Complex manifolds with split tangent bundle

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Abstract. Let X be a compact Kähler manifold. We ask whether any direct sum decomposition $T_X = \oplus_{i\in I} E_i$ of its tangent bundle comes from a splitting of the universal covering space of X as a product $\prod_{i\in I} U_i$, in such a way that the given decomposition $T_X = \oplus_{i\in I} E_i$ lifts to the canonical decomposition $T_{\prod U_i} = \oplus_i T_{U_i}$. We prove that this is the case when X is a Kähler-Einstein manifold or a Kähler surface, and discuss a general conjecture.

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Introduction

The theme of this note is to investigate when the tangent bundle of a compact complex manifold X splits as a direct sum of sub-bundles. This occurs typically when the universal covering space $\widetilde{X}$ of X splits as a product $\prod_{i\in I} U_i$ of manifolds on which the group $\pi_1(X)$ acts diagonally (that is, $\pi_1(X)$ acts on each U_i and its action on $\widetilde{X} = \prod U_i$ is the diagonal action $g.(u_i) = (gu_i)$): the vector bundles* T_{U_i} on $\widetilde{X}$ are stable under $\pi_1(X)$, hence the decomposition $T_{\widetilde{X}} = \oplus_i T_{U_i}$ descends to a direct sum decomposition of T_X. For *Kähler* manifolds, we ask whether the converse is true, namely whether a direct sum decomposition of the tangent bundle T_X gives rise to a splitting of the universal covering. We will show that this is indeed the case in three different situations:

a) X admits a Kähler-Einstein metric;

b) T_X is a direct sum of line bundles of negative degree;

c) X is a Kähler surface.

In case *a*) the properties of Hermite-Einstein metrics imply that the tangent bundle splits as a direct sum of *hermitian* sub-bundles; we then conclude with a

* Throughout the paper we will abuse notation and write T_{U_i} instead of $pr_i^*T_{U_i}$.

holonomy argument (a slightly less precise statement appears already in [Y]). Case *b*) is a small improvement of a uniformization result of Simpson [S]. To treat case *c*) we use the classification of surfaces and some simple remarks about connections. The result in this case is actually an easy consequence of the paper [KO], where the authors classify surfaces with a holomorphic conformal structure – this turns out to be closely related to the question we are studying here. However we found simpler and more enlightening to give an independent proof rather than extracting from [KO] the pieces of information that we need.

In § 2 we give examples which show that the Kähler assumption, as well as some integrability assumptions, are necessary, and we propose a general conjecture.

1. Kähler-Einstein manifolds

Theorem A. *Let X be a compact complex manifold admitting a Kähler-Einstein metric. Assume that the tangent bundle of X has a decomposition $T_X = \bigoplus_{i\in I} E_i$. Then the universal covering space of X is a product $\prod_{i\in I} U_i$ of complex manifolds, in such a way that the decomposition $T_X = \bigoplus_{i\in I} E_i$ lifts to the decomposition $T_{\prod U_i} = \bigoplus_{i\in I} T_{U_i}$; the group $\pi_1(X)$ acts diagonally on $\prod_{i\in I} U_i$.*

Proof. (1.1) A Kähler-Einstein metric on X is a *Hermite-Einstein* metric on the vector bundle T_X, that is a hermitian metric whose curvature endomorphism, contracted with the Kähler form ω, is scalar (a good reference for the properties of Hermite-Einstein metrics that we will use is [K]). By theorem V.8.3 of [K], the hermitian bundle T_X is the direct sum of a family $(F_j)_{j\in J}$ of ω-stable, hermitian vector bundles having the same slope as T_X. These bundles are preserved by the Levi-Civita connection, hence the holonomy representation of X is the direct sum of a family of representations corresponding to the F_j's. By the De Rham theorem, the universal covering space of X splits as a product $\prod_{j\in J} U_j$, such that the decomposition $T_X = \bigoplus_{j\in J} F_j$ pulls back to the decomposition $T_{\prod U_j} = \bigoplus_{j\in J} T_{U_j}$.

(1.2) We observe that the fact that the group $\pi_1(X)$ preserves the decomposition $T_{\prod U_j} = \bigoplus_{j\in J} T_{U_j}$ implies that it acts diagonally on $\prod_{j\in J} U_j$. Let indeed γ be an automorphism of $\prod U_i$; for $j\in I$, put $\gamma_j = pr_j\circ\gamma$. The condition $\gamma^* T_{U_j} = T_{U_j}$ means that the partial derivatives of γ_j in the directions of U_k for $k\neq j$ vanish, hence $\gamma_j((u_i)_{i\in I})$ depends only on u_j, which gives our claim.

(1.3) The bundles F_j are indecomposable, and we can assume that each E_i is indecomposable. By the Krull-Remak-Schmidt theorem, we can identify J to I

in such a way that F_i is isomorphic to E_i for every $i\in I$. We want to compare the decompositions $T_X=\bigoplus_{i\in I} E_i$ and $T_X=\bigoplus_{i\in I} F_i$.

Lemma 1.3. *If* $\mathrm{Hom}(F_i,F_j)\neq 0$ *for some distinct indices* i,j *in* I, *the bundles* F_i *and* F_j *are isomorphic and admit a holomorphic connection.*

In particular, all Chern classes of F_i vanish.

Proof. Since F_i and F_j are stable with the same slope, our hypothesis implies that F_i and F_j are isomorphic ([K], 7.11 and 7.12); this is equivalent to the existence of an isomorphism $\varphi : T_{U_i}\to T_{U_j}$ compatible with the actions of $\pi_1(X)$.

Recall that if $f:T\to S$ is a holomorphic map between two manifolds, and E a vector bundle on S, the bundle f^*E carries a canonical relative flat connection $\nabla_{T/S}:f^*E\to f^*E\otimes\Omega^1_{T/S}$, characterized by the property $\nabla_{T/S}(f^*s)=0$ for every local holomorphic section s of E; if moreover f is equivariant with respect to a group Γ acting on T, S and E, the connection $\nabla_{T/S}$ is Γ-equivariant. Applying this to the projection $\prod_i U_i\to U_i$ we obtain for each $k\neq i$ a partial, $\pi_1(X)$-equivariant, connection $\nabla_k : T_{U_i}\to T_{U_i}\otimes\Omega^1_{U_k}$. Similarly we have for each $k\neq j$ a partial connection $\nabla'_k : T_{U_j}\to T_{U_j}\otimes\Omega^1_{U_k}$. Put $\nabla_i=(\varphi\otimes 1)^{-1}\circ\nabla'_i\circ\varphi$; then $\sum_{k\in I}\nabla_k$ is a connection on T_{U_i} which is $\pi_1(X)$-equivariant, and therefore descends to a connection on F_i. □

(1.4) Let $i\in I$. If F_i does not admit any holomorphic connection, it follows from the Lemma that the only sub-bundle of T_X isomorphic to F_i is F_i itself, hence $E_i=F_i$.

Now assume that F_i has a holomorphic connection. Since F_i has the same slope as T_X, this can only occur if $c_1(X)=0$. According to the structure theorem for manifolds with $c_1=0$ ([B2], thm. 1), the set I splits into two subsets J and K, such that U_i is isomorphic to a vector space for $i\in J$ and is compact for $i\in K$; the vector bundle F_i has trivial Chern classes if and only if $i\in J$. Put $F=\bigoplus_{j\in J} F_j$; according to Lemma 1.2 we have $E_j\subset F$ for $j\in J$. We saw already that $E_k=F_k$ for $k\in K$, hence $\bigoplus_{j\in J} E_j=F$.

Put $V=\prod_{j\in J} U_j$, $M=\prod_{k\in K} U_k$. There exists a complex torus A with universal covering V and a finite étale covering $\pi : A\times M\to X$ (*loc. cit.*). We have $\pi^*F=T_A$; the decomposition $F=\bigoplus_{j\in J} E_j$ pulls back to a decomposition of the trivial bundle T_A, which corresponds to a decomposition $V=\bigoplus_{j\in J} V_j$ of V into vector subspaces. The splitting $\widetilde{X}=\prod_{j\in J} V_j\times\prod_{k\in K} U_k$ has the requested properties.

□

2. Discussion of the conjecture

Let us first show that the Kähler assumption is necessary.

(2.1) *Hopf manifolds*

Let $T = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ be a diagonal matrix, with $n \geq 2$ and $0 < |\alpha_i| < 1$ for each i. The cyclic group $T^{\mathbf{Z}}$ generated by T acts freely and properly on $\mathbf{C}^n - \{0\}$; the quotient X is a compact complex manifold, called a Hopf manifold. For each non-zero complex number θ, denote by L_θ the flat line bundle associated to the character of $\pi_1(X) = T^{\mathbf{Z}}$ mapping T to θ; in other words, L_θ is the quotient of the trivial line bundle $(\mathbf{C}^n - \{0\}) \times \mathbf{C}$ by the action of the automorphism (T, θ). By construction we have $T_X = \bigoplus_{i=1}^{n} L_{\alpha_i}$, but the universal covering space $\mathbf{C}^n - \{0\}$ of X is clearly not a product. Note that all direct sums $\bigoplus_{j \in J} L_{\alpha_j}$, for $J \subset [1, n]$, are integrable sub-bundles of T_X.

(2.2) *Integrability conditions*

Let X be a compact Kähler manifold. If a decomposition $T_X = \bigoplus_{i \in I} E_i$ is associated as above to a splitting $\widetilde{X} \cong \prod_{i \in I} U_i$ of the universal covering space of X, the vector bundles E_i and their direct sums $\bigoplus_{i \in J} E_i$, for every subset J of I, are integrable (that is, stable under the Lie bracket). It is easy to produce examples where the tangent bundle splits into non-integrable factors: take for instance $X = A \times \mathbf{P}^1$, where A is an abelian surface. Let (U, V) be a basis of $H^0(A, T_A)$, and S, T two vector fields on $\mathbf{P}^1$ which do not commute. The vector fields $U + S$ and $V + T$ span a (trivial) rank 2 sub-bundle of T_X, supplementary to $T_{\mathbf{P}^1}$, but not integrable.

In view of the above examples the natural conjecture is the following:

(2.3) *Let X be a compact Kähler manifold such that $T_X = \bigoplus_{i \in I} E_i$, each sub-bundle $\bigoplus_{i \in J} E_i$, for $J \subset I$, being integrable. Then the universal covering space of X is isomorphic to a product $\prod_{i \in I} U_i$, in such a way that the given decomposition $T_X = \bigoplus_{i \in I} E_i$ lifts to the canonical decomposition $T_{\prod U_i} = \bigoplus_i T_{U_i}$.*

In the case when all the E_i's are line bundles and X is projective, this conjecture has just been proved by S. Druel [D].

In the situations a), b), c) considered here it turns out that the integrability is automatic. One may ask whether this holds whenever the canonical bundle K_X is *nef*.

3. Simpson's uniformization result

The following lemma *, which is a variation on the Baum-Bott theorem [B-B], will allow us to slightly improve Simpson's result:

Lemma 3.1. *Let X be a complex manifold, and E a direct summand of T_X. The Atiyah class* $\mathrm{at}(E)\in H^1(X,\Omega^1_X\otimes\mathcal{E}nd(E))$ *comes from* $H^1(X,E^*\otimes\mathcal{E}nd(E))$. *In particular, any class in* $H^r(X,\Omega^r_X)$ *given by a polynomial in the Chern classes of* E *vanishes for* $r>\mathrm{rk}(E)$.

Proof. Write $T_X=E\oplus F$; let $p:T_X\to E$ be the corresponding projection. For sections U of E and V of F over some open subset of X, put $D_VU=p([V,U])$. This expression is $\mathcal{O}_X$-linear in V and satisfies the Leibnitz rule $D_V(fU)=fD_V(U)+(Vf)U$, so that D is a F-*connection* on E [B-B]: if we denote by $\mathcal{D}^1(E)$ the sheaf of differential operators $\Delta:E\to E$, of degree ≤ 1, whose symbol $\sigma(\Delta)$ is scalar, this means that D defines an $\mathcal{O}_X$-linear map $F\to\mathcal{D}^1(E)$ such that $\sigma(D_V)=V$ for all local sections V of F. Thus the exact sequence

$$0\to\mathcal{E}nd(E)\longrightarrow\mathcal{D}^1(E)\xrightarrow{\sigma}T_X\to 0$$

splits over the sub-bundle $F\subset T_X$; therefore its extension class $\mathrm{at}(E)\in H^1(X,\Omega^1_X\otimes\mathcal{E}nd(E))$ vanishes in $H^1(X,F^*\otimes\mathcal{E}nd(E))$, hence comes from $H^1(X,E^*\otimes\mathcal{E}nd(E))$. The last assertion follows from the definition of the Chern classes in terms of the Atiyah class. □

We denote as usual by $\mathbf{H}$ the Poincaré upper half-space.

Theorem B. *Let X be a compact Kähler manifold, with Kähler class ω. Assume that the tangent bundle T_X is a direct sum of line bundles $L_1,\ldots,L_n$ with $\omega^{n-1}.\,c_1(L_i)<0$ for each i. Then the universal covering space of X is $\mathbf{H}^n$, and the decomposition $T_X=\oplus L_i$ lifts to the canonical decomposition $T_{\mathbf{H}^n}=(T_{\mathbf{H}})^{\oplus n}$.*

Proof. Lemma 3.1 gives $c_1(L_i)^2=0$ for each i, hence $c_1(X)^2-2c_2(X)=0$. Then Cor. 9.7 of [S] shows that the universal covering space of X is $\mathbf{H}^n$. The assertion about the compatibility of decompositions is not explicitly stated in *loc. cit.*, but follows from the proof; or we can apply Theorem A. □

4. The surface case

Theorem C. *Let X be a compact complex surface. The tangent bundle of X splits as a direct sum of two line bundles if and only if one of the following occurs:*

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* F. Bogomolov reminded me that this lemma appears already in his IHES preprint *Kählerian varieties with trivial canonical class* (1981).

(a) *The universal covering space of* X *is a product* $U\times V$ *of two (simply-connected) Riemann surfaces and the group* $\pi_1(X)$ *acts diagonally on* $U\times V$ *; in that case the given splitting of* T_X *lifts to the direct sum decomposition* $T_{U\times V}=T_U\oplus T_V$ *.*

(b) X *is a Hopf surface, with universal covering space* $\mathbf{C}^2-\{0\}$ *. Its fundamental group is isomorphic to* $\mathbf{Z}\oplus\mathbf{Z}/m\mathbf{Z}$ *, for some integer* $m\geq 1$ *; it is generated by diagonal automorphisms* $(x,y)\mapsto(\alpha x,\beta y)$ *with* $|\alpha|\leq|\beta|<1$ *, and* $(x,y)\mapsto(\lambda x,\mu y)$ *where* λ *and* μ *are primitive* m*-th roots of* 1 *.*

As a corollary, for Kähler surfaces we see that any direct sum decomposition of the tangent bundle gives rise to a splitting of the universal covering, as announced in the introduction.

(4.1) Before starting the proof we will need a few preliminaries. From now on we denote by X a compact complex surface; we assume given a direct sum decomposition $\Omega^1_X\cong L\oplus M$. By lemma 3.1 (or by [B-B]) the Chern class $c_1(L)\in H^1(X,\Omega^1_X)$ belongs to the subspace $H^1(X,L)$, and similarly for M . As a consequence, we get:

(4.2) $L^2=M^2=0$, and therefore $c_1^2(X)=2\,L.M=2c_2(X)$.

The following consequence is less obvious.

Proposition 4.3. *Let* C *be a smooth rational curve in* X *. Then* $C^2\geq 0$ *.*

Proof. Put $C^2=-d$ and assume $d>0$. Since $H^1(C,\mathcal{O}_C(d+2))=0$, the exact sequence

$$0\to\mathcal{O}_C(d)\longrightarrow\Omega^1_{X\,|C}\longrightarrow\Omega^1_C\to 0$$

splits, providing an isomorphism $\Omega^1_{X\,|C}\cong\mathcal{O}_C(d)\oplus\mathcal{O}_C(-2)$. Thus one of the line bundles L or M , say L , satisfies $L_{|C}\cong\mathcal{O}_C(d)$. Consider the commutative diagram

$$\begin{array}{ccc} H^1(X,L) & \longrightarrow & H^1(X,\Omega^1_X) \\ \big\downarrow & & \big\downarrow \\ H^1(C,L_{|C}) & \longrightarrow & H^1(C,\Omega^1_C) \end{array} \quad ;$$

since $d>0$ we have $H^1(C,L_{|C})=0$; thus $c_1(L)$ goes to 0 in $H^1(C,\Omega^1_C)$, which means $d=0$, a contradiction. ◻

(4.4) We shall come across situations where the vector bundle $\Omega^1_X=L\oplus M$ appears as an extension

$$0\to P\longrightarrow\Omega^1_X\xrightarrow{\ p\ }Q\to 0$$

of two line bundles P and Q. In that case,
- either the restriction of p to one of the direct summands of Ω^1_X, say M, is surjective; then the exact sequence splits, Q is isomorphic to M and P to L;
- or the restriction of p to both L and M is not surjective; then there exists effective (non-zero) divisors A and B, whose supports do not intersect, such that $L \cong Q(-A)$, $M \cong Q(-B)$ and $P \cong Q(-A-B)$; the exact sequence does *not* split.

In particular, if $\mathrm{Hom}(P,Q)=0$, the exact sequence splits.

(4.5) Finally we will need some classical facts about connections (see [E]). Let $p: M \to B$ be a smooth holomorphic map between complex manifolds, whose fibres are isomorphic to a fixed variety F. A *connection* on p is a splitting of the exact sequence

$$0 \to p^*\Omega^1_B \longrightarrow \Omega^1_M \longrightarrow \Omega^1_{M/B} \to 0\ ,$$

that is a sub-bundle $L \subset \Omega^1_M$ mapping isomorphically onto $\Omega^1_{M/B}$; the connection is flat (or integrable) if $dL \subset L \wedge \Omega^1_M$ (this is automatic if B is a curve). In that case the group $\pi_1(B)$ acts on F by complex automorphisms, and M is the fibre bundle on B with fibre F associated to the universal covering $\widetilde{B} \to B$, that is the quotient of $\widetilde{B} \times F$ by the group $\pi_1(B)$ acting diagonally; the splitting $\Omega^1_M = p^*\Omega^1_B \oplus L$ pulls back to the decomposition $\Omega^1_{\widetilde{B}\times F} = \Omega^1_{\widetilde{B}} \oplus \Omega^1_F$.

5. Proof of Theorem C

(5.1) *Kodaira dimension* 2

If $\kappa(X)=2$, the canonical bundle K_X is ample by Prop. 4.3. The Aubin-Calabi-Yau theorem implies that X admits a Kähler-Einstein metric; we can therefore apply Theorem A.

(5.2) *Kodaira dimension* 1

If $\kappa(X)=1$, X admits an elliptic fibration $p: X \to B$. By 4.2 we have $c_2(X)=0$; this implies that the only singular fibres of p are multiples of smooth elliptic curves (see [B1], VI.4 and VI.5). For $b \in B$, we write $p^*[b] = m_b\, F_b$, where F_b is a smooth elliptic curve; we have $m_b \geq 1$ and $m_b = 1$ except for finitely many points. Put $\Delta = \sum_b (m_b - 1)\, F_b$. We have an exact sequence

$$0 \to p^*\Omega^1_B(\Delta) \longrightarrow \Omega^1_X \longrightarrow \omega_{X/B} \to 0 \tag{5.3}$$

where $\omega_{X/B}$ is the relative dualizing line bundle. Since $\chi(\mathcal{O}_X)=0$ by Riemann-Roch, we deduce from [BPV], V.12.2 and III.18.2, that $\omega_{X/B}$ is a torsion line bundle. Since $K_X = p^*\Omega^1_B(\Delta) \otimes \omega_{X/B}$, the hypothesis $\kappa(X)=1$ implies $\mathrm{Hom}(p^*\Omega^1_B(\Delta), \omega_{X/B}) = 0$, hence the exact sequence (5.3) splits by 4.4: one of the direct summands of Ω^1_X, say M, maps surjectively onto $\omega_{X/B}$.

Let $\rho: \widetilde{B} \to B$ be the orbifold universal covering of $(B,(m_b))$: this is a ramified Galois covering, with $\widetilde{B}$ simply-connected, such that the stabilizer of a point $\tilde{b} \in \widetilde{B}$ is a cyclic group of order $m_{\rho(\tilde{b})}$ (see for instance [KO], lemma 6.1; note that because of the hypothesis $\kappa(X)=1$ and the formula for K_X, there are at least 3 multiple fibers if B is of genus 0). Let $\widetilde{X}$ be the normalization of $X\times_B \widetilde{B}$. We have a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \pi\ } & X \\ {\scriptstyle \tilde{p}}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \widetilde{B} & \xrightarrow{\ \rho\ } & B \end{array}$$

where $\tilde{p}$ is smooth and π is étale ([B1], VI.7′). The exact sequence

$$0\to \tilde{p}^*\Omega^1_{\widetilde{B}} \longrightarrow \Omega^1_{\widetilde{X}} \longrightarrow \Omega^1_{\widetilde{X}/\widetilde{B}} \to 0$$

coincides with the pull back under π of the exact sequence (5.3); therefore p admits an integrable connection, given by the subbundle π^*M of $\Omega^1_{\widetilde{X}}$. The result follows from 4.5 and 1.2.

(5.4) *Kodaira dimension* 0

Assume $\kappa(X)=0$. By 4.2 and the classification of surfaces, X is either a complex torus, a bielliptic surface, or a Kodaira surface. Complex tori and bielliptic surfaces fall into case (a) of the theorem (a bielliptic surface is the quotient of a product $E\times F$ of elliptic curves by a finite abelian group acting diagonally).

A primary Kodaira surface has trivial canonical bundle and admits a smooth elliptic fibration $p:X\to B$. Thus the exact sequence (5.3) realizes Ω^1_X as an extension of $\mathcal{O}_X$ by $\mathcal{O}_X$. Since $h^{1,0}(X)=1$, this extension is non-trivial, and it follows from 4.4 that Ω^1_X does not split.

A secondary Kodaira surface admits a primary Kodaira surface as a finite étale cover, hence its tangent bundle cannot split either.

(5.5) *Ruled surfaces*

We consider the case when X is algebraic and $\kappa(X)=-\infty$. By 4.2 and 4.3, X is a geometrically ruled surface, that is a projective bundle $p:X\to B$ over a curve. We again consider the exact sequence

$$0\to p^*\Omega^1_B \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/B}\to 0\ ;$$

since $\Omega^1_{X/B}$ has negative degree on the fibres, we have $\mathrm{Hom}(p^*\Omega^1_B,\Omega^1_{X/B})=0$, hence by 4.4 the above exact sequence splits: one of the direct summands of Ω^1_X defines an integrable connection for p. The result follows then from 4.5.

(5.6) *Inoue surfaces*

We now assume that X is not algebraic and $\kappa(X)=-\infty$, so that X is what is usually called a surface of type VII_0. These surfaces have $b_1=h^{0,1}=1$ and therefore $c_1^2+c_2=12\chi(\mathcal{O}_X)=0$; in our case this gives $c_2=0$ in view of 4.2, and finally $b_2=0$. Moreover we have $H^0(X,\Omega^1_X\otimes L^{-1})\neq 0$. The surfaces with these properties have been completely classified by Inoue [I]: they are either Hopf surfaces, or belong to three classes of surfaces constructed by Inoue (*loc. cit.*).

We first consider the Inoue surfaces. The surfaces S_M of the first class are quotients of $\mathbf{H}\times\mathbf{C}$ by a group acting diagonally, hence they fall into case (a) of the theorem.

The surfaces $S^{(+)}_{N,p,q,r;t}$ of the second class are quotients of $\mathbf{H}\times\mathbf{C}$ by a group which does *not* act diagonally. This action leaves invariant the vector field $\partial/\partial z$ on $\mathbf{C}$, which therefore descends to a non-vanishing vector field v on X. This gives rise to an exact sequence

$$0\rightarrow K_X \xrightarrow{\ i(v)\ } \Omega^1_X \xrightarrow{\ i(v)\ } \mathcal{O}_X \rightarrow 0\ ,$$

which does not split since $h^{1,0}(X)=0$. We have $H^0(X,K_X^{-1})=0$, for instance because X contains no curves; we infer from 4.4 that Ω^1_X does not split.

The surfaces $S^{(-)}_{N,p,q,r}$ of the third class are quotients of certain surfaces of the second class by a fixed point free involution; therefore their tangent bundle does not split either.

(5.7) *Primary Hopf surfaces*

It remains to consider the class of Hopf surfaces, which are by definition the surfaces of class VII_0 whose universal covering space is $\mathbf{W}:=\mathbf{C}^2-\{0\}$. We consider first the *primary* Hopf surfaces, which are quotients of $\mathbf{W}$ by the infinite cyclic group generated by an automorphism T of $\mathbf{W}$. According to [Ko], § 10, there are two cases to consider:

a) $T(x,y)=(\alpha x,\beta y)$ for some complex numbers α,β with $0<|\alpha|\le|\beta|<1$;

b) $T(x,y)=(\alpha^m x+\lambda y^m,\alpha y)$ for some positive integer m and non-zero complex numbers α,λ with $|\alpha|<1$.

As in 2.1, we denote by L_θ, for $\theta\in\mathbf{C}$, the flat line bundle associated to the character of $\pi_1(X)$ mapping T to θ. In case *a)* we find $\Omega^1_X=L_\alpha^{-1}\oplus L_\beta^{-1}$, so the tangent bundle splits.

Let us consider case *b)*. The form dy on $\mathbf{W}$ satisfies $T^*dy=\alpha\,dy$, hence descends to a form $\overline{dy}$ in $H^0(X,\Omega^1_X\otimes L_\alpha)$; similarly the function y descends to a non-zero section of L_α. We have an exact sequence

$$0\rightarrow L_\alpha^{-1} \xrightarrow{\ \overline{dy}\ } \Omega^1_X \longrightarrow L_\alpha^{-m}\rightarrow 0\ .$$

Since L_α has a nonzero section, the space $\mathrm{Hom}(L_\alpha^{-1},L_\alpha^{-m})$ is zero for $m>1$. Hence if Ω^1_X splits, we deduce from 4.4 that the exact sequence splits. This means that there exists a form $\overline{\omega}\in H^0(X,\Omega^1_X\otimes L_\alpha^m)$ such that $\overline{\omega}\wedge\overline{dy}\neq 0$. Then $\overline{\omega}\wedge\overline{dy}$ is a generator of the trivial line bundle $K_X\otimes L_\alpha^{m+1}$, hence pulls back to $c\,dx\wedge dy$ on $\mathbf{W}$, for some constant $c\neq 0$. Therefore the pull back ω of $\overline{\omega}$ to $\mathbf{W}$ is of the form $c\,dx+f(x,y)dy$ for some holomorphic function f on $\mathbf{C}^2$. The flat line bundle L_α^m carries a flat holomorphic connection ∇; the 2-form $\nabla\overline{\omega}$, which is a global section of $K_X\otimes L_\alpha^m\cong L_\alpha^{-1}$, is zero. This implies $d\omega=0$, so the function $f(x,y)$ is independent of x; let us write it $f(y)$. Now the condition $T^*\omega=\alpha^m\omega$ reads $\alpha f(\alpha y)+c\lambda m y^{m-1}=\alpha^m f(y)$. Differentiating m times we find $f^{(m)}=0$, then differentiating $m-1$ times leads to a contradiction. □

(5.8) *Secondary Hopf surfaces*

A secondary Hopf surface X is the quotient of $\mathbf{W}$ by a group Γ acting freely, containing a central, finite index subgroup generated by an automorphism T of the above type. We assume that Ω^1_X splits. The primary Hopf surface $Y=\mathbf{W}/T^{\mathbf{Z}}$ is a finite étale cover of X, so Ω^1_Y also splits; it follows from (5.7) that T is of type *a*), and that Γ does not contain any transformation of type *b*). According to [Ka], § 3, this implies that after an appropriate change of coordinates, the group Γ acts *linearly* on $\mathbf{C}^2$.

We claim that Γ is contained in a maximal torus of $\mathbf{GL}(2,\mathbf{C})$. This is clear if $\alpha\neq\beta$, because T is central in Γ. If $\alpha=\beta$, the direct sum decomposition of Ω^1_X pulls back to a decomposition $\Omega^1_Y=L_\alpha^{-1}\oplus L_\alpha^{-1}$ (5.7), which for an appropriate choice of coordinates comes from the decomposition $\Omega^1_{\mathbf{W}}=\mathcal{O}_{\mathbf{W}}dx\oplus\mathcal{O}_{\mathbf{W}}dy$. Since Γ must preserve this decomposition, it is contained in the diagonal torus.

Thus we may identify Γ with a subgroup of $(\mathbf{C}^*)^2$; since it acts freely on $\mathbf{W}$, the first projection $\Gamma\to\mathbf{C}^*$ is injective. Therefore the torsion subgroup of Γ is cyclic, and we are in case (b) of the theorem. □

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