Complex Gaussian Ratio Distribution with Applications for Error Rate Calculation in Fading Channels with Imperfect CSI

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***Abstract*—Communications systems rarely have perfect channel state information (PCSI) when demodulating received symbols. This paper shows that the symbol error rate (SER) of a flat fading communications system can be expressed in closed form by expressing the demodulator outputs as random variable (RVs) that have a complex ratio distribution, which is the ratio of two correlated complex Gaussian RVs. To complete the analysis, the complex ratio probability density function (PDF) and cumulative distribution function (CDF) are both derived. Finally, using several scenarios based on M-QAM signaling, the SER performance of imperfect channel state information (ICSI) systems is analyzed.**

I. Introduction

The real-valued Gaussian ratio distribution (see [1], [2]) has been studied extensively for a variety of applications. The ratio of two zero-mean independent Gaussian random variables (RVs) has a Cauchy distribution, which is commonly used in physics and is sometimes referred to as the Cauchy−Lorentz distribution. Work in [3] determined a closed form expression for the ratio of two non-zero mean correlated Gaussian RVs. Most recently, in 2006 the author's of [4] derived an expression for the general real-valued ratio distribution in terms of the Hermite function and Kummer's confluent hypergeometric function to make evaluating the ratio density less computationally complex. All of the works cited above deal with the real-valued ratio distribution. Unfortunately, we were unable to find any literature on the ratio of two complex Gaussian RVs, which we refer to as the "complex ratio distribution."

While radio communication involves the transmission of radio frequency (RF) waves that have real-valued amplitude, most practical signal analysis is performed on a down-converted and sampled version of the RF signal in the base-band. Because of this, communications systems are often characterized using the baseband representation of the transmitted and received symbols. Such baseband symbols are conveniently described using complex-valued variables and complex-valued mathematical operations. Not surprisingly, the complex ratio distribution has applications in analyzing the performance of communications systems.

In this paper, we derive the complex ratio distribution between two zero-mean correlated complex Gaussian random variables and we use this result to analyze the performance of the zero-forcing equalizer in a fading channel scenario where the channel state information (CSI) is not perfectly known to the receiver (i.e. the receiver only has imperfect CSI (ICSI)). The case where perfect CSI (PCSI) is known is a special case of the ICSI case; therefore the results presented here also apply to PCSI systems.

Several studies have been done on the effect that ICSI has on the system performance. Recent work on error rate analysis for ICSI systems with error correction coding has been done in [5]. Earlier in 1999, the authors of [6] undertook a similar task of determining the error rate for ICSI systems. In that work, integral expressions for the bit error rate of QAM signals in fading were derived. To do this, the authors found the joint distribution of the true channel amplitude and the channel amplitude estimate as well as the joint distribution between the channel phase and the estimated channel phase. The result also assumes that the receiver uses the pilot symbol assisted modulation (PSAM) scheme by Cavers [7] to estimate the channel. The work in this paper is distinct because we derive and utilize the complex ratio distribution which allows for closed form expressions on the error rate for general QAM modulated signals.

II. Signal Model

To illustrate the effect of ICSI on symbols detections in communications systems, assume a simple flat-fading channel model, where, after demodulation, the received signal in the kth channel is

$$r_k = h_k q_k + \eta_k, \tag{1}$$

where $q_k \in \mathcal{A}$ are the modulating values drawn from a constellation set, $\mathcal{A}$, η_k is additive white Gaussian noise, and h_k is the channel response that is assumed to be complex Gaussian. The complex Gaussian distribution assumption for h_k is commonly referred to as a Rayleigh fading channel and is well-justified in rich scattering environments where multiple signal paths combine at the receiver. In this case, each path is modeled as an independent RV allowing the Central Limit Theorem to be invoked resulting in a complex Gaussian distribution on h_k [8]. The demodulation process takes place in baseband, so all of these variables are complex valued.

To decode q_k, h_k must be estimated by $\hat{h}_k$. This is typically done using a known preamble sequence that precedes the payload symbols. The receiver can use this known sequence to estimate the channel response (see [9] for a review of preamble channel estimation techniques). Using the zero-forcing equalizer the estimated symbol is

$$\hat{q}_k = \frac{r_k}{\hat{h}_k}. \tag{2}$$

In many analyses, it is assumed that $\hat{h}_k = h_k$, which is not generally true in practice. Instead, to generalize, assume the channel estimate $\hat{h}_k$ can be modeled as

$$\hat{h}_k = \alpha h_k + w_k, \tag{3}$$

where $w_k \sim CN(0, \sigma_w^2)$ is complex Gaussian distributed and where α is some deterministic complex-valued constant. Typically, to find the error rate performance in a fading channel, the AWGN error rate is derived and then integrated over the channel density function [10, p. 817].

In this paper, we take a different approach and calculate the non-Gaussian distribution of $\hat{q}_k$. By expanding (2), we find that

$$\hat{q}_k = \frac{h_k}{\alpha h_k + w_k} q_k + \frac{\eta_k}{\alpha h_k + w_k}. \tag{4}$$

We assume that $\eta_k \sim CN(0, \sigma_\eta^2)$ and $h_k \sim CN(0, \sigma_h^2)$, making the two quotients above ($\frac{h_k}{\alpha h_k + w_k}$ and $\frac{\eta_k}{\alpha h_k + w_k}$) RVs with a complex Gaussian ratio distribution, which is described in detail in the next section.

For an ICSI system in noise, when $\hat{q}_k$ in (4) crosses a predefined symbol boundary a symbol error occurs. In the ICSI noiseless case, a symbol error occurs when the quantity $\frac{h_k}{\alpha h_k + w_k} q_k$ crosses a detection boundary. Similarly in the PCSI case with additive noise, a symbol error occurs when $q_k + \frac{\eta_k}{h_k}$ crosses a detection boundary. In the following section, we derive the distribution of the ratio of two correlated complex Gaussian random variables and then use this distribution to find the symbol error rate of the fading system.

III. Ratio Distribution

From [11], define x and y to be zero-mean correlated complex Gaussian random variables having joint density

$$f_{x,y}(x, y) = \frac{1}{\pi^2 |\Sigma|} \exp\left(- \begin{bmatrix} x \\ y \end{bmatrix}^{\mathcal{H}} \Sigma^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right), \tag{5}$$

where

$$\begin{aligned} \Sigma &= \begin{bmatrix} E[x^* x] & E[x^* y] \\ E[x y^*] & E[y^* y] \end{bmatrix} &(6)\\ &= \begin{bmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho^* \sigma_x \sigma_y & \sigma_y^2 \end{bmatrix}. &(7) \end{aligned}$$

Notice that $\rho = \rho_r + j\rho_i$ is complex valued. In the appendix, we show that the new random variable $z = \frac{x}{y} = z_r + j z_i \in \mathbb{C}$ has probability density function (PDF)

$$f_{x/y}(z_r, z_i) = \frac{1 - |\rho|^2}{\pi \sigma_x^2 \sigma_y^2} \left(\frac{|z|^2}{\sigma_x^2} + \frac{1}{\sigma_y^2} - 2\frac{\rho_r z_r - \rho_i z_i}{\sigma_x \sigma_y} \right)^{-2}, \tag{8}$$

which we denote as $z \sim CR(\sigma_x^2, \sigma_y^2, \sigma_x \sigma_y \rho)$. When there is no correlation between the numerator x and the denominator y in the quotient, the ratio distribution simplifies to

$$f_{x/y}(z) = \frac{\sigma_x^2}{\pi \sigma_y^2} \left(|z|^2 + \frac{\sigma_x^2}{\sigma_y^2} \right)^{-2} \tag{9}$$

and is denoted as $z \sim CR(\sigma_x^2, \sigma_y^2, 0)$.

The integral of the PDF in (8) can be expressed in closed form as

$$\begin{aligned} \int\int f_{x/y}(z_r, z_i) dz_r dz_i &= g(z_r, z_i, \rho_r, \rho_i) &(10)\\ &\quad + g(z_i, z_r, -\rho_i, -\rho_r) &(11)\\ &= G(z_r, z_i), &(12) \end{aligned}$$

where

$$g(z_r, z_i, \rho_r, \rho_i) = \frac{\lambda(z_i, \rho_r, \rho_i)}{2\pi} \cdot \tan^{-1}\left(\frac{\sigma_y z_r - \rho_r \sigma_x}{\sqrt{(1 - \rho_r^2)\sigma_x^2 + 2\rho_i \sigma_x \sigma_y z_i + \sigma_y^2 z_i^2}} \right), \tag{13}$$

and

$$\lambda(z_i, \rho_r, \rho_i) = \frac{(\rho_i \sigma_x + \sigma_y z_i)}{\sqrt{(1 - \rho_r^2)\sigma_x^2 + 2\rho_i \sigma_x \sigma_y z_i + \sigma_y^2 z_i^2}}. \tag{14}$$

To avoid overly cumbersome notation, we excluded σ_x and σ_y from the argument list of $\lambda(\cdot)$, $g(\cdot)$ and $G(\cdot)$, but all three of these functions are also functions of σ_x and σ_y. Using these equations, the cumulative distribution function (CDF) is

$$F_{x/y}(z_r, z_i) = G(z_r, z_i) + \frac{1}{4}(\lambda(z_i, \rho_r, \rho_i) + \lambda(z_r, -\rho_i, -\rho_r) + 1). \tag{15}$$

The marginal CDFs can be calculated by

$$\begin{aligned} Fz_i(z_i, \rho_r, \rho_i) &= \lim_{z_r \to \infty} Fx/y(z_r, z_i) &(16)\\ &= \frac{1}{2}\left(\lambda(z_i, \rho_r, \rho_i) + 1\right). &(17) \end{aligned}$$

Similarly,

$$Fz_r(z_r, \rho_r, \rho_i) = Fz_i(z_r, -\rho_i, -\rho_r). \tag{18}$$

Fig. 1 is a plot of the CDF, PDF and marginal CDFs of the complex ratio distribution $CR(4, 1, 0)$. From the equation in (9), it is clear that the PDF is circularly symmetric in the complex plane, which is also evident in the plot. For comparison the $CR(4, 1, 1 + j\frac{1}{2})$ distribution functions are plotted in Fig. 2. The plot illustrates that for a non-zero value of ρ, the complex ratio distribution is no longer circularly symmetric.

IV. M-QAM Error Probabilities

A nice feature of the complex ratio distribution is that the sum of two complex ratios is also a complex ratio random variable. To calculate the error rate of a constellation using this channel model, we declare a symbol error when the difference

$$\hat{q}_k - q_k = \frac{((1 - \alpha)h_k - w_k)q_k + \eta_k}{\alpha h_k + w_k} \tag{19}$$

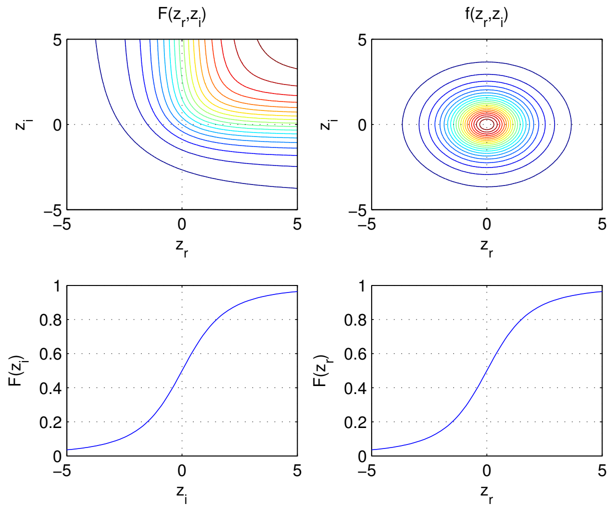

Fig. 1. Plot of CDF, PDF and marginal CDFs for ratio distribution. Distribution parameters are $\sigma_x = 2$, $\sigma_y = 1$, $\rho = 0$.

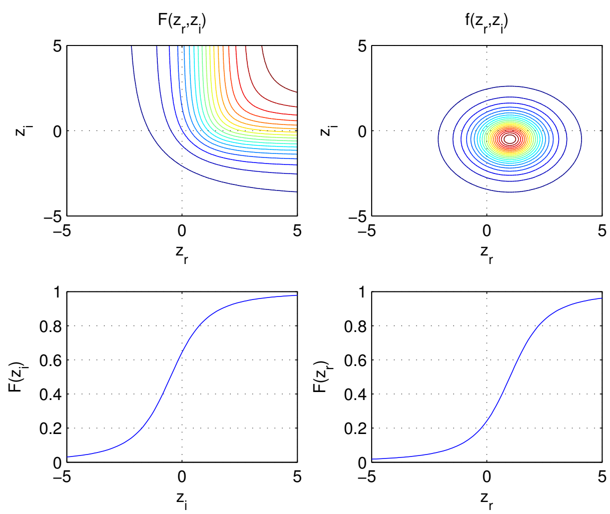

Fig. 2. Plot of CDF, PDF and marginal CDFs for ratio distribution. Distribution parameters are $\sigma_x = 2$, $\sigma_y = 1$, $\rho = 1/2 + j1/4$.

crosses a complex-valued threshold and resides in a region $\{\mathcal{R}_i\}_{i=1}^{|\mathcal{A}|} \subset \mathbb{C}$ that is dependent on the transmitted constellation point $a_i \in \mathcal{A}$ and is centered on the origin of the complex plane. Thus, the total symbol error is simply

$$SER = \sum_{i=1}^{|\mathcal{A}|} Pr(q_k = a_i) \int_{\mathcal{R}_i} f_{\hat{q}_k - q_k | q_k = a_i}(z) dz. \quad (20)$$

To compute this, we will compute the difference distribution of $\hat{q}_k - q_k$, which is given by

$$f_{\hat{q}_k - q_k}(z) \sim CR\Big(|q_k|^2 (\sigma_h^2 |1-\alpha|^2 + \sigma_w^2) + \sigma_\eta^2, \\ |\alpha|^2 \sigma_h^2 + \sigma_w^2, q_k^* (\alpha(1-\alpha)^* \sigma_h^2 + \sigma_w^2) \Big). \quad (21)$$

From the PDF, it is clear that the ratio distribution is not necessarily circularly symmetric. In fact, it is only symmetric when $\rho = 0$. For the channel/detection model in this paper, a value of $\rho = 0$ implies $\hat{h} = h$; i.e., the receiver has PCSI. Thus, for the ICSI case, it is not possible to simplify (20) by only evaluating the symbol errors for one quadrant of the constellation plane. The implication is that the SER depends on the transmitted constellation point and the channel is not symmetric across all symbols.

We can compute the M-QAM symbol error probabilities in closed form by simply enumerating all of the error events. For M-QAM, the error region from (20) is dependent on the constellation point a_i and is defined by a box around the origin, where $\mathcal{R} = \{\overline{z_r \cap z_i} : z_i \in [j\epsilon_{il}, j\epsilon_{ih}], z_r \in [\epsilon_{rl}, \epsilon_{rh}]\}$. Accordingly, the probability that a transmitted symbol is in $\mathcal{R}_i$ can be expressed as

$$\begin{aligned} p_e(\epsilon, a_i) &= \int_{\mathcal{R}_i} f_{\hat{q}_k - q_k | q_k = a_i}(z) dz \\ &= 1 - G(\epsilon_{rl}, \epsilon_{il}) - G(\epsilon_{rh}, \epsilon_{ih}) + \\ &\quad G(\epsilon_{rl}, \epsilon_{ih}) + G(\epsilon_{rh}, \epsilon_{il}), \end{aligned} \quad (22)$$

where $\epsilon = [\epsilon_{il}, \epsilon_{ih}, \epsilon_{rl}, \epsilon_{rh}]$.

For large M-QAM constellations, boundary effects contribute less to the SER. So for large M, the SER is well approximated by the probability of a symbol error occurring on one of the interior constellation points. For normalized M-QAM where $\sum_i |a_i|^2 = 1$, the bounding box on the interior constellation points is defined by $-\epsilon_{il} = \epsilon_{ih} = -\epsilon_{rl} = \epsilon_{rh} = \sqrt{\frac{3}{2(M-1)}}$. It is not clear from (22) because we shorten the argument list of $G(\cdot)$, but this probability of error depends on the moments σ_x, σ_y and ρ.

For the assumed channel/equalization model defined in Section II the resulting demodulated output has a distribution defined by (21), which is explicitly dependent on the transmitted value $q_k \in \mathcal{A}$. Therefore, each of the moments of $\hat{q}_k - q_k$ depend on q_k, which means that both $G(\cdot)$ and $p_e(\cdot)$ depend on the transmitted q_k. With this, the probability of error only requires evaluating (20) using the M-QAM assumption. Assuming equally likely symbols,

$$SER \leq \frac{1}{M} \sum_{i=1}^{|\mathcal{A}|} p_e(\epsilon, a_i). \quad (23)$$

This is an upper bound on the SER because the edge and corner constellation points are not bound by the same detection box as the interior constellation as assumed here. However, it is straightforward to use these results to derive the exact SER. We omit this derivation here because of space constraints, but the procedure simply involves finding an error expression similar to (22) for each constellation point. This precise SER expression along with the upper bound in (23) is evaluated in the simulations in the next section.

V. Simulations

As outlined in the previous sections, the complex ratio distribution allows for a concise representation of the SER in fading channels both when PCSI is available to the receiver and when only ICSI is available. In this section, several experiments are run to evaluate the performance of an ICSI system.

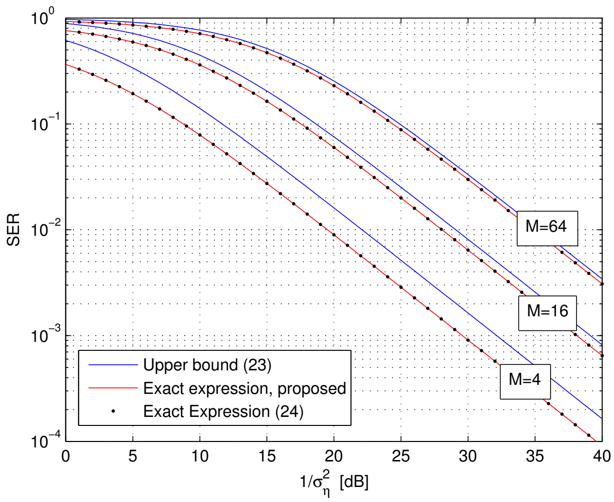

Fig. 3. SER for the proposed exact expression (same as (24)), SER given by the upper bound in (23), and SER from [12] in (24).

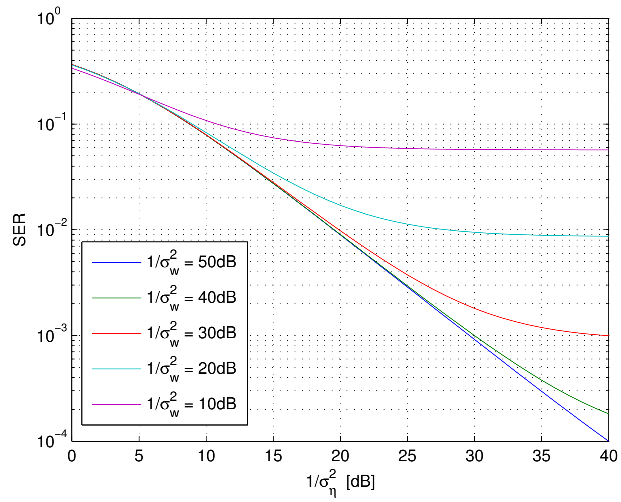

Fig. 4. SER with ICSI for $\alpha = 1$ and $\sigma_h = 1$.

Experiment 1: For the first experiment, it is verified that the PCSI SER result matches the commonly-cited expression in the literature given by [12, p. 254]. The authors of [12] derive the SER and show that it is exactly

$$SER_1 = 2A(1-B) - A^2(1 - B\ \tan^{-1}(B^{-1/2})), \quad (24)$$

where $A = \frac{\sqrt{M}-1}{\sqrt{M}}$ and $B = \sqrt{\frac{1.5}{\sigma_\eta^2(M-1)+1.5}}$.

In Fig. 3, the SER is plotted for the PCSI channel (assumes $\alpha = 1$, $\sigma_w = 0$ and $\sigma_h = 1$). While the SER expression has long been known, we have illustrated an alternative derivation and have shown that the SER result of the proposed method agrees with the truth data.

Experiment 2: In this experiment, the effect of ICSI is illustrated. Fig. 4 is a plot of the SER as the distortion noise on the channel estimate, σ_w is varied. For these results we assumed $\alpha = 1$ and $\sigma_h = 1$. It is clear from the plot that even a moderate noise on the channel estimate $\sigma_w = -20$dB, leads to a severe SER increase. The effect is seen as an SER floor that depends on the channel estimation error variance.

Experiment 3: In practice, there may be some residual phase shift between the true channel and the channel estimate. For the model proposed in this paper, this phase shift is manifested in the parameter α. To isolate the phase shift effect, we model $\alpha = e^{j2\pi\theta}$ and plot the effect of small non-zero values of θ. The SER results are independent of the sign of θ so only positive θ is examined. Fig. 5 is a plot of the SER as the channel estimate phase shift varies for two different distortion noise levels in the channel estimate, $\sigma_w = \{-\infty, -30\}$dB. For these results we assumed $\sigma_h = 1$. Unlike channel estimation noise that imparts an SER floor on the received symbols, small phase shifts have a less detrimental effect.

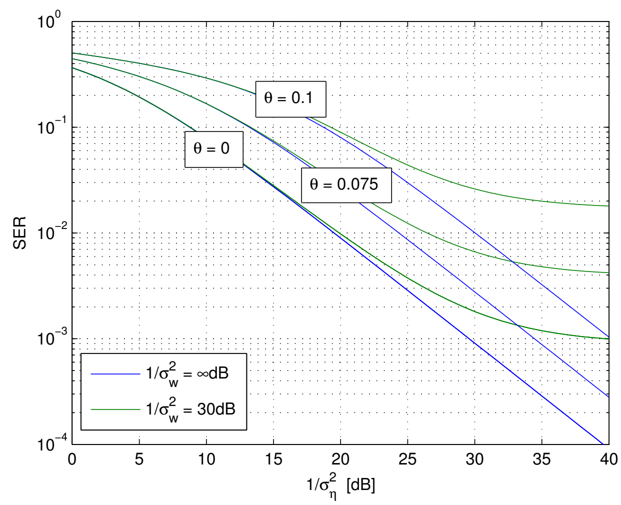

Fig. 5. SER with ICSI for $\alpha = e^{j2\pi\theta}$ and $\sigma_h = 1$.

VI. Conclusions

This paper offers two novel contributions. The first is the derivation of the distribution of the ratio of two zero-mean correlated complex Gaussian RVs. We call this the complex ratio distribution. The second contribution is the use of the complex ratio distribution in finding the exact closed-form SER expression of a communication system employing QAM with ICSI. The example experiments show that the SER is very sensitive to even small channel estimation variances and phase shifts.

Appendix

From [11], the correlated bivariate complex Gaussian PDF in (5) is expanded by writing $x = x_r + ix_i$ and $y = y_r + iy_i$. The density function for the zero-mean case (i.e. $\mu_x = \mu_y =$

0), becomes

$$f_{x,y}(x_r, x_i, y_r, y_i) = \frac{1}{\pi^2 \sigma_x \sigma_y (1-\rho^2)} \exp\left(\frac{-1}{(1-\rho^2)} \left(\frac{x_i^2 + x_r^2}{\sigma_x^2} + \frac{y_i^2 + y_r^2}{\sigma_y^2} - \frac{2\rho}{\sigma_x \sigma_y}(x_i y_i + x_r y_r) \right) \right). \quad (25)$$

The goal is to find the distribution of the ratio $z = x/y$. It is clear that the density function, $f_{x/y}(z)$, is defined by the following quadruple integral

$$f_{x/y}(z) = \int\int\int\int f_{x,y}(x_r, x_i, y_r, y_i) \delta\left(\frac{x_r + jx_i}{y_r + jy_i} - z \right) dx_r dx_i dy_r dy_i, \quad (26)$$

where the domain of integration is $(-\infty, \infty)$ for all four integrals and where $\delta(\cdot)$ is the complex valued delta function.

Following the lead of [2], the ratio density can be simplified with a change of variables so that $x = uv$ and $y = v$. Unlike the real-valued case, the complex-valued variable transformation has a Jacobian of $|v|^2$. Now, the cumulative distribution function (CDF) of z is defined as

$$F_{x/y}(z) = \int_{-\infty}^{z_r}\int_{-\infty}^{z_i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_i^2 + v_r^2) f_{x,y}(u_r v_r - u_i v_i, v_i u_r + u_i v_r, v_r, v_i) dv_r dv_i du_i du_r. \quad (27)$$

The goal is to find the density function $f_{x/y}(z)$, which by definition is the derivative of the CDF,

$$\begin{aligned} f_{x/y}(z_r, z_i) &= \frac{\partial^2}{\partial u_r \partial u_i} F_{x/y}(z_r, z_i) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_i^2 + v_r^2) f_{x,y}(z_r v_r - z_i v_i, \\ &\quad v_i z_r + z_i v_r, v_r, v_i) dv_r dv_i. \end{aligned} \quad (28)$$

With this, the problem entials evaluating the following integral

$$f_{x/y}(z_r, z_i) = \frac{1}{\pi^2 \sigma_x \sigma_y (1-\rho^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |v|^2 \exp\left(\frac{-1}{1-\rho^2} \left(\frac{(z_i^2 + z_r^2)|v|^2}{\sigma_x^2} + \frac{|v|^2}{\sigma_y^2} - 2\frac{\rho_r z_r - \rho_i z_i}{\sigma_x \sigma_y} |v|^2 \right) \right) dv_i dv_r.$$

Evaluating the integral by a change of variables to polar coordinates and normalizing so that the total probability is one results in the PDF expression given in (8).

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