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ORIGINAL ARTICLE

Complex neutrosophic set

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Abstract Complex fuzzy sets and complex intuitionistic fuzzy sets cannot handle imprecise, indeterminate, inconsistent, and incomplete information of periodic nature. To overcome this difficulty, we introduce complex neutrosophic set. A complex neutrosophic set is a neutrosophic set whose complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsehood membership functions are the combination of real-valued truth amplitude term in association with phase term, real-valued indeterminate amplitude term with phase term, and real-valued false amplitude term with phase term, respectively. Complex neutrosophic set is an extension of the neutrosophic set. Further set theoretic operations such as complement, union, intersection, complex neutrosophic product, Cartesian product, distance measure, and δ-equalities of complex neutrosophic sets are studied here. A possible application of complex neutrosophic set is presented in this paper. Drawbacks and failure of the current methods are shown, and we also give a comparison of complex neutrosophic set to all such methods in this paper. We also showed in this paper the dominancy of complex neutrosophic set to all current methods through the graph.

Keywords Fuzzy set · Intuitionistic fuzzy set · Complex fuzzy set · Complex intuitionistic fuzzy set · Neutrosophic set · Complex neutrosophic set

1 Introduction

Fuzzy sets were first proposed by Zadeh in the seminal paper [38]. This novel concept is used successfully in modeling uncertainty in many fields of real life. A fuzzy set is characterized by a membership function μ with the range [0,1]. Fuzzy sets and their applications have been extensively studied in different aspects from the last few decades such as control [19, 38], reasoning [44], pattern recognition [19, 44], and computer vision [44]. Fuzzy sets become an important area for the research in medical diagnosis [29], engineering [19], etc. A large amount of the literature on fuzzy sets can be found in [8, 9, 15, 21, 30, 40–43]. In fuzzy set, the membership degree of an element is single value between 0 and 1. Therefore, it may not always be true that the non-membership degree of an element in a fuzzy set is equal to 1 minus the membership degree because there is some degree of hesitation. Thus, Atanassov [2] introduced intuitionistic fuzzy sets in 1986 which incorporate the hesitation degree called hesitation margin. The hesitation margin is defining as 1 minus the sum of membership and non-membership. Therefore, the intuitionistic fuzzy set is characterized by a membership function μ and non-membership function v with range [0,1]. An intuitionistic fuzzy set is the generalization of fuzzy set. Intuitionistic fuzzy sets can successfully be applied in many fields such as medical diagnosis [29], modeling theories [11], pattern recognition [31], and decision making [17].

Ramot et al. [23] proposed an innovative concept to the extension of fuzzy sets by initiating the complex fuzzy sets

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where the degree of membership μ is traded by a complex-valued of the form

$r_s(x) \cdot e^{j\omega_S(x)},\ j = \sqrt{-1}$

where $r_S(x)$ and $\omega_S(x)$ are both belongs to [0,1] and $r_s(x) . e^{j\omega_S(x)}$ has the range in complex unit disk. Complex fuzzy set is completely a different approach from the fuzzy complex number discussed by Buckley [4–7], Nguyen et al. [21], and Zhang et al. [41]. The complex-valued membership function of the complex fuzzy set has an amplitude term with the combination of a phase term which gives wavelike characteristics to it. Depending on the phase term gives a constructive or destructive interference. Thus, complex fuzzy set is different from conventional fuzzy set [38], fuzzy complex set [23], type 2 fuzzy set [19], etc. due to the character of wavelike. The complex fuzzy set [23] still preserves the characterization of uncertain information through the amplitude term having value in the range of [0,1] with the addition of a phase term. Ramot et al. [23, 24] discussed several properties of complex fuzzy sets such as complement, union, and intersection. with sufficient amount of illustrative examples. Some more theory on complex fuzzy sets can be seen in [10, 35]. Ramot et al. [24] also introduced the concept of complex fuzzy logic which is a novel framework for logical reasoning. The complex fuzzy logic is a generalization of fuzzy logic, based on complex fuzzy set. In complex fuzzy logic [24], the inference rules are constructed and fired in such way that they are closely resembled to traditional fuzzy logic. Complex fuzzy logic [24] is constructed to hold the advantages of fuzzy logic while enjoying the features of complex numbers and complex fuzzy sets. Complex fuzzy logic is not only a linear extension to the conventional fuzzy logic but rather a natural extension to those problems that are very difficult or impossible to describe with one-dimensional grades of membership. Complex fuzzy sets have found their place in signal processing [23], physics [23], stock marketing [23] etc.

The concept of complex intuitionistic fuzzy set in short CIFS is introduced by Alkouri and Saleh in [1]. The complex intuitionistic fuzzy set is an extension of complex fuzzy set by adding complex-valued non-membership grade to the definition of complex fuzzy set. The complex intuitionistic fuzzy sets are used to handle the information of uncertainty and periodicity simultaneously. The complex-valued membership and non-membership function can be used to represent uncertainty in many corporal quantities such as wave function in quantum mechanics, impedance in electrical engineering, complex amplitude, and decision-making problems. The novel concept of phase term is extend in the case of complex intuitionistic fuzzy set which appears in several prominent concepts such as distance measure, Cartesian products, relations, projections, and cylindric extensions. The complex fuzzy set has only one extra phase term, while complex intuitionistic fuzzy set has two additional phase terms. Several properties of complex intuitionistic fuzzy sets have been studied such as complement, union, intersection, T-norm, and S-norm.

Smarandache [28] in 1998 introduced Neutrosophy that studies the origin, nature, and scope of neutralities and their interactions with distinct ideational spectra. A neutrosophic set is characterized by a truth membership function T, an indeterminacy membership function I and a falsehood membership function F. Neutrosophic set is powerful mathematical framework which generalizes the concept of classical sets, fuzzy sets [38], intuitionistic fuzzy sets [2], interval valued fuzzy sets [30], paraconsistent sets [28], dialetheist sets [28], paradoxist sets [28], and tautological sets [28]. Neutrosophic sets handle the indeterminate and inconsistent information that exists commonly in our daily life. Recently neutrosophic sets have been studied by several researchers around the world. Wang et al. [33] studied single-valued neutrosophic sets in order to use them in scientific and engineering fields that give an additional possibility to represent uncertainty, incomplete, imprecise, and inconsistent data. Hanafy et al. [13, 14] studied the correlation coefficient of neutrosophic set. Ye [35] studied the correlation coefficient of single-valued neutrosophic sets. Broumi and Smaradache presented the correlation coefficient of interval neutrosophic set in [3]. Salama et al. [26] studied neutrosophic sets and neutrosophic topological spaces. Some more literature on neutrosophic sets can be found in [12–14, 18, 20, 25, 27, 32, 34, 36, 37, 40].

Pappis [22] studied the notion of "proximity measure," with an attempt to show that "precise membership values should normally be of no practical significance." Pappis observed that the max–min compositional rule of inference is preserved with respect to "approximately equal" fuzzy sets. An important generalization of the work of Pappis proposed by Hong and Hwang [15] which is mainly based that the max–min compositional rule of inference is preserved with respect to "approximately equal fuzzy sets" and "approximately equal" fuzzy relation. But, Cai noticed that both the Pappis and Hong and Hwang approaches were confined to fixed ε. Therefore, Cai [8, 9] takes a different approach and introduced δ-equalities of fuzzy sets. Cai proposed that if two fuzzy sets are equal to an extent of δ, then they are said to be δ-equal. The notions of δ-equality are significance in both the fuzzy statistics and fuzzy reasoning. Cai [8, 9] applied them for assessing the robustness of fuzzy reasoning as well as in synthesis of real-time fuzzy systems. Cai also gave several reliability examples of δ-equalities [8, 9]. Zhang et al. [39] studied the δ-equalities of complex fuzzy set by following the philosophy of Ramot et al. [23, 24] and Cai [8, 9]. They mainly focus on the results of Cai's work [8, 9] to introduce δ-equalities of complex fuzz sets, and thus, they systematically develop distance measure, equality and similarity of complex fuzzy sets. Zhang et al. [39] then applied δ-equalities of complex fuzzy sets in a signal processing application.

This paper is an extension of the work of Ramot et al. [23], Alkouri and Saleh [1], Cai [8, 9], and Zhang et al. [39] to neutrosophic sets. Basically, we follow the philosophy of the work of Ramot et al. [23] to introduce complex neutrosophic set. The complex neutrosophic is characterized by complex-valued truth membership function, complex-valued indeterminate membership function, and complex-valued falsehood membership function. Further, complex neutrosophic set is the mainstream over all because it not only is the generalization of all the current frameworks but also describes the information in a complete and comprehensive way.

1.1 Why complex neutrosophic set can handle the indeterminate information in periodicity

As we can see that uncertainty, indeterminacy, incompleteness, inconsistency, and falsity in data are periodic in nature, to handle these types of problems, the complex neutrosophic set plays an important role. A complex neutrosophic set S is characterized by a complex-valued truth membership function $T_S(x)$, complex-valued indeterminate membership function $I_S(x)$, and complex-valued false membership function $F_S(x)$ whose range is extended from [0,1] to the unit disk in the complex plane. The complex neutrosophic sets can handle the information which is uncertain, indeterminate, inconsistent, incomplete, ambiguous, false because in $T_S(x)$, the truth amplitude term and phase term handle uncertainty and periodicity, in $I_S(x)$, the indeterminate amplitude term and phase term handle indeterminacy and periodicity, and in $F_S(x)$, the false amplitude term and phase term handle the falsity with periodicity. Complex neutrosophic set is an extension of the neutrosophic set with three additional phase terms.

Thus, the complex neutrosophic set deals with the information/data which have uncertainty, indeterminacy, and falsity that are in periodicity while both the complex fuzzy set and complex intuitionistic fuzzy sets cannot deal with indeterminacy, inconsistency, imprecision, vagueness, doubtfulness, error, etc. in periodicity.

The contributions of this paper are:

1. We introduced complex neutrosophic set which deals with uncertainty, indeterminacy, impreciseness, inconsistency, incompleteness, and falsity of periodic nature.
2. Further, we studied set theoretic operations of complex neutrosophic sets such as complement, union, intersection complex neutrosophic product, and Cartesian product.
3. We also introduced the novel concept "the game of winner, neutral, and loser" for phase terms.
4. We studied a distance measure on complex neutrosophic sets which we have used in the application.
5. We introduced δ-equalities of complex neutrosophic set and studied their properties.
6. We also gave an algorithm for signal processing using complex neutrosophic sets.
7. Drawbacks and failures of the current methods presented in this paper.
8. Finally, we gave the comparison of complex neutrosophic sets to the current methods.

The organization of this paper is as follows. In Sect. 2, we presented some basic and fundamental concepts of neutrosophic sets, complex fuzzy sets, and complex intuitionistic fuzzy sets. In the next section, we introduced complex neutrosophic sets and gave some interpretation of complex neutrosophic set for intuition. We also introduced the basic set theoretic operations of complex neutrosophic sets such as complement, union, intersection, complex neutrosophic product, and Cartesian product in the current section. Further, in this section, the game of winner, neutral, and loser is introduced for the phase terms in the case of union and intersection of two complex neutrosophic sets. It is completely an innovative approach for the phase terms. In Sect. 4, we introduced distance measure on complex neutrosophic sets, δ-equality on complex neutrosophic sets and studied some of their properties. An application in signal processing is presented for the possible utilization of complex neutrosophic set in the Sect. 5. In Sect. 6, we give the drawbacks of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, complex fuzzy sets, and complex intuitionistic fuzzy sets. We also give a comparison of different current methods to complex neutrosophic set in this section. Further, the dominancy of complex neutrosophic sets over other existing methods is shown in this section.

We now review some basic concepts of neutrosophic sets, single-valued neutrosophic set, complex fuzzy sets, and complex intuitionistic fuzzy sets.

2 Literature review

In this section, we present some basic material which will help in our later pursuit. The definitions and notions are taken from [1, 23, 28, 39].

Definition 2.1 [28] Neutrosophic set.

Let X be a space of points and let $x \in X$. A neutrosophic set S in X is characterized by a truth membership function T_S, an indeterminacy membership function I_S, and a falsity membership function F_S. $T_S(x)$, $I_S(x)$, and $F_S(x)$ are real standard or non-standard subsets of $]0^-,\ 1^+[$, and $T_S, I_S, F_S : X \to]0^-, 1^+[$. The neutrosophic set can be represented as

$$S = \{(x, T_S(x), I_S(x), F_S(x)) : x \in X\}$$

There is no restriction on the sum of $T_S(x)$, $I_S(x)$, and $F_S(x)$, so $0^- \le T_S(x) + I_S(x) + F_S(x) \le 3^+$.

From philosophical point view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-, 1^+[$. Thus, it is necessary to take the interval [0,1] instead of $]0^-, 1^+[$ for technical applications. It is difficult to apply $]0^-, 1^+[$ in the real-life applications such as engineering and scientific problems.

A single-valued neutrosophic set [23] is characterized by a truth membership function, $T_S(x)$, an indeterminacy membership function $I_S(x)$ and a falsity membership function $F_S(x)$ with $T_S(x)$, $I_S(x)$, $F_S(x) \in [0, 1]$ for all $x \in X$. If X is continuous, then

$$S = \int_X \frac{(T_S(x), I_S(x), F_S(x))}{x} \quad \text{for all} \quad x \in X.$$

If X is discrete, then

$$S = \sum_X \frac{(T_S(x), I_S(x), F_S(x))}{x} \quad \text{for all} \quad x \in X.$$

Actually, SVNS is an instance of neutrosophic set that can be used in real-life situations such as decision-making, scientific, and engineering applications. We will use single-valued neutrosophic set to define complex neutrosophic set.

We now give some set theoretic operations of neutrosophic sets.

Definition 2.2 [33] Complement of neutrosophic set.

The complement of a neutrosophic set S is denoted by $c(S)$ and is defined by

$$T_{c(S)}(x) = F_S(x), \quad I_{c(S)}(x) = 1 - I_S(x), \quad F_{c(S)}(x) = T_S(x) \quad \text{for all} \quad x \in X.$$

Definition 2.3 [23] Union of neutrosophic sets.

Let A and B be two complex neutrosophic sets in a universe of discourse X. Then, the union of A and B is denoted by $A \cup B$, which is defined by

$$A \cup B = \{(x, T_A(x) \vee T_B(x), I_A(x) \wedge I_B(x), F_A(x) \wedge F_B(x)) : x \in X\}$$

for all $x \in X$, and $\vee$ denote the max operator and $\wedge$ denote the min operator, respectively.

Definition 2.4 [23] Intersection of neutrosophic sets.

Let A and B be two complex neutrosophic sets in a universe of discourse X. Then, the intersection of A and B is denoted as $A \cap B$, which is defined by

$$A \cap B = \{(x, T_A(x) \wedge T_B(x), I_A(x) \vee I_B(x), F_A(x) \vee F_B(x)) : x \in X\}$$

for all $x \in X$.

The definitions and other notions of complex fuzzy sets are given as follows.

Definition 2.5 [23] Complex fuzzy set.

A complex fuzzy set S, defined on a universe of discourse X, is characterized by a membership function $\eta_S(x)$ that assigns any element $x \in X$ a complex-valued grade of membership in S. The values $\eta_S(x)$ all lie within the unit circle in the complex plane and thus all of the form $p_S(x).e^{j \cdot \mu_S(x)}$ where $p_S(x)$ and $\mu_S(x)$ are both real-valued and $p_S(x) \in [0, 1]$. Here, $p_S(x)$ is termed as amplitude term and $e^{j \cdot \mu_S(x)}$ is termed as phase term. The complex fuzzy set may be represented in the set form as

$$S = \{(x, \eta_S(x)) : x \in X\}.$$

The complex fuzzy set is denoted as CFS.

We now present set theoretic operations of complex fuzzy sets.

Definition 2.6 [23] Complement of complex fuzzy set.

Let S be a complex fuzzy set on X, and $\eta_S(x) = p_S(x).e^{j \cdot \mu_S(x)}$ its complex-valued membership function. The complement of S denoted as $c(S)$ and is specified by a function

$$\eta_{c(S)}(x) = p_{c(S)}(x).e^{j \cdot \mu_{c(S)}(x)} = (1 - p_S(x)).e^{j(2\pi - \mu_S(x))}.$$

Definition 2.7 [23] Union of complex fuzzy sets.

Let A and B be two complex fuzzy sets on X, and $\eta_A(x) = r_A(x).e^{j \cdot \mu_A(x)}$ and $\eta_B(x) = r_B(x).e^{j \cdot \mu_B(x)}$ be their membership functions, respectively. The union of A and B is denoted as $A \cup B$, which is specified by a function

$$\eta_{A \cup B}(x) = r_{A \cup B}(x).e^{j \cdot \mu_{A \cup B}(x)} = (r_A(x) \vee r_B(x)).e^{j(\mu_A(x) \vee \mu_B(x))}$$

where $\vee$ denote the max operator.

Definition 2.8 [23] Intersection of complex fuzzy sets.

Let A and B be two complex fuzzy sets on X, and $\eta_A(x) = r_A(x).e^{j \cdot \mu_A(x)}$ and $\eta_B(x) = r_B(x).e^{j \cdot \mu_B(x)}$ be their membership functions, respectively. The intersection of A and B is denoted as $A \cap B$, which is specified by a function

$$\eta_{A \cap B}(x) = r_{A \cap B}(x).e^{j \cdot \mu_{A \cap B}(x)} = (r_A(x) \wedge r_B(x)).e^{j(\mu_A(x) \wedge \mu_B(x))}$$

where $\wedge$ denote the max operator.

We now give some basic definitions and set theoretic operations of complex intuitionistic fuzzy sets.

Definition 2.9 [39] Let A and B be two complex fuzzy sets on X, and $\eta_A(x) = r_A(x).e^{j.\mu_A(x)}$ and $\eta_B(x) = r_B(x).e^{j.\mu_B(x)}$ be their membership functions, respectively. The complex fuzzy product of A and B, denoted as $A \circ B$, and is specified by a function

$$\eta_{A\circ B}(x) = r_{A\circ B}(x).e^{j.\mu_{A\circ B}(x)} = (r_A(x).r_B(x)).e^{j2\pi\left(\frac{\mu_A(x)}{2\pi}.\frac{\mu_B(x)}{2\pi}\right)}.$$

Definition 2.10 [39] Let A and B be two complex fuzzy sets on X, and $\eta_A(x) = r_A(x).e^{j.\mu_A(x)}$ and $\eta_B(x) = r_B(x).e^{j.\mu_B(x)}$ be their membership functions, respectively. Then, A and B are said to be δ equal if and only if $d(A,B) \leq 1 - \delta$, where $0 \leq \delta \leq 1$.

For more literature on δ-equality, we refer to [8, 9] and [35].

Definition 2.11 [1] Complex intuitionistic fuzzy set.

A complex intuitionistic fuzzy set S, defined on a universe of discourse X, is characterized by a membership function $\eta_S(x)$ and non-membership function $\zeta_S(x)$, respectively, that assign an element $x \in X$ a complex-valued grade to both membership and non-membership in S. The values of $\eta_S(x)$ and $\zeta_S(x)$ all lie with in the unit circle in the complex plane and are of the form $\eta_S(x) = p_S(x).e^{j.\mu_S(x)}$ and $\zeta_S(x) = r_S(x).e^{j.\omega_S(x)}$, where $p_S(x), r_S(x)$, $\mu_S(x)$, and $\omega_S(x)$ are all real-valued and $p_S(x)$, $r_S(x) \in [0,1]$ with $j = \sqrt{-1}$. The complex intuitionistic fuzzy set can be represented as

$$S = \{(x, \eta_S(x), \zeta_S(x)) : x \in U\}.$$

It is denoted as CIFS.

Definition 2.12 [1] Complement of complex intuitionistic fuzzy set.

Let S be a complex intuitionistic fuzzy set, and $\eta_S(x) = p_S(x).e^{j.\mu_S(x)}$ and $\zeta_S(x) = r_S(x).e^{j.\omega_S(x)}$ its membership and non-membership functions, respectively. The complement of S, denoted as $c(S)$, is specified by a function

$$\eta_{c(S)}(x) = p_{c(S)}(x).e^{j.\mu_{c(S)}(x)} = r_S(x).e^{j(2\pi-\mu_S(x))} \quad \text{and}$$
$$\zeta_{c(S)}(x) = r_{c(S)}(x).e^{j.\omega_{c(S)}(x)} = p_S(x).e^{j(2\pi-\omega_S(x))}$$

Definition 2.13 [1] Union of complex intuitionistic fuzzy sets.

Let A and B be two complex intuitionistic fuzzy sets on X, and $\eta_A(x) = p_A(x).e^{j.\mu_A(x)}$, $\zeta_A(x) = r_A(x).e^{j.\omega_A(x)}$ and $\eta_B(x) = p_B(x).e^{j.\mu_B(x)}$ and $\zeta_B(x) = r_B(x).e^{j.\omega_B(x)}$ be their membership and non-membership functions, respectively. The union of A and B is denoted as $A \cup B$, which is specified by a function

$$\eta_{A\cup B}(x) = p_{A\cup B}(x).e^{j.\mu_{A\cup B}(x)} = (p_A(x) \vee p_B(x)).e^{j(\mu_A(x)\vee\mu_B(x))}$$
and
$$\zeta_{A\cup B}(x) = r_{A\cup B}(x).e^{j.\omega_{A\cup B}(x)} = (r_A(x) \wedge r_B(x)).e^{j(\omega_A(x)\wedge\omega_B(x))}$$

where $\vee$ and $\wedge$ denote the max and min operator, respectively.

Definition 2.14 [1] Intersection of complex intuitionistic fuzzy sets.

Let A and B be two complex intuitionistic fuzzy sets on X, and $\eta_A(x) = p_A(x).e^{j.\mu_A(x)}$, $\zeta_A(x) = r_A(x).e^{j.\omega_A(x)}$ and $\eta_B(x) = p_B(x).e^{j.\mu_B(x)}$ and $\zeta_B(x) = r_B(x).e^{j.\omega_B(x)}$ be their membership and non-membership functions, respectively. The intersection of A and B is denoted as $A \cap B$, which is specified by a function

$$\eta_{A\cap B}(x) = p_{A\cap B}(x).e^{j.\mu_{A\cap B}(x)} = (p_A(x) \wedge p_B(x)).e^{j(\mu_A(x)\wedge\mu_B(x))} \quad \text{and}$$
$$\zeta_{A\cap B}(x) = r_{A\cap B}(x).e^{j.\omega_{A\cap B}(x)} = (r_A(x) \vee r_B(x)).e^{j(\omega_A(x)\vee\omega_B(x))}$$

where $\wedge$ and $\vee$ denote the min and max operators, respectively.

Next, the notion of complex neutrosophic set is introduced.

3 Complex neutrosophic set

In this section, we introduced the innovative concept of complex neutrosophic set. The definition of complex neutrosophic set is as follows.

Definition 3.1 Complex neutrosophic set.

A complex neutrosophic set S defined on a universe of discourse X, which is characterized by a truth membership function $T_S(x)$, an indeterminacy membership function $I_S(x)$, and a falsity membership function $F_S(x)$ that assigns a complex-valued grade of $T_S(x)$, $I_S(x)$, and $F_S(x)$ in S for any $x \in X$. The values $T_S(x)$, $I_S(x)$, $F_S(x)$ and their sum may all within the unit circle in the complex plane and so is of the following form,

$$T_S(x) = p_S(x).e^{j\mu_S(x)}, \quad I_S(x) = q_S(x).e^{j\nu_S(x)} \text{ and } F_S(x) = r_S(x).e^{j\omega_S(x)}$$

where $p_S(x)$, $q_S(x)$, $r_S(x)$ and $\mu_S(x)$, $\nu_S(x)$, $\omega_S(x)$ are, respectively, real valued and $p_S(x), q_S(x), r_S(x) \in [0,1]$ such that $^{-}0 \leq p_S(x) + q_S(x) + r_S(x) \leq 3^{+}$.

The complex neutrosophic set S can be represented in set form as

$$S = \{(x, T_S(x) = a_T, I_S(x) = a_I, F_S(x) = a_F) : x \in X\},$$

where $T_S{:}X \to \{a_T{:}a_T \in C, |a_T| \leq 1\}$, $I_S{:}X \to \{a_I{:}a_I \in C, |a_I| \leq 1\}$ and $F_S{:}X \to \{a_F{:}a_F \in C, |a_F| \leq 1\}$ and $|T_S(x) + I_S(x) + F_S(x)| \leq 3$.

Throughout the paper, complex neutrosophic set refers to a neutrosophic set with complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsity membership function while the term neutrosophic set with real-valued truth membership function, indeterminacy membership function, and falsity membership function.

3.1 Interpretation of complex neutrosophic set

The concept of complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsity membership function is not a simple task in understanding. In contrast, real-valued truth membership function, real-valued indeterminacy membership function, and real-valued falsity membership function in the interval [0,1] can be easily intuitive.

The notion of complex neutrosophic set can be easily understood from the form of its truth membership function, indeterminacy membership function, and falsity membership function which appears in above Definition 3.1.

$$T_S(x) = p_S(x).e^{j\mu_S(x)}, \quad I_S(x) = q_S(x).e^{j\nu_S(x)} \quad \text{and} \quad F_S(x) = r_S(x).e^{j\omega_S(x)}$$

It is clear that complex grade of truth membership function is defined by a truth amplitude term $p_S(x)$ and a truth phase term $\mu_S(x)$. Similarly, the complex grade of indeterminacy membership function is defined as an indeterminate amplitude term $q_S(x)$ and an indeterminate phase term $\nu_S(x)$, while the complex grade of falsity membership function is defined by false amplitude term $r_S(x)$ and a false phase term $\omega_S(x)$, respectively. It should be noted that the truth amplitude term $p_S(x)$ is equal to $|T_S(x)|$, the amplitude of $T_S(x)$. Also, the indeterminate amplitude term $q_S(x)$ is equal to $|I_S(x)|$ and the false amplitude term $r_S(x)$ is equal to $|F_S(x)|$.

Complex neutrosophic sets are the generalization of neutrosophic sets. It is a easy task to represent a neutrosophic set in the form of complex neutrosophic set. For instance, the neutrosophic set S is characterized by a real-valued truth membership function $\alpha_S(x)$, indeterminate membership function $\beta_S(x)$, and falsehood membership function $\gamma_S(x)$. By setting the truth amplitude term $p_S(x)$ equal to $\alpha_S(x)$, and the truth phase term $\mu_S(x)$ equal to zero for all x and similarly we can set the indeterminate amplitude term $q_S(x)$ equal to $\beta_S(x)$ and the indeterminate phase term equal to zero, while the false amplitude term $r_S(x)$ equal to $\gamma_S(x)$ with the false phase term equal to zero for all x. Thus, it has seen that a complex neutrosophic set can be easily transformed into a neutrosophic set. From this discussion, it is concluded that the truth amplitude term is equivalent to the real-valued grade of truth membership function, the indeterminate amplitude term is equivalent to the real-valued grade of indeterminate membership function, and the false amplitude term is essentially equivalent to the real-valued grade of false membership function. The only distinguishing factors are truth phase term, indeterminate phase term, and false phase term. This differs the complex neutrosophic set from the ordinary neutrosophic set. In simple words, if we omit all the three phase terms, the complex neutrosophic set will automatically convert into neutrosophic set effectively. All this discussion is supported by the reality that $p_S(x)$, $q_S(x)$, and $r_S(x)$ have range [0,1] which is for real-valued grade of truth membership, real-valued grade of indeterminate membership, and real-valued grade of false membership.

It should also be noted that complex neutrosophic sets are the generalization of complex fuzzy sets, conventional fuzzy sets, complex intuitionistic fuzzy sets and intuitionistic fuzzy sets, classical sets. This means that complex neutrosophic set is an advance generalization to all the existence methods and due to this feature, the concept of complex neutrosophic set is a distinguished and novel one.

3.2 Numerical example of a complex neutrosophic set

Example 3.2 Let $X = \{x_1, x_2, x_3\}$ be a universe of discourse. Then, S be a complex neutrosophic set in X as given below:

$$S = \frac{\left(0.6e^{j.0.8}, 0.3.e^{j.\frac{3\pi}{4}}, 0.5.e^{j.0.3}\right)}{x_1} + \frac{\left(0.7e^{j.0}0.2.e^{j.0.9}, 0.1.e^{j.\frac{2\pi}{3}}\right)}{x_2} + \frac{\left(0.9e^{j.0.1}, 0.4.e^{j.\pi}, 0.7.e^{j.0.7}\right)}{x_3}.$$

3.3 Set theoretic operations on complex neutrosophic set

Ramot et al. [23], calculated the complement of membership phase term $\mu_S(x)$ by several possible method such as $\mu_S^c(x) = \mu_S(x), 2\pi - \mu_S(x)$. Zhang [39] defined the complement of the membership phase term by taking the rotation of $\mu_S(x)$ by π radian as $\mu_S^c(x) = \mu_S(x) + \pi$.

To define the complement of a complex neutrosophic set, we simply take the neutrosophic complement [29] for the truth amplitude term $p_S(x)$, indeterminacy amplitude term $q_S(x)$, and falsehood amplitude term $r_S(x)$. For phase terms, we take the complements defined in [23]. We now proceed to define the complement of complex neutrosophic set.

Definition 3.3 Complement of complex neutrosophic set.

Let $S = \{(x, T_S(x), I_S(x), F_S(x)) : x \in X\}$ be a complex neutrosophic set in X. Then, the complement of a complex neutrosophic set S is denoted as $c(S)$ and is defined by

$$c(S) = \left\{\left(x, T_S^c(x), I_S^c(x), F_S^c(x)\right) : x \in X\right\},$$

where $T_S^c(x) = c\left(p_S(x).e^{j\cdot\mu_S(x)}\right)$, $I_S^c(x) = c\left(q_S(x).e^{j\cdot\nu_S(x)}\right)$, and $F_S^c(x) = c\left(r_S(x).e^{j\cdot\omega_S(x)}\right)$ in which $c\left(p_S(x).e^{j\cdot\mu_S(x)}\right) = c(p_S(x)).e^{j\cdot\mu_s^c(x)}$ is such that $c(p_S(x)) = r_S(x)$ and $\nu_S^c(x) = \nu_S(x), 2\pi - \nu_S(x)$ or $\nu_S(x) + \pi$. Similarly, $c(r_S(x).e^{j\cdot\mu_S(x)}) = c(r_S(x)).e^{j\cdot\omega_s^c(x)}$, where $c(q_S(x)) = 1 - q_S(x)$ and $\nu_S^c(x) = \nu_S(x), 2\pi - \nu_S(x)$ or $\nu_S(x) + \pi$.

Finally, $c\left(r_S(x).e^{j\cdot\mu_S(x)}\right) = c(r_S(x)).e^{j\cdot\omega_s^c(x)}$, where $c(r_S(x)) = p_S(x)$ and $\omega_S^c(x) = \omega_S(x), 2\pi - \omega_S(x)$ or $\omega_S(x) + \pi$.

Proposition 3.4 *Let A be a complex neutrosophic set on X. Then,* $c(c(A)) = A$.

Proof By definition 3.1, we can easily prove it.

Proposition 3.5 *Let A and B be two complex neutrosophic sets on X. Then,* $c(A \cap B) = c(A) \cup c(B)$.

Definition 3.6 Union of complex neutrosophic sets.

Ramot et al. [23] defined the union of two complex fuzzy sets A and B as follows.

Let $\mu_A(x) = r_A(x).e^{j\cdot\omega_A(x)}$ and $\mu_B(x) = r_B(x).e^{j\cdot\omega_B(x)}$ be the complex-valued membership functions of A and B, respectively. Then, the membership union of $A \cup B$ is given by $\mu_{A\cup B}(x) = [r_A(x) \oplus r_B(x)].e^{j\cdot\omega_{A\cup B}(x)}$. Since $r_A(x)$ and $r_B(x)$ are real-valued and belong to $[0, 1]$, the operators max and min can be applied to them. For calculating phase term $\omega_{A\cup B}(x)$, they give several methods.

Now we define the union of two complex neutrosophic sets as follows:

Let A and B be two complex neutrosophic sets in X, where

$$A = \{(x, T_A(x), I_A(x), F_A(X)) : x \in X\} \quad \text{and}$$
$$B = \{(x, T_B(x), I_B(x), F_B(X)) : x \in X\}.$$

Then the union of A and B is denoted as $A \cup_N B$ and is given as

$$A \cup_N B = \{(x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)) : x \in X\}$$

where the truth membership function $T_{A\cup B}(x)$, the indeterminacy membership function $I_{A\cup B}(x)$, and the falsehood membership function $F_{A\cup B}(x)$ are defined by

$$T_{A\cup B}(x) = [(p_A(x) \vee p_B(x))].e^{j\cdot\mu_{T_{A\cup B}}(x)},$$
$$I_{A\cup B}(x) = [(q_A(x) \wedge q_B(x))].e^{j\cdot\nu_{I_{A\cup B}}(x)},$$
$$F_{A\cup B}(x) = [(r_A(x) \wedge r_B(x))].e^{j\cdot\omega_{F_{A\cup B}}(x)},$$

where $\vee$ and $\wedge$ denote the max and min operators, respectively. To calculate phase the terms $e^{j\cdot\mu_{A\cup B}(x)}$, $e^{j\cdot\nu_{A\cup B}(x)}$, and $e^{j\cdot\omega_{A\cup B}(x)}$, we define the following:

Definition 3.7 Let A and B be two complex neutrosophic sets in X with complex-valued truth membership functions $T_A(x)$ and $T_B(x)$, complex-valued indeterminacy membership functions $I_A(x)$ and $I_B(x)$, and complex-valued falsehood membership functions $F_A(x)$ and $F_B(x)$, respectively. The union of the complex neutrosophic sets A and B is denoted by $A \cup_N B$ which is associated with the function:

$$\begin{aligned}\theta: &\{(a_T,a_I,a_F): a_T,a_I,a_F \in C, |a_T+a_I+a_F| \le 3, |a_T|,|a_I|,|a_F| \le 1\}\\ &\times \{(b_T,b_I,b_F): b_T,b_I,b_F \in C, |b_T+b_I+b_F| \le 3, |b_T|,|b_I|,|b_F| \le 1\}\\ &\to \{(d_T,d_I,d_F): d_T,d_I,d_F \in C, |d_T+d_I+d_F| \le 3, |d_T|,|d_I|,|d_F| \le 1\},\end{aligned}$$

where a, b, d, a', b', d', and a'', b'', d'' are the complex truth membership, complex indeterminacy membership, and complex falsity membership of A, B, and $A \cup B$, respectively.

A complex value is assigned by θ, that is, for all $x \in X$,

$$\theta(T_A(x), T_B(x)) = T_{A\cup B}(x) = d_T,$$
$$\theta(I_A(x), I_B(x)) = I_{A\cup B}(x) = d_I \quad \text{and}$$
$$\theta(F_A(x), F_B(x)) = F_{A\cup B}(x) = d_F.$$

This function θ must obey at least the following axiomatic conditions.

For any $a, b, c, d, a', b', c', d', a'', b'', c'', d'' \in \{x : x \in C, |x| \le 1\}$:

- Axiom 1: $|\theta_T(a, 0)| = |a|, |\theta_I(a', 1)| = |a'|$ and $|\theta_F(a'', 1)| = |a''|$ (boundary condition).
- Axiom 2: $\theta_T(a, b) = \theta_T(b, a), \theta_I(a', b') = \theta_I(b', a')$ and $\theta_F(a'', b'') = \theta_F(b'', a'')$ (commutativity condition).
- Axiom 3: if $|b| \le |d|$, then $|\theta_T(a, b)| \le |\theta_T(a, d)|$ and if $|b'| \le |d'|$, then $|\theta_I(a', b')| \le |\theta_I(a', d')|$ and if $|b''| \le |d''|$, then $|\theta_F(a'', b'')| \le |\theta_F(a'', d'')|$ (monotonic condition).
- Axiom 4: $\theta_T(\theta_T(a, b), c) = \theta_T(a, \theta_T(b, c))$, $\theta_I(\theta_I(a', b'), c') = \theta_I(a', \theta_I(b', c'))$ and $\theta_F(\theta_F(a'', b''), c'') = \theta_F(a'', \theta_F(b'', c''))$ (associative condition).

It may be possible in some cases that the following are also hold:

- Axiom 5: θ is continuous function (continuity).
- Axiom 6: $|\theta_T(a,a)| > |a|$, $|\theta_I(a',a')| < |a'|$ and $|\theta_F(a'',a'')| < |a''|$ (superidempotency).
- Axiom 7: $|a| \le |c|$ and $|b| \le |d|$, then $|\theta_T(a,b)| \le |\theta_T(c,d)|$, also $|a'| \ge |c'|$ and $|b'| \ge |d'|$, then $|\theta_I(a',b')| \ge |\theta_I(c',d')|$ and $|a''| \ge |c''|$ and $|b''| \le |d''|$, then $|\theta_F(a'',b'')| \ge |\theta_F(c',d')|$ (strict monotonicity).

The phase term of complex truth membership function, complex indeterminacy membership function, and complex falsity membership function belongs to $(0, 2\pi)$. To define the phase terms, we suppose that $\mu_{T_{A\cup B}}(x) = \mu_{A\cup B}(x)$, $\nu_{I_{A\cup B}}(x) = \nu_{A\cup B}(x)$, and $\omega_{F_{A\cup B}}(x) = \omega_{A\cup B}(x)$. Now we take those forms which Ramot et al. presented in [23] to define the phase terms of $T_{A\cup B}(x)$, $I_{A\cup B}(x)$, and $F_{A\cup B}(x)$, respectively.

(a) Sum:

$$\begin{aligned}\mu_{A\cup B}(x) &= \mu_A(x) + \mu_B(x),\\ \nu_{A\cup B}(x) &= \nu_A(x) + \nu_B(x),\\ \omega_{A\cup B}(x) &= \omega_A(x) + \omega_B(x).\end{aligned}$$

(b) Max:

$$\begin{aligned}\mu_{A\cup B}(x) &= \max(\mu_A(x), \mu_B(x)),\\ \nu_{A\cup B}(x) &= \max(\nu_A(x), \nu_B(x)),\\ \omega_{A\cup B}(x) &= \max(\omega_A(x), \omega_B(x)).\end{aligned}$$

(c) Min:

$$\begin{aligned}\mu_{A\cup B}(x) &= \min(\mu_A(x), \mu_B(x)),\\ \nu_{A\cup B}(x) &= \min(\nu_A(x), \nu_B(x)),\\ \omega_{A\cup B}(x) &= \min(\omega_A(x), \omega_B(x)).\end{aligned}$$

(d) "The game of winner, neutral, and loser":

$$\begin{aligned}\mu_{A\cup B}(x) &= \begin{cases}\mu_A(x) & \text{if } p_A > p_B\\ \mu_B(x) & \text{if } p_B > p_A\end{cases},\\ \nu_{A\cup B}(x) &= \begin{cases}\nu_A(x) & \text{if } q_A < q_B\\ \nu_B(x) & \text{if } q_B < q_A\end{cases},\\ \omega_{A\cup B}(x) &= \begin{cases}\omega_A(x) & \text{if } r_A < r_B\\ \omega_B(x) & \text{if } r_B < r_A\end{cases}.\end{aligned}$$

The game of winner, neutral, and loser is a novel concept, and it is the generalization of the concept "winner take all" introduced by Ramot et al. [23] for the union of phase terms.

Example 3.8 Let $X = \{x_1, x_2, x_3\}$ be a universe of discourse. Let A and B be two complex neutrosophic sets in X as shown below:

$$A = \frac{(0.6e^{j.0.8}, 0.3.e^{j.\frac{3\pi}{4}}, 0.5.e^{j.0.3})}{x_1} + \frac{(0.7e^{j.0}0.2.e^{j.0.9}, 0.1.e^{j.\frac{2\pi}{3}})}{x_2} + \frac{(0.9e^{j.0.1}, 0.4.e^{j.\pi}, 0.7.e^{j.0.7})}{x_3},$$

and

$$B = \frac{(0.8e^{j.0.9}, 0.1.e^{j.\frac{\pi}{4}}, 0.4.e^{j.0.5})}{x_1} + \frac{(0.6e^{j.0.1}, 1.e^{j.0.6}, 0.01.e^{j.\frac{4\pi}{3}})}{x_2} + \frac{(0.2e^{j.0.3}, 0.e^{j.2\pi}, 0.5.e^{j.0.5})}{x_3},$$

Then

$$A \cup_N B = \frac{(0.8.e^{j.0.9}, 0.3.e^{j.\frac{3\pi}{4}}, 0.5.e^{j.0.5})}{x_1}, \frac{(0.6.e^{j.0.1}, 0.2.e^{j.0.9}, 0.01, e^{j.\frac{4\pi}{3}})}{x_2}, \frac{(0.2.e^{j.0.3}, 0.e^{j.2\pi}, 0.5.e^{j.0.7})}{x_3}.$$

Definition 3.9 Intersection of complex neutrosophic sets.

Let A and B be two complex neutrosophic sets in X, where

$$A = \{(x, T_A(x), I_A(x), F_A(X)) : x \in X\} \quad \text{and}$$
$$B = \{(x, T_B(x), I_B(x), F_B(X)) : x \in X\}.$$

Then, the intersection of A and B is denoted as $A \cap_N B$ and is defined as

$$A \cap_N B = \{(x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)) : x \in X\},$$

where the truth membership function $T_{A\cap B}(x)$, the indeterminacy membership function $I_{A\cap B}(x)$, and the falsehood membership function $F_{A\cap B}(x)$ are given as:

$$\begin{aligned}T_{A\cap B}(x) &= [(p_A(x) \wedge p_B(x))].e^{j.\mu_{T_{A\cap B}}(x)},\\ I_{A\cap B}(x) &= [(q_A(x) \vee q_B(x))].e^{j.\nu_{I_{A\cap B}}(x)},\\ F_{A\cap B}(x) &= [(r_A(x) \vee r_B(x))].e^{j.\omega_{F_{A\cap B}}(x)},\end{aligned}$$

where $\vee$ and $\wedge$ denote the max and min operators, respectively. We calculate phase terms $e^{j.\mu_{A\cap B}(x)}$, $e^{j.\nu_{A\cap B}(x)}$, and $e^{j.\omega_{A\cap B}(x)}$ after define the following:

Definition 3.10 Let A and B be two complex neutrosophic sets in X with complex-valued truth membership functions $T_A(x)$ and $T_B(x)$, complex-valued indeterminacy membership functions $I_A(x)$ and $I_B(x)$, and complex-valued falsehood membership functions $F_A(x)$ and $F_B(x)$, respectively. The intersection of the complex neutrosophic sets A and B is denoted by $A \cap_N B$ which is associated with the function:

$$\phi:\{(a_T,a_I,a_F):a_T,a_I,a_F \in C,|a_T+a_I+a_F|\le 3,|a_T|,|a_I|,|a_F|\le 1\} \times \{(b_T,b_I,b_F):b_T,b_I,b_F \in C,|b_T+b_I+b_F|\le 3,|b_T|,|b_I|,|b_F|\le 1\} \to \{(d_T,d_I,d_F):d_T,d_I,d_F \in C,|d_T+d_I+d_F|\le 3,|d_T|,|d_I|,|d_F|\le 1\},$$

where a,b,d, a',b',d', and a'',b'',d'' are the complex truth membership, complex indeterminacy membership, and complex falsity membership of A, B, and $A \cap B$, respectively. ϕ assigned a complex value, that is, for all $x \in X$,

$$\phi(T_A(x),T_B(x)) = T_{A\cap B}(x) = d_T,$$
$$\phi(I_A(x),I_B(x)) = I_{A\cap B}(x) = d_I \text{ and}$$
$$\phi(F_A(x),F_B(x)) = F_{A\cap B}(x) = d_F.$$

ϕ must satisfy at least the following axiomatic conditions.

For any $a,b,c,d,a',b',c',d',a'',b'',c'',d'' \in \{x : x \in C, |x| \le 1\}$:

- Axiom 1: If $|b| = 1$, then $|\phi_T(a,b)| = |a|$. If $|b'| = 0$, then $|\phi_I(a',b')| = |a'|$ and if $|\phi_F(a'',b'')| = |a''|$ (boundary condition).
- Axiom 2: $\phi_T(a,b) = \phi_T(b,a)$, $\phi_I(a',b') = \phi_I(b',a')$, and $\phi_F(a'',b'') = \phi_F(b'',a'')$ (commutative condition).
- Axiom 3: if $|b| \le |d|$, then $|\phi_T(a,b)| \le |\phi_T(a,d)|$ and if $|b'| \le |d'|$, then $|\phi_I(a',b')| \le |\phi_I(a',d')|$ and if $|b''| \le |d''|$, then $|\phi_F(a'',b'')| \le |\phi_F(a'',d'')|$ (monotonic condition).
- Axiom 4: $\phi_T(\phi_T(a,b),c) = \phi_T(a,\phi_T(b,c))$, $\phi_I(\phi_I(a',b'),c') = \phi_I(a',\phi_I(b',c'))$, and $\phi_F(\phi_F(a'',b''),c'') = \phi_F(a'',\phi_F(b'',c''))$ (associative condition).

The following axioms also hold in some cases.

- Axiom 5: ϕ is continuous function (continuity).
- Axiom 6: $|\phi_T(a,a)| > |a|$, $|\phi_I(a',a')| < |a'|$, and $|\phi_F(a'',a'')| < |a''|$ (superidempotency).
- Axiom 7: $|a| \le |c|$ and $|b| \le |d|$, then $|\phi_T(a,b)| \le |\phi_T(c,d)|$, also $|a'| \ge |c'|$ and $|b'| \ge |d'|$, then $|\phi_I(a',b')| \ge |\phi_I(c',d')|$ and $|a''| \ge |c''|$ and $|b''| \le |d''|$, then $|\phi_F(a'',b'')| \ge |\phi_F(c'',d'')|$ (strict monotonicity).

We can easily calculate the phase terms $e^{j\cdot\mu_{A\cap B}(x)}$, $e^{j\cdot v_{A\cap B}(x)}$, and $e^{j\cdot\omega_{A\cap B}(x)}$ on the same lines by winner, neutral, and loser game.

Proposition 3.11 *Let A,B,C be three complex neutrosophic sets on X. Then,*

1. $(A \cup B) \cap C = (A \cap C) \cup (A \cap B)$,
2. $(A \cap B) \cup C = (A \cup C) \cap (A \cup B)$.

Proof Here we only prove part 1. Let A,B,C be three complex neutrosophic sets in X and $T_A(x),I_A(x),F_A(x)$, $T_B(x),I_B(x),F_B(x)$ and $T_C(x),I_C(x),F_V(x)$, respectively, be their complex-valued truth membership function, complex-valued indeterminate membership function, and complex-valued falsehood membership functions. Then, we have

$$\begin{aligned} T_{(A\cup B)\cap C}(x) &= p_{(A\cup B)\cap C}(x).e^{j\cdot\mu_{(A\cup B)\cap C}(x)} \\ &= \min(p_{A\cup B}(x),p_C(x)).e^{j\cdot\min(\mu_{A\cup B}(x),\mu_C(x))}, \\ &= \min(\max(p_A(x),p_B(x)),p_C(x)) \\ &\quad .e^{j\cdot\min(\max(\mu_A(x),\mu_B(x)),\mu_C(x))}, \\ &= \max(\min(p_A(x),p_c(x)),\min(p_B(x),p_C(x))) \\ &\quad .e^{j.max(\min(\mu_A(x),\mu_C(x)),\min(\mu_B(x),\mu_C(x)))}, \\ &= \max(p_{A\cap C}(x),p_{B\cap C}(x)).\, e^{j.max(\mu_{A\cap C}(x),\mu_{B\cap C}(x))}, \\ &= p_{(A\cap C)\cup(B\cap C)}(x).e^{j\cdot\mu_{(A\cap C)\cup(B\cap C)}(x)} = T_{(A\cap C)\cup(B\cap C)}(x). \end{aligned}$$

Similarly, on the same lines, we can show it for $I_{(A\cup B)\cap C}(x)$ and $F_{(A\cup B)\cap C}(x)$, respectively. □

Proposition 3.12 *Let A and B be two complex neutrosophic sets in X. Then,*

1. $(A \cup B) \cap A = A$,
2. $(A \cap B) \cup A = A$.

Proof We prove it for part 1. Let A and B be two complex neutrosophic sets in X and $T_A(x),I_A(x),F_A(x)$ and $T_B(x),I_B(x),F_B(x)$, respectively, be their complex-valued truth membership function, complex-valued indeterminate membership function, and complex-valued falsehood membership functions. Then,

$$\begin{aligned} T_{(A\cup B)\cap A}(x) &= p_{(A\cup B)\cap A}(x).\, e^{j\cdot\mu_{(A\cup B)\cap A}(x)} \\ &= \min(p_{A\cup B}(x),p_A(x)) \\ &\quad .e^{i.\min(\mu_{A\cup B}(x),\mu_A(x))}, \\ &= \min(\max(p_A(x),p_B(x)),p_A(x)) \\ &\quad .e^{i.\min(\max(\mu_A(x),\mu_B(x)),\mu_A(x))} \\ &= T_A(x). \end{aligned}$$

Similarly, we can show it for $I_{(A\cup B)\cap A}(x)$ and $F_{(A\cup B)\cap A}(x)$, respectively. □

Definition 3.13 Let A and B be two complex neutrosophic sets on X, and $T_A(x) = p_A(x).e^{j\cdot\mu_A(x)}$, $I_A(x) = q_A(x).e^{j\cdot v_A(x)}$, $F_A(x) = r_A(x).e^{j\cdot\omega_A(x)}$ and $T_B(x) = p_B(x).e^{j\cdot\mu_B(x)}$, $I_B(x) = q_B(x).e^{j\cdot v_B(x)}$, $F_B(x) = r_B(x).e^{j\cdot\omega_B(x)}$, respectively, be their complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsity membership function.

The complex neutrosophic product of A and B denoted as $A \circ B$ and is specified by the functions,

$$T_{A\circ B}(x) = p_{A\circ B}(x).e^{j\cdot\mu_{A\circ B}(x)} = (p_A(x).p_B(x)).e^{j.2\pi\left(\frac{\mu_A(x)}{2\pi}.\frac{\mu_B(x)}{2\pi}\right)},$$
$$I_{A\circ B}(x) = q_{A\circ B}(x).e^{j\cdot\mu_{A\circ B}(x)} = (q_A(x).q_B(x)).e^{j.2\pi\left(\frac{v_A(x)}{2\pi}.\frac{v_B(x)}{2\pi}\right)},$$
$$F_{A\circ B}(x) = r_{A\circ B}(x).e^{j\cdot\mu_{A\circ B}(x)} = (r_A(x).r_B(x)).e^{j.2\pi\left(\frac{\omega_A(x)}{2\pi}.\frac{\mu_B(x)}{2\pi}\right)}.$$

Example 3.14 Let $X = \{x_1, x_2, x_3\}$ and let

$$A = \frac{\left(0.6e^{j1.2\pi}, 0.3e^{j0.5\pi}, 1.0e^{j0.1\pi}\right)}{x_1} + \frac{\left(1.0e^{j2\pi}, 0.2e^{j.3\pi}, 0.5e^{j0.4\pi}\right)}{x_2} + \frac{\left(0.8e^{j1.6\pi}, 0.1e^{j1.2}, 0.6e^{j0.1\pi}\right)}{x_3},$$

$$B = \frac{\left(0.6e^{j1.2\pi}, 0.1e^{j0.4\pi}, 1.0e^{j0.1\pi}\right)}{x_1} + \frac{\left(1.0e^{j1.2\pi}, 0.3e^{j.2\pi}, 0.7e^{j0.5\pi}\right)}{x_2} + \frac{\left(0.2e^{j1.6\pi}, 0.2e^{j1.3\pi}, 0.7e^{j0.1\pi}\right)}{x_3}$$

Then

$$A \circ B = \frac{\left(0.36e^{j0.72\pi}, 0.3e^{j0.1\pi}, 1.0e^{j0.0025\pi}\right)}{x_1}, \frac{\left(1.0e^{j1.2\pi}, 0.06e^{j3\pi}, 0.35e^{j0.1\pi}\right)}{x_3}, \frac{\left(0.16e^{j1.28\pi}, 0.02e^{j0.78\pi}, 0.42e^{j0.005\pi}\right)}{x_3}$$

Definition 3.15 Let A_n be N complex neutrosophic sets on X $(n = 1, 2, \ldots, N)$, and $T_{A_n}(x) = p_{A_n}(x).e^{j\cdot\mu_{A_n}(x)}$, $I_{A_n}(x) = q_{A_n}(x).e^{j\cdot v_{A_n}(x)}$, and $F_{A_n}(x) = r_{A_n}(x).e^{j.\omega_{A_n}(x)}$ be their complex-valued membership function, complex-valued indeterminacy membership function and complex-valued non-membership function, respectively. The Cartesian product of A_n, denoted as $A_1 \times A_2 \times \cdots \times A_N$, specified by the function

$$\begin{aligned} T_{A_1\times A_2\times\cdots\times A_N}(x) &= p_{A_1\times A_2\times\cdots\times A_N}(x).e^{j\cdot\mu_{A_1\times A_2\times\cdots\times A_N}(x)} \\ &= \min(p_{A_1}(x_1), p_{A_2}(x_2), \ldots, p_{A_N}(x_N)) \\ &\quad .\, e^{j\min\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_N}(x_N)\right)}, \end{aligned}$$

$$\begin{aligned} I_{A_1\times A_2\times\cdots\times A_N}(x) &= q_{A_1\times A_2\times\cdots\times A_N}(x).e^{j\cdot v_{A_1\times A_2\times\cdots\times A_N}(x)} \\ &= \max(q_{A_1}(x_1), q_{A_2}(x_2), \ldots, q_{A_N}(x_N)) \\ &\quad .\, e^{j\max\left(v_{A_1}(x_1), v_{A_2}(x_2), \ldots, v_{A_N}(x_N)\right)}, \end{aligned}$$

and

$$\begin{aligned} F_{A_1\times A_2\times\cdots\times A_N}(x) &= r_{A_1\times A_2\times\cdots\times A_N}(x).e^{j.\omega_{A_1\times A_2\times\cdots\times A_N}(x)} \\ &= \max(r_{A_1}(x_1), r_{A_2}(x_2), \ldots, r_{A_N}(x_N)) \\ &\quad .\, e^{j\max\left(\omega_{A_1}(x_1), \omega_{A_2}(x_2), \ldots, \omega_{A_N}(x_N)\right)}, \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_N) \in \underbrace{X \times X \times \cdots \times X}_{N}$.

4 Distance measure and δ-equalities of complex neutrosophic sets

In this section, we introduced distance measure and other operational properties of complex neutrosophic sets.

Definition 4.1 Let $\mathrm{CN}(X)$ be the collection of all complex neutrosophic sets on X and $A, B \in \mathrm{CN}(X)$. Then, $A \subseteq B$ if and only if $T_A(x) \le T_B(x)$ such that the amplitude terms $p_A(x) \le p_B(x)$ and the phase terms $\mu_A(x) \le \mu_B(x)$, and $I_A(x) \ge I_B(x)$ such that the amplitude terms $q_A(x) \ge q_B(x)$ and the phase terms $v_A(x) \ge v_B(x)$ whereas $F_A(x) \ge F_B(x)$ such that the amplitude terms $r_A(x) \ge r_B(x)$ and the phase terms $\omega_A(x) \ge \omega_B(x)$.

Definition 4.2 Two complex neutrosophic sets A and B are said to equal if and only if $p_A(x) = p_B(x)$, $q_A(x) = q_B(x)$, and $r_A(x) = r_B(x)$ for amplitude terms and $\mu_A(x) = \mu_B(x)$, $v_A(x) = v_B(x)$, $\omega_A(x) = \omega_B(x)$ for phase terms (arguments).

Definition 4.3 A distance of complex neutrosophic sets is a function $d_{\mathrm{CNS}}:\mathrm{CN}(X) \times \mathrm{CN}(X) \to [0, 1]$ such that for any $A, B, C \in \mathrm{CN}(X)$

1. $0 \le d_{\mathrm{CNS}}(A, B) \le 1$,
2. $d_{\mathrm{CNS}}(A, B) = 0$ if and only if $A = B$,
3. $d_{\mathrm{CNS}}(A, B) = d_{\mathrm{CNS}}(B, A)$,
4. $d_{\mathrm{CNS}}(A, B) \le d_{\mathrm{CNS}}(A, C) + d_{\mathrm{CNS}}(C, B)$.

Let $d_{\mathrm{CNS}}:\mathrm{CN}(X) \times \mathrm{CN}(X) \to [0, 1]$ be a function which is defined as

$$d_{\mathrm{CNS}}(A, B) = \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x) - p_B(x)|, \sup_{x\in X}|q_A(x) - q_B(x)|, \sup_{x\in X}|r_A(x) - r_B(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|v_A(x) - v_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)|) \end{pmatrix}$$

Theorem 4.4 *The function $d_{\mathrm{CNS}}(A, B)$ defined above is a distance function of complex neutrosophic sets on X.*

Proof The proof is straightforward. □

Definition 4.5 Let A and B be two complex neutrosophic sets on X, and $T_A(x) = p_A(x).e^{j.\mu_A(x)}$, $I_A(x) = q_A(x).e^{j.\nu_A(x)}$, $F_A(x) = r_A(x).e^{j.\omega_A(x)}$ and $T_B(x) = p_A(x).e^{j.\mu_B(x)}$, $I_B(x) = q_B(x).e^{j.\nu_B(x)}$, $F_B(x) = r_B(x).e^{j.\omega_B(x)}$ are their complex-valued truth membership, complex-valued indeterminacy membership, and complex-valued falsity membership functions, respectively. Then, A and B are said to be δ-equal, if and only if $d_{\mathrm{CNS}}(A, B) \leq 1 - \delta$, where $0 \leq \delta \leq 1$. It is denoted by $A = (\delta)B$.

Proposition 4.6 *For complex neutrosophic sets A, B, and C, the following holds.*

1. $A = (0)B$,
2. $A = (1)B$ if and only if $A = B$,
3. If $A = (\delta)B$ *if and only if* $B = (\delta)A$,
4. $A = (\delta_1)B$ *and* $\delta_2 \leq \delta_1$, *then* $A = (\delta_2)B$,
5. If $A = (\delta_\alpha)B$, *then* $A = (\sup_{\alpha \in J} \delta_\alpha)B$ *for all* $\alpha \in J$, *where J is an index set*,
6. If $A = (\delta')B$ *and there exist a unique* δ *such that* $A = (\delta)B$, *then* $\delta' \leq \delta$ *for all* A, B
7. *If* $A = (\delta_1)B$ and $B = (\delta_2)C$, *then* $A = (\delta)C$, *where* $\delta = \delta_1 * \delta_2$.

Proof 4.7 Properties 1–4, 6 can be proved easily. We only prove 5 and 7.

5. Since $A = (\delta_\alpha)B$ for all $\alpha \in J$, we have

$$d_{\mathrm{CNS}}(A, B) = \max\begin{pmatrix} \max(\sup_{x \in X}|p_A(x) - p_B(x)|, \sup_{x \in X}|q_A(x) - q_B(x)|, \sup_{x \in X}|r_A(x) - r_B(x)|), \\ \max(\frac{1}{2\pi}\sup_{x \in X}|\mu_A(x) - \mu_B(x)|, \frac{1}{2\pi}\sup_{x \in X}|\nu_A(x) - \nu_B(x)|, \frac{1}{2\pi}\sup_{x \in X}|\omega_A(x) - \omega_B(x)|) \end{pmatrix} \leq 1 - \delta_\alpha$$

Therefore,

$$\sup_{x \in X}|p_A(x) - p_B(x)| \leq 1 - \sup_{\alpha \in J} \delta_\alpha,$$
$$\sup_{x \in X}|q_A(x) - q_B(x)| \leq 1 - \sup_{\alpha \in J} \delta_\alpha,$$
$$\sup_{x \in X}|r_A(x) - r_B(x)| \leq 1 - \sup_{\alpha \in J} \delta_\alpha, \quad \text{and}$$
$$\frac{1}{2\pi}\sup_{x \in X}|\mu_A(x) - \mu_B(x)| \leq 1 - \sup_{\alpha \in J} \delta_\alpha,$$
$$\frac{1}{2\pi}\sup_{x \in X}|\nu_A(x) - \nu_B(x)| \leq 1 - \sup_{\alpha \in J} \delta_\alpha$$
$$\frac{1}{2\pi}\sup_{x \in X}|\omega_A(x) - \omega_B(x)| \leq 1 - \sup_{\alpha \in J} \delta_\alpha.$$

Thus,

$$\begin{aligned} d_{\mathrm{CNS}}(A, B) = \max&\begin{pmatrix} \max(\sup_{x \in X}|p_A(x) - p_B(x)|, \sup_{x \in X}|q_A(x) - q_B(x)|, \sup_{x \in X}|r_A(x) - r_B(x)|), \\ \max\left(\frac{1}{2\pi}\sup_{x \in X}|\mu_A(x) - \mu_B(x)|, \frac{1}{2\pi}\sup_{x \in X}|\nu_A(x) - \nu_B(x)|, \frac{1}{2\pi}\sup_{x \in X}|\omega_A(x) - \omega_B(x)|\right) \end{pmatrix} \\ &\leq 1 - \sup_{\alpha \in J} \delta_\alpha \end{aligned}$$

Hence, $A = (\sup_{\alpha \in J} \delta_\alpha)B$.

7. Since $A = (\delta_1)B$, we have

$$d_{\mathrm{CNS}}(A,B) = \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x) - p_B(x)|, \sup_{x\in X}|q_A(x) - q_B(x)|, \sup_{x\in X}|r_A(x) - r_B(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)|) \end{pmatrix} \leq 1 - \delta_1$$

which implies

$$\sup_{x\in X}|p_A(x) - p_B(x)| \leq 1 - \delta_1,$$

$$\sup_{x\in X}|q_A(x) - q_B(x)| \leq 1 - \delta_1,$$

$$\sup_{x\in X}|r_A(x) - r_B(x)| \leq 1 - \delta_1 \text{ and}$$

$$\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_B(x)| \leq 1 - \delta_1,\ \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_B(x)| \leq 1 - \delta_1$$

$$\frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)| \leq 1 - \delta_1.$$

Also we have $B = (\delta_2)C$, so

$$d_{\mathrm{CNS}}(B,C) = \max\left(\begin{matrix} \max(\sup_{x\in X}|p_B(x) - p_C(x)|, \sup_{x\in X}|q_B(x) - q_C(x)|, \sup_{x\in X}|r_B(x) - r_C(x)|), \\ \max\left(\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x) - \mu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x) - \nu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x) - \omega_C(x)|\right) \end{matrix}\right) \leq 1 - \delta_2$$

which implies

$$\sup_{x\in X}|p_B(x) - p_C(x)| \leq 1 - \delta_2,$$

$$\sup_{x\in X}|q_B(x) - q_C(x)| \leq 1 - \delta_2,$$

$$\sup_{x\in X}|r_B(x) - r_C(x)| \leq 1 - \delta_1 \text{ and}$$

$$\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x) - \mu_C(x)| \leq 1 - \delta_2,\ \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x) - \nu_C(x)| \leq 1 - \delta_2,$$

$$\frac{1}{2\pi}\sup_{x\in X}|\omega_B(x) - \omega_C(x)| \leq 1 - \delta_2.$$

Now,

$$
\begin{aligned}
d_{\mathrm{CNS}}(A,C) &= \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x)-p_C(x)|, \sup_{x\in X}|q_A(x)-q_C(x)|, \sup_{x\in X}|r_A(x)-r_C(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_C(x)|) \end{pmatrix} \\
&\le \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x)-p_B(x)|, \sup_{x\in X}|q_A(x)-q_B(x)|, \sup_{x\in X}|r_A(x)-r_B(x)|)+ \\ \max(\sup_{x\in X}|p_B(x)-p_C(x)|, \sup_{x\in X}|q_B(x)-q_C(x)|, \sup_{x\in X}|r_B(x)-r_C(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_B(x)|)+ \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x)-\mu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x)-\nu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x)-\omega_C(x)|) \end{pmatrix} \\
&\le \max((1-\delta_1)+(1-\delta_2),(1-\delta_1)+(1-\delta_2)) = (1-\delta_1)+(1-\delta_2) = 1-(\delta_1+\delta_2-1),
\end{aligned}
$$

From Definition 4.3, $d_{\mathrm{CNS}}(A,C)\le 1$. Therefore, $d_{\mathrm{CNS}}(A,C)\le 1-\delta_1*\delta_2 = 1-\delta$, where $\delta=\delta_1*\delta_2$. Thus, $A=(\delta)C$. □

Theorem 4.8 *If* $A=(\delta)B$, *then* $c(A)=(\delta)c(B)$, *where* $c(A)$ and $c(B)$ *are the complement of the complex neutrosophic sets A and B*.

Proof Since

$$
\begin{aligned}
d_{\mathrm{CNS}}(c(A),c(B)) &= \max\begin{pmatrix} \max\left(\sup_{x\in X}\left|p_{C(A)}(x)-p_{c(B)}(x)\right|, \sup_{x\in X}\left|q_{c(A)}(x)-q_{c(B)}(x)\right|, \sup_{x\in X}\left|r_{c(A)}(x)-r_{c(B)}(x)\right|\right), \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\mu_{c(A)}(x)-\mu_{c(B)}(x)\right|, \frac{1}{2\pi}\sup_{x\in X}\left|\nu_{c(A)}(x)-\nu_{c(B)}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\omega_{c(A)}(x)-\omega_{c(B)}(x)\right| \end{pmatrix} \end{pmatrix} \\
&= \max\begin{pmatrix} \max(\sup_{x\in X}|r_A(x)-r_B(x)|, \sup_{x\in X}|(1-q_A(x))-(1-q_B(x))|, \sup_{x\in X}|p_A(x)-p_B(x)|), \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}|(2\pi-\mu_A(x))-(2\pi-\mu_B(x))|, \frac{1}{2\pi}\sup_{x\in X}|(2\pi-\nu_A(x))-(2\pi-\nu_B(x))|, \\ \frac{1}{2\pi}\sup_{x\in X}|(2\pi-\omega_A(x))-(2\pi-\omega_B(x))| \end{pmatrix} \end{pmatrix} \\
&= \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x)-p_B(x)|, \sup_{x\in X}|q_A(x)-q_B(x)|, \sup_{x\in X}|r_A(x)-r_B(x)|), \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_B(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_B(x)| \end{pmatrix} \end{pmatrix} = d_{\mathrm{CNS}}(A,B) \le 1-\delta
\end{aligned}
$$

□

5 Application of complex neutrosophic set in signal processing

The complex neutrosophic set and δ-equalities of complex neutrosophic sets are applied in signal processing application which demonstrates to point out a particular signal of interest out of a large number of signals that are received by a digital receiver. This is the example which Ramot et al. [23] discussed for complex fuzzy set. We now apply complex neutrosophic set to this example.

Suppose that there are L' different signals, $S_1(t), S_2(t), \ldots, S_{L'}(t)$. These signals have been detected and sampled by a digital receiver and each of which is sampled N times. Suppose that $S_{l'}(k')$ denote the k'th of the l'-th signal, where $1\le k'\le N$ and $1\le l'\le L'$. Now we form the following algorithm for this application.

5.1 Algorithm

Step 1. Write the discrete Fourier transforms of the L' signals in the form of complex neutrosophic set,

$$S_{l'}(k') = \frac{1}{N}.\sum_{n=1}^{N}\left(C_{l',n}, D_{l',n}, E_{l',n}\right).e^{\frac{2\pi j(n-1)(k'-1)}{N}} \tag{1}$$

where $C_{l',n}, D_{l',n}, E_{l',n}$ are the complex-valued Fourier coefficients of the signals and $1\le n\le N$.

The above sum may be written as

$$S_{l'}(k') = \frac{1}{N}.\sum_{n=1}^{N}\left(U_{l',n}, V_{l',n}, W_{l',n}\right).e^{\frac{j(2\pi(n-1)(k'-1)+\alpha_{l',n})}{N}} \tag{2}$$

where $C_{l',n} = U_{l',n}.e^{j\alpha_{l',n}}$, $D_{l',n} = V_{l',n}.e^{j\alpha_{l',n}}$, and $E_{l',n} = W_{l',n}.e^{j\alpha_{l',n}}$ with $U_{l',n}$, $V_{l',n}$, $W_{l',n}\ge 0$, and $\alpha_{l',n}$ are real-valued for all n.

The purpose of this application is to point out the reference signals R' of the L' signals. This reference signal R\prime has been also sampled N times $(1 \le n \le N)$.

Step 2. Write the Fourier coefficient of R' in the form of complex neutrosophic set,

$$R'(k') = \frac{1}{N} \cdot \sum_{n=1}^{N} \left(C_{R',n}, D_{R',n}, E_{R',n} \right) . e^{\frac{j2\pi(n-1)(k'-1)}{N}} \tag{3}$$

where $C_{R',n}, D_{R',n}, E_{R',n}$ are the complex Fourier coefficients of the reference signals.

The above expression can be rewritten as

$$R'(k') = \frac{1}{N} \cdot \sum_{n=1}^{N} \left(U_{R',n}, V_{R',n}, W_{R',n} \right) . e^{\frac{j(2\pi(n-1)(k'-1) + \alpha_{R',n})}{N}} \tag{4}$$

where $C_{R',n} = U_{R',n}.e^{j\alpha_{R',n}}$, $D_{R',n} = V_{R',n}.e^{j\alpha_{R',n}}$, and $E_{R',n} = W_{R',n}.e^{j\alpha_{R',n}}$ with $U_{R',n}$, $V_{R',n}$, $W_{R',n} \ge 0$, and $\alpha_{R',n}$ are real-valued for all n.

Step 3 Since the sum of truth amplitude term, indeterminate amplitude term, and falsity amplitude term (in the case when they are crisp numbers, not sets) is not necessarily equal to 1, the normalization is not required and we can keep them un-normalized. But if the normalization is needed, we can normalize the amplitude terms of $S_{l'}(k')$ and $R'(k')$, respectively, as follows:

$$\underset{l',n}{\tilde{U}} = \frac{U_{l',n}}{U_{l',n} + V_{l',n} + W_{l',n}}, \quad \underset{l',n}{\tilde{V}} = \frac{V_{l',n}}{U_{l',n} + V_{l',n} + W_{l',n}},$$
$$\underset{l',n}{\tilde{W}} = \frac{W_{l',n}}{U_{l',n} + V_{l',n} + W_{l',n}} \text{ and } \underset{R',n}{\tilde{U}} = \frac{U_{R',n}}{U_{R',n} + V_{R',n} + W_{R',n}},$$
$$\underset{R',n}{\tilde{V}} = \frac{V_{R',n}}{U_{R',n} + V_{R',n} + W_{R',n}}, \quad \underset{R',n}{\tilde{W}} = \frac{W_{R',n}}{U_{R',n} + V_{R',n} + W_{R',n}}.$$

Step 4 Calculate the similarity/distances between the signals $R'(k')$ and the signals $S_{l'}(k')$ as follows.

$$d_{\text{CNS}}(S_{l'}(k'), R'(k')) = \max \begin{pmatrix} \max\left(\sup_{x\in X} \left| U_{l',n} - U_{R',n} \right|, \sup_{x\in X} \left| V_{l',n} - V_{R',n} \right|, \sup_{x\in X} \left| W_{l',n} - W_{R',n} \right| \right), \\ \max\left(\frac{1}{2\pi} \sup_{x\in X} \left| \frac{(2\pi(n-1)(k'-1) + \alpha_{l',n})}{N} - \frac{(2\pi(n-1)(k'-1) + \alpha_{R',n})}{N} \right| \right) \end{pmatrix}$$

Step 5 In order to identify $S_{l'}$ as R', compare $1 - d_{CNS}(S_{l'}, R'(k'))$ to a threshold δ, where $1 \le l' \le L'$

If $1 - d_{\text{CNS}}\left(S'_l(k'), R'(k')\right)$ exceeds the threshold, identify $S_{l'}$ as R'.

The similarity between two signals can be measured by this method. By this method, we can find the right signals which have not only uncertain but also indeterminate, inconsistent, false because when the signals are received by a digital receiver, there is a chance for the right signals, chance for the indeterminate signals, and the chance that the signals are not the right one. Thus, by using a complex neutrosophic set, we can find the correct reference signals by taking all the chances, while the complex fuzzy set and complex intuitionistic fuzzy set cannot find the correct reference signals if we take all the chances because they are not able to deal with the chance of indeterminacy.

This method can be effectively used for any application in signal analysis in which the chance of indeterminacy is important.

6 Drawbacks of the current methods

The complex fuzzy sets [23] are used to represents the information with uncertainty and periodicity simultaneously. The novelty of complex fuzzy sets appears in the phase term with membership term (amplitude term). The main problem with complex fuzzy set is that it can only handle the problems of uncertainty with periodicity in the form of amplitude term (real-valued membership function) which handle uncertainty and an additional term called phase term to represent periodicity, but the complex fuzzy set cannot deal with inconsistent, incomplete, indeterminate, false etc. information which appears in a periodic manner in our real life. For example, in quantum mechanics, a wave particle such as an electron can be in two different positions at the same time. Thus, the complex fuzzy set is not able to deal with this phenomenon.

Complex intuitionistic fuzzy set [1] represents the information involving two or more answers of type: yes, no, I do not know, I am not sure, and so on, which is happening repeatedly over a period of time. CIFS can represent the information on people's decision which happens periodically. In CIFS, the novelty also appears in the phase term but for both membership and non-membership functions in some inherent concepts in contrast to CFS which is only characterized by a membership function. The complex fuzzy set [23] has only one additional phase term, but in CIFS [1], we have two additional phase terms. This confers more range values to represent the uncertainty and periodicity semantics simultaneously, and to define the values of belongingness and non-belongingness for any object in these complex-valued functions. The failure of the CIFS appears in the inconsistent, incomplete, indeterminate information which happening repeatedly.

The current research (complex fuzzy set) cannot solve this problem because the complex fuzzy set is not able to deal with indeterminate, incomplete, and inconsistent date which is in periodicity. The weaknesses of complex fuzzy set are that it deals only with uncertainty, but indeterminacy and falsity are far away from the scope of complex fuzzy sets. Similarly, the complex intuitionistic fuzzy set cannot handle the inconsistent, indeterminate, incomplete data in periodicity simultaneously. Thus, both the approaches are unable to deal with inconsistent, indeterminate, and incomplete data of periodic nature. For example, both the methods fail to deal with the information which is true and false at the same time or neither true nor false at the same time.

It is a fundamental fact that some information has not only a certain degree of truth, but also a falsity degree as well as indeterminacy degree that are independent from each other. This indeterminacy exits both in a subjective and an objective sense in a periodic nature. What should we do if we have the following situation? For instance [16], a 20° temperature means a cool day in summer and a warm day in winter. But if we assume this situation as in the following manner, a 20° temperature means cool day in summer and a warm day in winter but neither cool nor warm day in spring. The question is that why we ignore this situation? How we can handle it? Why the current methods fail to handle it? We cannot ignore this kind of situation of daily life. This phenomenon indicates that information is not only of semantic uncertainty and periodicity but also of semantic indeterminacy and periodicity.

7 Discussion

In the Table 1, we showed comparison of different current approaches to complex neutrosophic sets. In the Table 1, from 1, we mean that the corresponding method can handle the uncertain, false, indeterminate, uncertainty with periodicity, falsity with periodicity, and indeterminacy with periodicity, while from 0, we mean the corresponding method fails. It is clear from the Table 1 that how complex neutrosophic sets are dominant over all the current methods.

Consider two voting process for some attribute ρ. In the first voting process, 0.4 voters say "yes," 0.3 say "no," and 0.3 are undecided. Similarly, in second voting process, 0.5 voters say "yes," 0.3 say "no" and 0.2 are undecided for the same attribute ρ. These two voting processes held on two different dates.

We now apply all these mentioned methods in the table one by one to show that which method is suitable to describe the situation of above mentioned voting process best and what is the failure of the rest of the methods. It is clear that the fuzzy set cannot handle this situation because it only represents the membership 0.4 voters while it fails to tell about the non-membership 0.3 and indeterminate membership 0.3 simultaneously in first voting process. Similar is the situation in second voting process. Now when we apply intuitionistic fuzzy set to both the voting process, it tells us only about the membership 0.4 and non-membership 0.3 in first voting process, but cannot tell anything about the 0.3 undecided voters in first process. Thus, intuitionistic fuzzy set also fails to handle this situation. We now apply neutrosophic set. The neutrosophic set tells about the membership 0.4 voters, non-membership 0.3 voters, and indeterminate membership or undecided 0.3 voters in the first round, and similarly, it tells about the second round but neutrosophic set cannot describe both the voting process simultaneously. By applying complex fuzzy set to both the voting process, if we set that the amplitude term represents the membership 0.4 in first voting process and the phase term represents 0.5 voters in second process which form complex-valued membership function to represent in both the voting process for an attribute ρ. But complex fuzzy set remains unsuccessful to describe the non-membership and indeterminacy in both the process. The complex intuitionistic fuzzy set only handle complex-valued membership and complex-valued non-membership in both the process by setting 0.4 and 0.3 as amplitude membership and amplitude non-membership in process one and setting 0.5 and 0.3 as phase terms in second process. But clearly it fails to identify the indeterminacy (undecidedness) in both the voting process. Finally, by applying the complex neutrosophic set to both the voting process by considering the votes in process one as amplitude terms of membership, non-membership and indeterminate membership, and setting the second process vote as phase terms of membership, non-membership, and indeterminacy. Therefore, the amplitude term of membership in first process and the phase term in second process forms complex-valued truth membership function. Similarly, the amplitude term of non-membership in process one and the phase term of non-membership in second process form complex-valued falsity membership function. Also, the amplitude term of undecidedness in first process and the phase term of indeterminacy in second process form the complex-valued indeterminate membership function. Thus, both the voting process forms a complex neutrosophic set as whole which is shown below:

$$S = \left\{ \left(\rho, T_S(\rho) = 0.4.e^{j2\pi(0.5)}, I_S(\rho) = 0.3.e^{j2\pi(0.3)}, F_S(\rho) = 0.3.e^{j2\pi(0.2)} \right) \right\}$$

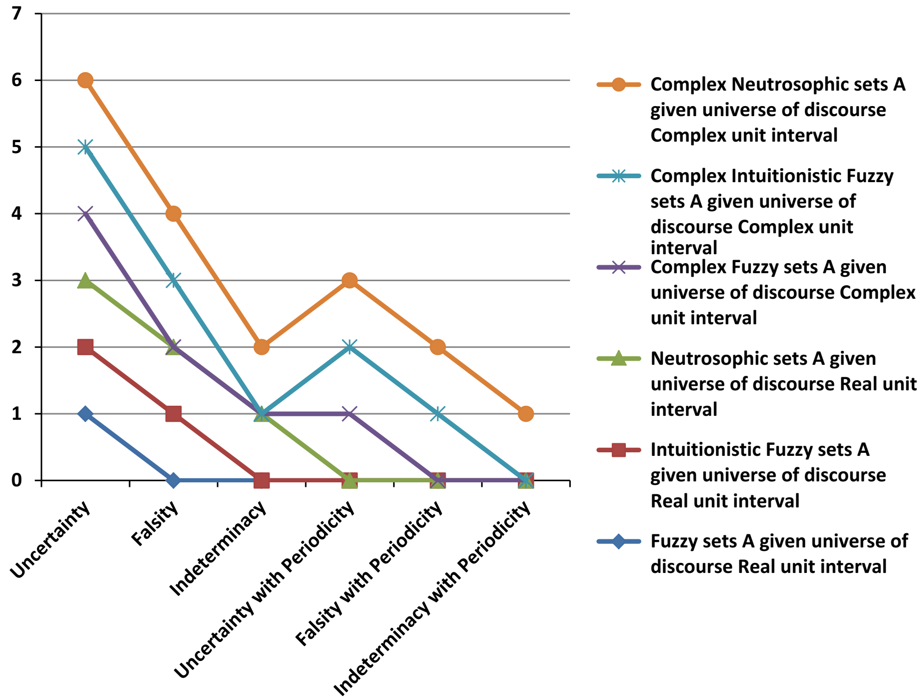

Fig. 1 Dominancy of complex neutrosophic sets to all other current approaches

Therefore, complex neutrosophic set represent both the situations in a single set simultaneously, whereas all the other mentioned methods in the table are not able to handle this situation as whole.

The graphical representation in Fig. 1 shows the dominancy of the complex neutrosophic set to all other existing methods. The highest value indicates the ability of the approach to handle all type of uncertain, incomplete, inconsistent, imprecise information or data in our real-life problems. Each value on the left vertical line shows the value of the ability of the corresponding method on the horizontal line in the graph.

8 Conclusion

An extended form of complex fuzzy set and complex intuitionistic fuzzy set is presented in this paper, so-called complex neutrosophic set. Complex neutrosophic set can handle the redundant nature of uncertainty, incompleteness, indeterminacy, inconsistency, etc. A complex neutrosophic set is defined by a complex-valued truth membership function, complex-valued indeterminate membership function, and a complex-valued falsehood membership function. Therefore, a complex-valued truth membership function is a combination of traditional truth membership function with the addition of an extra term. The traditional truth membership function is called truth amplitude term, and the additional term is called phase term. Thus, in this way, the truth amplitude term represents uncertainty and the phase term represents periodicity in the uncertainty. Thus, a complex-valued truth membership function represents uncertainty with periodicity as a whole. Similarly, complex-valued indeterminate membership function represents indeterminacy with periodicity and complex-valued falsehood membership function represents falsity with periodicity. Further, we presented an interpretation of complex neutrosophic set and also discussed some of the basic set theoretic properties such as complement, union, intersection, complex neutrosophic product, Cartesian product in this paper. We also presented δ-equalities of complex neutrosophic set and then using these δ-equalities in the application of signal processing. Drawbacks of the current methods are discussed and a comparison of all these methods to complex neutrosophic sets was presented in this paper.

This paper is an introductory paper of complex neutrosophic sets, and indeed, much research is still needed for the full comprehension of complex neutrosophic sets. The complex neutrosophic set presented in this paper is an entire general concept which is not limited to a specific application.

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Appendix

Comparison of complex neutrosophic sets to fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, complex fuzzy sets, and complex intuitionistic fuzzy sets is listed below (Table 1).

Table 1 Comparison of complex neutrosophic sets to the current approaches

Sets/logics	Domain	Co-domain	Uncertainty	Falsity	Indeterminacy	Uncertainty with periodicity	Falsity with periodicity	Indeterminacy with periodicity
Fuzzy sets	A given universe of discourse	Real unit interval	1	0	0	0	0	0
Intuitionistic fuzzy sets	A given universe of discourse	Real unit interval	1	1	0	0	0	0
Neutrosophic sets	A given universe of discourse	Real unit interval	1	1	1	0	0	0
Complex fuzzy sets	A given universe of discourse	Complex unit interval	1	0	0	1	0	0
Complex intuitionistic fuzzy sets	A given universe of discourse	Complex unit interval	1	1	0	1	1	0
Complex neutrosophic sets	A given universe of discourse	Complex unit interval	1	1	1	1	1	1

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