the $\mathcal{CPT}$ operator becomes $\mathcal{T}$, which is just complex conjugation. As a consequence, the inner product (13) defined with respect to the $\mathcal{CPT}$ conjugation reduces to the complex conjugate inner product of conventional quantum mechanics when $\epsilon \to 0$. Similarly, in this limit (14) reduces to the usual statement of completeness $\sum_n \phi_n(x)\phi_n^*(y) = \delta(x-y)$.

The $\mathcal{CPT}$ inner-product (13) is independent of the choice of integration contour C so long as C lies inside the asymptotic wedges associated with the boundary conditions for the Sturm–Liouville problem (2). Path independence follows from Cauchy's theorem and the analyticity of the integrand. In ordinary quantum mechanics, where the positive-definite inner product has the form $\int dx\, f^*(x)g(x)$, the integral must be taken along the real axis and the path of the integration cannot be deformed into the complex plane because the integrand is not analytic[5]. The $\mathcal{PT}$ inner product (6) shares with (13) the advantage of analyticity and path independence, but suffers from nonpositivity. We find it surprising that a positive-definite metric can be constructed using $\mathcal{CPT}$ conjugation without disturbing the path independence of the inner-product integral.

Finally, we explain why $\mathcal{PT}$-symmetric theories are unitary. Time evolution is determined by the operator e^{-iHt}, whether the theory is expressed in terms of a $\mathcal{PT}$-symmetric Hamiltonian or just an ordinary Hermitian Hamiltonian. To establish the global unitarity of a theory we must show that as a state vector evolves its norm does not change in time. If $\psi_0(x)$ is a prescribed initial wave function belonging to the Hilbert space spanned by the energy eigenstates, then it evolves into the state $\psi_t(x)$ at time t according to $\psi_t(x) = e^{-iHt}\psi_0(x)$. With respect to the $\mathcal{CPT}$ inner product defined in (13), the norm of the vector $\psi_t(x)$ does not change in time, $\langle\psi_t|\psi_t\rangle = \langle\psi_0|\psi_0\rangle$, because the Hamiltonian H commutes with the $\mathcal{CPT}$ operator.

Establishing unitarity at a local level is harder. Here, we must show that in coordinate space there exists a local probability density that satisfies a continuity equation so that the probability does not leak away. This is a subtle result because the probability current flows in the complex plane rather than along the real axis as in conventional Hermitian quantum mechanics. Preliminary numerical studies indeed indicate that the continuity equation is fulfilled [12].

[5]If a function satisfies a linear ordinary differential equation, then the function is analytic wherever the coefficient functions of the differential equation are analytic. The Schrödinger equation (3) is linear and its coefficients are analytic except for a branch cut at the origin; this branch cut can be taken to run up the imaginary axis. We choose the integration contour for the inner product (7) so that it does not cross the positive imaginary axis. Path independence occurs because the integrand of the inner product (7) is a product of analytic functions.

3 Illustrative example: A 2×2 matrix Hamiltonian

We illustrate the above results concerning $\mathcal{PT}$-symmetric quantum mechanics using the finite-dimensional 2×2 matrix Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \tag{15}$$

where the three parameters r, s, and θ are real. This Hamiltonian is not Hermitian in the usual sense, but it is $\mathcal{PT}$ symmetric, where the parity operator is [13]

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

and $\mathcal{T}$ performs complex conjugation.

There are two parametric regions for this Hamiltonian. When $s^2 < r^2 \sin^2\theta$, the energy eigenvalues form a complex conjugate pair. This is the region of broken $\mathcal{PT}$ symmetry. On the other hand, if $s^2 \geq r^2 \sin^2\theta$, then the eigenvalues $\varepsilon_\pm = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta}$ are real. This is the region of unbroken $\mathcal{PT}$ symmetry. In the unbroken region the simultaneous eigenstates of the operators H and $\mathcal{PT}$ are

$$|\varepsilon_+\rangle = \frac{1}{\sqrt{2\cos\alpha}} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} \qquad \text{and} \qquad |\varepsilon_-\rangle = \frac{i}{\sqrt{2\cos\alpha}} \begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2} \end{pmatrix}, \tag{17}$$

where we set $\sin\alpha = (r/s)\sin\theta$. It is easily verified that $(\varepsilon_\pm, \varepsilon_\pm) = \pm 1$ and that $(\varepsilon_\pm, \varepsilon_\mp) = 0$, recalling that $(u, v) = (\mathcal{PT}u)\cdot v$. Therefore, with respect to the $\mathcal{PT}$ inner product, the resulting vector space spanned by energy eigenstates has a metric of signature $(+,-)$. The condition $s^2 > r^2\sin^2\theta$ ensures that $\mathcal{PT}$ symmetry is not broken. If this condition is violated, the states (17) are no longer eigenstates of $\mathcal{PT}$ because α becomes imaginary[6].

Next, we construct the operator $\mathcal{C}$:

$$\mathcal{C} = \frac{1}{\cos\alpha} \begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}. \tag{18}$$

Note that $\mathcal{C}$ is distinct from H and $\mathcal{P}$ and has the key property that $\mathcal{C}|\varepsilon_\pm\rangle = \pm|\varepsilon_\pm\rangle$. The operator $\mathcal{C}$ commutes with H and satisfies $\mathcal{C}^2 = 1$. The eigenvalues of $\mathcal{C}$ are precisely the signs of the $\mathcal{PT}$ norms of the corresponding eigenstates. Using the operator $\mathcal{C}$ we construct the new inner product structure $\langle u|v\rangle = (\mathcal{CPT}u)\cdot v$. This inner product is positive definite because $\langle\varepsilon_\pm|\varepsilon_\pm\rangle = 1$. Thus, the two-dimensional Hilbert space spanned by $|\varepsilon_\pm\rangle$, with inner product $\langle\cdot|\cdot\rangle$, has a Hermitian structure with signature $(+,+)$.

Let us demonstrate explicitly that the $\mathcal{CPT}$ norm of any vector is positive. For the arbitrary vector $\psi = \binom{a}{b}$, where a and b are any complex numbers, we see that $\mathcal{T}\psi = \binom{a^*}{b^*}$, that $\mathcal{PT}\psi = \binom{b^*}{a^*}$, and that $\mathcal{CPT}\psi = \frac{1}{\cos\alpha}\binom{a^* + ib^*\sin\alpha}{b^* - ia^*\sin\alpha}$. Thus, $\langle\psi|\psi\rangle = (\mathcal{CPT}\psi)\cdot\psi = \frac{1}{\cos\alpha}[a^*a + b^*b + i(b^*b - a^*a)\sin\alpha]$. Now let $a = x + iy$ and $b = u + iv$, where x, y, u, and v are real. Then

$$\langle\psi|\psi\rangle = \left(x^2 + v^2 + 2xv\sin\alpha + y^2 + u^2 - 2yu\sin\alpha\right)/\cos(\alpha), \tag{19}$$

which is explicitly positive and vanishes only if $x = y = u = v = 0$.

Since $\langle u|$ denotes the $\mathcal{CPT}$-conjugate of $|u\rangle$, the completeness condition reads

$$|\varepsilon_+\rangle\langle\varepsilon_+| + |\varepsilon_-\rangle\langle\varepsilon_-| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{20}$$

[6]When $\mathcal{PT}$ symmetry is broken, we find that the $\mathcal{PT}$ norm of the energy eigenstate vanishes.

Furthermore, using the $\mathcal{CPT}$ conjugate $\langle\varepsilon_\pm|$, we have $\mathcal{C}$ as $\mathcal{C} = |\varepsilon_+\rangle\langle\varepsilon_+| - |\varepsilon_-\rangle\langle\varepsilon_-|$, instead of the representation in (11), which uses the $\mathcal{PT}$ conjugate.

We represent an observable in this theory by a $\mathcal{CPT}$-invariant operator; that is, one that commutes with $\mathcal{CPT}$. Thus, if $\mathcal{CPT}$ symmetry is not broken, the eigenvalues of the observable are real. The operator $\mathcal{C}$ satisfies this requirement, so it is an observable. For the two-state system, if we set $\theta = 0$, then the Hamiltonian (15) becomes Hermitian. However, $\mathcal{C}$ then reduces to the parity operator $\mathcal{P}$. As a consequence, the requirement of $\mathcal{CPT}$ invariance reduces to the standard condition of Hermiticity for a symmetric matrix, namely, that $H = H^*$. This is why the hidden symmetry $\mathcal{C}$ was not noticed previously. The operator $\mathcal{C}$ emerges only when we extend a real symmetric Hamiltonian into the complex domain.

We have also calculated the $\mathcal{C}$ operator in infinite-dimensional quantum mechanical models. For an $x^2 + ix^3$ potential $\mathcal{C}$ can be obtained from the summation in (11) using perturbative methods and for an $x^2 - x^4$ potential $\mathcal{C}$ can be calculated using nonperturbative methods [11].

4 Applications and possible observable consequences

Could non-Hermitian, $\mathcal{PT}$-symmetric Hamiltonians be used to describe experimentally observable phenomena? Non-Hermitian Hamiltonians have *already* been used to describe interacting systems. Wu showed that the ground state of a Bose system of hard spheres is described by a non-Hermitian Hamiltonian [14]. Wu found that the ground-state energy of this system is real and conjectured that all energy levels were real. Hollowood showed that even though the Hamiltonian of a complex Toda lattice is non-Hermitian, the energy levels are real [15]. Non-Hermitian Hamiltonians of the form $H = p^2 + ix^3$ also arise in various Reggeon field theory models that exhibit real positive spectra [16]. In these examples the fact that a non-Hermitian Hamiltonian had a real spectrum appeared mysterious at the time, but now we know why: In each case the non-Hermitian Hamiltonian is $\mathcal{PT}$-symmetric. In each case the Hamiltonian constructed so that the position operator x or the field operator ϕ is always multiplied by i.

An experimental signal of a complex Hamiltonian might be found in the context of condensed matter physics. Consider the complex crystal lattice whose potential is $V(x) = i\sin x$. While the Hamiltonian $H = p^2 + i\sin x$ is not Hermitian, it is $\mathcal{PT}$-symmetric, and all of the energy bands are *real*. However, at the edge of the bands the wave function of a particle in such a lattice is always bosonic (2π-periodic) and, unlike the case of ordinary crystal lattices, the wave function is never fermionic (4π-periodic) [17]. Direct observation of such a band structure would give unambiguous evidence of a $\mathcal{PT}$-symmetric Hamiltonian.

There are many opportunities for the use of non-Hermitian Hamiltonians in the study of quantum field theory. For example, a scalar quantum field theory with a cubic self-interaction described by the Lagrangian $\mathcal{L} = \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + g\varphi^3$ is physically unacceptable because the energy spectrum is not bounded below. However, the cubic scalar quantum field theory that corresponds to H in (1) with $\epsilon = 1$ is given by the Lagrangian density $\mathcal{L} = \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + ig\varphi^3$. This is a new, physically acceptable quantum field theory. Moreover, the theory that corresponds to H in (1) with $\epsilon = 2$ is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 - \frac{1}{4}g\varphi^4. \tag{21}$$

This "wrong-sign" field theory is remarkable because, in addition to the energy spectrum being real and positive, the one-point Green's function (the vacuum expectation value of the field φ) is *nonzero* [18]. Also, the field theory is renormalizable, and in four dimensions is asymptotically free (and thus nontrivial) [19]. Based on these features, we believe that the theory may provide a setting to describe the dynamics of the Higgs sector in the standard model.

Other field theory models whose Hamiltonians are non-Hermitian and $\mathcal{PT}$-symmetric have also been studied. For example, $\mathcal{PT}$-symmetric electrodynamics is particularly interesting because it is asymptotically free (unlike ordinary electrodynamics) and because the direction of the Casimir force is the negative of that in ordinary electrodynamics [20]. This theory is remarkable because it can determine its own coupling constant. Supersymmetric $\mathcal{PT}$-symmetric quantum field theories have also been studied [21].

We have found that $\mathcal{PT}$-symmetric quantum theories exhibit surprising and new phenomena. For example, for g sufficiently small the $-g\varphi^4$ theory described by the Lagrangian (21) possesses bound states (the conventional $g\varphi^4$ theory does not because the potential is repulsive). The bound states occur for all dimensions $0 \leq D < 3$ [22], but for purposes of illustration we describe the bound states in the context of one-dimensional quantum field theory (quantum mechanics). For the conventional anharmonic oscillator, which is described by the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + \frac{1}{4}gx^4 \qquad (g > 0), \tag{22}$$

the small-g Rayleigh-Schrödinger perturbation series for the kth energy level E_k is

$$E_k \sim m\left[k + \frac{1}{2} + \frac{3}{4}(2k^2 + 2k + 1)\nu + \mathrm{O}\left(\nu^2\right)\right] \qquad (\nu \to 0^+), \tag{23}$$

where $\nu = g/(4m^3)$. The *renormalized mass* M is defined as the first excitation above the ground state: $M \equiv E_1 - E_0 \sim m\big[1 + 3\nu + \mathrm{O}\big(\nu^2\big)\big]$ as $\nu \to 0^+$.

To determine if the two-particle state is bound, we examine the second excitation above the ground state using (23). We define

$$B_2 \equiv E_2 - E_0 \sim m\left[2 + 9\nu + \mathrm{O}\left(\nu^2\right)\right] \qquad (\nu \to 0^+). \tag{24}$$

If $B_2 < 2M$, then a two-particle bound state exists and the (negative) binding energy is $B_2 - 2M$. If $B_2 > 2M$, then the second excitation above the vacuum is interpreted as an unbound two-particle state. From (24) we see that in the small-coupling region, where perturbation theory is valid, the conventional anharmonic oscillator does not possess a bound state. Indeed, using WKB, variational methods, or numerical calculations, one can show that there is no two-particle bound state for any value of $g > 0$. Because there is no bound state the gx^4 interaction may be considered to represent a repulsive force[7].

The perturbation series for the non-Hermitian, $\mathcal{PT}$-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 - \frac{1}{4}gx^4 \qquad (g > 0), \tag{25}$$

is obtained from the perturbation series for the conventional anharmonic oscillator by replacing $\nu \to -\nu$. Thus, while the conventional anharmonic oscillator does not possess a two-particle bound state, the $\mathcal{PT}$-symmetric oscillator does indeed possess such a state. We measure the binding energy of this state in units of the renormalized mass M and we define the *dimensionless* binding energy Δ_2 by

$$\Delta_2 \equiv (B_2 - 2M)/M \sim -3\nu + \mathrm{O}\left(\nu^2\right) \qquad (\nu \to 0^+). \tag{26}$$

This bound state disappears when ν increases beyond $\nu = 0.0465$. As ν continues to increase, Δ_2 reaches a maximum value of 0.427 at $\nu = 0.13$ and then approaches 0.28 as $\nu \to \infty$.

[7] In general, a repulsive force in a quantum field theory is represented by an energy dependence in which the energy of a two-particle state decreases with separation. The conventional anharmonic oscillator Hamiltonian corresponds to a field theory in one space-time dimension, where there cannot be any spatial dependence. In this case the repulsive nature of the force is understood to mean that the energy B_2 needed to create two particles at a given time is more than twice the energy M needed to create one particle.

In the $\mathcal{PT}$-symmetric anharmonic oscillator, there are not only two-particle bound states but also k-particle bound states for all $k \geq 2$. The dimensionless binding energies are

$$\Delta_k \equiv (B_k - kM)/M \sim -3k(k-1)\nu/2 + \mathrm{O}\left(\nu^2\right) \qquad (\nu \to 0+). \tag{27}$$

The coefficient of ν is negative. Since the dimensionless binding energy becomes negative as ν increases from 0, there is a k-particle bound state. The higher multiparticle bound states cease to be bound for smaller values of ν; starting with the three-particle bound state, the binding energy of these states becomes positive as ν increases past 0.039, 0.034, 0.030, and 0.027.

Thus, for any value of ν there are always a finite number of bound states and an infinite number of unbound states. The number of bound states decreases with increasing ν until there are no bound states at all. There is a range of ν for which there are only two- and three-particle bound states, just like the physical world in which one observes only states of two and three bound quarks. In this range of ν if one has an initial state containing a number of particles (renormalized masses), these particles will clump together into bound states, releasing energy in the process. Depending on the value of ν, the final state will consist either of two- or of three-particle bound states, whichever is energetically favored. There is a special value of ν for which two- and three-particle bound states can exist in thermodynamic equilibrium.

How does a $g\varphi^3$ theory compare with a $g\varphi^4$ theory? A $g\varphi^3$ theory has an attractive force. Bound states arising as a consequence of this force can be found by using the Bethe–Salpeter equation. However, the $g\varphi^3$ field theory is unacceptable because the spectrum is not bounded below. If we replace g by ig, the spectrum becomes real and positive, but now the force becomes repulsive and there are no bound states. The same is true for a two-scalar theory with interaction of the form $ig\varphi^2\chi$, which is a model of scalar electrodynamics that has no analog of positronium.

Another feature of $\mathcal{PT}$-symmetric quantum field theory that distinguishes it from conventional quantum field theory is the commutation relation between the $\mathcal{P}$ and $\mathcal{C}$ operators. If we write $\mathcal{C} = \mathcal{C}_{\mathrm{R}} + i\mathcal{C}_{\mathrm{I}}$, where $\mathcal{C}_{\mathrm{R}}$ and $\mathcal{C}_{\mathrm{I}}$ are real, then $\mathcal{C}_{\mathrm{R}}\mathcal{P} = \mathcal{P}\mathcal{C}_{\mathrm{R}}$ and $\mathcal{C}_{\mathrm{I}}\mathcal{P} = -\mathcal{P}\mathcal{C}_{\mathrm{I}}$. These commutation and anticommutation relations suggest the possibility of interpreting $\mathcal{PT}$-symmetric quantum field theory as describing both bosonic and fermionic degrees of freedom, an idea analogous to the supersymmetric quantum theories. The distinction here, however, is that the supersymmetry can be broken; that is, bosonic and fermionic counterparts can have different masses without breaking the $\mathcal{PT}$ symmetry. Therefore, another possible observable experimental consequence might be the breaking of the supersymmetry.

5 Concluding remarks

We have described an alternative to the axiom of standard quantum mechanics that the Hamiltonian must be Hermitian. We have shown that Hermiticity may be replaced by the more physical condition of $\mathcal{PT}$ (space-time reflection) symmetry. Space-time reflection symmetry is distinct from the condition of Hermiticity, so it is possible to consider new quantum theories, such as quantum field theories whose self-interaction potentials are $ig\varphi^3$ or $-g\varphi^4$. Such theories have previously been thought to be mathematically and physically unacceptable because the spectrum might not be real and because the time evolution might not be unitary.

These new theories are complex extensions of ordinary quantum mechanics; they are continuations of real symmetric Hamiltonians to complex Hamiltonians. The idea of analytically continuing a Hamiltonian was first discussed by Dyson, who argued heuristically that perturbation theory for quantum electrodynamics diverges [23]. Dyson's argument involves rotating the electric charge e into the complex plane $e \to ie$. Applied to the anharmonic oscillator (22), Dyson's argument goes: If the coupling constant g is continued in the complex-g plane to $-g$, then the potential is no longer bounded below, so the resulting theory has no ground state. Thus, the ground-state energy $E_0(g)$ has an abrupt transition at $g = 0$. As a series in powers

of g, $E_0(g)$ must have a zero radius of convergence because $E_0(g)$ is singular at $g = 0$. Hence, the perturbation series must diverge for all $g \neq 0$. The perturbation series does indeed diverge, but this argument is flawed because the spectrum of the Hamiltonian (25) remains ambiguous until the boundary conditions that the wave functions must satisfy are specified. The spectrum depends crucially on how this Hamiltonian with a negative coupling constant is obtained.

There are two ways to obtain H in (25). First, one can substitute $g = |g|e^{i\theta}$ into (22) and rotate from $\theta = 0$ to $\theta = \pi$. Under this rotation, the ground-state energy $E_0(g)$ becomes complex. Evidently, $E_0(g)$ is real and positive when $g > 0$ and complex when $g < 0$[8]. Second, one can obtain (25) as a limit of the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + \frac{1}{4}gx^2(ix)^\epsilon \qquad (g > 0) \tag{28}$$

as $\epsilon : 0 \to 2$. The spectrum of this Hamiltonian is real, positive, and discrete. The spectrum of the limiting H in (25) obtained in this manner is similar in structure to that of H in (22).

How can H in (25) possess two such astonishingly different spectra? The answer lies in the boundary conditions on the wave functions $\phi_n(x)$. In the first case, in which $\theta = \arg g$ is rotated in the complex-g plane from 0 to π, $\psi_n(x)$ vanishes in the complex-x plane as $|x| \to \infty$ inside the wedges $-\pi/3 < \arg x < 0$ and $-4\pi/3 < \arg x < -\pi$. In the second case, in which the exponent ϵ ranges from 0 to 2, $\phi_n(x)$ vanishes in the complex-x plane as $|x| \to \infty$ inside the wedges $-\pi/3 < \arg x < 0$ and $-\pi < \arg x < -2\pi/3$. In this second case the boundary conditions hold in wedges that are symmetric with respect to the imaginary axis; these boundary conditions enforce the $\mathcal{PT}$ symmetry of H and are responsible for the reality of the energy spectrum.

Apart from the spectra, there is another striking difference between the two theories corresponding to H in (25). The one-point Green's function $G_1(g)$ is defined as the expectation value of the operator x in the ground-state wave function $\phi_0(x)$,

$$G_1(g) = \langle 0|x|0\rangle / \langle 0|0\rangle \equiv \int_C dx\, x\psi_0^2(x) \Big/ \int_C dx\, \psi_0^2(x), \tag{29}$$

where C is a contour that lies in the asymptotic wedges described above. The value of $G_1(g)$ for H in (25) depends on the limiting process by which we obtain H. If we substitute $g = g_0e^{i\theta}$ into the Hamiltonian (22) and rotate from $\theta = 0$ to $\theta = \pi$, we find that $G_1(g) = 0$ for all g on the semicircle in the complex-g plane. Thus, this rotation in the g plane preserves parity symmetry ($x \to -x$). However, if we define H in (25) by using the Hamiltonian in (28) and by allowing ϵ to range from 0 to 2, we find that $G_1(g) \neq 0$. Indeed, $G_1(g) \neq 0$ for *all* values of $\epsilon > 0$. Thus, in this theory $\mathcal{PT}$ symmetry (reflection about the imaginary axis, $x \to -x^*$) is preserved, but parity symmetry is permanently broken. Hence, one might be able to describe the dynamics of the Higgs sector by using a $-g\varphi^4$ quantum field theory.

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[1] Bender C.M., Boettcher S. and Meisinger P.N., $\mathcal{PT}$-symmetric quantum mechanics, *J. Math. Phys.*, 1999, V.40, 2201–2229.

[2] Bender C.M., Meisinger P.N. and Wang Q., All Hermitian Hamiltonians have parity, *J. Phys. A: Math. Gen.*, 2003, V.36, 1029–1031.

[8] Rotating from $\theta = 0$ to $\theta = -\pi$, we obtain the same Hamiltonian as in (25) but the spectrum is the complex conjugate of the spectrum obtained when we rotate from $\theta = 0$ to $\theta = \pi$.

[3] Bender C.M. and Boettcher S., Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry, *Phys. Rev. Lett.*, 1998, V.80, 5243–5246. In this reference there is an extended discussion of the wedges in the complex-x plane and the contour along which the differential equation (3) is solved. See especially Fig. 2 in this reference.

[4] Dorey P., Dunning C. and Tateo R., Supersymmetry and the spontaneous breakdown of $\mathcal{PT}$ symmetry *J. Phys. A: Math. Gen.*, 2001, V.34, L391–L400; Spectral equivalences, Bethe ansatz equations, and reality properties in $\mathcal{PT}$-symmetric quantum mechanics, *J. Phys. A: Math. Gen.*, 2001, V.34, 5679–5704.

[5] Lévai G. and Znojil M., Systematic search for $\mathcal{PT}$ symmetric potentials with real energy spectra, *J. Phys. A: Math. Gen.*, 2000, V.33, 7165–7180; Bender C.M., Dunne G.V., Meisinger P.N. and Şimşek M., Quantum complex Henon–Heiles potentials, *Phys. Lett. A*, 2001, V.281, 311–316; Bagchi B. and Quesne C., Non-Hermitian Hamiltonians with real and complex eigenvalues in a Lie-algebraic framework, *Phys. Lett. A*, 2002, V.300, 18–26; Trinh D.T., PhD Thesis, University of Nice-Sophia Antipolis, 2002, and references therein.

[6] Mostafazadeh A., Pseudo-Hermiticity versus $\mathcal{PT}$ symmetry. I, II, III, *J. Math. Phys.*, 2002, V.43, 205–214; 2814–2816; 3944–3951; math-ph/0203041; Ahmed Z., Pseudo-Hermiticity of Hamiltonians under gauge-like transformation: real spectrum of non-Hermitian Hamiltonians, *Phys. Lett. A*, 2002, V.294, 287–291; Japaridze G.S., Space of state vectors in $\mathcal{PT}$-symmetric quantum mechanics, *J. Phys. A: Math. Gen.*, 2002, V.35, 1709–1718; Znojil M., Conservation of pseudo-norm in $\mathcal{PT}$ symmetric quantum mechanics, math-ph/0104012; Ramirez A. and Mielnik B., The challenge of non-Hermitian structures in physics, *Rev. Mex. Fis.*, 2003, V.49, 130–133.

[7] Bender C.M., Boettcher S. and Savage V.M., Conjecture on the interlacing of zeros in complex Sturm-Liouville problems, *J. Math. Phys.*, 2000, V.41, 6381–6387; Bender C.M., Boettcher S., Meisinger P.N. and Wang Q., Two-point Green's function in $\mathcal{PT}$-symmetric theories, *Phys. Lett. A*, 2002, V.302, 286–290.

[8] Mezincescu G.A., Some properties of eigenvalues and eigenfunctions of the cubic oscillator with imaginary coupling constant, *J. Phys. A: Math. Gen.*, 2000, V.33, 4911–4916; Bender C. M. and Wang Q., Comment on a recent paper by Mezincescu, *J. Phys. A: Math. Gen.*, 2001, V.34, 3325–3328.

[9] Dirac P.A.M., The physical interpretation of quantum mechanics, *Proc. R. Soc. London A*, 1942, V.180, 1–40.

[10] Bender C.M., Brody D.C. and Jones H.F., Complex extension of quantum mechanics, *Phys. Rev. Lett.*, 2002, V.89, 270401; Must a Hamiltonian be Hermitian, *Am. J. Phys.*, 2003, V.71, 1095–1102.

[11] Bender C.M., Meisinger P.N. and Wang Q., Perturbative calculation of the hidden symmetry operator in $\mathcal{PT}$-symmetric quantum mechanics, *J. Phys. A: Math. Gen.*, 2003, V.36, 1973–1983.

[12] Bender C.M., Meisinger P.N. and Wang Q., Probability density and probability flux in $\mathcal{PT}$-symmetric quantum mechanics, in preparation.

[13] Bender C.M., Berry M.V. and Mandilara A., Generalized $\mathcal{PT}$ symmetry and real spectra, *J. Phys. A: Math. Gen.*, 2002, V.35, L467–L471.

[14] Wu T.T., Ground state of a Bose system of hard spheres, *Phys. Rev.*, 1959, V.115, 1390–1404.

[15] Hollowood T., Solitons in affine Toda field theories, *Nucl. Phys. B*, 1992, V.384, 523–540.

[16] Brower R., Furman M. and Moshe M., Critical exponents for the Reggeon quantum spin model, *Phys. Lett. B*, 1978, V.76, 213–219.

[17] Bender C.M., Dunne G.V. and Meisinger P.N., Complex periodic potentials with real band structure, *Phys. Lett. A*, 1999, V.252, 272–276.

[18] Bender C.M., Meisinger P. and Yang H., Calculation of the one-point Green's function for a $-g\phi^4$ quantum field theory, *Phys. Rev. D*, 2001, V.63, 45001.

[19] Bender C.M., Milton K.A. and Savage V.M., Solution of Schwinger–Dyson equations for $\mathcal{PT}$-symmetric quantum field theory, *Phys. Rev. D*, 2000, V.62, 85001.

[20] Bender C.M. and Milton K.A., A nonunitary version of massless quantum electrodynamics possessing a critical point, *J. Phys. A: Math. Gen.*, 1999, V.32, L87–L92.

[21] Bender C.M. and Milton K.A., Model of supersymmetric quantum field theory with broken parity symmetry, *Phys. Rev. D*, 1998, V.57, 3595–3608.

[22] Bender C.M., Boettcher S., Jones H.F., Meisinger P.N. and Şimşek M., Bound states of non-Hermitian quantum field theories, *Phys. Lett. A*, 2001, V.291, 197–202.

[23] Dyson F.J., Divergence of perturbation theory in quantum electrodynamics, *Phys. Rev.*, 1952, V.85, 631–632.