COMPLEX REFLECTION GROUPS, BRAID GROUPS, HECKE ALGEBRAS

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Abstract. Presentations "à la Coxeter" are given for all (irreducible) finite complex reflection groups. They provide presentations for the corresponding generalized braid groups (for all but six cases), which allow us to generalize some of the known properties of finite Coxeter groups and their associated braid groups, such as the computation of the center of the braid group and the construction of deformations of the finite group algebra (Hecke algebras). We introduce monodromy representations of the braid groups which factorize through the Hecke algebras, extending results of Cherednik, Opdam, Kohno and others.

Summary

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Introduction

In [DeMo] (17.20), Deligne and Mostow raised the following question :

"Let $W \subset \mathrm{GL}(V)$ be an irreducible finite group generated by complex reflections, and let V' be the complement in V of the fixed hyperplanes of the complex reflections in W. For H the fixed hyperplane of a complex reflection in W, let s_H be the generator of the monodromy around the image of H in V/W. It is well defined up to conjugacy in $\pi := \pi_1(V'/W)$. The fundamental group π is an extension of W by the fundamental group of V', and s_H projects in W to the inverse of the generator of the fixer of H, with non trivial eigenvalue of the form $\exp(2\pi i/n_H)$.

Question 3. *For each conjugacy class of each hyperplane H fixed by a complex reflection, let $q_H(t)$ be a path in $\mathbb{C}^\times$, starting at $\exp(-2\pi i/n_H)$. Is it uniquely possible to deform with t a representation ρ_t of π, starting at $t = 0$ with the given representation of the quotient W of π, so that $\rho_t(s_H)$ is a complex reflection with non trivial eigenvalue $q_H(t)$?"*

As noticed by Deligne and Mostow, in the case where W is a Coxeter group, the existence of the Hecke algebra as an image of the group algebra of the braid group π provides a positive answer to their question.

It is one of our purposes here to give a positive answer to the preceding question, at least for all infinite series of irreducible finite complex reflection groups, and for some exceptional ones (a more partial answer, without proofs, had been announced in [BMR]). We shall get this answer by exhibiting a generalized Hecke algebra for these groups again as an image of the group algebra of the associated "braid group" π.

Through recent work on representations of reductive finite groups and related topics (like representations of finite Coxeter groups and associated Hecke algebras)[1] it has become clearer and clearer that finite "complex reflection groups" (*i.e.,* linear groups generated by pseudo-reflections) behave very much like Coxeter groups, or even like Weyl groups.

- Many of them behave as if they were the Weyl group of a reductive algebraic group : in particular, they determine families of polynomials which share many properties of the set of generic degrees of the unipotent characters of a reductive algebraic group.
- Through suitable presentations by generators and relations, it has become possible to deform the complex group algebra of most complex reflection groups in a way which generalizes the construction of classical Hecke algebras of finite Coxeter groups.

Here we prove in particular that these presentations are naturally associated to presentations of the corresponding braid groups, thus providing a more intrinsic definition.

It should be noticed that some other nice properties of "Coxeter braid groups" extend to this more general setting.

For example, generalizing a result of Deligne and Brieskorn–Saito valid for Coxeter groups, we check here, in most cases, that the centers of braid groups associated to irreducible complex reflection groups are cyclic.

[1] see for example [AlLu], [Ari], [ArKo], [BreMa], [BrMa], [BrMi], [BMM], [Lu], [Ma1], [Ma2].

Also (at least in the case of the infinite series) the pure braid group of an r–dimensional irreducible complex reflection group has a natural structure as an r–fold iterated semidirect product of free groups (of finite rank).

Nevertheless, it should be emphasized that this must only be the beginning of a long story which is still to be discovered. Our results are – almost – general, but few of our proofs are. Moreover, new questions emerge now : how to characterize the distinguished generators and the diagrams representing the relations ? How to explain the natural "diagram invariants" like degrees, codegrees, zeta function (see §5 below) ?

1. Complex reflection groups and their presentations

A. Background from complex reflection groups.

For all the results quoted here, we refer the reader to the classical literature on complex reflections groups, such as [Bou], [Ch], [Co], [ShTo], [Sp], and also to the more recent fundamental work on the subject by Orlik, Solomon and Terao (see [OrSo1], [OrSo2], [OrTe]).

Let V be a complex vector space of dimension r. A *pseudo–reflection* of $\mathrm{GL}(V)$ is a non trivial element s of $\mathrm{GL}(V)$ which acts trivially on a hyperplane, called the *reflecting hyperplane* of s. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo–reflections. The pair (V, W) is called a *complex reflection group*.

A *parabolic subgroup* of W is by definition the subgroup of elements of W which act trivially on a subspace of V. The following result is due to Steinberg ([St], Theorem 1.5) — cf. also exercises 7 and 8 in [Bou], Ch. v, §6.

1.1. Theorem. *Let V' be a subspace of V. Then the parabolic subgroup $W_{V'}$, consisting of all elements of W which fix V' pointwise, is still generated by pseudo–reflections : $W_{V'}$ is generated by those pseudo–reflections in W whose reflecting hyperplane contains V'.*

We denote by $\mathcal{A}$ the set of reflecting hyperplanes of (V, W), and we set $N := |\mathcal{A}|$. We denote by N^* the number of pseudo-reflections in W (note that for real reflection groups we have $N = N^*$).

For $H \in \mathcal{A}$, we denote by W_H the pointwise stabilizer of H, and we set $e_H := |W_H|$. The group W_H is a minimal non trivial parabolic subgroup of W. All its non trivial elements are pseudo-reflections. The group W_H is cyclic : if s_H denotes the element of W_H with determinant $\exp(2i\pi/e_H)$, we have $W_H = \langle s_H \rangle$, the group generated by s_H.

The centralizer $C_W(W_H)$ of W_H in W is also its normalizer, as well as the normalizer (setwise stabilizer) of H.

For $\mathcal{C} \in \mathcal{A}/W$ an orbit of hyperplanes, we denote by $N_{\mathcal{C}}$ its cardinality. We have $N_{\mathcal{C}} = |W : C_W(W_H)|$ for $H \in \mathcal{C}$. We also set $e_{\mathcal{C}} := e_H$ for $H \in \mathcal{C}$.

We denote by S the symmetric algebra of V, by $R = S^W$ the algebra of invariants of W, by R_+ the ideal of R consisting of elements of positive degree, and we set $S_W := S/R_+S$.

The following facts are known (they are introduced here in an order which is convenient for the exposition, but not necessarily for their proof).

• *Degrees.*

There is a family of r integers $d_1, d_2, \ldots, d_r$ called the *degrees* of (V, W), defined as follows : the Poincaré polynomial of the graded module $(V \otimes S_W)^W$ is

$$q^{d_1 - 1} + q^{d_2 - 1} + \cdots + q^{d_r - 1}\,.$$

We have

$$(q + d_1 - 1)(q + d_2 - 2) \cdots (q + d_r - 1) = \sum_{w \in W} q^{\dim V^{\langle w \rangle}}$$

(where $V^{\langle w \rangle}$ denotes the space of fixed points of w). It follows that

$$\sum_{j=1}^{j=r} (d_j - 1) = \sum_{H \in \mathcal{A}} (e_H - 1) = \sum_{\mathcal{C} \in \mathcal{A}/W} N_{\mathcal{C}}(e_{\mathcal{C}} - 1) = N^*\,.$$

• *Codegrees.*

There is a family of r integers $d_1^*, d_2^*, \ldots, d_r^*$ called the *codegrees* of (V, W), defined by the following condition: the Poincaré polynomial of the graded module $(V^* \otimes S_W)^W$ is

$$q^{d_1^* + 1} + q^{d_2^* + 1} + \cdots + q^{d_r^* + 1}\,.$$

We have

$$(q - d_1^* - 1)(q - d_2^* - 1) \cdots (q - d_r^* - 1) = \sum_{w \in W} \det_V(w) q^{\dim V^{\langle w \rangle}}\,.$$

It follows that

$$\sum_{j=1}^{j=r} (d_j^* + 1) = \sum_{H \in \mathcal{A}} 1 = \sum_{\mathcal{C} \in \mathcal{A}/W} N_{\mathcal{C}} = N\,.$$

and so

$$N + N^* = \sum_{j=1}^{j=r} (d_j + d_j^*) = \sum_{\mathcal{C} \in \mathcal{A}/W} N_{\mathcal{C}} e_{\mathcal{C}}\,.$$

Remark. The "codegrees" have not been introduced as such in the quoted literature. Nevertheless, the sets of degrees and the codegrees are related to the sets of *exponents* $\{m_1, m_2, \ldots, m_r\}$ and *coexponents* $\{m_1^*, m_2^*, \ldots, m_r^*\}$ (which are defined in [OrSo2]) by the formulae

$$m_j = d_j - 1 \quad \text{and} \quad m_j^* = d_j^* + 1 \quad (j = 1, 2, \ldots, r)\,.$$

• *Algebra of invariants – More on degrees.*

The algebra of invariants R is generated by r algebraically independent homogeneous elements of S respectively of degrees $d_1, d_2, \ldots, d_r$.

The order of W is $|W| = d_1 d_2 \cdots d_r$.

If W is irreducible, its center $Z(W)$ has order $|Z(W)| = d_1 \wedge d_2 \wedge \cdots \wedge d_r$ (where we denote $d_1 \wedge d_2 \wedge \cdots \wedge d_r := \gcd\{d_1, d_2, \ldots, d_r\}$).

• *Cohomology of the hyperplane complements – More on codegrees.*

We set $\mathcal{M} := V - \bigcup_{H\in\mathcal{A}} H$. For $H \in \mathcal{A}$, let us denote by α_H a linear form on V with kernel H, and let us define the holomorphic differential form ω_H on $\mathcal{M}$ by the formula

$$\omega_H := \frac{1}{2i\pi}\frac{d\alpha_H}{\alpha_H}\,,$$

which we also write $\omega_H = \dfrac{1}{2i\pi} d\mathrm{Log}(\alpha_H)$. We denote by $[\omega_H]$ the corresponding de Rham cohomology class.

Brieskorn (cf. [Br2], Lemma 5) has proved the following result.

1.2. *Let $\mathbb{C}[(\omega_H)_{H\in\mathcal{A}}]$ (resp. $\mathbb{Z}[(\omega_H)_{H\in\mathcal{A}}]$) be the $\mathbb{C}$-subalgebra (resp. the $\mathbb{Z}$-subalgebra) of the $\mathbb{C}$-algebra of holomorphic differential forms on $\mathcal{M}$ which is generated by $\{\omega_H\}_{H\in\mathcal{A}}$. Then the map $\omega_H \mapsto [\omega_H]$ induces an isomorphism between $\mathbb{C}[(\omega_H)_{H\in\mathcal{A}}]$ and the cohomology algebra $\mathrm{H}^*(\mathcal{M},\mathbb{C})$ (resp. an isomorphism between $\mathbb{Z}[(\omega_H)_{H\in\mathcal{A}}]$ and the singular cohomology algebra $\mathrm{H}^*(\mathcal{M},\mathbb{Z})$).*

From now on, we write ω_H instead of $[\omega_H]$.

Orlik and Solomon (cf. [OrSo1]) have given a description of the algebra $\mathrm{H}^*(\mathcal{M},\mathbb{C})$. Before stating their result, we need to introduce more notation.

- Let $\mathbb{C}\mathcal{A} := \bigoplus_{H\in\mathcal{A}} \mathbf{e}_H$ be the vector space with basis indexed by $\mathcal{A}$, and let $\Lambda\mathcal{A}$ be its exterior algebra, endowed with the usual Koszul differential map $\delta\colon \Lambda\mathcal{A} \to \Lambda\mathcal{A}$ of degree -1.
- For $\mathcal{B} = \{H_1, H_2, \dots, H_k\} \subset \mathcal{A}$, we denote by $D_\mathcal{B}$ the line generated by $\mathbf{e}_{H_1} \wedge \mathbf{e}_{H_2} \wedge \cdots \wedge \mathbf{e}_{H_k}$.
- We say that $\mathcal{B}$ is *dependent* if $\mathrm{codim}(\bigcap_{H\in\mathcal{B}} H) < |\mathcal{B}|$.
- We denote by $\mathrm{I}\Lambda\mathcal{A}$ the (graded) ideal of $\Lambda\mathcal{A}$ generated by the $\delta(D_\mathcal{B})$ where $\mathcal{B}$ runs overs the set of all dependent subsets of $\mathcal{A}$.

1.3. Theorem. *(Orlik and Solomon) The map $\mathbf{e}_H \mapsto \omega_H$ induces an isomorphism of graded algebras between $\Lambda\mathcal{A}/\mathrm{I}\Lambda\mathcal{A}$ and $\mathrm{H}^*(\mathcal{M},\mathbb{C})$.*

Let $\mathrm{Int}(\mathcal{A})$ be the set of intersections of elements of $\mathcal{A}$. For $X \in \mathrm{Int}(\mathcal{A})$, we set $\mathrm{H}^{(X)}(\mathcal{M},\mathbb{C}) := \sum D_\mathcal{B}$ where the summation is taken over all $\mathcal{B} \subset \mathcal{A}$, $|\mathcal{B}| = \mathrm{codim}(X)$, $\bigcap_{H\in\mathcal{B}} H = X$, and where $D_\mathcal{B}$ is the complex line generated by $\omega_{H_1} \wedge \omega_{H_2} \wedge \cdots \wedge \omega_{H_k}$ if $\mathcal{B} = (H_1, H_2, \dots, H_k)$.

Then it follows from Theorem 1.3 that

1.4. *for any integer n, we have*

$$\mathrm{H}^n(\mathcal{M},\mathbb{C}) = \bigoplus_{\substack{(X\in\mathrm{Int}(\mathcal{A}))\\(\mathrm{codim}(X)=n)}} \mathrm{H}^{(X)}(\mathcal{M},\mathbb{C})\,.$$

Moreover, we see that

1.5.

(1) *the family $(\omega_H)_{H\in\mathcal{A}}$ is a basis of $\mathrm{H}^1(\mathcal{M},\mathbb{C})$,*
(2) *for X an element of $\mathrm{Int}(\mathcal{A})$ with codimension 2, if H_X denotes a fixed element of $\mathcal{A}$ which contains X,*

- *whenever H and H' are two elements of $\mathcal{A}$ which contain X, we have* $\omega_H \wedge \omega_{H'} = \omega_{H_X} \wedge \omega_{H'} - \omega_{H_X} \wedge \omega_H$,
- *the family* $(\omega_{H_X} \wedge \omega_H)_{(H \supset X)(H \neq H_X)}$ *is a basis of* $\mathrm{H}^{(X)}(\mathcal{M}, \mathbb{C})$.

The codegrees are determined by the arrangement $\mathcal{A}$, by the following consequence of Theorem 1.3.

1.6. *The Poincaré polynomial* $P_{\mathcal{M}}(q) := \sum_n q^n \dim(\mathrm{H}^n(\mathcal{M}, \mathbb{C}))$ *of the cohomology algebra* $\mathrm{H}^*(\mathcal{M}, \mathbb{C})$ *is given by the following formulae :*

$$\begin{aligned} P_{\mathcal{M}}(q) &= (1+(1+d_1^*)q)(1+(1+d_2^*)q)\cdots(1+(1+d_r^*)q) \\ &= \sum_{w \in W} \det_V(w)(-q)^{\mathrm{codim}(V^{\langle w \rangle})} . \end{aligned}$$

B. Presentations.

The tables in Appendix 2 provide a complete list of the irreducible finite pseudo–reflection groups, as classified by Shephard and Todd, together with presentations of these groups symbolized by diagrams "à la Coxeter", as well as some of the data attached to these groups. Many of these presentations were previously known. This is the case of the rank r groups which are generated by r reflections, studied by Coxeter [Cx]. Some others (the ones corresponding to the infinite series) occurred in [BrMa] or were inspired by [Ari].

The reader may refer to Appendix 2 to understand what follows.

Isomorphisms between diagrams.

We may notice that the only isomorphisms between the diagrams of our tables are between the diagrams of $G(2,1,2)$ and $G(4,4,2)$, between the diagrams of $\mathfrak{S}_4$ and $G(2,2,3)$, between the diagrams of $\mathfrak{S}_3$ and $G(3,3,2)$, and between the diagrams of $\mathfrak{S}_2$ and $G(2,1,1)$.

Coxeter diagrams.

Note (see tables) the following correspondence of notation :

- $\mathfrak{S}_{r+1}$ $(r \geq 0)$ is the Coxeter group of type A_r,
- $G(2,1,r)$ $(r \geq 2)$ is the Coxeter group of type B_r,
- $G(2,2,r)$ $(r \geq 3)$ is the Coxeter group of type D_r.

Indeed (see table 2 for notation) since $e = 2$, t_2 and t_2' commute, and it is enough to show that the " double–link" braid relation $t_3t_2't_2t_3t_2't_2 = t_2't_2t_3t_2't_2t_3$ is a consequence of the other relations.

Applying successively the fact that t_2 and t_2' commute, the braid relation between t_3 and t_2', and the braid relation between t_3 and t_2, we get $t_3t_2't_2t_3t_2't_2 = t_3t_2t_2't_3t_2't_2 = t_3t_2t_3t_2't_3t_2 = t_2t_3t_2t_2't_3t_2 = t_2t_3t_2't_2t_3t_2 = t_2t_3t_2't_3t_2t_3 = t_2t_2't_3t_2't_2t_3 = t_2't_2t_3t_2't_2t_3$.

- $G(e,e,2)$ $(e \geq 3)$ is the dihedral group of order $2e$,
- G_{28} is the Coxeter group of type F_4,
- G_{35} is the Coxeter group of type E_6,
- G_{36} is the Coxeter group of type E_7,
- G_{37} is the Coxeter group of type E_8,
- G_{23} is the Coxeter group of type H_3,
- G_{30} is the Coxeter group of type H_4.

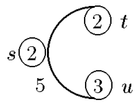
Admissible subdiagrams and parabolic subgroups.

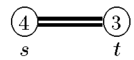
Let $\mathcal{D}$ be one of the diagrams. Let us define an equivalence relation between nodes by $s \sim s$ and, for $s \neq t$,

$$s \sim t \quad \Longleftrightarrow \quad s \text{ and } t \text{ are not in a homogeneous relation with support } \{s,t\}\,.$$

Then we see that the equivalence classes have 1 or 3 elements, and that there is at most one class with 3 elements.

If there is no class with 3 elements, the rank r of the group is the number of nodes of the diagram, while it is this number minus 1 in case there is a class with 3 elements.

Thus 

has rank 2, as well as 

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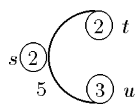
Remark. One must point out that, in the first of the preceding two diagrams, s, t and u must be considered as linked by a line (so t and u *do not* commute).

An **admissible subdiagram** is a full subdiagram of the same type, namely a diagram with 1 or 3 elements per class.

Thus, the diagram 

has five admissible subdiagrams, namely the empty diagram, the three diagrams consisting of one node, and the whole diagram.

1.7. Fact. *Let $\mathcal{D}$ be the diagram of W as given in tables 1 to 4 in Appendix 2 below.*

(1) *If $\mathcal{D}'$ is an admissible subdiagram of $\mathcal{D}$, it gives a presentation of the corresponding subgroup $W(\mathcal{D}')$ of W. This subgroup is a parabolic subgroup.*

(2) *Assume W is neither G_{27}, G_{29}, G_{33} nor G_{34}. If $P_1 \subseteq P_2 \subseteq \cdots P_n$ is a chain of parabolic subgroups of W, there exist $g \in W$ and a chain $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots \mathcal{D}_n$ of admissible subdiagrams of $\mathcal{D}$ such that*

$$(P_1, P_2, \ldots, P_n) = {}^g(W(\mathcal{D}_1), W(\mathcal{D}_2), \ldots, W(\mathcal{D}_n))\,.$$

Remark.

For groups G_{27} and G_{29}, all isomorphism classes of parabolic subgroups are represented by admissible subdiagrams of our diagrams, but not all *conjugacy classes* of parabolics subgroups are represented by admissible subdiagrams, as noticed by Orlik.

For groups G_{33} and G_{34}, not all isomorphism classes of parabolic subgroups are represented by admissible subdiagrams of our diagrams. In these cases, it seems that a second diagram should be introduced, as suggested by [Hu]. Then all parabolic subgroups can be found somewhere inside one of the two diagrams given.

More precisely, for G_{33}, the second diagram is only needed for parabolic subgroups of type D_4, while for G_{34} it is needed for parabolic subgroups of type D_4, D_5 and the second copy of A_5.

2. Braid groups and their diagrams

For X a topological space, we denote by $\mathcal{P}(X)$ its fundamental groupoid, where the composition of (classes of) paths is defined so that, if γ_1 is a path going from x_0 to x_1 and γ_2 is a path going from x_1 to x_2, then the composite map going from x_0 to x_2 is denoted by $\gamma_2 \cdot \gamma_1$.

Given a point $x_0 \in X$, we denote by $\pi_1(X, x_0)$ (or $\pi_1(X)$ if the choice of x_0 is clear) the fundamental group with base point x_0. So we have $\pi_1(X, x_0) = \mathrm{End}_{\mathcal{P}(X)}(x_0)$. If $f\colon X \to Y$ is a continuous map, we denote by $\mathcal{P}(f)$ the corresponding functor from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$. We also denote by $\pi_1(f, x_0)$ (or $\pi_1(f)$) the group homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, f(x_0))$ induced by $\mathcal{P}(f)$.

We choose, once for all, a square root of (-1) in $\mathbb{C}$, which is denoted by i. Moreover, for every $z \in \mathbb{C}^\times$, we identify $\pi_1(\mathbb{C}^\times, z)$ with $\mathbb{Z}$ by sending onto 1 the loop $\lambda_z\colon [0,1] \to \mathbb{C}^\times$ defined by $\lambda_z(t) := z\exp(2i\pi t)$.

A. Generalities about hyperplane complements.

What follows is probably well known to specialists of hyperplane complements and topologists. We include it for the convenience of the reader, and because of the lack of convenient references.

Let $\mathcal{A}$ be a finite set of affine hyperplanes (*i.e.,* affine subspaces of codimension one) in a finite dimensional complex vector space V. We set $\mathcal{M} := V - \bigcup_{H\in\mathcal{A}} H$.

Let $x_0 \in \mathcal{M}$. We shall give now some properties of the fundamental group $\pi_1(\mathcal{M}, x_0)$.

Generators of the monodromy around the hyperplanes.

In Appendix 1, we explain what we mean by the *generator* $\rho_{[\gamma]}$ *of the monodromy around* H, associated to a path γ "from x_0 to an affine hyperplane" $H \in \mathcal{A}$.

For $H \in \mathcal{A}$, let α_H be an affine map $V \to \mathbb{C}$ such that $H = \{x \in V \mid \alpha_H(x) = 0\}$. Its restriction to $\mathcal{M} \to \mathbb{C}^\times$ induces a functor $\mathcal{P}(\alpha_H)\colon \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathbb{C}^\times)$, and in particular a group homomorphism $\pi_1(\alpha_H, x_0)\colon \pi_1(\mathcal{M}, x_0) \to \mathbb{Z}$

2.1. Lemma. *For $H, H' \in \mathcal{A}$ and γ a path from x_0 to H (see Appendix 1), we have*

$$\pi_1(\alpha_{H'})(\rho_{[\gamma]}) = \delta_{H,H'}\,.$$

Proof of 2.1.

Let us set $\mathcal{M}_H := H - \bigcup_{\substack{H'\in\mathcal{A}\\ H'\neq H}} H'$. Let $x_\gamma := \gamma(1)$ and let B be an open ball with center x_γ contained in $\mathcal{M} \cup \mathcal{M}_H$. Let $u \in [0,1[$ such that $\gamma(t) \in B$ for $t \geq u$. We set $x_1 := \gamma(u)$. Then, the restriction of α_H to $B \cap \mathcal{M}$ induces an isomorphism $\pi_1(\alpha_H)\colon \pi_1(B \cap \mathcal{M}, x_1) \to \mathbb{Z}$. Let λ be a loop in $B \cap \mathcal{M}$, with origin x_1, whose image under $\pi_1(\alpha_H)$ is 1. Let γ_u be the "restriction" of γ to $[0,u]$, defined by $\gamma_u(t) := \gamma(ut)$ for all $t \in [0,1]$. Define $\rho_{\gamma,\lambda} := {\gamma_u}^{-1} \cdot \lambda \cdot \gamma_u$. Then the loop $\rho_{\gamma,\lambda}$ induces the generator of the monodromy $\rho_{[\gamma]}$ (see Appendix 1), and

$$\pi_1(\alpha_{H'})(\rho_{\gamma,\lambda}) = \pi_1(\alpha_{H'})(\lambda) = \delta_{H,H'}\,.$$

□

2.2. Proposition.

(1) The fundamental group $\pi_1(\mathcal{M}, x_0)$ is generated by all the generators of the monodromy around the affine hyperplanes $H \in \mathcal{A}$.

(2) Let $\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}}$ denote the largest abelian quotient of $\pi_1(\mathcal{M}, x_0)$. For $H \in \mathcal{A}$, we denote by ρ_H^{ab} the image of $\rho_{H,\gamma}$ in $\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}}$. Then

$$\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}} = \prod_{H \in \mathcal{A}} \langle \rho_H^{\mathrm{ab}} \rangle ,$$

where each $\langle \rho_H^{\mathrm{ab}} \rangle$ is infinite cyclic. Dually, we have

$$\mathrm{Hom}(\pi_1(\mathcal{M}, x_0), \mathbb{Z}) = \prod_{H \in \mathcal{A}} \langle \pi_1(\alpha_H) \rangle .$$

Proof of 2.2.

(1) is a special case of Proposition A1 (see Appendix 1 below).

(2) is immediate and left to the reader. $\square$

Remark. Let us recall that we have natural isomorphisms

$$\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{H}_1(\mathcal{M}, \mathbb{Z}) \quad \text{and} \quad \mathrm{Hom}(\pi_1(\mathcal{M}, x_0), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{M}, \mathbb{Z}) .$$

Moreover, the duality between $\pi_1(\mathcal{M}, x_0)^{\mathrm{ab}}$ and $\mathrm{H}^1(\mathcal{M}, \mathbb{Z})$ may be seen as follows. For γ a loop in $\mathcal{M}$ with origin x_0 and for ω a holomorphic differential 1-form on $\mathcal{M}$, we set $<\gamma, \omega> := \int_\gamma \omega$. It is then clear that, under the isomorphism

$$\mathrm{Hom}(\pi_1(\mathcal{M}, x_0), \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{M}, \mathbb{Z}) ,$$

the element $\pi_1(\alpha_H)$ is sent onto the class of the 1-form $\omega_H = \dfrac{1}{2i\pi} \dfrac{d\alpha_H}{\alpha_H}$ (see 1.2 for more details).

About the center of the fundamental group.

In this part, we assume the hyperplanes in $\mathcal{A}$ to be *linear*.

2.3. Notation. *We denote by $\boldsymbol{\pi}$ the loop $[0,1] \to \mathcal{M}$ defined by*

$$\boldsymbol{\pi} \colon t \mapsto x_0 \exp(2i\pi t) .$$

2.4. Lemma.

(1) *$\boldsymbol{\pi}$ belongs to the center $Z(\pi_1(\mathcal{M}, x_0))$ of the fundamental group $\pi_1(\mathcal{M}, x_0)$.*

(2) *For all $H \in \mathcal{A}$, we have $\pi_1(\alpha_H)(\boldsymbol{\pi}) = 1$.*

Proof of 2.4.

(1) results from a more general lemma, for which we need to introduce more notation.

Let $z = |z|e^{i\theta}$ be a complex number with argument θ, chosen so that $-\pi < \theta \leq \pi$. For $t \in [0,1]$, we set $z^t := |z|^t e^{ti\theta}$. For $x \in \mathcal{M}$, we denote by $\gamma_{z,x}$ the path in $\mathcal{M}$, with initial point x and terminal point zx, defined by

$$\gamma_{z,x} \colon [0,1] \to \mathcal{M} \ , \ t \mapsto z^t x .$$

2.5. Lemma. *Let γ be a path in $\mathcal{M}$, with initial point x and terminal point y. Then the paths $\gamma_{z,y}\cdot\gamma$ and $z\gamma\cdot\gamma_{z,x}$ are homotopy equivalent (where $z\gamma$ denotes the path defined by $t\mapsto z\gamma(t)$).*

The proof of Lemma 2.5 is easy and left to the reader. Note that Lemma 2.5 holds whenever $\mathcal{M}$ is a subset of V which is stable under multiplication by $\mathbb{C}^\times$.

(2) is immediate since $\pi_1(\alpha_H)(\boldsymbol{\pi}) = \dfrac{1}{2\pi i}\int_0^1 \dfrac{d(\alpha_H(\boldsymbol{\pi}(t)))}{\alpha_H(\boldsymbol{\pi}(t))} = \int_0^1 dt\,.$ $\square$

2.6. Proposition. *Let $\overline{\mathcal{M}}$ be the image of $\mathcal{M}$ in $(V-\{0\})/\mathbb{C}^\times$, and let $\overline{x}_0$ denote the image of x_0 in $\overline{\mathcal{M}}$.*

(1) *The map $\pi_1(\mathcal{M},x_0)\to\pi_1(\overline{\mathcal{M}},\overline{x}_0)$ is surjective, and its kernel is $\langle\boldsymbol{\pi}\rangle$.*
(2) *The center of $\pi_1(\mathcal{M},x_0)$ is $\langle\boldsymbol{\pi}\rangle$ if and only if the center of $\pi_1(\overline{\mathcal{M}},\overline{x}_0)$ is trivial.*

Proof of 2.6.
(1) Since the $\boldsymbol{\pi}(t)$ are scalar multiples of x_0, it is clear that $\boldsymbol{\pi}$ belongs to the kernel of the map $\pi_1(\mathcal{M},x_0)\to\pi_1(\overline{\mathcal{M}},\overline{x}_0)$.

The homotopy exact sequence $\cdots\to\pi_1(\mathbb{C}^\times)\to\pi_1(\mathcal{M},x_0)\to\pi_1(\overline{\mathcal{M}},\overline{x}_0)\to 1$ shows that the morphism $\pi_1(\mathcal{M},x_0)\to\pi_1(\overline{\mathcal{M}},\overline{x}_0)$ is surjective, and that its kernel is cyclic. Since $\boldsymbol{\pi}$ belongs to this kernel, it suffices to prove that $\boldsymbol{\pi}$ is a primitive element of $\pi_1(\mathcal{M},x_0)$, *i.e.*, that $\boldsymbol{\pi}$ has no proper root in $\pi_1(\mathcal{M},x_0)$. But this results from Lemma 2.4, (2).

(2) Let us notice that, by Lemma 2.4, (2), the group $\langle\boldsymbol{\pi}\rangle$ maps injectively into the largest abelian quotient of $\pi_1(\mathcal{M},x_0)$. So it suffices to prove the following elementary lemma.

2.7. Lemma. *Let G be a group, and let H a normal subgroup of G which maps injectively into the largest abelian quotient $G/[G,G]$ of G. Then the natural morphism $G\to G/H$ sends the center of G onto the center of G/H.*

Proof of 2.7. Indeed, let z be an element of G which becomes central in G/H. Then $[z,G]\subset H$. But by hypothesis we have $H\cap[G,G]=1$. Thus we have $[z,G]=1$. $\square$

$\square$

Generating with one loop per hyperplane.

With a little more work, Proposition 2.2 can be made more precise ; one (well-chosen) generator of the monodromy around each affine hyperplane suffices to generate the fundamental group :

2.8. Proposition. *There is a set $R=\{\rho_H\}_{H\in\mathcal{A}}$ of generators of $\pi_1(\mathcal{M},x_0)$, where ρ_H is a generator of the monodromy around H.*

Proof of 2.8.
We may assume that $\mathcal{A}$ is not empty. We prove the proposition by induction on the dimension of V.

- *The linear case.*

Let us first consider the case where the intersection of the affine hyperplanes of $\mathcal{A}$ is non trivial. Up to translation, we can assume that 0 is contained in this intersection, *i.e.*, the hyperplanes of $\mathcal{A}$ are linear.

Let H_2 be a hyperplane of $\mathcal{A}$ and H_1 be the affine hyperplane of V parallel to H_2 and containing x_0.

We consider the conic projection on H_1 with center 0 :

$$f : (V - H_2) \longrightarrow H_1 \ , \ x \longmapsto \mathbb{C}x \cap H_1.$$

Both f and its restriction $\mathcal{M} \to \mathcal{M}_1 = H_1 \cap \mathcal{M}$ are locally trivial fibrations (see for example [Spa], chap. 2) with fiber $F \simeq \mathbb{C}^\times x$.

The associated exact sequence of fundamental groups is

$$\pi_1(F, x_0) \to \pi_1(\mathcal{M}, x_0) \xrightarrow{\pi_1(f)} \pi_1(\mathcal{M}_1, x_0) \to 1\,.$$

Let $\mathcal{A}' = \mathcal{A} - \{H_2\}$. By induction, we can assume the proposition holds for the affine hyperplane arrangement $\mathcal{A}_1 = \{H \cap H_1\}_{H\in\mathcal{A}'}$ in H_1 : there is a set $\{\rho_L\}_{L\in\mathcal{A}_1}$ of generators of $\pi_1(\mathcal{M}_1, f(x_0))$ where ρ_L is a generator of the monodromy around L.

Let i be the inclusion $\mathcal{M}_1 \to \mathcal{M}$. Then, $\rho_H = \pi_1(i)(\rho_{H\cap H_1})$ is a generator of the monodromy around $H \in \mathcal{A}'$. Note that fi is the identity on $\mathcal{M}_1$, hence $\pi_1(f)\pi_1(i) = 1$. In particular, the exact sequence shows that $\pi_1(\mathcal{M}, x_0)$ is generated by the set $\{\rho_H\}_{H\in\mathcal{A}'}$, together with $\boldsymbol{\pi}$, the image of the positive generator of $\pi_1(F, x_0)$, which is central in $\pi_1(\mathcal{M}, x_0)$.

Let ρ_{H_2} be a generator of the monodromy around H_2. Then, there exists α in the subgroup generated by $\{\rho_H\}_{H\in\mathcal{A}'}$ such that $\pi_1(f)(\rho_{H_2}\alpha) = 1$, that is, $\rho_{H_2}\alpha = \boldsymbol{\pi}^r$ for some integer r. Since $\pi_1(\alpha_{H_2})(\rho_{H_2}) = 1$, $\pi_1(\alpha_{H_2})(\rho_H) = 0$ for $H \in \mathcal{A}'$ and $\pi_1(\alpha_{H_2})(\boldsymbol{\pi}) = 1$ by Lemmas 2.1 and 2.4, we obtain $r = 1$. Hence, $\boldsymbol{\pi}$ is in the subgroup generated by $\{\rho_H\}_{H\in\mathcal{A}}$ and this proves that $\{\rho_H\}_{H\in\mathcal{A}}$ generates $\pi_1(\mathcal{M}, x_0)$.

- *The affine case.*

Let $\mathcal{A}'$ be a finite set of affine hyperplanes of V disjoint from $\mathcal{A}$ and let $\mathcal{M}' = V - \bigcup_{H\in\mathcal{A}\cup\mathcal{A}'} H$. Assume $x_0 \in \mathcal{M}'$. Since one gets $\mathcal{M}'$ by removing a sub–variety of (real) codimension 2 from $\mathcal{M}$, the injection $\mathcal{M}' \hookrightarrow \mathcal{M}$ induces a surjection $\pi_1(\mathcal{M}', x_0) \twoheadrightarrow \pi_1(\mathcal{M}, x_0)$ (see for example [Go], chap. x, th. 2.3.). Under this morphism, a generator of the monodromy around a hyperplane $H' \in \mathcal{A}'$ becomes trivial. Hence, if the proposition holds for $\mathcal{M}'$, then it holds for $\mathcal{M}$. Note also that we can change the base point x_0 in order to prove the result.

We choose an affine hyperplane H_1 of V outside $\mathcal{A}$, a new origin $0 \in V - H_1$ for V and a new base point $x_0 \in H_1 \cap \mathcal{M}$ such that

- there is an open ball Ω with center 0, containing x_0 and which doesn't intersect any of the non-linear hyperplanes of $\mathcal{M}$,
- the line $\mathbb{C}x_0$ never intersects two distinct affine hyperplanes of $\mathcal{A}$ at the same point, and
- no translate of the line $\mathbb{C}x_0$ lies in an affine hyperplane of $\mathcal{A}$.

Then, adding to $\mathcal{A}$

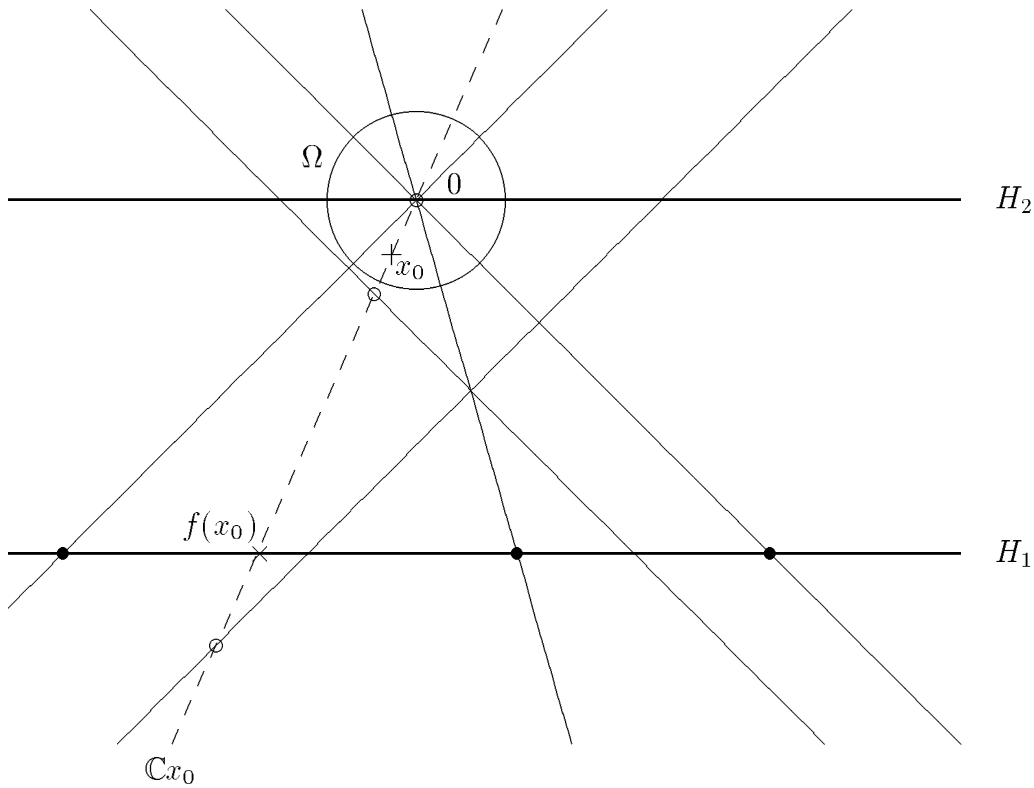
- the linear hyperplane H_2 parallel to H_1,
- the linear hyperplanes parallel to the affine hyperplanes of $\mathcal{A}$,
- given two distinct non-linear hyperplanes H, H' of $\mathcal{A}$, the linear hyperplane containing $H \cap H'$,

we may and will assume that $\mathcal{A}$ satisfies the following assumption :

Let $\mathcal{A}'$ be the set of linear hyperplanes of $\mathcal{A}$ distinct from H_2, $\mathcal{A}''$ the set of non-linear hyperplanes of $\mathcal{A}$, $\mathcal{A}_1 = \{H_1 \cap H\}_{H \in \mathcal{A}'}$ and $\mathcal{M}_1 = H_1 - \bigcup_{L \in \mathcal{A}_1} L$. Then, the map

$$\begin{aligned} f : \mathcal{M} &\longrightarrow \mathcal{M}_1 \\ x &\longmapsto \mathbb{C}x \cap H_1 \end{aligned}$$

is a locally trivial fibration.



Note that the restriction of f to $\Omega \cap \mathcal{M} \to \mathcal{M}_1$ is also a locally trivial fibration.

The associated exact sequences of fundamental groups give rise to the commutative diagram :

$$\begin{tikzcd} \pi_1(\Omega \cap (\mathbb{C}x_0 - \{0\}), x_0) \arrow[r] \arrow[d] & \pi_1(\Omega \cap \mathcal{M}, x_0) \arrow[r] \arrow[d] & \pi_1(\mathcal{M}_1, f(x_0)) \arrow[r] \arrow[d, equal] & 1 \\ \pi_1(F, x_0) \arrow[r] & \pi_1(\mathcal{M}, x_0) \arrow[r] & \pi_1(\mathcal{M}_1, f(x_0)) \arrow[r] & 1 \end{tikzcd}$$

where $F = f^{-1}(f(x_0)) = \mathbb{C}x_0 - (\{0\} \cup \{\mathbb{C}x_0 \cap H\}_{H \in \mathcal{A}''})$.

The study of the linear case above shows that there is a set $\{\rho_h\}_{H \in \mathcal{A}' \cup H_2}$ of generators of the monodromy around the linear hyperplanes in $\Omega \cap \mathcal{M}$ which generates $\pi_1(\Omega \cap \mathcal{M}, x_0)$.

There are generators of the monodromy ρ_H around the points $\mathbb{C}x_0 \cap H$ $(H \in \mathcal{A}'')$ in F, such that, together with the image of $\pi_1(\Omega \cap (\mathbb{C}x_0 - \{0\}), x_0)$, they generate $\pi_1(F, x_0)$.

Now, the set $\{\rho_H\}_{H \in \mathcal{A}}$ generates $\pi_1(\mathcal{M}, x_0)$. $\square$

B. Generalities about the braid groups.

More notation.

We go back to notation introduced in §1. In particular, $\mathcal{A}$ is now the set of reflecting hyperplanes of a finite subgroup W of $\mathrm{GL}(V)$ generated by pseudo–reflections. We denote by $p\colon \mathcal{M} \to \mathcal{M}/W$ the canonical surjection.

Let $x_0 \in \mathcal{M}$. We introduce the following notation for the fundamental groups:

$$P := \pi_1(\mathcal{M}, x_0) \quad \text{and} \quad B := \pi_1(\mathcal{M}/W, p(x_0))\,,$$

and we call B and P respectively the *braid group* (at x_0) and the *pure braid group* (at x_0) associated to W. We shall often write $\pi_1(\mathcal{M}/W, x_0)$ for $\pi_1(\mathcal{M}/W, p(x_0))$.

The covering $\mathcal{M} \to \mathcal{M}/W$ is Galois by Steinberg's theorem (see Theorem 1.1 above), hence the projection p induces a surjective map $B \twoheadrightarrow W$, $\sigma \mapsto \overline{\sigma}$, as follows :

Let $\tilde{\sigma}\colon [0,1] \to \mathcal{M}$ be a path in $\mathcal{M}$, such that $\tilde{\sigma}(0) = x_0$, which lifts σ. Then $\overline{\sigma}$ is defined by the equality $\overline{\sigma}(x_0) = \tilde{\sigma}(1)$.

The map $\sigma \mapsto \overline{\sigma}$ is an *anti–morphism*. Indeed, if σ' is another loop in $\mathcal{M}/W$ with origin $\overline{x}_0$, and if σ' is lifted onto a path $\tilde{\sigma}'$ with origin x_0 in $\mathcal{M}$, we may lift the loop $(\sigma'\sigma)$ onto the path $\overline{\sigma}(\tilde{\sigma}') \cdot \tilde{\sigma}$, whose image in W is clearly $\overline{\sigma\sigma'}$ (here we set $\overline{\sigma}(\tilde{\sigma}')(t) := \overline{\sigma}(\tilde{\sigma}'(t))$).

Denoting by W^{op} the group opposite to W, we have the following short exact sequence :

$$1 \to P \to B \to W^{\mathrm{op}} \to 1\,, \tag{2.9}$$

where the map $B \to W^{\mathrm{op}}$ is defined by $\sigma \mapsto \overline{\sigma}$.

The spaces $\mathcal{M}$ and $\mathcal{M}/W$ are conjectured to be $K(\pi,1)$-spaces.

The following result is due to Fox and Neuwirth [FoNe] for the type A_n, to Brieskorn [Br2] for Coxeter groups of type different from H_3, H_4, E_6, E_7, E_8, to Deligne [De1] for general Coxeter groups. The case of the infinite series of complex reflection groups $G(de,e,r)$ has been solved by Nakamura [Na]. For the non-real Shephard groups (non-real groups with Coxeter braid diagrams), this has been proven by Orlik and Solomon [OrSo3]. Note that the rank 2 case is trivial.

2.10. Theorem. *Assume W has no irreducible component of type G_{24}, G_{27}, G_{29}, G_{31}, G_{33} or G_{34}. Then, $\mathcal{M}$ and $\mathcal{M}/W$ are $K(\pi,1)$-spaces.*

Generators of the monodromy around the hyperplanes.

For $H \in \mathcal{A}$, we set $\zeta_H := \exp(2i\pi/e_H)$, and we denote by s_H the pseudo-reflection in W with reflecting hyperplane H and determinant ζ_H. We set

$$L_H := \mathrm{im}(s_H - \mathrm{Id}_V)\,.$$

For $x \in V$, we set $x = \mathrm{pr}_H(x) + \mathrm{pr}_H^{\perp}(x)$ with $\mathrm{pr}_H(x) \in H$ and $\mathrm{pr}_H^{\perp}(x) \in L_H$.

Thus, we have $s_H(x) = \zeta_H \mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x)$.

If $t \in \mathbb{R}$, we set $\zeta_H^t := \exp(2i\pi t/e_H)$, and we denote by s_H^t the element of $\mathrm{GL}(V)$ (a pseudo–reflection if $t \neq 0$) defined by :

$$s_H^t(x) = \zeta_H^t \mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x)\,. \tag{2.11}$$

For $x \in V$, we denote by $\sigma_{H,x}$ the path in V from x to $s_H(x)$, defined by :

$$\sigma_{H,x}\colon [0,1] \to V \ , \ t \mapsto s_H^t(x) \, .$$

For any path γ in $\mathcal{M}$, with initial point x_0 and terminal point x_H, the path defined by $s_H(\gamma^{-1})\colon t \mapsto s_H(\gamma^{-1}(t))$ is a path in $\mathcal{M}$ going from $s_H(x_H)$ to $s_H(x_0)$.

Whenever γ is a path in $\mathcal{M}$, with initial point x_0 and terminal point x_H, we define the path $\sigma_{H,\gamma}$ from x_0 to $s_H(x_0)$ as follows :

(2.12)
$$\sigma_{H,\gamma} := s_H(\gamma^{-1}) \cdot \sigma_{H,x_H} \cdot \gamma \, .$$

It is not difficult to see that, provided x_H is chosen “close to H”, the path $\sigma_{H,\gamma}$ is in $\mathcal{M}$, and its homotopy class does not depend on the choice of x_H, and the element it induces in the braid group B is actually a generator of the monodromy around the image of H in $\mathcal{M}/W$ (see Appendix 1 below).

The following properties are immediate.

2.13. Lemma.

(1) *The image of* $\mathbf{s}_{H,\gamma}$ *in* W *is* s_H.

(2) *Whenever* γ' *is a path in* $\mathcal{M}$, *with initial point* x_0 *and terminal point* x_H, *if* τ *denotes the loop in* $\mathcal{M}$ *defined by* $\tau := {\gamma'}^{-1}\gamma$, *one has*

$$\sigma_{H,\gamma'} = \tau \cdot \sigma_{H,\gamma} \cdot \tau^{-1}$$

and in particular $\mathbf{s}_{H,\gamma}$ *and* $\mathbf{s}_{H,\gamma'}$ *are conjugate in* P.

(3) *The path* $\prod_{j=e_H-1}^{j=0} \sigma_{H,s_H^j(\gamma)}$, *a loop in* $\mathcal{M}$, *induces the element* $\mathbf{s}_{H,\gamma}^{e_H}$ *in the braid group* B, *and belongs to the pure braid group* P. *It is homotopy equivalent, as a loop in* $\mathcal{M}$, *to the generator* $\rho_{[\gamma]}$ *of the monodromy around* H *in* P *(see Appendix 1).*

2.14. Definition.

• *A* distinguished pseudo–reflection *in* W *is a pseudo–reflection* s *with the following property : if* H *denotes its reflecting hyperplane, and if* e_H *is the order of the minimal parabolic subgroup* W_H, *then* s *is the element of* W_H *with determinant* $e^{2i\pi/e_H}$.

• *Let* s *be a distinguished pseudo–reflection in* W, *with reflecting hyperplane* H. *An* s–generator of the monodromy *is a generator of the monodromy* $\mathbf{s}$ *around the image of* H *in* $\mathcal{M}/W$ *such that* $\overline{\mathbf{s}} = s$.

The discriminants.

Let $\mathcal{C}$ be an orbit of W on $\mathcal{A}$. Recall that we denote by $e_\mathcal{C}$ the (common) order of the pointwise stabilizer W_H for $H \in \mathcal{C}$. We call *discriminant at* $\mathcal{C}$ and we denote by $\delta_\mathcal{C}$ the element of the symmetric algebra of V^* defined (up to a non zero scalar multiplication) by

$$\delta_\mathcal{C} := (\prod_{H \in \mathcal{C}} \alpha_H)^{e_\mathcal{C}} \, .$$

Since (see for example [Co], 1.8) $\delta_\mathcal{C}$ is W-invariant, it induces a continuous function $\delta_\mathcal{C}\colon \mathcal{M}/W \to \mathbb{C}^\times$, hence induces a functor $\mathcal{P}(\delta_\mathcal{C})\colon \mathcal{P}(\mathcal{M}/W) \to \mathcal{P}(\mathbb{C}^\times)$, and in particular it induces a group homomorphism $\pi_1(\delta_\mathcal{C})\colon B \to \mathbb{Z}$.

2.15. Proposition. *For any $H \in \mathcal{A}$, we have*

$$\pi_1(\delta_{\mathcal{C}})(\mathbf{s}_{H,\gamma}) = \begin{cases} 1 & \textit{if } H \in \mathcal{C}\,, \\ 0 & \textit{if } H \notin \mathcal{C}\,. \end{cases}$$

Proof of 2.15.

Let us set $\mathcal{C}^{\#} := \mathcal{C} - \{H\}$ (so $\mathcal{C}^{\#} = \mathcal{C}$ if $H \notin \mathcal{C}$), and $\delta_{\mathcal{C}^{\#}} := (\prod_{H' \in \mathcal{C}^{\#}} \alpha_{H'})^{e_{\mathcal{C}}}$.

Recall that W_H denotes the (parabolic) subgroup of W generated by s_H. Then the maps

$$\delta_{\mathcal{C}^{\#}}\,,\ \alpha_H^{e_H} : \mathcal{M} \to \mathbb{C}^\times$$

are both W_H-invariant, and so define maps

$$\delta_{\mathcal{C}^{\#}}\,,\ \alpha_H^{e_H} : \mathcal{M}/W_H \to \mathbb{C}^\times\,.$$

The following diagram summarizes where the maps are defined :

$$\begin{tikzcd}
 & & \mathcal{M}/W \arrow[ddr, "\delta_{\mathcal{C}}"] & \\
 & \mathcal{M}/W_H \arrow[ur] \arrow[drr, "(\alpha_H)^{e_H}\,,\,\delta_{\mathcal{C}^{\#}}"] & & \\
\mathcal{M} \arrow[ur] \arrow[rrr, "\alpha_{H'}"] & & & \mathbb{C}^\times
\end{tikzcd}$$

The computation of $\pi_1(\delta_{\mathcal{C}}(\mathbf{s}_{H,\gamma}))$ may be performed at the level $\mathcal{M}/W_H$, and so it suffices to check

(1) $\pi_1(\delta_{\mathcal{C}^{\#}})(\mathbf{s}_{H,\gamma}) = 0$,
(2) $\pi_1(\alpha_H^{e_H})(\mathbf{s}_{H,\gamma}) = 1$.

Let us check (1) and (2).

(1) It suffices to check that $\pi_1(\delta_{\mathcal{C}^{\#}})(\mathbf{s}_{H,\gamma}^{e_H}) = 0$, and this follows from Lemmas 2.1 and 2.13, (3).

(2) We have

$$\pi_1(\alpha_H^{e_H})(\mathbf{s}_{H,\gamma}) = \frac{1}{2i\pi}\int_0^1 \frac{d(\alpha_H(s_H^t(x_H))^{e_H})}{\alpha_H(s_H^t(x_H))^{e_H}}\,.$$

Since

$$\begin{aligned} \alpha_H(s_H^t(x_H))^{e_H} &= \alpha_H(\mathrm{pr}_H(x_H) + \zeta_H^t \mathrm{pr}_H^{\perp}(x_H))^{e_H} \\ &= \zeta_H^{te_H}\alpha_H(\mathrm{pr}_H^{\perp}(x_H))^{e_H} \\ &= \exp(2i\pi t)\alpha_H(\mathrm{pr}_H^{\perp}(x_H))^{e_H}\,, \end{aligned}$$

we see that $\pi_1(\alpha_H^{e_H})(\mathbf{s}_{H,\gamma}) = \int_0^1 dt = 1$. $\square$

Generators and abelianization of B.

2.16. Theorem.

(1) The group B is generated by the generators $\{\mathbf{s}_{H,\gamma}\}$ (for all hyperplanes $H \in \mathcal{A}$ and all paths γ from x_0 to H in $\mathcal{M}$) of the monodromy (in B) around the elements of $\mathcal{A}$.

(2) We denote by B^{ab} the largest abelian quotient of B. For $\mathcal{C} \in \mathcal{A}/W$, we denote by $\mathbf{s}^{\mathrm{ab}}_{\mathcal{C}}$ the image of $\mathbf{s}_{H,\gamma}$ in B^{ab} for $H \in \mathcal{C}$. Then

$$B^{\mathrm{ab}} = \prod_{\mathcal{C}\in\mathcal{A}/W} \langle \mathbf{s}^{\mathrm{ab}}_{\mathcal{C}} \rangle \,,$$

where each $\langle \mathbf{s}^{\mathrm{ab}}_{\mathcal{C}} \rangle$ is infinite cyclic. Dually, we have

$$\mathrm{Hom}(B,\mathbb{Z}) = \prod_{\mathcal{C}\in\mathcal{A}/W} \langle \pi_1(\delta_{\mathcal{C}}) \rangle \,.$$

Remark. We have natural isomorphisms

$$B^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{H}_1(\mathcal{M}/W,\mathbb{Z}) \quad \text{and} \quad \mathrm{Hom}(B,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{M}/W,\mathbb{Z})\,,$$

and, under the second isomorphism, we have

$$\pi_1(\delta_{\mathcal{C}}) \mapsto e_{\mathcal{C}} \sum_{H\in\mathcal{C}} \frac{1}{2i\pi}\frac{d\alpha_H}{\alpha_H} = \frac{1}{2i\pi} d\mathrm{Log}(\delta_{\mathcal{C}})\,.$$

Proof of 2.16. The second assertion is immediate by the first one and by Proposition 2.15. Let us sketch a proof of (1).

Since W is generated by the set $\{s_H\}_{(H\in\mathcal{A})}$ and since we have the exact sequence (2.9), it is enough to prove that the pure braid group P is generated by all the conjugates in P of the elements $\mathbf{s}^{e_H}_{H,\gamma}$. This is a consequence of Proposition 2.2, (1). □

Let us denote by $\mathrm{Gen}(B)$ the set of all generators of the monodromy in B (see Definition 2.14 above). For $\mathbf{s} \in \mathrm{Gen}(B)$, we denote by $e_{\mathbf{s}}$ the order of $\overline{\mathbf{s}}$.

In other words, if $\mathbf{s}$ is a generator of the monodromy around the reflecting hyperplane $H \in \mathcal{A}$, we set now (using the notation of Definition 2.14) : $e_{\mathbf{s}} := e_H$.

The following property is a consequence of general results recalled in Appendix 1 below.

2.17. Proposition.

(1) *The pure braid group P is generated by $\{\mathbf{s}^{e_{\mathbf{s}}}\}_{\mathbf{s}\in\mathrm{Gen}(B)}$.*

(2) *We have*

$$W \simeq B/\langle \mathbf{s}^{e_{\mathbf{s}}} \rangle_{\mathbf{s}\in\mathrm{Gen}(B)}\,.$$

Proof of 2.17. The two assertions are obviously equivalent. The first one results from Propositions A2 and A3, (2) (see Appendix 1 below). □

Length.

Let

$$\delta := \prod_{\mathcal{C} \in \mathcal{A}/W} \delta_{\mathcal{C}}$$

be the discriminant, and let $\pi_1(\delta) \colon B \to \mathbb{Z}$ be the corresponding group morphism.

Let $b \in B$. By Theorem 2.16 above, there exists an integer k and for $1 \leq j \leq k$, $H_j \in \mathcal{A}$, a path γ_j from x_0 to H_j and an integer n_j such that

$$b = \mathbf{s}_{H_1,\gamma_1}^{n_1} \mathbf{s}_{H_2,\gamma_2}^{n_2} \cdots \mathbf{s}_{H_k,\gamma_k}^{n_k} .$$

The following proposition results from Proposition 2.15 above.

2.18. Proposition. *We have*

$$\pi_1(\delta)(b) = \sum_{j=1}^{j=k} n_j .$$

We call *the length* of b and we denote by $\ell(b)$ the integer $\pi_1(\delta)(b)$.

If $\{\mathbf{s}\}$ is a set of generators of the monodromy around hyperplanes which generates B, let us denote by B^+ the sub–monoid of B generated by $\{\mathbf{s}\}$. Then for $b \in B^+$, its length $\ell(b)$ coincide with its length on the distinguished set of generators $\{\mathbf{s}\}$ of the monoid B^+.

About the center of B.

2.19. Notation. *We denote by $\boldsymbol{\beta}$ the path $[0,1] \to \mathcal{M}$ defined by*

$$\boldsymbol{\beta} \colon t \mapsto x_0 \exp(2i\pi t/|Z(W)|) .$$

The following result is a consequence of Corollary 2.25. Notice that it generalizes a result of Deligne [De1], (4.21) (see also [BrSa]), from which it follows that if W is a Coxeter group, then $\ell(\boldsymbol{\pi}) = 2N$. It was noticed "experimentally" in [BrMi], (4.8).

2.20. Corollary. *We have $\ell(\boldsymbol{\beta}) = (N + N^*)/|Z(W)|$ and $\ell(\boldsymbol{\pi}) = N + N^*$.*

From now on, we assume that W *acts irreducibly on* V. Note that, since W is irreducible on V, it results from Schur's lemma that

$$Z(W) = \{\exp(2i\pi k/|Z(W)|) \mid (k \in \mathbb{Z})\} ,$$

and so in particular $\boldsymbol{\beta}$ defines an element of B, which we will still denote by $\boldsymbol{\beta}$.

2.21. Lemma.

(1) *The image $\overline{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ in W is the scalar multiplication by $\exp(2i\pi/|Z(W)|)$. It is a generator of the center $Z(W)$ of W.*

(2) *We have $\boldsymbol{\beta} \in Z(B)$, $\boldsymbol{\pi} \in Z(P)$, and $\boldsymbol{\pi} = \boldsymbol{\beta}^{|Z(W)|}$.*

Proof of 2.21. We only have to check that $\boldsymbol{\beta} \in Z(B)$. This results from Lemma 2.5. $\square$

2.22. Proposition. *Let $\overline{\mathcal{M}}$ be the image of $\mathcal{M}$ in $(V-\{0\})/\mathbb{C}^\times$. Then, we have a commutative diagram, where all short sequences are exact :*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \langle \boldsymbol{\pi} \rangle \arrow[r] \arrow[d] & \langle \boldsymbol{\beta} \rangle \arrow[r] \arrow[d] & Z(W) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\mathcal{M}, x_0) \arrow[r] \arrow[d] & \pi_1(\mathcal{M}/W, x_0) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\overline{\mathcal{M}}, \overline{x}_0) \arrow[r] \arrow[d] & \pi_1(\overline{\mathcal{M}}/W, \overline{x}_0) \arrow[r] \arrow[d] & W/Z(W) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Proof of 2.22.

It is clear that $\boldsymbol{\beta}$ belongs to the kernel of the map $\pi_1(\mathcal{M}/W, x_0) \to \pi_1(\overline{\mathcal{M}}/W, \overline{x}_0)$.

By Lemma 2.21, we know that the map $\langle \boldsymbol{\beta} \rangle \to Z(W)$ is onto. The three horizontal sequences are exact, as well as the last vertical one. So it suffices to check that the first vertical sequence is exact, *i.e.*, to show that $\langle \boldsymbol{\pi} \rangle$ is equal to the kernel of the map $\pi_1(\mathcal{M}, x_0) \to \pi_1(\overline{\mathcal{M}}, \overline{x}_0)$. This is Proposition 2.6, (1). $\square$

The following statement is known for Coxeter groups (see [De1] or [BrSa]). The result holds as well for G_{25}, G_{26}, G_{32}, since the corresponding braid groups are the same as braid groups of Coxeter groups. We shall prove it for all the infinite series in §3 below (see Propositions 3.4, 3.10, 3.33), and we give below a proof for the particular case of groups in dimension 2.

We conjecture it is still true in the case of G_{31}, as well as for G_{24}, G_{27}, G_{29}, G_{33}, G_{34}.

2.23. Theorem. *Assume W different from G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}.*

The center $Z(B)$ of B is infinite cyclic and generated by $\boldsymbol{\beta}$, the center $Z(P)$ of P is infinite cyclic and generated by $\boldsymbol{\pi}$, and the short exact sequence (2.9) induces a short exact sequence

$$1 \to Z(P) \to Z(B) \to Z(W) \to 1\,.$$

Note that, by Propositions 2.22 and 2.6, (2), Theorem 2.23 is equivalent to the following statement :

2.24. *The center of the "projective braid group" $\pi_1(\overline{\mathcal{M}}, \overline{x}_0)$ is trivial.*

Proof of 2.23 in dimension 2. Assume that V has dimension 2. The space $\overline{\mathcal{M}}$ is homeomorphic to $\mathbb{P}^1(\mathbb{C})$ minus N points, so $\pi_1(\overline{\mathcal{M}}, \overline{x}_0)$ is isomorphic to a free group F_{N-1} on $N-1$ generators. Since W is irreducible, we have $N > 2$ and so 2.23 is proved. $\square$

2.25. Corollary. *Let $\boldsymbol{\beta}^{\mathrm{ab}}$ be the image in B^{ab} of the central element $\boldsymbol{\beta}$ of B. Then we have*

$$\boldsymbol{\beta}^{\mathrm{ab}} = \prod_{\mathcal{C}\in\mathcal{A}/W} (\mathbf{s}_{\mathcal{C}}^{\mathrm{ab}})^{e_{\mathcal{C}} N_{\mathcal{C}}/|Z(W)|}\,.$$

Proof of 2.25. It suffices to prove that, for all $\mathcal{C} \in \mathcal{A}/W$, we have $\pi_1(\delta_{\mathcal{C}})(\boldsymbol{\beta}^{\mathrm{ab}}) = e_{\mathcal{C}} N_{\mathcal{C}}/|Z(W)|$. This is immediate :

$$\begin{aligned}\pi_1(\delta_{\mathcal{C}})(\boldsymbol{\beta}^{\mathrm{ab}}) &= \sum_{H\in\mathcal{C}} \frac{e_{\mathcal{C}}}{2i\pi}\int_0^1 \frac{d\alpha_H(x_0\exp(2i\pi t/|Z(W)|))}{\alpha_H(x_0\exp(2i\pi t/|Z(W)|))} \\ &= \frac{e_{\mathcal{C}}}{2i\pi}\sum_{H\in\mathcal{C}} \frac{2i\pi}{|Z(W)|}\int_0^1 dt = e_{\mathcal{C}} N_{\mathcal{C}}/|Z(W)|\,.\end{aligned}$$

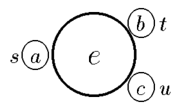
□

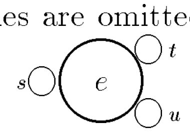
C. The braid diagrams.

Let us first introduce some notation.

Let (V, W) be a finite irreducible complex reflection group. As previously, we set $\mathcal{M} := V - \bigcup_{H\in\mathcal{A}} H$, $B := \pi_1(\mathcal{M}/W, x_0)$, and we denote by $\sigma \mapsto \overline{\sigma}$ the antimorphism $B \twoheadrightarrow W$ defined by the Galois covering $\mathcal{M} \twoheadrightarrow \mathcal{M}/W$.

Let $\mathcal{D}$ be one of the diagrams given in tables 1, 2, 3 (Appendix 2 below) symbolizing a set of relations as described in Appendix 2.

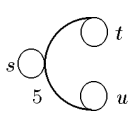
• We denote by $\mathcal{D}_{\mathrm{br}}$ and we call *braid diagram associated to* $\mathcal{D}$ the set of nodes of $\mathcal{D}$ subject to all relations of $\mathcal{D}$ but the orders of the nodes, and we represent the braid diagram $\mathcal{D}_{\mathrm{br}}$ by the same picture as $\mathcal{D}$ where numbers insides the nodes are omitted. Thus, if $\mathcal{D}$ is the diagram 

, then $\mathcal{D}_{\mathrm{br}}$ is the diagram 

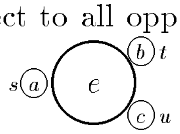
and represents the relations

$$\underbrace{stustu\cdots}_{e\text{ factors}} = \underbrace{tustus\cdots}_{e\text{ factors}} = \underbrace{ustust\cdots}_{e\text{ factors}}\,.$$

Note that this braid diagram for $e = 3$ is the braid diagram associated to $G(2d, 2, 2)$ $(d \geq 2)$, as well as G_7, G_{11}, G_{19}. Also, for $e = 4$, this is the braid diagram associated to G_{12} and for $e = 5$, the braid diagram associated to G_{22}. Similarly, the braid diagram

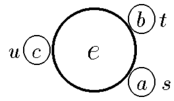


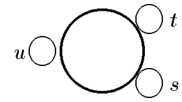
is associated to the diagrams of both G_{15} and $G(4d, 4, 2)$.

• We denote by $\mathcal{D}^{\mathrm{op}}$ and we call *opposite diagram associated to* $\mathcal{D}$ the set of nodes of $\mathcal{D}$ subject to all opposite relations (words in reverse order) of $\mathcal{D}$. Thus, if $\mathcal{D}$ is the diagram 

, then $\mathcal{D}^{\mathrm{op}}$ represents the relations

$$s^a = t^b = u^c = 1 \text{ and } \underbrace{utsuts\cdots}_{e\text{ factors}} = \underbrace{sutsut\cdots}_{e\text{ factors}} = \underbrace{tsutsu\cdots}_{e\text{ factors}}\,.$$

Note that $\mathcal{D}^{\mathrm{op}}$ is the diagram 

. Finally, we denote by $\mathcal{D}^{\mathrm{op}}_{\mathrm{br}}$ the braid diagram associated with $\mathcal{D}^{\mathrm{op}}$. Thus, in the above case, $\mathcal{D}^{\mathrm{op}}_{\mathrm{br}}$ is the diagram 

.
Note that if $\mathcal{D}_{\mathrm{br}}$ is a Coxeter type diagram, then it is equal to $\mathcal{D}^{\mathrm{op}}_{\mathrm{br}}$.

The following statement is well known for Coxeter groups (see for example [Br1] or [De1]). It has been noticed by Orlik and Solomon (see [OrSo3], 3.7) for the case of non real Shephard groups (*i.e.*, non real complex reflection groups whose braid diagram – see above – is a Coxeter diagram). We shall prove it below for all the infinite series. We also checked it case by case for all the exceptional groups but G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34} – for the case of groups of rank 2, we made use of [Ba].

We conjecture it still holds for G_{31}. The question whether it is possible to find right diagrams for G_{24}, G_{27}, G_{29}, G_{33}, G_{34} is still open (see remark below).

2.26. Theorem. *Let W be a finite irreducible complex reflection group, different from G_{24}, G_{27}, G_{29}, G_{33}, G_{34} – and also different from G_{31} for which the following assertions are still conjectural.*

Let $\mathcal{N}(\mathcal{D})$ be the set of nodes of the diagram $\mathcal{D}$ for W given in tables 1–3 below, identified with a set of pseudo-reflections in W. For each $s \in \mathcal{N}(\mathcal{D})$, there exists an s–generator of the monodromy $\mathbf{s}$ in B (cf. Definition 2.14) such that the set $\{\mathbf{s}\}_{s\in\mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}^{\mathrm{op}}_{\mathrm{br}}$, is a presentation of B.

2.27. Questions. *Let W be a finite irreducible complex reflection group, different from G_{24}, G_{27}, G_{29}, G_{33}, G_{34}. We denote by B^+ the monoid defined by generators and relations as follows : a set of generators is $\{\mathbf{s}\}_{s\in\mathcal{N}(\mathcal{D})}$, subject to the braid relations represented by $\mathcal{D}^{\mathrm{op}}_{\mathrm{br}}$.*

(1) *Is the natural morphism $B^+ \to B$ injective ?*
(2) *Do we have*
$$B = \{\boldsymbol{\pi}^n b \mid (n \in \mathbb{Z})(b \in B^+)\}\,?$$

Remark. This is true for Coxeter groups (see [De1]). But the answers to the above questions are negative for diagrams given above for G_{24}, G_{27}, G_{29}, G_{33}, G_{34}.

3. Proofs of the main theorems for the braid groups $B(de, e, r)$

In this paragraph, we shall prove Theorems 2.26 and 2.23 for the infinite series of irreducible complex reflection groups $G(de, e, r)$.

A. Notation and prerequisites.

Notation.

Let d, e and r be positive integers. We denote by $(z_1, z_2, \dots, z_r)$ a general element of $\mathbb{C}^r$. Let $G(de, e, r)$ be the subgroup of $GL_r(\mathbb{C})$ whose elements are :

$$[\underline{a}, \sigma] : z_j \mapsto a_j z_{\sigma(j)}$$

for $\sigma \in \mathfrak{S}_r$ and $\underline{a} = (a_1, \dots, a_r)$ where $a_j \in \mathbb{C}$, $a_j^{de} = 1$ and $(a_1 \cdots a_r)^d = 1$.

The group $G(de,e,r)$ is a subgroup of index e of $G(de,1,r)$ and $G(de,1,r) \simeq (\mathbb{Z}/de\mathbb{Z}) \wr \mathfrak{S}_r$.

For $de \neq 1$ and $(d,e,r) \neq (1,2,2)$, the group $G(de,e,r)$ acts irreducibly on $\mathbb{C}^r$, while $G(1,1,r)$ is isomorphic to $\mathfrak{S}_r$ in its permutation action on $\mathbb{C}^r$.

Note that the center $Z(G(de,e,r))$ of $G(de,e,r)$ is cyclic, of order $d(e \wedge r)$. We denote by $\Delta(de,e,r)$ the abelian normal subgroup of $G(de,e,r)$ given by

$$\Delta(de,e,r) := \{[\underline{a},1]\}\,.$$

The group $\Delta(de,e,r)$ is of order $d(de)^{r-1}$.

For the following notation, we assume that $de \neq 1$ and $(d,e,r) \neq (1,2,2)$ (*i.e.*, that $G(de,e,r)$ acts irreducibly on $\mathbb{C}^r$).

For $m \in \mathbb{N} - \{0\}$, let $\zeta_m := \exp(2i\pi/m)$. We set

$$\begin{aligned} s_m &:= [(\zeta_m, 1, \dots, 1), 1] \\ t'_2(m) &:= [(\zeta_m^{-1}, \zeta_m, 1, \dots, 1), (1,2)] \\ t_j &:= [\underline{1}, (j-1,j)] \quad \text{for } 2 \leq j \leq r\,. \end{aligned} \tag{3.1}$$

Let $S(de,e,r)$ denote the set of pseudo-reflections of $G(de,e,r)$ given by

$$S(de,e,r) := \begin{cases} \{s_d, t'_2(de), t_2, \dots, t_r\} \text{ when } e \neq 1\,, d \neq 1 \\ \{s_d, t_2, \dots, t_r\} \text{ when } e = 1\,, \\ \{t'_2(e), t_2, \dots, t_r\} \text{ when } d = 1\,. \end{cases}$$

The following result is proved, for example, in [Ari].

3.2. Proposition. *The set $S(de,e,r)$, together with the relations described in Appendix 2 and its tables 1 and 2, give a presentation by generators and relations of $G(de,e,r)$.*

Note that $S(de,e,r)$ consists of distinguished pseudo-reflections (see Definition 2.14 above) for $G(de,e,r)$.

Reflecting hyperplanes.

The following lemma is well known and easy to check.

3.3. Lemma. *Let m be a positive integer.*

(1) *For $e \mid m$ and $e < m$, the complement in $\mathbb{C}^r$ of the union of the reflecting hyperplanes of $G(m,e,r)$ is*

$$\mathcal{M}^{\#}(m,r) := \{(z_1, z_2, \dots, z_r) \mid (\forall j,k, 1 \leq j \neq k \leq r)(\forall a \in \mathbb{Z})(z_j \neq 0)(z_j \neq \zeta_m^a z_k)\}\,.$$

(2) *For all $e \in \mathbb{N}$, the complement in $\mathbb{C}^r$ of the union of the reflecting hyperplanes of $G(e,e,r)$ is*

$$\mathcal{M}(m,r) := \{(z_1, z_2, \dots, z_r) \mid (\forall j,k, 1 \leq j < k \leq r)(\forall a \in \mathbb{Z})(z_j \neq \zeta_m^a z_k)\}\,.$$

Choosing an appropriate base point, we denote by $B(de,e,r)$ and $P(de,e,r)$ respectively the corresponding braid group and pure braid group associated with $G(de,e,r)$.

Remark. By Lemma 3.3 above, $P(de,e,r)$ depends only on (de,r) :

$$P(de,e,r) = P(de,1,r) \quad (\text{for all } d \neq 1)\,.$$

On the other hand, we shall prove (see Proposition 3.8 below) that $B(de,e,r)$ depends only on (e,r) for $d \neq 1$ (so that $B(de,e,r) = B(2e,e,r)$).

Preliminary : the case of the symmetric group.

Here we quote some well known results about the usual braid groups, mainly due to Artin [Ar] – see also [Bi], th. 1.8.

Let us introduce some specific notation.

We set $\mathcal{M}(r) := \mathcal{M}(1,r)$ and $\mathcal{M}^{\#}(r) := \mathcal{M}^{\#}(1,r)$ (see Proposition 3.3 above).

For all $j \leq r$, we denote by $H_j^{(r+1)}$ the hyperplane of $\mathbb{C}^{r+1}$ defined by the equation $z_j = z_{j+1}$, and we denote by $s_j^{(r+1)}$ (or simply s_j) the reflection in $\mathbb{C}^{r+1}$ with respect to $H_j^{(r+1)}$. The set $\{s_j^{(r+1)}\}_{(1\leq j\leq r)}$ generates a subgroup of $\mathrm{GL}_{r+1}(\mathbb{C})$ which we identify with the symmetric group $\mathfrak{S}_{r+1}$, and the set $\mathcal{M}(r+1)$ is the complement of the union of the reflecting hyperplanes of $\mathfrak{S}_{r+1}$.

We choose a base point $x \in \mathbb{R}^{r+1}$ with coordinates $x_1, x_2, \ldots, x_{r+1}$ such that $x_1 < x_2 < \cdots < x_{r+1}$. Note that x is in one of the alcôves of $\mathcal{M}(r+1) \cap \mathbb{R}^{r+1}$ delimited by (the real part of) the hyperplanes $H_j^{(r+1)}$.

We set

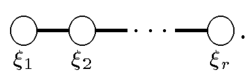
$$P(r+1) := \pi_1(\mathcal{M}(r+1), x) \quad \text{and} \quad B(r+1) := \pi_1(\mathcal{M}(r+1)/\mathfrak{S}_{r+1}, x)\,.$$

For each $j \leq r$, we denote by $\xi_j^{(r+1)}$ (or simply ξ_j) the generator of the monodromy around $H_j^{(r+1)}$ in $\mathcal{M}(r+1)/\mathfrak{S}_{r+1}$ associated to a path contained in $\mathbb{R}^{r+1}/\mathfrak{S}_{r+1}$.

The following well known proposition establishes Theorem 2.26 for the case where $G = \mathfrak{S}_{r+1}$. The second assertion has been proved by Chow [Cho].

3.4. Proposition.

(1) *The group $B(r+1)$ has a presentation described by the following diagram*



(2) *Let $\boldsymbol{\pi}(r+1)$ be the element of $B(r+1)$ defined by*

$$\boldsymbol{\pi}(r+1) := (\xi_1\xi_2\cdots\xi_r)^{r+1}\,.$$

For $r \geq 2$, we have

$$Z(B(r+1)) = Z(P(r+1)) = \langle \boldsymbol{\pi}(r+1) \rangle\,.$$

We will often consider $B(r) = \pi_1(\mathcal{M}(r)/\mathfrak{S}_r, (x_2, \ldots, x_{r+1}))$ as a subgroup of $B(r+1)$ through the injection $\xi_j^{(r)} \mapsto \xi_{j+1}^{(r+1)}$. This induces an injection of the pure braid group $P(r) = \pi_1(\mathcal{M}(r), (x_2, \ldots, x_{r+1}))$ into $P(r+1)$, as well as an injection of $\mathfrak{S}_r$ in $\mathfrak{S}_{r+1}$ as the subgroup fixing the first coordinate.

3.5. Proposition. *The map*

$$p_r \colon \mathcal{M}(r+1) \to \mathcal{M}(r)\,,\ (z_1, z_2, \dots, z_{r+1}) \mapsto (z_2, \dots, z_{r+1})$$

is a locally trivial fibration, and it induces a short exact sequence

$$1 \to F(r) \to P(r+1) \to P(r) \to 1\,,$$

where $F(r)$ is a free subgroup, on the set of generators

$$\{\xi_1^2, \xi_2\xi_1^2\xi_2^{-1}, \dots, \xi_r \cdots \xi_3\xi_2\xi_1^2\xi_2^{-1}\xi_3^{-1} \cdots \xi_r^{-1}\}\,.$$

We have $P(r+1) = F(r) \rtimes P(r)$.

B. Computation of $B(de, e, r)$ and of its center for $d \neq 1$.

B1. Proof of Theorem 2.26.

Let us use the notation introduced above about $B(r+1)$ and $P(r+1)$, as well as notation introduced in (3.1).

3.6. Theorem. *Assume $d \neq 1$.*

(1) For s equal to respectively $s_d, t_2, t_3, \dots, t_r$, there exist s-generators of the monodromy denoted respectively by $\sigma, \tau_2, \tau_3, \dots, \tau_r$ in $B(d,1,r)$ and an injective group morphism

$$\phi_{(d,1,r)} \colon B(d,1,r) \hookrightarrow B(r+1) \quad , \quad \phi_{(d,1,r)} \colon \begin{cases} \sigma \mapsto \xi_1^2 \\ \tau_j \mapsto \xi_j \ \textit{for}\ j \geq 2 \end{cases}$$

which induces an isomorphism of $B(d,1,r)$ onto the subgroup of $B(r+1)$ generated by $\{\xi_1^2, \xi_2, \xi_3, \dots, \xi_r\}$.

(2) This isomorphism, as well as the isomorphism between $B(r)$ and the subgroup of $B(r+1)$ generated by $\{\xi_2, \xi_3, \dots, \xi_r\}$, induce the following commutative diagram :

$$\begin{tikzcd}
 & 1 \arrow[dd] & \\
1 \arrow[dd] & & 1 \arrow[dd] \\
 & P(r) \arrow[dl, hook'] \arrow[dr, hook] \arrow[dd] & \\
P(d,1,r) \arrow[rr, hook] \arrow[dd] & & P(r+1) \arrow[dd] \\
 & B(r) \arrow[dl, hook'] \arrow[dr, hook] \arrow[dd] & \\
B(d,1,r) \arrow[rr, hook] \arrow[dd] & & B(r+1) \arrow[dd] \\
 & \mathfrak{S}_r \arrow[dr, hook] \arrow[dd] & \\
G(d,1,r) \arrow[ur, two heads] \arrow[rr] \arrow[dd] & & \mathfrak{S}_{r+1} \arrow[dd] \\
 & 1 & \\
1 & & 1
\end{tikzcd}$$

Proof of Theorem 3.6.

The map

$$(z_1, z_2, \ldots, z_r) \mapsto (z_1^d, z_2^d, \ldots, z_r^d)$$

identifies the quotient of $\mathcal{M}^{\#}(d,r)$ by the action of the diagonal group $\Delta(d,1,r)$ with the space $\mathcal{M}^{\#}(r)$.

The map

$$f : \mathcal{M}(r+1) \to \mathcal{M}^{\#}(r)\ ,\ (z_1, z_2, \ldots, z_{r+1}) \mapsto (z_1 - z_2, z_1 - z_3, \ldots, z_1 - z_{r+1})$$

is a trivial fibration with fiber $\mathbb{C}$, which is $\mathfrak{S}_r$-equivariant with respect to the action of $\mathfrak{S}_r$ on $\mathcal{M}(r+1)$ defined by the embedding of $\mathfrak{S}_r$ into $\mathfrak{S}_{r+1}$ as the pointwise stabilizer of the first coordinate.

Since

$$G(d,1,r) = \Delta(d,1,r) \rtimes \mathfrak{S}_r\,,$$

we have the following commutative diagram :

$$\begin{tikzcd}
\mathcal{M}^{\#}(d,r) \arrow[d] & & \\
\mathcal{M}^{\#}(d,r)/\Delta(d,1,r) \arrow[r, "\sim"] \arrow[d] & \mathcal{M}^{\#}(r) \arrow[d] & \mathcal{M}(r+1) \arrow[l, "\mathbb{C}-\text{fibration}"'] \arrow[d] \\
\mathcal{M}^{\#}(d,r)/G(d,1,r) \arrow[r, "\sim"] & \mathcal{M}^{\#}(r)/\mathfrak{S}_r & \mathcal{M}(r+1)/\mathfrak{S}_r \arrow[l] \arrow[d] \\
 & & \mathcal{M}(r+1)/\mathfrak{S}_{r+1}
\end{tikzcd}$$

The horizontal arrows induce isomorphisms between fundamental groups.

Let $y \in \mathcal{M}^{\#}(d,r)$ with image $f(x)$ in $\mathcal{M}^{\#}(r)$. Let ψ be the isomorphism

$$\pi_1(\mathcal{M}^{\#}(d,r)/G(d,1,r), y) \to \pi_1(\mathcal{M}(r+1)/\mathfrak{S}_r, x)$$

and $\phi(d,r)$ be the injection

$$\pi_1(\mathcal{M}^{\#}(d,r)/G(d,1,r), y) \to \pi_1(\mathcal{M}(r+1)/\mathfrak{S}_r, x) \to \pi_1(\mathcal{M}(r+1)/\mathfrak{S}_{r+1}, x)\,.$$

Note that ξ_1^2 is a generator of the monodromy around $H_1^{(r+1)}$ in $\mathcal{M}(r+1)/\mathfrak{S}_r$ (Proposition A3, Appendix 1). Since $P(r+1)$ is contained in the subgroup of $B(r+1)$ generated by $\xi_1^2, \xi_2, \ldots, \xi_r$ and since the image of this subgroup in $\mathfrak{S}_{r+1}$ is $\mathfrak{S}_r$, it follows that $\pi_1(\mathcal{M}(r+1)/\mathfrak{S}_r, x)$ is the subgroup of $\pi_1(\mathcal{M}(r+1)/\mathfrak{S}_{r+1}, x)$ generated by $\xi_1^2, \xi_2, \ldots, \xi_r$.

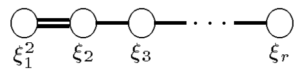
Let $\sigma = \psi^{-1}(\xi_1^2)$: this is a generator of the monodromy around the hyperplane $z_1 = 0$. Let $\tau_j = \psi^{-1}(\xi_j)$ for $j \geq 2$: this is a generator of the monodromy around the hyperplane $H_j^{(r)}$. Then, $\pi_1(\mathcal{M}^{\#}(d,r)/G(d,1,r), y)$ is generated by $\sigma, \tau_2, \ldots, \tau_r$ and Theorem 3.6 follows. $\square$

Let us now explain why Theorem 3.6 above implies Theorem 2.26 (for the case which is presently considered, namely the case of $B(de,e,r)$ with $d > 1$).

1. The case $e = 1$.

By Theorem 3.6, the group $B(d,1,r)$ is isomorphic to the subgroup of $B(r+1)$ generated by $\{\xi_1^2, \xi_2, \xi_3, \dots, \xi_r\}$, and from now on we identify $B(d,1,r)$ with this subgroup.

In particular, it follows from Theorem 3.6 above that $P(r+1) \subset B(d,1,r)$. Since the image of $B(d,1,r)$ in $\mathfrak{S}_{r+1}$ (recall that $\mathfrak{S}_{r+1} = B(r+1)/P(r+1)$) is isomorphic to $\mathfrak{S}_r$, the index of $B(d,1,r)$ in $B(r+1)$ is $(r+1)$. Hence the set $\{1, \xi_1, \xi_1\xi_2, \dots, \xi_1\xi_2\cdots\xi_r\}$ is a set of right coset representatives of $B(d,1,r)$ in $B(r+1)$. By construction, it is actually a *Schreier* set of right coset representatives, and it results from the Reidemeister–Schreier method (see [MKS], Theorem 2.8) that the braid relations defined by the diagram



are indeed defining relations for $B(d,1,r)$.

Remark. The group $G(2,1,r)$ is actually a Coxeter group, since it is isomorphic to the Weyl group of type B_r. So we have reproved in this case a result which is known for all Coxeter groups by [De1] or [Br1].

Note also that

3.7. *the injection $B(d,1,r) \hookrightarrow B(r+1)$ induces an injection*

$$P(d,1,r) \hookrightarrow P(r+1)\,.$$

2. The general case $e > 1$.

Let us set $\xi_2' := \xi_1^2\xi_2\xi_1^{-2}$. Note that (with notation introduced in Part A above) that ξ_2' is a $t_2'(de)$–generator of the monodromy in the braid group $B(de,e,r)$.

Note also that, since we have the following coverings

$$\mathcal{M}^{\#}(de,r) \to \mathcal{M}^{\#}(de,r)/G(de,e,r) \to \mathcal{M}^{\#}(de,r)/G(de,1,r)\,,$$

it results from Proposition A3 (Appendix 1 below) that ξ_1^{2e} is an s_d–generator of the monodromy in the braid group $B(de,e,r)$.

By Lemma 3.3 above we may identify $P(de,1,r)$ and $P(de,e,r)$ for $d \neq 1$. Further, $G(de,e,r)$ acts as a subgroup of $G(de,1,r)$ on $\mathcal{M}^{\#}(de,r)$, so we have the natural embeddings

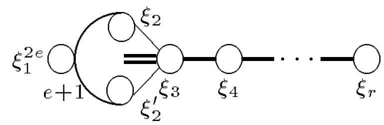
$$P(de,1,r) \hookrightarrow B(de,e,r) \hookrightarrow B(de,1,r)\,,$$

and the index of the latter embedding equals e. Let $\kappa : B(de,1,r) \to G(de,1,r)$ denote the canonical epimorphism. Set $\xi := \xi_1^2$. Now $\{1, \kappa(\xi), \dots, \kappa(\xi^{e-1})\}$ is a set of right coset representatives of $G(de,e,r)$ in $G(de,1,r)$, so, since

$$[\kappa(B(de,1,r)) : \kappa(B(de,e,r))] = e = [B(de,1,r) : B(de,e,r)]\ ,$$

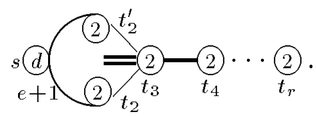
the set $\{1, \xi, \dots, \xi^{e-1}\}$ is a right (Schreier) transversal of $B(de,e,r)$ in $B(de,1,r)$.

An application of the Reidemeister–Schreier method then proves, starting from the presentation for $B(de,1,r)$ on the set $\xi_1^2, \xi_2, \ldots, \xi_r$ (proved above) that the braid relations defined by the diagram



are indeed defining relations for $B(de,e,r)$. This proves Theorem 2.26 for $B(de,e,r)$ and $d>1$ assuming the corresponding statement for $B(de,1,r)$. $\square$

Note that the above diagram is indeed the opposite diagram to the braid diagram describing the relations between the set $S(de,e,r)$ of the corresponding family of distinguished generators of the finite group $G(de,e,r)$, namely



It will be useful to note that we have proved for $G(de,e,r)$ a statement similar to (and more general than) Theorem 3.6, (1), namely :

3.8. Proposition. *For s equal to respectively $s_d, t_2'(de), t_2, t_3, \ldots, t_r$, there exist s-generators of the monodromy denoted respectively by $\sigma^e, \tau_2', \tau_2, \tau_3, \ldots, \tau_r$ in $B(de,e,r)$ and an injective group morphism*

$$\phi_{(de,e,r)} \colon B(de,e,r) \hookrightarrow B(r+1) \quad , \quad \phi_{(de,e,r)} \colon \begin{cases} \sigma^e \mapsto \xi_1^{2e} \\ \tau_2' \mapsto \xi_2' \quad (\textit{where } \xi_2' = \xi_1^2 \xi_2 \xi_1^{-2}) \\ \tau_j \mapsto \xi_j \textit{ for } j \geq 2 \end{cases}$$

which induces an isomorphism of $B(de,e,r)$ onto the subgroup of $B(r+1)$ generated by $\{\xi_1^{2e}, \xi_2', \xi_2, \xi_3, \ldots, \xi_r\}$.

On the pure braid group.

Let us note a result about the structure of $P(d,1,r)$ which is analogous to Proposition 3.5.

Let $F(r)$ be the free subgroup of $P(r+1)$ introduced in Proposition 3.5. Let us set

$$\begin{aligned} &\varphi_1 := \xi_1^2 \quad \text{and} \\ &\varphi_j := (\xi_j \cdots \xi_3 \xi_2) \xi_1^2 (\xi_2^{-1} \xi_3^{-1} \cdots \xi_j^{-1}) \quad \text{for } j = 2, \ldots, r\,. \end{aligned}$$

Then $F(r)$ is the free group on $\{\varphi_1, \varphi_2, \ldots, \varphi_r\}$.

The map $\pi_1(\mathcal{M}(r+1), x) \simeq \pi_1(\mathcal{M}^{\#}(r), f(x)) \to \Delta(d,1,r)$ provides by restriction a morphism $F(r) \to \Delta(d,1,r)$. We denote by $F(d,r)$ the kernel of this morphism. Thus we have a short exact sequence

$$1 \to F(d,r) \to F(r) \to \Delta(d,1,r) \to 1\,.$$

3.9. Lemma.

(1) We have $F(d,r) = F(r) \cap P(d,1,r)$, and $F(d,r)$ is a free group on $((r-1)d^r + 1)$ generators.

(2) We have $P(d,1,r) = F(d,r) \rtimes P(r)$.

Proof of 3.9.

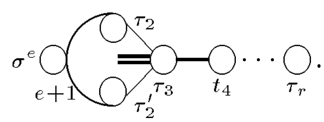
The equality $F(d,r) = F(r) \cap P(d,1,r)$ expresses the definition of $F(d,r)$. Since $F(d,r)$ is a subgroup of index d^r of the free group $F(r)$, it is a free group on $((r-1)d^r+1)$ generators.

Since $P(r+1) = F(r) \rtimes P(r)$, the second assertion follows from the fact that $P(r) \subset P(d,1,r) \subset P(r+1)$. $\square$

B2. The center of $B(de,e,r)$ for $d \neq 1$.

Let us denote by $\boldsymbol{\beta}(de,e,r)$ the central element of $B(de,e,r)$ defined as in 2.19.

The following proposition proves Theorem 2.23 for the braid groups $B(de,e,r)$ with $d > 1$. We use the notation introduced in Proposition 3.8. In particular, $B(de,e,r)$ is defined by generators and relations represented by the diagram



3.10. Proposition. *We have :*

(1) $\boldsymbol{\beta}(de,e,r) = \sigma^{er/(e\wedge r)}(\tau_2'\tau_2\tau_3\cdots\tau_r)^{e(r-1)/(e\wedge r)}$,

(2) $Z(B(de,e,r)) = \langle \boldsymbol{\beta}(de,e,r)\rangle$,

(3) $Z(P(de,e,r)) = Z(B(de,e,r)) \cap P(de,e,r) = \langle \boldsymbol{\beta}(de,e,r)^{d(e\wedge r)}\rangle$.

Proof of 3.10. Note that for $e = 1$ the result is already known by [De1], since by Theorem 3.6 we have $B(d,1,r) = B(2,1,r)$, the braid group associated to a Weyl group.

In what follows, we identify $B(de,e,r)$ with its image in $B(r+1)$ (see Proposition 3.8 above).

Step 1. We prove that

$$Z(P(de,e,r)) \subset \langle \boldsymbol{\pi}(r+1)\rangle \,. \tag{3.11}$$

Let $z \in Z(P(de,e,r))$.

Since $P(r) \subset P(de,e,r) \subset P(r+1)$, the element z belongs to $P(r+1)$ and centralizes $P(r)$. Since (cf. Proposition 3.5) $P(r+1) = F(r) \rtimes P(r)$, where $F(r)$ is the normal closure in $P(r+1)$ of the subgroup generated by ξ_1^2, in order to prove that $z \in Z(P(r+1))$ it suffices to prove that z centralizes ξ_1^2. But z centralizes ξ_1^{2de}. Thus the elements $z\xi_1^2 z^{-1}$ and ξ_1^2 both belong to the free group $F(r)$, and their (de)-th powers are equal. This implies that they are equal (see for example [MKS], 1.4, ex. 2). This proves (3.11).

Thus we have

$$(Z(B(de,e,r)) \cap P(de,e,r)) \subset Z(P(de,e,r)) \subset \langle \boldsymbol{\pi}(r+1)\rangle \cap B(de,e,r) \,. \tag{3.12}$$

Step 2. Let us now prove that

$$\langle \boldsymbol{\pi}(r+1)\rangle \cap B(de,e,r) = \langle \boldsymbol{\pi}(r+1)^{e/(e\wedge r)}\rangle\,. \tag{3.13}$$

We have (see [Bi], 1.8.4) :

$$\boldsymbol{\pi}(r+1) = \xi_1^2(\xi_2'\xi_1^2\xi_2')(\xi_3\xi_2'\xi_1^2\xi_2'\xi_3)\cdots(\xi_r\cdots\xi_3\xi_2'\xi_1^2\xi_2'\xi_3\cdots\xi_r)\,. \tag{3.14}$$

Since $\xi_2 = \xi_1^{-2}\xi_2'\xi_1^2$, we have $\xi_2'\xi_1^2 = \xi_1^2\xi_2$, and (3.14) becomes

$$\boldsymbol{\pi}(r+1) = \xi_1^2(\xi_1^2\xi_2\xi_2')(\xi_3(\xi_1^2\xi_2\xi_2')\xi_3)\cdots(\xi_r\cdots\xi_3(\xi_1^2\xi_2\xi_2')\xi_3\cdots\xi_r)\,. \tag{3.15}$$

Since ξ_1^2 commutes with $(\xi_2\xi_2')$, as well as with ξ_j for $j\geq 3$, we deduce from (3.15) that

$$\boldsymbol{\pi}(r+1) = \xi_1^{2r}(\xi_2\xi_2')(\xi_3(\xi_2\xi_2')\xi_3)\cdots(\xi_r\cdots\xi_3(\xi_2\xi_2')\xi_3\cdots\xi_r)\,. \tag{3.16}$$

and then for all $n\in\mathbb{N}$,

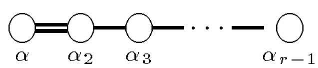
$$\boldsymbol{\pi}(r+1)^n = \xi_1^{2rn}\left((\xi_2\xi_2')(\xi_3(\xi_2\xi_2')\xi_3)\cdots(\xi_r\cdots\xi_3(\xi_2\xi_2')\xi_3\cdots\xi_r)\right)^n\,. \tag{3.17}$$

Since, for $e'\in\mathbb{N}$, we have $\langle B(de,e,r),\xi_1^{2e'}\rangle = B(de,(e\wedge e'),r)$, it follows from (3.17) that $\boldsymbol{\pi}(r+1)^n\in B(de,e,r)$ if and only if e divides rn, *i.e.,* if and only if $e/(e\wedge r)$ divides n, which proves (3.13).

Step 3. Let us now check that

$$(\xi_2\xi_2')(\xi_3(\xi_2\xi_2')\xi_3)\cdots(\xi_r\cdots\xi_3(\xi_2\xi_2')\xi_3\cdots\xi_r) = (\xi_2\xi_2'\xi_3\cdots\xi_r)^{r-1}\,. \tag{3.18}$$

Let us introduce the group $B(2,1,r-1)$ together with its distinguished set of generators $\{\alpha,\alpha_2,\dots,\alpha_{r-1}\}$, which satisfy the relations described by the diagram :



Then the map

$$\alpha\mapsto\xi_2\xi_2'\ ,\ \ \alpha_j\mapsto\xi_{j+1}\quad\text{for } j\geq 2$$

defines a morphism $B(2,1,r-1)\to B(de,e,r)$. Thus, in order to prove (3.18), it suffices to prove that

$$\alpha(\alpha_2\alpha\alpha_2)(\alpha_3\alpha_2\alpha\alpha_2\alpha_3)\cdots(\alpha_{r-1}\cdots\alpha_3\alpha_2\alpha\alpha_2\alpha_3\cdots\alpha_{r-1}) = (\alpha\alpha_2\alpha_3\cdots\alpha_{r-1})^{r-1}\,.$$

This last equality expresses a known property of reduced expressions of the longest element in the Weyl group $G(2,1,r-1)$, *i.e.,* the Weyl group of type B_{r-1} (see for example [Bou], chap. v, §6, ex. 2). The proof of (3.18) is complete.

Last step. Let us (temporarily) set $\boldsymbol{\beta}' := \boldsymbol{\pi}(r+1)^{e/(e\wedge r)}$. By (3.12) and (3.13), we see that

$$Z(B(de,e,r))\cap P(de,e,r)\subset\langle\boldsymbol{\beta}'\rangle\,, \tag{3.19}$$

and by (3.18) and (3.16), we see that

$$\boldsymbol{\beta}' = \xi_1^{2re/(e\wedge r)}(\xi_2\xi_2'\xi_3\cdots\xi_r)^{e(r-1)/(e\wedge r)}\,,$$

or, with the identification made in Proposition 3.8,

$$\boldsymbol{\beta}' = \sigma^{er/(e\wedge r)}(\tau_2'\tau_2\tau_3\cdots\tau_r)^{e(r-1)/(e\wedge r)}\,. \tag{3.20}$$

On the other hand, it is not difficult to check that

3.21. *the canonical epimorphism $\kappa\colon B(de,e,r)\to G(de,e,r)$ sends $\boldsymbol{\beta}'$ onto the scalar multiplication in V by* $\exp(2i\pi/|Z(G(de,e,r))|)$.

Since the map $\kappa\colon Z(B(de,e,r))\to Z(G(de,e,r))$ is onto, and since (by (3.19)) its kernel is contained in $\langle\boldsymbol{\beta}'\rangle$, it follows from 3.21 that

$$Z(B(de,e,r)) = \langle\boldsymbol{\beta}'\rangle\,. \tag{3.22}$$

Using (3.11), it is then easy to prove that $Z(P(de,e,r)) = \langle{\boldsymbol{\beta}'}^{d(e\wedge r)}\rangle$.

It remains to check now that $\boldsymbol{\beta}' = \boldsymbol{\beta}(de,e,r)$. This follows from the fact that,

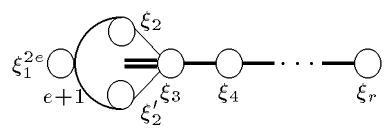
- on one hand, we have (by (3.22) and Lemma 2.21, (2)) $\boldsymbol{\beta}(de,e,r)\in\langle\boldsymbol{\beta}'\rangle$,
- on the other hand, $\boldsymbol{\beta}(de,e,r)$ and $\boldsymbol{\beta}'$ have both the same image under the discriminant map $\pi_1(\delta)\colon B(de,e,r)\to\mathbb{Z}$ (cf. §2, B above). $\square$

C. Computation of $B(e,e,r)$ and of its center, $e\neq 1$.

In this section we study the braid group of type $B(e,e,r)$ and in particular prove Theorems 2.26 and 2.23 for type $G(e,e,r)$.

Note that the construction in the proof of Theorem 3.6 above gives an identification of $\pi_1(\mathcal{M}^\#(e,r)/G(e,e,r))$ with the subgroup of the braid group $B(r+1)$ generated by $\xi_1^{2e},\xi_2',\xi_2,\xi_3,\ldots,\xi_r$ with presentation

(3.23)



Since $\mathcal{M}^\#(e,r)$ is obtained from $\mathcal{M}(e,r)$ by removing the hyperplanes $\{z_j=0\}$ for $1\le j\le r$ we have a natural map

$$\psi:\pi_1(\mathcal{M}^\#(e,r)/G(e,e,r))\to\pi_1(\mathcal{M}(e,r)/G(e,e,r))\,,$$

which is surjective since the complement $\mathcal{M}(e,r)/G(e,e,r)-\mathcal{M}^\#(e,r)/G(e,e,r)$ has complex codimension 1.

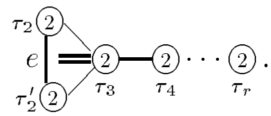
The following proposition follows from Proposition A1 in Appendix 1.

Proposition 3.24. *The kernel of ψ is the normal closure of the subgroup generated by ξ_1^{2e} in the group $\pi_1(\mathcal{M}^\#(e,r)/G(e,e,r))$.*

Note that Theorem 2.26 follows immediately from this. Indeed, by the above Proposition the presentation of $B(e,e,r)$ is obtained from (3.23) by suppressing the node corresponding to ξ_1^{2e}.

Complements on $B(e,e,r)$.

Theorem 3.25. *Let $e, r \geq 2$, and let $B(e,e,r)$ be the braid group of type $G(e,e,r)$, on standard generators $\tau_2, \tau_2', \tau_3, \dots, \tau_r$ ordered such that $(\tau_2\tau_2'\tau_3)^2 = (\tau_3\tau_2\tau_2')^2$:*



Let $\tilde{B}(e,1,r-1)$ be the preimage of the subgroup $G(e,1,r-1)$ of $G(e,e,r)$ fixing the first coordinate. Then $\tilde{B}(e,1,r-1)$ has index r in $B(e,e,r)$ and has a presentation on generators

$$\{\alpha_j, \beta_{j,l}, \mid 2 \leq j \leq r,\ 0 \leq l \leq e-1\}$$

subject to

$$\alpha_i^{-1}\beta_{j,l}\alpha_i = \begin{cases} \beta_{j,l} & \text{if } i \neq 2, j, j+1 \\ \beta_{j+1,l} & \text{if } i = j+1 \\ \beta_{j,l+1}^{-1}\beta_{j-1,l+1}\beta_{j,l} & \text{if } i = j \neq 2 \\ \beta_{2,l+1}\beta_{j,l}\beta_{2,l}^{-1} & \text{if } i = 2 < j \\ \beta_{j,l+2} & \text{if } i = j = 2 \end{cases} \tag{3.26}$$

(where the subscript l of $\beta_{j,l}$ is taken modulo e),

$$\beta_{j,e-1}\beta_{j,e-2}\cdots\beta_{j,0} = 1 \qquad \text{for } 2 \leq j \leq r, \tag{3.27}$$

and $\alpha_2, \dots, \alpha_r$ satisfy the relations of the standard generators of $B(e,1,r-1)$.

In terms of the generators of $B(e,e,r)$ we may take

$$\alpha_2 = \tau_2\tau_2', \qquad \alpha_i = \tau_i \text{ for } 3 \leq i \leq r, \qquad \beta_{2,0} = \tau_2'\tau_2^{-1}, \qquad \beta_{2,1} = \tau_2'^{-1}\tau_2\,.$$

In particular, $\tilde{B}(e,1,r-1)$ has a semidirect product decomposition

$$\tilde{B}(e,1,r-1) = F_{(e-1)(r-1)} \rtimes B(e,1,r-1)\,,$$

where $F_{(e-1)(r-1)}$ denotes the free group on the $(e-1)(r-1)$ generators $\beta_{2,0}, \dots, \beta_{r,e-1}$.

Proof. The assertion can be proved by the Reidemeister–Schreier method (see for example [MKS], Theorem 2.8). Assume first that $r = 2$. Then $B(e,e,2)$ is generated by $\{\tau_2, \tau_2'\}$ subject to the single relation $\tau_2\tau_2'\tau_2\cdots = \tau_2'\tau_2\tau_2'\cdots$ with e factors on each side. A right transversal for $\tilde{B}(e,1,1)$ in $B(e,e,2)$ is given by $T := \{1, \tau_2\}$. Let $\rho : B(e,e,2) \to T$ be the transversal map which to every element of $B(e,e,2)$ associates its coset representative in T. Then by the Reidemeister-Schreier theorem $\tilde{B}(e,1,1)$ is generated by the elements $(tg)^{-1}\rho(tg)$ where $t \in T$ and g runs over the generators of $B(e,e,2)$. In our case this yields the generators

$$\alpha_2 := \tau_2\tau_2',\ \beta_{2,0} := \tau_2'\tau_2^{-1},\ \gamma_2 := \tau_2^2. \tag{3.28}$$

Furthermore, by the Reidemeister-Schreier algorithm the relations for $B(e,e,2)$ yield

$$\alpha_2^{(e+1)/2} = \beta_{2,0}(\gamma_2\beta_{2,0})^{(e-1)/2}\alpha_2 = (\gamma_2\beta_{2,0})^{(e-1)/2}\gamma_2 \quad \text{if } e \text{ is odd,} \tag{3.29}$$

$$\alpha_2^{e/2} = (\gamma_2\beta_{2,0})^{e/2} = (\beta_{2,0}\gamma_2)^{e/2} \quad \text{if } e \text{ is even,} \tag{3.30}$$

as defining relations for $\tilde{B}(e,1,1)$. By introducing $\beta_{2,1} := {\tau_2'}^{-1}\tau_2 = \alpha_2^{-1}\gamma_2$ we may eliminate γ_2 and arrive at the statement of the theorem in this case.

Now assume that $r = 3$. Here a transversal is given by $T := \{1, \tau_2, \tau_2\tau_3\}$. We obtain generators

$$\alpha_3 := \tau_3,\ \gamma_3 := \tau_2\tau_3^2\tau_2^{-1},\ \delta := \tau_2{\tau_2'}^{-1}\tau_3\tau_2'\tau_2^{-1},$$

and $\alpha_2, \gamma_2, \beta_{2,0}$ from (3.28) above, subject to the relations

$$\begin{aligned} &(\alpha_2\alpha_3)^2 = (\alpha_3\alpha_2)^2, \quad \alpha_3^{-1}\gamma_2\alpha_3 = \gamma_3, \quad \alpha_3\gamma_3\gamma_2 = \gamma_3\gamma_2\alpha_3, \\ &\beta_{2,0}\delta = \alpha_3\beta_{2,0}, \quad \alpha_2\alpha_3\beta_{2,0} = \delta\gamma_3, \quad \delta\gamma_3\alpha_2 = \gamma_3\alpha_2\alpha_3, \\ &\gamma_2\beta_{2,0}\alpha_3\delta\gamma_3 = \alpha_3\delta\gamma_3\gamma_2\beta_{2,0} = \gamma_3\gamma_2\beta_{2,0}\alpha_3\delta, \end{aligned} \tag{3.31}$$

and the relations (3.29) respectively (3.30). We may eliminate $\gamma_3 = \alpha_3^{-1}\gamma_2\alpha_3$ and $\delta = \beta_{2,0}^{-1}\alpha_3\beta_{2,0}$. With $\beta_{2,1} := \alpha_2^{-1}\gamma_2$ as above we find the stated presentation.

Finally, for $r \geq 4$ it follows easily from the presentation for $B(e,e,r)$ given in Theorem 2.26 (proved by Proposition 3.24) that

$$T := \{1, \tau_2, \tau_2\tau_3, \ldots, \tau_2\tau_3\cdots\tau_r\}$$

is a right transversal for $\tilde{B}(e,1,r-1)$ in $B(e,e,r)$. This yields the generators

$$\alpha_i := \tau_i \ \text{ and } \ \gamma_i := \tau_2\tau_3\cdots\tau_{i-1}\tau_i^2\tau_{i-1}^{-1}\cdots\tau_3^{-1}\tau_2^{-1}\ (3 \leq i \leq r),$$

$$\alpha_2 := \tau_2\tau_2',\ \gamma_2 := \tau_2^2,\ \beta_{2,0} := \tau_2'\tau_2^{-1},\ \delta := \tau_2{\tau_2'}^{-1}\tau_3\tau_2'\tau_2^{-1}.$$

Furthermore, by the Reidemeister-Schreier algorithm the relations for $B(e,e,r)$ yield as relations for $\tilde{B}(e,1,r-1)$ the relations of the standard generators for $B(e,1,r-1)$ on $\alpha_2, \ldots, \alpha_r$, the commutator rules

$$[\gamma_i, \alpha_j] = 1 \text{ for } j \neq i, i+1, \quad \alpha_{i+1}^{-1}\gamma_i\alpha_{i+1} = \gamma_{i+1} \text{ for } 2 \leq i,$$

and

$$\alpha_{i+1}\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\alpha_{i+1} \text{ for } 2 \leq i, ,$$

as well as

$$[\beta_{2,0}, \alpha_j] = 1 \text{ for } j \geq 4, \quad [\delta, \alpha_j] = 1 \text{ for } j > 4, \quad [\delta, \gamma_j] = 1 \text{ for } j \geq 4,$$

$$\delta\alpha_4\delta = \alpha_4\delta\alpha_4, \quad (\alpha_3\delta\alpha_4)^2 = (\alpha_4\alpha_3\delta)^2.$$

and the relations (3.31), (3.29) respectively (3.30). Note that the generators $\gamma_3, \ldots, \gamma_r$ may be eliminated from this presentation using the relation $\gamma_{i+1} = \alpha_{i+1}^{-1}\gamma_i\alpha_{i+1}$. This reduces the assertion to the case $r = 3$. $\square$

This result has some nice consequences, like the following analogue of Proposition 3.5 and Lemma 3.9.

Corollary 3.32. *The pure braid group $P(e,e,r)$ is a semidirect product*

$$P(e,e,r) \cong F_{(e-1)(r-1)} \rtimes P(e,1,r-1)$$

of the free group of rank $(e-1)(r-1)$ with the pure braid group of type $G(e,1,r-1)$.

This follows immediately by descent to the pure braid groups (see [Na], p. 6, for a related result). We also obtain Theorem 2.23 for type $G(e,e,r)$:

Corollary 3.33. *For $e,r \geq 2$, $(e,r) \neq (2,2)$, the center of the braid group $B(e,e,r)$ is generated by $(\tau_2 \cdots \tau_r)^{e(r-1)/(e\wedge r)}$.*

Proof. Let $\kappa : B(e,e,r) \to G(e,e,r)$ be the canonical projection. If z is central in $B(e,e,r)$ then so is $\kappa(z)$ in $G(e,e,r)$. Hence $\kappa(z) = (\tau_2 \cdots \tau_r)^n$ with n a multiple of $e(r-1)/(e \wedge r)$, and $z = (\tau_2 \cdots \tau_r)^n w$ for some w in $\ker(\kappa)$. But then we already have $z \in \tilde{B}(e,1,r-1)$ (defined as above).

Let λ: $\tilde{B}(e,1,r-1) \to B(e,1,r-1)$ be the canonical projection with kernel $F := F_{(e-1)(r-1)}$ the free group on $(e-1)(r-1)$ generators emanating from Theorem 3.25. Since the center of $\lambda(\tilde{B}(e,1,r-1)) = B(e,1,r-1)$ is generated by $(\alpha_2 \cdots \alpha_r)^r = (\tau_2 \cdots \tau_r)^r$ we deduce that $z = (\tau_2 \cdots \tau_r)^n w$ for some $w \in F$. But $(\tau_2 \cdots \tau_r)^n$ is central in $\tilde{B}(e,1,r-1)$, while the center of the free group F is trivial (note that $(e-1)(r-1) \geq 2$). Thus the center of $\tilde{B}(e,1,r-1)$ is generated by $(\tau_2 \cdots \tau_r)^{e(r-1)/(e\wedge r)}$. □

Remark 3.34. For the braid group $B(2,2,r)$ of Coxeter type D_r Theorem 3.25 specializes to the following: $\tilde{B}(2,1,r-1)$ has a presentation on

$$\{\alpha_i, \beta_j, \mid 2 \leq i,j \leq r\}$$

subject to

$$\alpha_i^{-1}\beta_j\alpha_i = \begin{cases} \beta_j & \text{if } i \neq 2, j, j+1 \\ \beta_{j+1} & \text{if } i = j+1 \\ \beta_j\beta_{j-1}^{-1}\beta_j & \text{if } i = j \neq 2 \\ \beta_2^{-1}\beta_j\beta_2^{-1} & \text{if } i = 2 < j \\ \beta_j & \text{if } i = j = 2 \end{cases}$$

and $\alpha_2, \ldots, \alpha_r$ satisfy the relations of the standard generators of $B(2,1,r-1)$.

Remark 3.35. The subgroup $\tilde{B}(e,1,1)$ of index 2 of the braid group $B(e,e,2)$ of Coxeter type $I_2(e)$ has a presentation on $\{\alpha_2, \beta_l, \mid 0 \leq l \leq e-1\}$ subject to

$$\alpha_2^{-1}\beta_l\alpha_2 = \beta_{l+2} \qquad \text{for } 0 \leq l \leq e-1$$

(where the subscript of β_l has to be taken mod e), and

$$\beta_{e-1}\beta_{e-2}\cdots\beta_0 = 1\,.$$

Remark 3.36. The action of $B(e,1,r-1)$ on $F_{(e-1)(r-1)}$ in Theorem 3.25 can be extended to an action of the Artin braid group $B(r)$. More precisely, let $\tilde{\alpha}_2, \alpha_3, \ldots, \alpha_r$ be the standard generators of $B(r)$. Then $B(e,1,r-1)$ is isomorphic to the subgroup generated by $\tilde{\alpha}_2^2, \alpha_3, \ldots, \alpha_r$ by Theorem 3.6. We extend the action (3.26) of $\alpha_2, \ldots, \alpha_r$ to an action of $B(r)$ on $F_{(e-1)(r-1)} = \langle \beta_{j,l} \mid \prod_{l=0}^{e-1} \beta_{j,l} = 1 \rangle$ by

$$\tilde{\alpha}_2^{-1} \beta_{j,l} \tilde{\alpha}_2 = \begin{cases} \beta_{2,l} \beta_{j,l}^{-1} & \text{if } j \neq 2, \\ \beta_{j,l+1} & \text{if } j = 2. \end{cases}$$

It is easy to verify that this does in fact extend the action of $B(e,1,r-1)$.

On the other hand the action (3.26) can be viewed as an action on the free group $F_{e(r-1)}$ on free generators $\langle \beta_{j,l} \mid 2 \leq j \leq r, 0 \leq l \leq e-1 \rangle$ by just omitting relations (3.27). The homomorphism defined by

$$\phi : F_{e(r-1)} \to \langle t \rangle \cong \mathbb{Z}, \qquad \beta_{j,l} \mapsto t \quad \text{for all } j,l\,,$$

is $B(e,1,r-1)$-equivariant (with trivial action on the right side), so it gives rise to a Magnus representation (see [Bi], Th. 3.9)

$$\Phi : B(e,1,r-1) \to \mathrm{GL}_{e(r-1)}(\mathbb{Z}(t))$$

with

$$\Phi(\alpha_i)_{(j,l),(k,m)} = \begin{cases} \delta_{jk} \delta_{lm} & \text{if } i \neq 2, j, j+1 \\ \delta_{j+1,k} \delta_{lm} & \text{if } i = j+1 \\ (\delta_{j-1,k} \delta_{l+1,m} - \delta_{jk} \delta_{l+1,m}) t^{-1} + \delta_{jk} \delta_{lm} & \text{if } i = j \neq 2 \\ \delta_{2k} \delta_{l+1,m} + (\delta_{jk} \delta_{lm} - \delta_{2k} \delta_{lm}) t & \text{if } i = 2 < j \\ \delta_{jk} \delta_{l+2,m} & \text{if } i = j = 2 \end{cases}$$

(where $2 \leq i,j,k \leq r$, $0 \leq l,m \leq e-1$) of $B(e,1,r-1)$.

A general statement for pure braid groups.

Let us first introduce some notation.

- We make the convention that $G(1,1,r) := \mathfrak{S}_{r+1}$, and we denote by $P(1,1,r)$ the corresponding pure braid group.
- Let $m_r^*(de,e,r)$ be the co-exponent (see §1.A above) of $G(de,e,r)$ such that the set of co-exponents of $G(de,e,r)$ consists of the set of coexponents of $G(de,1,r-1)$, together with $m_r^*(de,e,r)$. We have $m_r^*(de,e,r) = (r-1)de+1$ for $d \neq 1$ and $m_r^*(e,e,r) = (r-1)(e-1)$.
- For any natural integer m, let F_m be the free group on m generators.

3.37. Proposition. *For all positive integers d,e,r, we have a split short exact sequence*

$$0 \to F_{m_r^*(de,e,r)} \to P(de,e,r) \to P(de,1,r-1) \to 0\,.$$

In particular, $P(de,e,r)$ is the semidirect product of a free group on $m_r^*(de,e,r)$ generators by the pure braid group associated with a complex reflection subgroup of $G(de,e,r)$ of rank $(r-1)$.

Proof of 3.37. We assume $d \neq 1$, since for $d = 1$, the result was proven in Corollary 3.32.

Consider the map $f : \mathcal{M}^{\#}(de,r) \to \mathcal{M}^{\#}(de,r-1)$, $(z_1,\dots,z_r) \mapsto (z_1,\dots,z_{r-1})$. This is a locally trivial fibration, with fiber isomorphic to $\mathbb{C}$ minus $(r-1)de+1$ points. By Theorem 2.10, $\mathcal{M}^{\#}(de,r-1)$ is a $K(\pi,1)$-space. Hence, we have a short exact sequence of fundamental groups associated to the fibration :

$$0 \to F_{(r-1)de+1} \to P(de,e,r) \to P(de,1,r-1) \to 0\,.$$

The locally trivial fibration $\mathcal{M}^{\#}(de,r) \to \mathcal{M}^{\#}(r)$, $(z_1,\dots,z_r) \mapsto (z_1^{de},\dots,z_r^{de})$ induces a commutative diagram with exact rows and columns :

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & F_{(r-1)de+1)} \arrow[r] \arrow[d] & P(de,e,r) \arrow[r] \arrow[d] & P(de,e,r-1) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & F_r \arrow[r] \arrow[d] & P(r) \arrow[r] \arrow[d] & P(r-1) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}/de\mathbb{Z} \arrow[r] \arrow[d] & \Delta(de,1,r) \arrow[r] \arrow[d] & \Delta(de,1,r-1) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

The splitting of the map $P(r) \to P(r-1)$ together with the splitting of $\Delta(de,1,r) \to \Delta(de,1,r-1)$ given by identifying $\Delta(de,1,r-1)$ with the subgroup of $\Delta(de,1,r)$ of elements acting trivially on the last coordinate give then a splitting of $P(de,e,r) \to P(de,e,r-1)$ and the proposition follows. $\square$

4. Hecke algebras

We extend to the case of complex reflection groups the construction of generalized Knizhnik–Zamolodchikov connections for Weyl groups due to Cherednik ([Ch1], [Ch2], [Ch3]; see also the constructions of Dunkl [Du], Opdam [Op] and Kohno [Ko1]).[2] This allows us to construct explicit isomorphisms between the group algebra of a complex reflection group and its Hecke algebra.

A. Background from differential equations and monodromy.

What follows is well known, and is introduced here at an elementary level for the convenience of the unexperienced reader, since we only need this elementary approach. For a more general approach, see for example [De2].

We go back to the setting of §1. Let A be a finite dimensional complex vector space. We denote by 1 a chosen non zero point of A — in the applications, A we will be an

[2]This construction has also been noticed independently by Opdam, who is able to deduce from it some important consequences concerning the "generalized fake degrees" of a complex reflection group. We thank him for useful and friendly conversations.

algebra. We set $E := \mathrm{End}(A)$. Let ω be a holomorphic differential form on $\mathcal{M}$ with values in E, *i.e.,* a holomorphic map $\mathcal{M} \to \mathrm{Hom}(V, E)$, where $\mathrm{Hom}(V, E)$ denotes the space of linear maps from V into E, such that (see 1.2 and 1.5, (1)) we have

$$\omega = \sum_{H \in \mathcal{A}} f_H \omega_H \,,$$

with $\omega_H = \dfrac{1}{2i\pi}\dfrac{d\alpha_H}{\alpha_H}$, and $f_H \in E$. For $x \in \mathcal{M}$ and $v \in V$, we have $\omega(x)(v) = \dfrac{1}{2i\pi}\sum_{H\in\mathcal{A}} \dfrac{\alpha_H(v)}{\alpha_H(x)} f_H$.

We consider the following linear differential equation

$$(\mathrm{Eq}(\omega)) \qquad dF = \omega(F)\,,$$

where F is a holomorphic function defined on an open subset of $\mathcal{M}$ with values in A. In other words, for x in this open subset, we have $dF(x) \in \mathrm{Hom}(V, A)$, and we want F to satisfy, for all $v \in V$, $dF(x)(v) = \omega(x)(v)(F(x))$, or in other words $dF(x)(v) = \dfrac{1}{2i\pi}\sum_{H\in\mathcal{A}} \dfrac{\alpha_H(v)}{\alpha_H(x)} f_H(F(x))$.

For $y \in \mathcal{M}$, let us denote by $\mathcal{V}(y)$ the largest open ball with center y contained in $\mathcal{M}$. The existence and unicity theorem for linear differential equations shows that for each $y \in \mathcal{M}$, there exists a unique function

$$F_y \colon \mathcal{V}(y) \to A\,,\ \ x \mapsto F_y(x)\,,$$

solution of $\mathrm{Eq}(\omega)$ and such that $F_y(y) = 1$. From now on, we set

$$F(x, y) := F_y(x)\,.$$

Assume now that the finite group W acts linearly on A through a morphism $\varphi \colon W \to \mathrm{GL}(A)$. Then it induces an action of W on the space of differential forms on $\mathcal{M}$ with values in E, and an easy computation shows that ω is W-stable if and only if, for all $w \in W$,

$$\omega(w(x)) = \varphi(w)(\omega(x) \cdot w^{-1})\varphi(w^{-1})\,, \tag{4.1}$$

which can also be written, for all $x \in \mathcal{M}$ and $v \in V$:

$$\sum_{H\in\mathcal{A}} \omega_H(wx)(v) f_H = \sum_{H\in\mathcal{A}} \omega_H(x)(w^{-1}(v))\varphi(w) f_H \varphi(w^{-1})\,.$$

An easy computation shows that this is equivalent to

$$\sum_{H\in\mathcal{A}} f_{w(H)} \frac{d\alpha_{w(H)}}{\alpha_H} = \sum_{H\in\mathcal{A}} \varphi(w) f_H \varphi(w^{-1}) \frac{d\alpha_{w(H)}}{\alpha_H}\,. \tag{4.2}$$

In particular we see that

4.3. *If $f_{w(H)} = \varphi(w) f_H \varphi(w^{-1})$ for all $H \in \mathcal{A}$ and $w \in W$, then the form ω is W-stable.*

From (4.1) (and from the existence and unicity theorem), it follows that

4.4. *If ω is W-stable, then for all $y \in \mathcal{M}$, $x \in \mathcal{V}(y)$ and $w \in W$, the solution $x \mapsto F(x,y)$ satisfies*

$$\varphi(w)(F(x,y)) = F(w(x), w(y)) .$$

The case of an interior W-algebra.

The following hypothesis and notation will be in force for the rest of this chapter.

From now on, we assume that A is endowed with a structure of $\mathbb{C}$-algebra with unity, and that ω takes its values in the subalgebra of E consisting of the multiplications by the elements of A – which, by abuse of notation, we still denote by A. With this abuse of notation, we may assume that

$$\omega = \sum_{H \in \mathcal{A}} a_H \omega_H \,,$$

where $a_H \in A$, and the equation $\mathrm{Eq}(\omega)$ is written

$$dF = \omega F \;\text{ or }\; dF(x)(v) = \frac{1}{2i\pi} \sum_{H \in \mathcal{A}} \frac{\alpha_H(v)}{\alpha_H(x)} a_H F(x) .$$

Let γ be a path in $\mathcal{M}$. From the existence and unicity of local solutions of $\mathrm{Eq}(\omega)$, it results that the solution $x \mapsto F(x, \gamma(0))$ has an analytic continuation $t \mapsto (\gamma^* F)(t, \gamma(0))$ along γ, which satisfies the following properties.

Let us say that a sequence of real numbers $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n = 1$ is adapted to $(\gamma, \mathrm{Eq}(\omega))$ if for all $1 \le j \le n$, we have $\gamma([t_{j-1}, t_j]) \subset \mathcal{V}(\gamma(t_j))$.

Then :

(1) there exists $\varepsilon > 0$ such that $(\gamma^* F)(t, \gamma(0)) = F(\gamma(t), \gamma(0))$ for $0 \le t \le \varepsilon$,

(2) whenever $t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n = 1$ is adapted to $(\gamma, \mathrm{Eq}(\omega))$, we have $(\gamma^* F)(t_j, \gamma(0)) = F(\gamma(t_j), \gamma(t_{j-1}))(\gamma^* F)(t_{j-1}, \gamma(0)) \quad \text{for all } j > 0 .$

We see that

$$(\gamma^* F)(1, \gamma(0)) = \prod_{j=n}^{j=1} F(\gamma(t_j), \gamma(t_{j-1})) . \tag{4.5}$$

Note that there is always an adapted sequence for $(\gamma, \mathrm{Eq}(\omega))$.

The case of an integrable form.

We recall that the form ω is said to be integrable if $d\omega + \omega \wedge \omega = 0$.

The following fact was noticed, for example, by Kohno (see [Ko2], 1.2).

4.6. Lemma. *The form $\omega = \sum_{H \in \mathcal{A}} a_H \omega_H$ is integrable if and only if, for all subspaces X of V with codimension 2, and for all $H \in \mathcal{A}$ such that $X \subset H$, a_H commutes with $\sum_{\substack{(H' \in \mathcal{A}) \\ (H' \supset X)}} a_{H'}$.*

Indeed, this is an immediate consequence of 1.5, (2).

If ω is integrable, the value $(\gamma^*F)(1,\gamma(0))$ depends only on the homotopy class of γ. By (4.5), we see that we get a *covariant functor*

$$S\colon \begin{cases} \mathcal{P}(\mathcal{M}) \to A^\times \\ \gamma \mapsto (\gamma^*F)(1,\gamma(0)) \end{cases}$$

Action of W.

Assume now that A is an interior W-algebra, *i.e.,* that there is a group morphism $W \to A^\times$ (through which the image of $w \in W$ is still denoted by w), which defines a linear operation φ of W on A by composition with the injection $A^\times \hookrightarrow \mathrm{GL}(A)$. So, with our convention, for $w \in W$ and $a \in A$ we have $\varphi(w)(a) = wa$.

The form ω is then W-stable if and only if, for all $w \in W$ and $x \in \mathcal{M}$,

$$\omega(w(x)) = w(\omega(x) \cdot w^{-1})w^{-1}\,,$$

which can also be written, for all $x \in \mathcal{M}$ and $v \in V$:

$$\sum_{H\in\mathcal{A}} \omega_H(wx)(v)a_H = \sum_{H\in\mathcal{A}} \omega_H(x)(w^{-1}(v))wa_Hw^{-1}\,.$$

By 4.3, we have the following criterion.

4.7. *If, for all $H \in \mathcal{A}$ and $w \in W$, we have $a_{w(H)} = wa_Hw^{-1}$, then the form ω is W-stable.*

By 4.4, the solution F of $\mathrm{Eq}(\omega)$ then satisfies

$$wF(x,y)w^{-1} = F(w(x),w(y))\,. \tag{4.8}$$

4.9. Definition–Proposition. *Assuming that ω is W-stable, we define a group morphism*

$$T\colon \pi_1(\mathcal{M}/W, x_0) \to (A^\times)^{\mathrm{op}}$$

(or, in other words, a group anti–morphism $T\colon \pi_1(\mathcal{M}/W,x_0) \to A^\times$), called the monodromy morphism associated with ω, as follows.

For $\sigma \in B$, with image $\overline{\sigma}$ in W through the natural anti–morphism $B \to W$ (see 2.B above), we denote by $\tilde{\sigma}$ a path in $\mathcal{M}$ from x_0 to $\overline{\sigma}(x_0)$ which lifts σ. Then we set

$$T(\sigma) := S(\tilde{\sigma}^{-1})\overline{\sigma}\,.$$

Let us check that T is a group anti–morphism.

Notice first that, by (4.8) and by (4.5), for $w \in W$ and γ a path in $\mathcal{M}$, we have

$$wS(\gamma)w^{-1} = S(w(\gamma))\,.$$

Thus we have

$$\begin{aligned} T(\sigma_2)T(\sigma_1) &= S(\tilde{\sigma_2}^{-1})\overline{\sigma}_2 S(\tilde{\sigma_1}^{-1})\overline{\sigma}_1 \\ &= S(\tilde{\sigma_2}^{-1})S(\overline{\sigma}_2(\tilde{\sigma_1}^{-1}))\overline{\sigma}_2\overline{\sigma}_1 \\ &= S(\tilde{\sigma_2}^{-1}\overline{\sigma}_2(\tilde{\sigma_1}^{-1}))\overline{\sigma}_2\overline{\sigma}_1 \\ &= S((\overline{\sigma}_2(\tilde{\sigma_1})\tilde{\sigma_2})^{-1})\overline{\sigma_1\sigma_2} \end{aligned}$$

which proves that $T(\sigma_2)T(\sigma_1) = T(\sigma_1\sigma_2)$, since $\overline{\sigma}_2(\tilde{\sigma_1})\tilde{\sigma_2}$ is indeed a path in $\mathcal{M}$ with origin x_0 which lifts $(\sigma_1\sigma_2)$. $\square$

Dependence of parameters.

Suppose the form ω depends holomorphically on m parameters $t_1, \ldots, t_m$. Denoting by $\mathcal{O}$ the ring of holomorphic functions of the variables $t_1, \ldots, t_m$, we have $\omega = \sum_{H\in\mathcal{A}} f_H\omega_H$ where $f_H \in \mathcal{O}\otimes_{\mathbb{C}} E$. Then, for $y \in \mathcal{M}$, the function F_y is a holomorphic function of $t_1, \ldots, t_m$, *i.e.*, F_y has values in $\mathcal{O}\otimes_{\mathbb{C}} A$.

Then, given a path γ in $\mathcal{M}$, the analytic continuation $t \mapsto (\gamma^* F)(t, \gamma(0))$ depends holomorphically of $t_1, \ldots, t_m$.

If ω is integrable and W-stable, then the monodromy morphism depends holomorphically on the parameters $t_1, \ldots, t_m$. It follows that we have a monodromy morphism

$$T : \pi_1(\mathcal{M}/W, x_0)^{\mathrm{op}} \to (\mathcal{O}\otimes_{\mathbb{C}} A)^{\times}.$$

B. A family of monodromy representations of the braid group.

From now on, we assume that $A = \mathbb{C}W$.

Notation and hypothesis.

We denote by $\mathcal{O}$ the ring of holomorphic functions of a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ variables $\mathbf{z} = (z_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}$.

Let

$$\mathbf{t} := (t_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}$$

be a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ complex numbers. For $H \in \mathcal{C}$, we set $t_{H,j} := t_{\mathcal{C},j}$.

We put

$$q_{\mathcal{C},j} = \exp(-t_{\mathcal{C},j}/e_{\mathcal{C}}) \quad \text{for } \mathcal{C}\in\mathcal{A}/W\,,\ 0\leq j\leq e_{\mathcal{C}}-1\,.$$

For $H \in \mathcal{C}$, we set $q_{H,j} := q_{\mathcal{C},j}$.

Let $\mathcal{C} \in \mathcal{A}/W$ and let $H \in \mathcal{C}$. For $0 \leq j \leq e_{\mathcal{C}} - 1$, we denote by $\varepsilon_j(H)$ the primitive idempotent of the group algebra $\mathbb{C}W_H$ associated with the character $\det_V^j$ of the group W_H. Thus we have

$$\varepsilon_j(H) = \frac{1}{e_{\mathcal{C}}}\sum_{k=0}^{k=e_{\mathcal{C}}-1} \exp(\frac{-2i\pi jk}{e_{\mathcal{C}}})s_H^k\,.$$

We set

$$a_H := \sum_{j=0}^{j=e_H-1} t_{H,j}\varepsilon_j(H) \quad\text{and}\quad \omega := \sum_{H\in\mathcal{A}} a_H\omega_H\,.$$

In other words, we have

$$\omega = \sum_{\mathcal{C}\in\mathcal{A}/W}\ \sum_{j=0}^{j=e_{\mathcal{C}}-1}\ \sum_{H\in\mathcal{C}} t_{\mathcal{C},j}\varepsilon_j(H)\omega_H\,.$$

The following lemma is clear.

4.10. *The map $\mathcal{A}\to A$, $H\mapsto a_H$ has the following properties :*

(1) *it is W-stable, i.e., for all $w \in W$ and $H \in \mathcal{A}$, we have $a_{w(H)} = wa_Hw^{-1}$,*
(2) *for all $H \in \mathcal{A}$, a_H belongs to the image of $\mathbb{C}W_H$ in A.*

The following property follows from 4.10.

4.11. Lemma. *The form ω is W-stable and integrable.*

Proof of 4.11. The form ω is W-stable by 4.7. It is integrable by 4.6. Indeed, let X be a codimension 2 subspace of V and let H be an element of $\mathcal{A}$ containing X. By 4.10 above, it is enough to check that, if $w \in W_H$, then w commutes with $\sum_{(H' \in \mathcal{A})(H' \supset X)} a_{H'}$. This is the case since w centralizes X, hence normalizes $\{H' \in \mathcal{A} \mid (H' \supset X)\}$. $\square$

The main theorem.

4.12. Theorem. *We denote by $T \colon B^{\mathrm{op}} \to (\mathbb{C}W)^\times$ the monodromy morphism associated with the differential form ω on $\mathcal{M}$. For all $H \in \mathcal{C}$, we have*

$$\prod_{j=0}^{j=e_{\mathcal{C}}-1} (T(\mathbf{s}_{H,\gamma}) - q_{H,j}\det_V(s_H)^j) = 0\,.$$

Furthermore, T depends holomorphically on the parameters $t_{\mathcal{C},j}$, i.e., arises by specialization from a morphism $\mathbf{T} \colon B^{\mathrm{op}} \to (\mathcal{O}W)^\times$.

Proof of 4.12.

First step : case of rank 1

Here we assume $\dim(V) = 1$. So we may assume that W is the cyclic group of order e generated by the multiplication s by $\exp(2i\pi/e)$. We have $\mathcal{M} = \mathbb{C}^\times$.

For $0 \le j \le e-1$, let ε_j be the primitive idempotent of $\mathbb{C}W$ corresponding to the character of W which sends s onto $\exp(2i\pi j/e)$.

There are e complex numbers $t_0, t_1, \dots, t_{e-1}$ such that

$$\omega = \sum_{j=0}^{j=e-1} \frac{t_j}{2i\pi} \varepsilon_j \frac{dz}{z}\,.$$

A function $F \colon \mathbb{C}^\times \to \mathbb{C}W$ may be written

$$F = \sum_{j=0}^{j=e-1} F_j \varepsilon_j$$

where $F_j \colon \mathbb{C}^\times \to \mathbb{C}$.

The equation $\mathrm{Eq}(\omega)$ becomes

$$\frac{dF_j}{dz} = \frac{t_j}{2i\pi} \frac{F_j(z)}{z} \quad \text{for } 0 \le j \le e-1\,.$$

Hence the solution $F(x,1)$ is given by the formula

$$F(x,1) = \sum_{j=0}^{j=e-1} x^{t_j/2i\pi} \varepsilon_j\,.$$

The analytic continuation of F along the path $\sigma : t \mapsto \exp(2i\pi t/e)$ gives

$$S(\sigma) = \sum_{j=0}^{j=e-1} \exp(t_j/e)\varepsilon_j$$

hence

$$T(\mathbf{s}) = S(\sigma)^{-1}s = \sum_{j=0}^{j=e-1} \exp(-t_j/e)\exp(2i\pi j/e)\varepsilon_j \,.$$

Thus we see that, with $q_j := \exp(-t_j/e)$, we have

$$\prod_{j=0}^{j=e-1} (T(\mathbf{s}) - q_j \exp(2i\pi j/e)) = 0\,, \tag{4.13}$$

as claimed.

Second step : towards the reduction to the case of rank 1

We are back to the general case. Here we use notation introduced in §2.B.

1. First we prove that, to compute the relation satisfied by the monodromy of $\mathbf{s}_{H,\gamma}$, we may assume that x_0 is "close to H", namely that $x_0 = x_H$.

Let us denote by

$$T_{x_H} : \pi_1(\mathcal{M}/W, x_H)^{\mathrm{op}} \to A$$

the monodromy morphism associated with ω, and let us denote by $\mathbf{s}_H$ the element of $\pi_1(\mathcal{M}/W, x_H)$ defined by the path σ_{H,x_H}.

4.14. Lemma. *For any path γ from x_0 to $s_H(x_0)$, we have*

$$T(\mathbf{s}_{H,\gamma}) = S(\gamma)^{-1}T_{x_H}(\mathbf{s}_H)S(\gamma)\,.$$

Proof of 4.14. By (2.12), we have $\sigma_{H,\gamma} := s_H(\gamma^{-1})\cdot\sigma_{H,x_H}\cdot\gamma$, from which it follows that

$$\begin{aligned} T(\mathbf{s}_{H,\gamma}) &= S(\sigma_{H,\gamma}^{-1})s_H \\ &= S(\gamma^{-1})S(\sigma_{H,x_H}^{-1})S(s_H(\gamma))s_H \\ &= S(\gamma^{-1})S(\sigma_{H,x_H}^{-1})s_H S(\gamma) \\ &= S(\gamma^{-1})T_{x_H}(\mathbf{s}_H)S(\gamma) \end{aligned}$$

□

2. Now we prove that we may reduce to rank one.

Choose and fix $H \in \mathcal{A}$. We still use notation introduced in §2.

The elements of the affine line $(x_H + L_H)$ are the elements $x_H(z) := \mathrm{pr}_H(x_H) + z\mathrm{pr}_H^{\perp}(x_H)$ with $z \in \mathbb{C}$. We may adjust the choice of x_H so that, if

$$D_H^{\times} := \{x_H(z) \mid 0 < |z| < 2\}\,,$$

we have $D_H^\times \subset \mathcal{M}$. Note that $D_H^\times$ is stable by the operation of the group W_H.

We have $\alpha_H(x_H(z)) = z\alpha_H(\mathrm{pr}_H^\perp(x_H))$. For $H' \neq H$, we set $\alpha_{H'}(\mathrm{pr}_H^\perp(x_H)) = u_{H'}$, and $\alpha_{H'}(\mathrm{pr}_H(x_H)) = v_{H'}$. Recall that x_H has been chosen so that

$$zu_{H'} + v_{H'} \neq 0 \quad \text{on } D_H \quad \text{if } H' \neq H\,.$$

Then the function

$$R_H F\colon D_H^\times \to \mathbb{C}W\ ,\ x_H(z) \mapsto F(x_H(z), x_H)\,,$$

satisfies the following differential equation :

$$\frac{d(R_H F)}{dz} = \left(\frac{1}{2i\pi}\frac{a_H}{z} + \sum_{H' \neq H}\frac{1}{2i\pi}\frac{u_{H'}}{u_{H'}z + v_{H'}}a_{H'}\right) R_H F(x_H(z))\,.$$

In other words, $R_H F$ satisfies the differential equation associated with the differential form $R_H\omega$ defined on $D_H^\times$ by

$$R_H\omega := \frac{1}{2i\pi}\left(\frac{a_H}{z} + \sum_{H' \neq H}\frac{u_{H'}}{u_{H'}z + v_{H'}}a_{H'}\right) dz\,. \tag{4.15}$$

Note that $R_H\omega$ is W_H-stable.

3. Now we reduce to the case of the action of the cyclic group W_H on $D_H^\times$.

Let $R_H S\colon \mathcal{P}(D_H^\times) \to \mathbb{C}W$ be the monodromy functor associated with the form $R_H\omega$. By the existence and unicity theorem for linear differential equations, since the loop σ_H takes its values in $D_H^\times$, we see that

$$S(\sigma_H) = R_H S(\sigma_H)\,. \tag{4.16}$$

Let us still denote by $\mathbf{s}_H$ the image of the path σ_H in $\pi_1(D_H^\times/W_H)$. Let

$$R_H T_{x_H}\colon \pi_1(D_H^\times/W_H)^{\mathrm{op}} \to A$$

be the monodromy morphism associated with the differential form $R_H\omega$. Then it results from (4.16) and from Lemma 4.14 that

4.17. *$T(\mathbf{s}_{H,\gamma})$ is conjugate (in $(\mathbb{C}W)^\times$) to $R_H T_{x_H}(\mathbf{s}_H)$.*

Third step : reduction to the case of rank 1

Let $T_H\colon \pi_1(D_H^\times/W_H, x_H)^{\mathrm{op}} \to \mathbb{C}W_H$ be the monodromy morphism associated with the W_H-stable differential form defined on $D_H^\times$ by $a_H\omega_H$. By (4.13), we know that the characteristic polynomial of $T_H(\mathbf{s}_H)$ (viewed as acting on $\mathbb{C}W_H$ by left multiplication) is

$$P_H(t) := \prod_{j=0}^{j=e_{\mathcal{C}}-1}(t - q_{\mathcal{C},j}\exp(2i\pi j/e_{\mathcal{C}}))\,,$$

where $\mathcal{C}$ denotes the W-orbit of H. We want to prove that $P_H(R_H T_{x_H}(\mathbf{s}_H)) = 0$.

By the first two steps, it is clear that our problem may be reformulated as follows.

We set $D := \{z \in \mathbb{C} \mid (|z| < 2)\}$ and $D^\times := D - \{0\}$, and we view D and $D^\times$ as endowed with the action of W_H defined by $(w, z) \mapsto \det_V(w)z$ for $w \in W_H$ and $z \in D$.

In order to simplify the notation, we also set $e := e_{\mathcal{C}}(= e_H)$, and for $0 \leq j \leq e-1$, $\varepsilon_j := \varepsilon_j(H), t_j := t_{\mathcal{C},j}, q_j := q_{\mathcal{C},j}$, and $a := a_H = \sum_{j=0}^{e-1} t_j \varepsilon_j$. We define a holomorphic function b on D by the formula $b(z) := \dfrac{1}{2i\pi} \sum_{H' \neq H} \dfrac{u_{H'}}{u_{H'}z + v_{H'}} a_{H'}$ (note that $u_{H'}z + v_{H'} \neq 0$ on D).

We have two differential equations for holomorphic functions (locally) defined on $D^\times$ with values in $\mathbb{C}W$:

$$(\mathrm{Eq}(a)) \qquad \frac{dF_H}{dz}(z) = \frac{a}{z} F_H(z)$$

$$(\mathrm{Eq}(a,b)) \qquad \frac{dF}{dz}(z) = (\frac{a}{z} + b(z))F(z)\,.$$

Both these equations correspond to W_H-stable differential forms on $D^\times$: a commutes with the action of W_H, and we have $b(w.z) = \det_V(w)^{-1} w b(z) w^{-1}$ for $w \in W_H$ and $z \in D$.

We denote by S_H and S the functors from $\mathcal{P}(D^\times)$ to $(\mathbb{C}W)^\times$ associated respectively to the equations $\mathrm{Eq}(a)$ and $\mathrm{Eq}(a,b)$, and by T_H , T: $\pi_1(D^\times/W_H, 1) \to (\mathbb{C}W)^\times$ the corresponding morphisms.

4.18. Proposition. *Assume that, for all j,k, $0 \leq j,k \leq e-1$, $q_j q_k^{-1}$ is not an e-th root of the unity. Then there exists an invertible element u of $\mathbb{C}W$ such that, for all $\sigma \in \pi_1(D^\times/W_H, 1)$, we have $T(\sigma) = uT_H(\sigma)u^{-1}$.*

Proof of 4.18.

1. *Equivalence of* $\mathrm{Eq}(a)$ *and* $\mathrm{Eq}(a,b)$.

Here we follow [Ha], 1.4.

Let us consider the following differential equation

$$(\mathrm{Eq}'(a,b)) \qquad \frac{d\Phi}{dz}(z) = \frac{1}{z}(a\Phi(z) - \Phi(z)a) + b(z)\Phi(z)\,,$$

for Φ a function (locally) defined on $D^\times$ with values in $\mathbb{C}W$. The following assertion is proved, for example, in [Ha], 1.4. (see in particular 1.4.2, and proof of 1.4.1). Here we use the fact that the spectrum of the multiplication by a in $\mathbb{C}W$ is $\{\dfrac{t_0}{2i\pi}, \dfrac{t_1}{2i\pi}, \ldots, \dfrac{t_{e-1}}{2i\pi}\}$ (each $\dfrac{t_j}{2i\pi}$ with multiplicity $|W : W_H|$).

4.19. Lemma. *Assume that, for all j,k, $0 \leq j,k \leq e-1$, we have $t_j - t_k \notin 2i\pi\mathbb{Z} - \{0\}$. Then there is a unique solution Φ of $\mathrm{Eq}'(a,b)$ satisfying the following two conditions :*

(1) *Φ is holomorphic on D,*
(2) $\Phi(0) = 1$.

Now it is immediate to check that, if $z \mapsto F_H(z, z_0)$ is the solution of Eq(a), defined in the neighbourhood of z_0, and such that $F_H(z_0, z_0) = 1$, then the function $z \mapsto F(z, z_0) := \Phi(z)F_H(z, z_0)\Phi(z_0)^{-1}$ is the solution of Eq(a, b), defined in the neighbourhood of z_0, and such that $F(z_0, z_0) = 1$.

By the formula (4.5), we see that, for all homotopy classes of paths γ in $D^\times$, we have then

$$S(\gamma) = \Phi(\gamma(1))S_H(\gamma)\Phi(\gamma(0))^{-1} . \tag{4.20}$$

2. W_H-equivariance and proof of Proposition 4.18

By the unicity property of Φ (see Lemma 4.19), it follows from the W_H invariance of a and b that $\Phi(\det_V(w)z) = w\Phi(z)w^{-1}$ for all $w \in W_H$ and $z \in D$. Then the formula (4.20), together with the definition 4.9 of the monodromy functors, imply Proposition 4.18 with $u := \Phi(1)$, *i.e.*, for all $\sigma \in \pi_1(D^\times/W_H, 1)$, we have

$$T(\sigma) = \Phi(1)T_H(\sigma)\Phi(1)^{-1} .$$

$\square$

Conclusion : end of proof of Theorem 4.12

From what precedes, we see that Theorem 4.12 is proved provided the family

$$\mathbf{q} := (q_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_\mathcal{C}-1)}$$

satisfies the condition that (for all $\mathcal{C}$ and all j, k) $q_{\mathcal{C},j}q_{\mathcal{C},k}^{-1}$ is not an $e_\mathcal{C}$-th root of the unity, *i.e.*, if the family $\mathbf{t}$ has the property that $t_{\mathcal{C},j} - t_{\mathcal{C},k}$ is not a non-zero integer, for all $\mathcal{C}$ and all j, k. Since the set of such families is a dense open subset in the space $\mathbb{C}^{\sum_{\mathcal{C}\in\mathcal{A}} e_\mathcal{C}}$ of all families $\mathbf{t}$, we see that Theorem 4.12 follows by continuity, since the solution $x \mapsto F(x, y)$ is a holomorphic function of $\mathbf{t}$. $\square$

C. Hecke algebras.

We define a set

$$\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_\mathcal{C}-1)}$$

of $\sum_{\mathcal{C}\in\mathcal{A}/W}(e_\mathcal{C})$ indeterminates. We denote by $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ the ring of Laurent polynomials in the indeterminates $\mathbf{u}$.

Let $\mathfrak{I}$ be the ideal of the group algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ generated by the elements

$$(\mathbf{s}_{H,\gamma} - u_{\mathcal{C},0})(\mathbf{s}_{H,\gamma} - u_{\mathcal{C},1}) \cdots (\mathbf{s}_{H,\gamma} - u_{\mathcal{C},e_\mathcal{C}-1})$$

where $\mathcal{C} \in \mathcal{A}/W$, $H \in \mathcal{C}$, $\mathbf{s}_{H,\gamma}$ is a generator of the monodromy around H in B (cf. (2.12)) and s is the image of $\mathbf{s}_{H,\gamma}$ in W.

4.21. Definition. *The Hecke algebra $\mathcal{H}_\mathbf{u}(W)$ is the $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B/\mathfrak{I}$.*

Now assume that W is a finite irreducible complex reflection group (see §2.C above for notation and references). Let $\mathcal{D}$ be the diagram of W, and let $\mathbf{s} \in \mathcal{N}(\mathcal{D})$ be a node of $\mathcal{D}$. We set $u_{\mathbf{s},j} := u_{\mathcal{C},j}$ for $j = 0, 1, \ldots, e_\mathcal{C} - 1$, where $\mathcal{C}$ denotes the orbit under W of the reflecting hyperplane of $\mathbf{s}$.

The following proposition is an immediate consequence of Theorem 2.26 :

4.22. Proposition. *Assume W is different from G_{24}, G_{27}, G_{29}, G_{33}, G_{34} – and also different from G_{31} for which the following assertion is still conjectural. The Hecke algebra $\mathcal{H}_{\mathbf{u}}(W)$ is isomorphic to the $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-algebra generated by elements $(T_{\mathbf{s}})_{\mathbf{s}\in\mathcal{N}(\mathcal{D})}$ such that*

- *the elements T_s satisfy the braid relations defined by $\mathcal{D}_{\mathrm{br}}^{\mathrm{op}}$,*
- *we have $(T_{\mathbf{s}}-u_{\mathbf{s},0})(T_{\mathbf{s}}-u_{\mathbf{s},1})\cdots(T_{\mathbf{s}}-u_{\mathbf{s},e_{\mathbf{s}}-1})=0$.*

Notice that through the specialization $u_{\mathbf{s},j}\mapsto \det_V(s)^j$ (for $\mathbf{s}\in\mathcal{N}(\mathcal{D})$ and $0\leq j\leq e_{\mathbf{s}}-1$), the algebra $\mathcal{H}_{\mathbf{u}}(W)$ becomes the group algebra of W^{op} over a suitable cyclotomic extension of $\mathbb{Z}$.

Hecke algebras and monodromy representations.

By Theorem 4.12, we see that the monodromy representation $\mathbf{T}$ factors through $\mathcal{H}_{\mathbf{u}}(W)$. Indeed, let us set

$$t_{\mathcal{C},j}:=\det_V(s)^j u_{\mathcal{C},j}$$

for all $(\mathcal{C},j)$, and

$$\mathbf{t}:=(t_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_{\mathcal{C}}-1)}\,,$$

and let us denote by $\mathcal{O}$ the ring of holomorphic functions of the set of variables $\mathbf{t}$. Then we have the following commutative diagram :

$$\begin{tikzcd}
\mathcal{O}B \arrow[rr, "\mathbf{T}"] \arrow[dr] & & \mathcal{O}W^{\mathrm{op}} \\
& \mathcal{O}\otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]}\mathcal{H}_{\mathbf{u}}(W) \arrow[ur] &
\end{tikzcd}$$

Let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$.

The following lemma is a key point to understand the structure of $\mathcal{H}_{\mathbf{u}}(W)$. It is well-known to hold for Coxeter groups. For the infinite series of complex reflection groups, see [ArKo] for $G(d,1,r)$, [BrMa], (4.12) for $G(2d,2,r)$ and [Ari], Proposition 1.4 for the general case (it has been also checked for many of the remaining groups of small rank — see for example [BrMa], Satz 4.7). We conjecture it is true for all complex reflection groups.

4.23. Lemma. *Assume W is Coxeter group or a complex reflection group in the infinite series.*

The $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-module $\mathcal{H}_{\mathbf{u}}(W)$ can be generated by $|W|$ elements.

From this lemma, we can now deduce the following

4.24. Theorem. *Assume W is Coxeter group or a complex reflection group in the infinite series.*

The monodromy representation $\mathbf{T}$ induces an isomorphism of $\mathcal{K}$-algebras

$$\mathcal{K}\otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]}\mathcal{H}_{\mathbf{u}}(W)\xrightarrow{\sim}\mathcal{K}W^{\mathrm{op}}\,.$$

Furthermore, $\mathcal{H}_{\mathbf{u}}(W)$ is a free $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-module of rank $|W|$.

Proof. By Lemma 4.23, there is a surjective morphism of $\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]$-modules

$$\phi\colon \mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]^{|W|}\to\mathcal{H}_{\mathbf{u}}(W)\,.$$

Let $\mathfrak{m}$ be the ideal of $\mathcal{O}$ of the functions vanishing at the point $(t_{\mathcal{C},j} = 1)$. The morphism $\mathcal{O}_\mathfrak{m} \otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]} \mathcal{H}_\mathbf{u} \to \mathcal{O}_\mathfrak{m} W$ induced by the monodromy is surjective by Nakayama's lemma, since it becomes an isomorphism after tensoring by $(\mathcal{O}_\mathfrak{m})/\mathfrak{m}$. Composing with $1_{\mathcal{O}_\mathfrak{m}} \otimes \phi$, we obtain an epimorphism $\mathcal{O}_\mathfrak{m}^{|W|} \to \mathcal{O}_\mathfrak{m} W$: this must be an isomorphism. Hence, $\ker \phi = 0$, *i.e.,* ϕ is an isomorphism and $\mathcal{H}_\mathbf{u}$ is free of rank $|W|$ over $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$.

Since the morphism $K \otimes_{\mathbb{Z}[\mathbf{u},\mathbf{u}^{-1}]} \mathcal{H}_\mathbf{u} \to KW$ is a surjective morphism between two K-modules with same dimensions, it is an isomorphism and Theorem 4.24 follows. □

5. Diagrams and tables

Information provided by the tables: invariants of braid diagrams.

Let us recall that a diagram where the orders of the nodes are "forgotten" and where only the braid relations are kept is called a *braid diagram* for the corresponding group.

The groups have been ordered by their diagrams, by collecting groups with the same braid diagram. Thus, for example,

- G_{15} has the same braid diagram as the groups $G(4d, 4, 2)$ for all $d \geq 2$,
- G_4, G_8, G_{16}, G_{25}, G_{32} all have the same braid diagrams as groups $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$,
- G_5, G_{10}, G_{18} have the same braid diagram as the groups $G(d, 1, 2)$ for all $d \geq 2$,
- G_7, G_{11}, G_{19} have the same braid diagram as the groups $G(2d, 2, 2)$ for all $d \geq 2$,
- G_{26} has the same braid diagram as $G(d, 1, 3)$ for $d \geq 2$.

The element β (generator of $Z(W)$) is given in the last column of our tables. Notice that the knowledge of degrees and codegrees allows then to find the order of $Z(W)$, which is not explicitely provided in the tables.

The tables provide diagrams and data for all irreducible reflection groups.

- Tables 1 and 2 collect groups corresponding to infinite families of braid diagrams,
- Table 3 collects groups corresponding to exceptional braid diagrams (notice that the fact that the diagram for G_{31} provides a braid diagram is only conjectural), but G_{24}, G_{27}, G_{29}, G_{33}, G_{34},
- The last table (table 4) provides diagrams for the remaining cases (G_{24}, G_{27}, G_{29}, G_{33}, G_{34}). It is not known nor conjectural whether these diagrams provide braid diagrams for the corresponding braid groups.

Degrees and codegrees of a braid diagram.

The following property may be noticed on the tables. It generalizes a property already noticed by Orlik and Solomon for the case of Coxeter–Shephard groups (see [OrSo3], (3.7)).

5.1. Theorem. *Let $\mathcal{D}$ be a braid diagram of rank r. There exist two families*

$$(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r) \quad \text{and} \quad (\mathbf{d}_1^*, \mathbf{d}_2^*, \ldots, \mathbf{d}_r^*)$$

of r integers, depending only on $\mathcal{D}$, and called respectively the degrees and the codegrees of $\mathcal{D}$, with the following property: whenever W is a complex reflection group with $\mathcal{D}$ as a braid diagram, its degrees and codegrees are given by the formulae

$$d_j = |Z(W)|\mathbf{d}_j \quad \text{and} \quad d_j^* = |Z(W)|\mathbf{d}_j^* \quad (j = 1, 2, \ldots, r).$$

The zeta function of a braid diagram.

In [DeLo], Denef and Loeser compute the *zeta function of local monodromy of the discriminant* of a complex reflection group W, which is the element of $\mathbb{Q}[q]$ defined by the formula

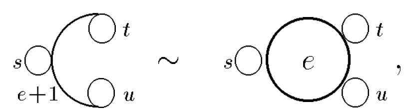
$$Z(q,W) := \prod_j \det(1 - q\mu, H^j(F_0, \mathbb{C}))^{(-1)^{j+1}} ,$$

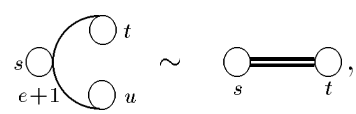
where F_0 denotes the Milnor fiber of the discriminant at 0 and μ denotes the monodromy automorphism (see [DeLo]).

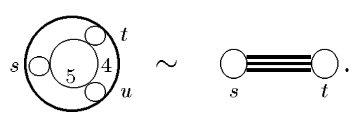
Putting together the tables of [DeLo] and our braid diagrams, one may notice the following fact.

5.2. Theorem. *The zeta function of local monodromy of the discriminant of a complex reflection group W depends only on the braid diagram of W.*

Remark. Two different braid diagrams may be associated to isomorphic braid groups. For example, this is the case for the following rank 2 diagrams (where the sign "$\sim$" means that the corresponding groups are isomorphic) :

For e even, 

for e odd, 

and 

It should be noticed, however, that the above pairs of diagrams do not have the same degrees and codegrees, nor do they have the same zeta function. Thus, degrees, codegrees and zeta functions are indeed attached to the braid diagrams, not to the braid groups.

Appendix 1 : Generators of the monodromy around an irreducible divisor

We define here what we mean by a "generator of the monodromy around an irreducible divisor" and recall some well known properties.

Let Y be a smooth connected complex algebraic variety, I a finite family of irreducible codimension 1 closed subvarieties (irreducible divisors) and $Z := \cup_{D\in I} D$. Let $X := Y - Z$ and $x_0 \in X$.

For $D \in I$, let D_s be the smooth part of D and $\tilde{D} := D_s - \left(D_s \cap \bigcup_{D'\in I, D'\neq D} D'\right)$.

"*A path from x_0 to D in X*" is by definition a path γ in Y such that $\gamma(0) = x_0$, $\gamma(1) \in \tilde{D}$ and $\gamma(t) \in X$ for $t \neq 1$.

Let γ' be another path from x_0 to D in X. We say that γ and γ' are *D-homotopic* if there is a continuous map $T : [0,1]\times[0,1] \to Y$ such that $T(t,0) = \gamma(t)$ and $T(t,1) = \gamma'(t)$ for $t \in [0,1]$, $T(0,u) = x_0$ and $T(1,u) \in \tilde{D}$ for all $u \in [0,1]$ and $T(t,u) \in X$ for $t \in [0,1[$ and $u \in [0,1]$. We denote by $[\gamma]$ the D-homotopy class of γ.

Given a path γ from x_0 to D in X, let B be a connected open neighbourhood of $\gamma(1)$ in $X \cup \tilde{D}$ such that $B \cap X$ has a fundamental group free abelian of rank 1. Let $u \in [0,1[$ such that $\gamma(t) \in B$ for $t \geq u$. Put $x_1 := \gamma(u)$. The orientation of $B \cap X$ coming from the orientation of X gives an isomorphism $f\colon \pi_1(B\cap X, x_1) \xrightarrow{\sim} \mathbb{Z}$. Let λ be a loop in $B \cap X$ from x_1 such that $f([\lambda]) = 1$.

Let γ_u be the "restriction" of γ to $[0,u]$, defined by $\gamma_u(t) := \gamma(ut)$ for all $t \in [0,1]$. Define $\rho_{\gamma,\lambda} := {\gamma_u}^{-1}\cdot\lambda\cdot\gamma_u$. Then, the homotopy class of $\rho_{\gamma,\lambda}$ in $\pi_1(\mathcal{M}, x_0)$ depends only on the D-homotopy class of γ and is denoted by $\rho_{[\gamma]}$. We call it *the generator of the monodromy around D associated to* $[\gamma]$.

Given two paths γ and γ' from x_0 to D, the generators of monodromy $\rho_{[\gamma]}$ and $\rho_{[\gamma']}$ are conjugate.

A1. Proposition. *Let i be the injection of an irreducible divisor D in a smooth connected complex variety Y and $x_0 \in Y - D$. Then, the kernel of the morphism $\pi_1(i)\colon \pi_1(Y-D, x_0) \to \pi_1(Y, x_0)$ is generated by all the generators of the monodromy around D.*

Sketch of proof of A1. Note that the singular points of D form a closed subvariety D_{sing} of D, distinct from D, hence of (complex) codimension at least 2 in Y. Therefore (see for example [Go], chap. x, 2.3) the natural morphism $\pi_1(Y-D-D_{\mathrm{sing}}, x_0) \to \pi_1(Y-D, x_0)$ is an isomorphism, and in order to prove A1 we may assume D is smooth, which we do now.

The lemma then follows from the fact that given a locally constant sheaf $\mathcal{F}$ over $Y - D$, its extension $i_*\mathcal{F}$ to Y is locally constant if and only if every generator of the monodromy around D acts trivially on $\mathcal{F}$. $\square$

A2. Proposition. *Suppose that Y is simply connected. Then the fundamental group $\pi_1(X, x_0)$ is generated by all the generators of the monodromy around the divisors $D \in I$.*

Proof of A2.

This follows immediately from Proposition A1 by induction on $|I|$. $\square$

Lifting generators of the monodromy.

Let $p\colon Y \to \overline{Y}$ be a finite covering between two smooth connected complex varieties. Let D be the branch locus of p and $\overline{D} = p(D)$. We assume $\overline{D}$ is an irreducible divisor. We set $\overline{X} := \overline{Y} - \overline{D}$ and $X := Y - D$.

We shall see that a generator of the monodromy around $\overline{D}$ (associated to a path $\overline{\gamma}$ from $\overline{x}_0$ to $\overline{D}$ in $\overline{Y}$) may be naturally lifted to an element of $\mathcal{P}(X)$ (which depends only on the $\overline{D}$–homotopy class of $\overline{\gamma}$).

Indeed, let γ be the path from x_0 to an irreducible component, say D_γ, of D, which lifts $\overline{\gamma}$. Let $\overline{B}$ be an open neighbourhood of $\overline{x}_0$ in $\overline{Y}$ such that the fundamental group of $\overline{B} \cap \overline{X}$ is free abelian of rank 1 and $B \cap (X \cup \tilde{D}_\gamma) \to \overline{B}$ is unramified outside D_γ. Let $u \in [0, 1[$ such that $\overline{\gamma}(t) \in \overline{B}$ for $t \geq u$. Let $\overline{\lambda}$ be a loop in $\overline{B} \cap \overline{X}$ with origin $\overline{\gamma}(u)$ which is a positive generator of $\pi_1(\overline{B} \cap \overline{X}, \overline{\gamma}(u))$.

Let λ be the path from $\gamma(u)$ which lifts $\overline{\lambda}$. Let γ_u be the restriction of γ to $[0, u]$. Let $\gamma_u^\vee$ be the path from $\lambda(1)$ which lifts $(\overline{\gamma}_u)^{-1}$, where $\overline{\gamma}_u$ is the “restriction” of $\overline{\gamma}$ to $[0, u]$.

The proof of the following proposition is left to the reader.

A3. Proposition. *We define $\rho_\gamma := \gamma_u^\vee \cdot \lambda \cdot \gamma_u$.*

(1) *The homotopy class of ρ_γ in $\mathcal{P}(X)$ depends only on the $\overline{D}$–homotopy class of $\overline{\gamma}$.*
(2) *Let e_D denote the ramification index of p on $\tilde{D}$. Then $\rho_\gamma^{e_D}$ is the generator of the monodromy around D_γ associated to γ.*

Appendix 2 : tables 1 to 5

Here are some definitions, notation, conventions, which will allow the reader to understand the diagrams.

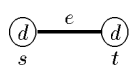
The groups have presentations given by diagrams $\mathcal{D}$ such that

- the nodes correspond to pseudo-reflections in W, the order of which is given inside the circle representing the node,
- two distinct nodes which do not commute are related by "homogeneous" relations with the same "support" (of cardinality 2 or 3), which are represented by links beween two or three nodes, or circles between three nodes, weighted with a number representing the degree of the relation (as in Coxeter diagrams, 3 is omitted, 4 is represented by a double line, 6 is represented by a triple line). These homogeneous relations are called the *braid relations* of $\mathcal{D}$.

More details are provided below.

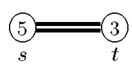
Meaning of the diagrams.

This paragraph provides a list of examples which illustrate the way in which diagrams provide presentations for the attached groups.

- The diagram 

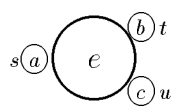
corresponds to the presentation

$$s^d = t^d = 1 \text{ and } \underbrace{ststs\cdots}_{e \text{ factors}} = \underbrace{tstst\cdots}_{e \text{ factors}}$$

- The diagram 

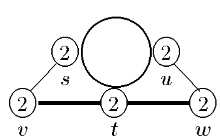
corresponds to the presentation

$$s^5 = t^3 = 1 \text{ and } stst = tsts\,.$$

- The diagram 

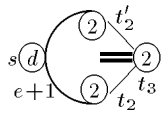
corresponds to the presentation

$$s^a = t^b = u^c = 1 \text{ and } \underbrace{stustu\cdots}_{e \text{ factors}} = \underbrace{tustus\cdots}_{e \text{ factors}} = \underbrace{ustust\cdots}_{e \text{ factors}}\,.$$

- The diagram 

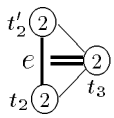
corresponds to the presentation

$$\begin{aligned}
&s^2 = t^2 = u^2 = v^2 = w^2 = 1\,,\\
&uv = vu\,,\ sw = ws\,,\ vw = wv\,,\\
&sut = uts = tsu\,,\\
&svs = vsv\,,\ tvt = vtv\,,\ twt = wtw\,, wuw = uwu\,.
\end{aligned}$$

- The diagram 

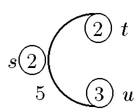
corresponds to the presentation

$$\begin{aligned}
&s^d = {t_2'}^2 = t_2^2 = t_3^2 = 1\,,\ st_3 = t_3 s\,,\\
&st_2't_2 = t_2't_2 s\,,\\
&t_2't_3t_2' = t_3t_2't_3\,,\ t_2t_3t_2 = t_3t_2t_3\,,\ t_3t_2't_2t_3t_2't_2 = t_2't_2t_3t_2't_2t_3\,,\\
&\underbrace{t_2st_2't_2t_2't_2t_2'\cdots}_{e+1\ \text{factors}} = \underbrace{st_2't_2t_2't_2t_2't_2\cdots}_{e+1\ \text{factors}}\,.
\end{aligned}$$

- The diagram 

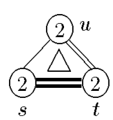
corresponds to the presentation

$$\begin{aligned}
&{t_2'}^2 = t_2^2 = t_3^2 = 1\,,\\
&t_2't_3t_2' = t_3t_2't_3\,,\ t_2t_3t_2 = t_3t_2t_3\,,\ t_3t_2't_2t_3t_2't_2 = t_2't_2t_3t_2't_2t_3\,,\\
&\underbrace{t_2t_2't_2t_2't_2t_2'\cdots}_{e\ \text{factors}} = \underbrace{t_2't_2t_2't_2t_2't_2\cdots}_{e\ \text{factors}}\,.
\end{aligned}$$

- The diagram 

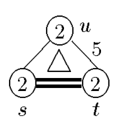
corresponds to the presentation

$$s^2 = t^2 = u^3 = 1\,,\ stu = tus\,,\ ustut = stutu\,.$$

- The diagram 

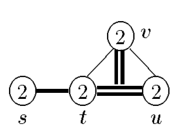
corresponds to the presentation

$$s^2 = t^2 = u^2 = 1\,,\ stst = tsts\,,\ tutu = utut\,,\ utusut = sutusu\,,\ sus = usu\,.$$

- The diagram 

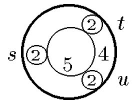
corresponds to the presentation

$$s^2 = t^2 = u^2 = 1\,,\ stst = tsts\,,\ tutut = ututu\,,\ utusut = sutusu\,,\ sus = usu\,.$$

- The diagram 

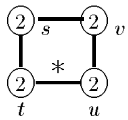
corresponds to the presentation

$$\begin{aligned}
&s^2 = t^2 = u^2 = v^2 = 1\,,\ sv = vs\,,\ su = us\,,\\
&sts = tst\,,\ vtv = tvt\,,\ uvu = vuv\,,\ tutu = utut\,,\ vtuvtu = tuvtuv\,.
\end{aligned}$$

- The diagram 

corresponds to the presentation

$$s^2 = t^2 = u^2 = 1\,,\ ustus = stust\,,\ tust = ustu\,.$$

- The diagram 

corresponds to the presentation

$$s^2 = t^2 = u^2 = v^2 = 1\,,\ su = us\,,\ tv = vt\,,$$
$$sts = tst\,,\ tut = utu\,,\ uvu = vuv\,,\ vsv = svs\,,\ stuvstuvs = tuvstuvst\,.$$

In the following tables, we denote by $H \rtimes K$ a group which is a non-trivial split extension of K by H. We denote by $H \cdot K$ a group which is a non-split extension of K by H. We denote by p^n an elementary abelian group of order p^n.

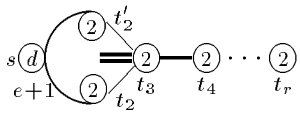
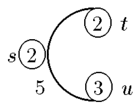
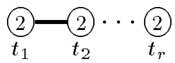
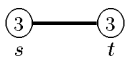
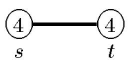
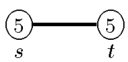
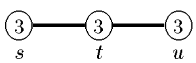
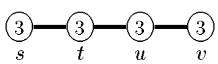
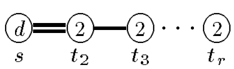
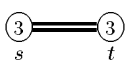
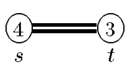
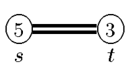
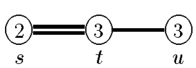
name	**diagram**	**degrees**	**codegrees**	β	**field**	$G/Z(G)$
$G(de,e,r)$ $e\geq 2, d, r\geq 2$		$(ed,2ed,\ldots,(r-1)ed,rd)$	$(0,ed,\ldots,(r-1)ed)$	$s^{\frac{r}{(e\wedge r)}}(t_2't_2t_3\cdots t_r)^{\frac{e(r-1)}{(e\wedge r)}}$	$\mathbb{Q}(\zeta_{de})$	
G_{15}		$12, 24$	$0, 24$	$ustut=s(tu)^2$	$\mathbb{Q}(\zeta_{24})$	$\mathfrak{S}_4$
$\mathfrak{S}_{r+1}$		$(2,3,\ldots,r+1)$	$(0,1,\ldots,r-1)$	$(t_1\cdots t_r)^{r+1}$	$\mathbb{Q}$	
G_4		$4, 6$	$0, 2$	$(st)^3$	$\mathbb{Q}(\zeta_3)$	$\mathfrak{A}_4$
G_8		$8, 12$	$0, 4$	$(st)^3$	$\mathbb{Q}(i)$	$\mathfrak{S}_4$
G_{16}		$20, 30$	$0, 10$	$(st)^3$	$\mathbb{Q}(\zeta_5)$	$\mathfrak{A}_5$
G_{25}		$6, 9, 12$	$0, 3, 6$	$(stu)^4$	$\mathbb{Q}(\zeta_3)$	$3^2\rtimes SL_2(3)$
G_{32}		$12,18,24,30$	$0,6,12,18$	$(stuv)^5$	$\mathbb{Q}(\zeta_3)$	$PSp_4(3)$
$G(d,1,r)$ $d\geq 2$		$(d,2d,\ldots,rd)$	$(0,d,\ldots,(r-1)d)$	$(st_2t_3\cdots t_r)^r$	$\mathbb{Q}(\zeta_d)$	
G_5		$6, 12$	$0, 6$	$(st)^2$	$\mathbb{Q}(\zeta_3)$	$\mathfrak{A}_4$
G_{10}		$12, 24$	$0, 12$	$(st)^2$	$\mathbb{Q}(\zeta_{12})$	$\mathfrak{S}_4$
G_{18}		$30, 60$	$0, 30$	$(st)^2$	$\mathbb{Q}(\zeta_{15})$	$\mathfrak{A}_5$
G_{26}		$6, 12, 18$	$0, 6, 12$	$(stu)^3$	$\mathbb{Q}(\zeta_3)$	$3^2\rtimes SL_2(3)$

TABLE 1

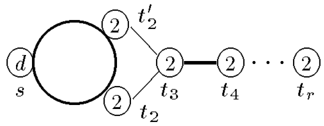
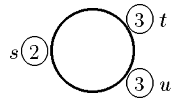
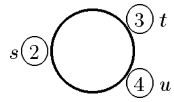
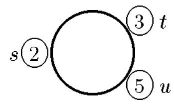
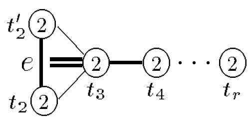
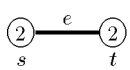
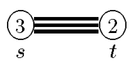
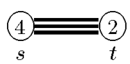
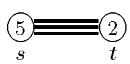
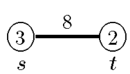
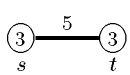
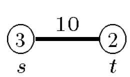
name	**diagram**	**degrees**	**codegrees**	β	**field**	$G/Z(G)$
$G(2d,2,r)$ $d,r\geq 2$		$(2d,4d,\ldots,2(r-1)d,rd)$	$(0,2d,\ldots,2(r-1)d)$	$s^{\frac{r}{(2\wedge r)}}(t_2't_2t_3\cdots t_r)^{\frac{2(r-1)}{(2\wedge r)}}$	$\mathbb{Q}(\zeta_{2d})$	
G_7		$12,12$	$0,12$	stu	$\mathbb{Q}(\zeta_{12})$	$\mathfrak{A}_4$
G_{11}		$24,24$	$0,24$	stu	$\mathbb{Q}(\zeta_{24})$	$\mathfrak{S}_4$
G_{19}		$60,60$	$0,60$	stu	$\mathbb{Q}(\zeta_{60})$	$\mathfrak{A}_5$
$G(e,e,r)$ $e\geq 2,r>2$		$(e,2e,\ldots,(r-1)e,r)$	$(0,e,\ldots,(r-2)e,(r-1)e-r)$	$(t_2't_2t_3\cdots t_r)^{\frac{e(r-1)}{(e\wedge r)}}$	$\mathbb{Q}(\zeta_e)$	
$G(e,e,2)$ $e\geq 3$		$2,e$	$0,e-2$	$(st)^{e/(e\wedge 2)}$	$\mathbb{Q}(\zeta_e+\zeta_e^{-1})$	
G_6		$4,12$	$0,8$	$(st)^3$	$\mathbb{Q}(\zeta_{12})$	$\mathfrak{A}_4$
G_9		$8,24$	$0,16$	$(st)^3$	$\mathbb{Q}(\zeta_8)$	$\mathfrak{S}_4$
G_{17}		$20,60$	$0,40$	$(st)^3$	$\mathbb{Q}(\zeta_{20})$	$\mathfrak{A}_5$
G_{14}		$6,24$	$0,18$	$(st)^4$	$\mathbb{Q}(\zeta_3,\sqrt{-2})$	$\mathfrak{S}_4$
G_{20}		$12,30$	$0,18$	$(st)^5$	$\mathbb{Q}(\zeta_3,\sqrt{5})$	$\mathfrak{A}_5$
G_{21}		$12,60$	$0,48$	$(st)^5$	$\mathbb{Q}(\zeta_{12},\sqrt{5})$	$\mathfrak{A}_5$

TABLE 2

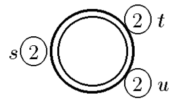
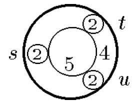
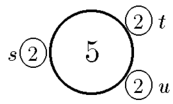
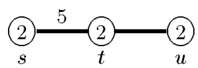
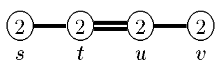
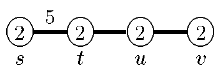
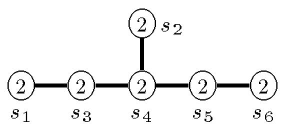
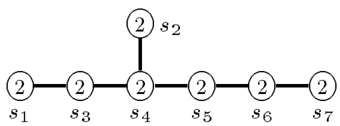
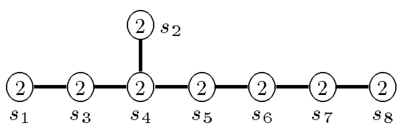
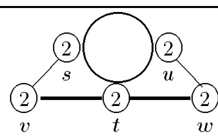
name	diagram	degrees	codegrees	β	field	$G/Z(G)$
G_{12}		6,8	0,10	$(stu)^4$	$\mathbb{Q}(\sqrt{-2})$	$\mathfrak{S}_4$
G_{13}		8,12	0,16	$(stu)^3$	$\mathbb{Q}(\zeta_8)$	$\mathfrak{S}_4$
G_{22}		12,20	0,28	$(stu)^5$	$\mathbb{Q}(i,\sqrt{5})$	$\mathfrak{A}_5$
G_{23}		2,6,10	0,4,8	$(stu)^5$	$\mathbb{Q}(\sqrt{5})$	$\mathfrak{A}_5$
G_{28}		2,6, 8,12	0,4, 6,10	$(stuv)^6$	$\mathbb{Q}$	$2^4{\rtimes}(\mathfrak{S}_3{\times}\mathfrak{S}_3)$ †
G_{30}		2,12, 20,30	0,10, 18,28	$(stuv)^{15}$	$\mathbb{Q}(\sqrt{5})$	$(\mathfrak{A}_5{\times}\mathfrak{A}_5){\rtimes}2$ ‡
G_{35}		2,5,6,8, 9,12	0,3,4,6, 7,10	$(s_1 \cdots s_6)^{12}$	$\mathbb{Q}$	$SO_6^-(2)'$
G_{36}		2,6,8, 10,12, 14,18	0,4,6, 8,10, 12,16	$(s_1 \cdots s_7)^9$	$\mathbb{Q}$	$SO_7(2)$
G_{37}		2,8,12, 14,18,20, 24,30	0,6,10, 12,16,18, 22,28	$(s_1 \cdots s_8)^{15}$	$\mathbb{Q}$	$SO_8^+(2)$
G_{31}		8,12, 20,24	0,12, 16,28	$(stuvw)^6$	$\mathbb{Q}(i)$	$2^4 \rtimes \mathfrak{S}_6$ ⋆

TABLE 3

It is still conjectural whether the corresponding braid diagram for G_{31} provides a presentation for the associated braid group.

† The action of $\mathfrak{S}_3 \times \mathfrak{S}_3$ on 2^4 is irreducible.

‡ The automorphism of order 2 of $\mathfrak{A}_5 \times \mathfrak{A}_5$ permutes the two factors.

⋆ The group $G_{31}/Z(G_{31})$ is not isomorphic to the quotient of the Weyl group D_6 by its center.

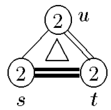
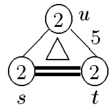
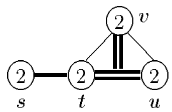
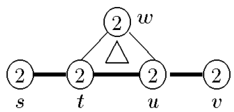
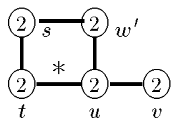
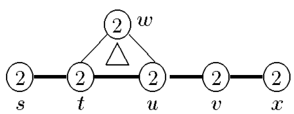
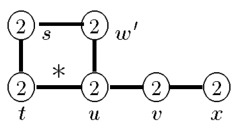
name	diagram	degrees	codegrees	β	field	$G/Z(G)$
G_{24}		4,6,14	0,8,10	$(stu)^7$	$\mathbb{Q}(\sqrt{-7})$	$GL_3(2)$
G_{27}		6,12,30	0,18,24	$(stu)^5$	$\mathbb{Q}(\zeta_3, \sqrt{5})$	$\mathfrak{A}_6$
G_{29}		4,8,12,20	0,8,12,16	$(stuv)^5$	$\mathbb{Q}(i)$	$2^4 \rtimes \mathfrak{S}_5$ †
G_{33}		4,6,10, 12,18	0,6,8, 12,14	$(ustvw)^9$	$\mathbb{Q}(\zeta_3)$	$SO_5(3)'$
						
G_{34}		6,12,18,24, 30,42	0,12,18,24, 30,36	$(stuvwx)^7$	$\mathbb{Q}(\zeta_3)$	$PSO_6^-(3)'\cdot 2$
						

Table 4

These diagrams provide presentations for the corresponding finite groups. It is not known nor conjectural whether they provide presentations for the corresponding braid groups.

† The group $G_{29}/Z(G_{29})$ is not isomorphic to the Weyl group D_5.

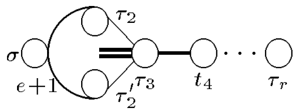
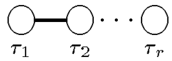
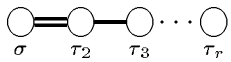
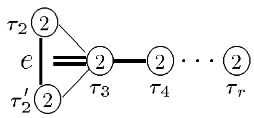
name	diagram	degrees	codegrees	β
$B(de,e,r)$ $e\geq 2, r\geq 2, d>1$		$e, 2e, \ldots, (r-1)e, r$	$0, e, \ldots, (r-1)e$	$\sigma^{\frac{r}{(e\wedge r)}}(\tau_2\tau_2'\tau_3\cdots\tau_r)^{\frac{e(r-1)}{(e\wedge r)}}$
$B(1,1,r)$		$2, 3, \ldots, r+1$	$0, 1, \ldots, r-1$	$(\tau_1\cdots\tau_r)^{r+1}$
$B(d,1,r)$ $d>1$		$1, 2, \ldots, r$	$0, , \ldots, (r-1)$	$(\sigma\tau_2\tau_3\cdots\tau_r)^r$
$B(e,e,r)$ $e\geq 2, r\geq 2$		$e, 2e, \ldots, (r-1)e, r$	$0, e, \ldots, (r-2)e, (r-1)e-r$	$(\tau_2\tau_2'\tau_3\cdots\tau_r)^{\frac{e(r-1)}{(e\wedge r)}}$

Table 5 : braid diagrams

This table provides a complete list of the infinite families of braid diagrams and corresponding data. Note that the braid diagram $B(de,e,r)$ for $e=2$, $d>1$ can also be described by a diagram as the one used for $G(2d,2,r)$ in Table 2. Similarly, the diagram for $B(e,e,r)$, $e=2$, can also be described by the Coxeter diagram of type D_r. The list of exceptional diagrams (but those associated with G_{24}, G_{27}, G_{29}, G_{33}, G_{34}) is identical with table 3.

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