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Complex space approach to perfectly matched layers: a review and some new developments

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SUMMARY

We discuss the interpretation of the perfectly matched layer (PML) absorbing boundary condition (ABC) as an analytic continuation of the coordinate space to a complex variables spatial domain (complex space). The generalization of the PML to curvilinear coordinates and to general linear media using this rationale is reviewed and summarized. The analytic continuation is shown to be equivalent to a change on the metric of the space. By using such geometric viewpoint on the PML, we then discuss the various PML formulations in connection with fundamental symmetries of Maxwell's equations. Copyright © 2000 John Wiley & Sons, Ltd.

1. INTRODUCTION

When applied to open region problems, partial differential equation (PDE) solvers for electromagnetic wave propagation problems, such as the finite-element method (FEM) and the finite-difference time-domain (FDTD) method, must be truncated by means of an absorbing boundary condition (ABC). An efficient ABC should simulate the radiation condition by minimizing reflections from the lattice truncation. This reduces the size of the computational domain and increases the dynamical range of the numerical solution. In addition, an efficient ABC should preserve the (lower) computational complexity associated with the PDE solvers.

The perfectly matched layer (PML), introduced by Berenger [1], is a material ABC which provides reflectionless absorption of electromagnetic waves. Since then, it has been worked on intensely (for a partial list, see References [1–38] and references therein). On the lattice, the PML is not reflectionless anymore, but the reflection is solely caused by the discretization procedure and the reflection levels incurred are order of magnitudes smaller than previously employed ABC.

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Being a local ABC, the PML preserves the computational complexity of PDE solvers and is particularly well suited for parallel computations [3].

As derived in [1], the reflectionless absorption of electromagnetic waves inside the PML is achieved by the introduction of matched electric and magnetic artificial conductivities and by the splitting of the electric and magnetic field components into subcomponents. The reflectionless property is then established through an analysis of the reflection coefficient.

Later, alternative formulations of the PML were developed where no field-splitting was necessary and the PML was represented as an artificial anisotropic medium with time- and field-dependent sources. These formulations support a Maxwellian interpretation in the sense that the resultant fields inside the PML can be associated with physical fields in a medium with properly chosen constitutive parameters [6,7,10–15,20–22,27].

Also, an alternative interpretation for the PML was given in terms of a complex coordinate stretching of the spatial variables of the Fourier domain Maxwell's equations [3]. This was later recognized as an analytic continuation of the co-ordinate space of Maxwell's equations to a complex variables spatial domain (complex space) [16,17]. This interpretation allowed the identification of field solutions inside the PML region as ordinary field solutions of Maxwell's equations subject to such an analytic continuation [16,17,19–21,27]. The reflectionless absorption is then just a direct consequence of the fact that propagating solutions (eigenfunctions) are continuously mapped into exponentially decaying solutions inside the PML. This interpretation motivated the extension of the PML concept to non-planar lattice terminations, such as cylindrical [16,17,19,20], spherical [16,19,20] and conformal [21,38] terminations (alternative approaches for the cylindrical and spherical cases are also considered in References [15,18]), and to more general media, such as (bi-)anisotropic media [27].

In this work, we review the complex space approach to derive perfectly matched layers for the various geometries and media, discuss the relationship between various PML formulations, and relate it with fundamental symmetries of Maxwell's equations. As its title indicates, our review on the PML will be mainly restricted to the complex space approach and, therefore, it is not intended to be complete. Throughout the paper, we work on the Fourier domain with the $e^{-i\omega t}$ convention assumed.

2. COMPLEX SPACE APPROACH TO THE PML

2.1. Analytic viewpoint: the PML as a change of variables

The first use of complex space in electromagnetics apparently harks back to Deschamps, who applied the concept of a point source in complex space to simulate Gaussian beams [39], Subsequently, he used complex extensions of scatterers to trace complex rays to describe scattering from targets in lossy media [40]. Other authors have also applied similar ideas [41–43].

The general connection between the perfectly matched layer and the solutions of Maxwell's equations on a complex space was independently established in References [16,17]. There, it was recognized that, by a simple change of variables, the Fourier domain modified Maxwell's equations for PML media reduce to the ordinary Fourier domain Maxwell's equations in a complex variables spatial domain. This allowed an interpretation of the PML concept as a *mapping* of ordinary solutions of Maxwell's equations in the real coordinate space to a complex coordinate space, i.e. an analytic continuation of the solutions. Under this analytic continuation, the original propagating eigenfunctions are mapped to exponentially decaying eigenfunctions and, in the new boundary value problem, the standard Sommerfeld radiation condition is replaced by a simple uniform boundness condition.

In Cartesian co-ordinates, the analytic continuations is carried out through the following change of variables:

$$\zeta \to \tilde{\zeta} = \int_0^{\zeta} s_\zeta(\zeta')\, \mathrm{d}\zeta' \tag{1}$$

where s_ζ, with $\zeta = x, y, z$, are the complex co-ordinate stretching variables introduced in Reference [3]. Note that, from Equation (1), the new complex-space variables, $\tilde{\zeta}$, are continuous functions of ζ. By choosing

$$s_\zeta(\zeta) = 1 + \mathrm{i}\,\frac{\sigma_\zeta(\zeta)}{\omega} \tag{2}$$

and substituting the above expressions into the Maxwell's equations in Cartesian co-ordinates, we recover the Berenger's PML formulation. The variables σ_ζ are zero in the physical domain and increase monotonically inside the PML [3]. Other frequency dependences are also possible for s_ζ as long as they lead to the absorptive effect. In particular, the following dependence in the Fourier domain was suggested for low-frequency problems in the FEM [28]

$$s_\zeta(\zeta) = 1 + \frac{\sigma_\zeta(\zeta)}{1 - \mathrm{i}\alpha\omega} \tag{3}$$

with $\alpha > 0$, implying a finite $s_\zeta(\zeta)$ at $\omega = 0$. This choice, however, leads to more complicated equations for the PML if a time domain implementation is sought. The use of $\mathrm{Re}[s_\zeta(\zeta)] = a_\zeta(\zeta) \geqslant 1$ has also been employed to achieve additional decay of evanescent waves (which are not affected by the original Berenger's PML) inside the PML. Each different choice for $s_\zeta(\omega)$ leads to a different path on the complex plane over which the analytic continuation on the spatial coordinates is effected. It should also be emphasized that, due to the ω dependence in Equations (2) and (3), each spectral component is analytic continued over a different path.

2.2. *Geometric viewpoint: the PML as a change on the metric of space*

Under the analytic continuation of the co-ordinate space, the elementary arclength of the complex space is written as

$$(\mathrm{d}\tilde{s})^2 = (\mathrm{d}\tilde{x})^2 + (\mathrm{d}\tilde{y})^2 + (\mathrm{d}\tilde{z})^2 \tag{4}$$

or, using Equation (1),

$$(\mathrm{d}\tilde{s})^2 = s_x^2(\mathrm{d}x)^2 + s_y^2(\mathrm{d}y)^2 + s_z^2(\mathrm{d}z)^2 \tag{5}$$

where we used the fact that s_ζ is a function of ζ only. From an analytic viewpoint, this is just an application of the Leibniz rule, but Equation (5) allows us to give a geometric interpretation in terms of a change in the metric of space. From the Euclidean metric tensor,

$$\bar{\mathbf{G}} = \bar{\boldsymbol{\delta}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

we are led to a *complex* metric tensor given in terms of covariant components by

$$\tilde{\bar{\mathbf{G}}}(x, y, z) = \bar{\boldsymbol{\sigma}}(x, y, z) \cdot \bar{\boldsymbol{\delta}} \cdot \bar{\boldsymbol{\sigma}}(x, y, z) \tag{7}$$

where

$$\bar{\boldsymbol{\sigma}}(x, y, z) = \begin{bmatrix} s_x(x) & 0 & 0 \\ 0 & s_y(y) & 0 \\ 0 & 0 & s_z(z) \end{bmatrix} \tag{8}$$

Therefore, in the Fourier domain, the PML may be interpreted as a complexification of the metric tensor of space. The passage from an analytic viewpoint on the PML to a geometric viewpoint is useful in establishing new results, as will become clear later.

2.3. PML in curvilinear coordinates

In this section, we discuss the application of the complex-space approach to produce PMLs in curvilinear mesh terminations, or *conformal* PML. The conformal PML was introduced in Reference [21]. A numerical implementation is presented in Reference [38]. In an orthogonal curvilinear co-ordinate system (u, v, w), if we choose w to be analytically continued as

$$w \to \tilde{w} = \int_0^w s_w(w')\, \mathrm{d}w' \tag{9}$$

and if the original metric tensor are given in terms of the metric coefficients h_i as

$$\bar{\mathbf{G}} = \begin{bmatrix} (h_u)^2 & 0 & 0 \\ 0 & (h_v)^2 & 0 \\ 0 & 0 & (h_w)^2 \end{bmatrix} \tag{10}$$

then this metric tensor under $w \to \tilde{w}$ is mapped to

$$\bar{\mathbf{G}} \to \tilde{\bar{\mathbf{G}}} = \begin{bmatrix} (\tilde{h}_u)^2 & 0 & 0 \\ 0 & (\tilde{h}_v)^2 & 0 \\ 0 & 0 & (\tilde{h}_w)^2 \end{bmatrix} \tag{11}$$

where $\tilde{h}_u = h_u(u, v, \tilde{w})$, $\tilde{h}_v = h_v(u, v, \tilde{w})$, and $\tilde{h}_w = s_w(w) h_w(u, v, \tilde{w})$. The new metric can be recast as

$$\tilde{\bar{\mathbf{G}}}(u, v, w) = \bar{\boldsymbol{\sigma}}(u, v, w) \cdot \bar{\mathbf{G}}(u, v, w) \cdot \bar{\boldsymbol{\sigma}}(u, v, w) \tag{12}$$

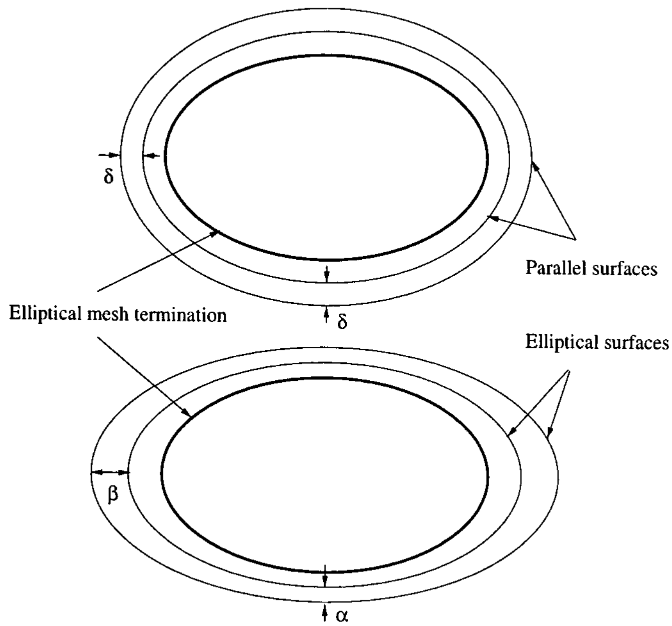

Figure 1. (Upper): Depiction of PML layers in curvilinear co-ordinates (conformal PML). A *local* co-ordinate system is attached to the termination surface (an ellipse in the Figure), with local metric coefficients depending on the local geometry. The analytic continuation is carried out along the normal co-ordinate. The resulting PML is defined over layers which comprise *parallel surfaces* to the mesh termination. (Lower): A system of elliptical surfaces, defined through a *global* elliptical system. The resulting surfaces are not parallel anymore.

with

$$\bar{\boldsymbol{\sigma}}(u,v,w) = \begin{bmatrix} (\tilde{h}_u/h_u) & 0 & 0 \\ 0 & (\tilde{h}_v/h_v) & 0 \\ 0 & 0 & (\tilde{h}_w/h_w) \end{bmatrix} \tag{13}$$

The PML for doubly curved mesh terminations (or *conformal* PML) [21] is derived by first attaching a *local* co-ordinate system to the termination surface, with orthogonal co-ordinates along the normal to the surface, and along the principal curvatures, and then enforcing the analytic continuation on the normal co-ordinate so that outward propagating modes are mapped into exponentially decaying (along the normal direction) modes. If w is the normal co-ordinate to the mesh termination (defined by $w = 0$), and if we set $h_3 = 1$, the conformal PML is built over *parallel surfaces* to the mesh termination (i.e. $w = c, c \geqslant 0$), as illustrated in Figure 1. The transverse metric coefficients h_u, h_v on this co-ordinate system are given as $h_u = (r_u + w)/r_u$, $h_v = (r_v + w)/r_v$, where r_u, r_v are the local radii of curvature of the termination surface and u, v, the principal directions on the termination surface.

In this case, the original metric tensor of such a co-ordinate system

$$\bar{\mathbf{G}}(u,v,w)=\begin{bmatrix}(h_u)^2 & 0 & 0\\ 0 & (h_v)^2 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{14}$$

is mapped to

$$\tilde{\bar{\mathbf{G}}}(u,v,w)=\bar{\boldsymbol{\sigma}}(u,v,w)\cdot\bar{\mathbf{G}}(u,v,w)\cdot\bar{\boldsymbol{\sigma}}(u,v,w) \tag{15}$$

inside the PML, with

$$\bar{\boldsymbol{\sigma}}(u,v,w)=\begin{bmatrix}(\tilde{h}_u/h_u) & 0 & 0\\ 0 & (\tilde{h}_v/h_v) & 0\\ 0 & 0 & s_w\end{bmatrix} \tag{16}$$

where $\tilde{h}_u=(r_u+\tilde{w})/r_u$, $\tilde{h}_v=(r_v+\tilde{w})/r_v$, and $\tilde{h}_w=s_w$ since $h_w=1$. It is important to observe that, in order to achieve a proper homotopy between the ordinary fields and the resultant fields after the analytic continuation—so that the fields inside the PML can be written as the original fields after a change of variable as in Equation (9) and no reflection is incurred—it is important to have the metric coefficients properly modified according to Equation (16), since they are, in general, function of the co-ordinates themselves. This peculiarity is not encountered in the Cartesian PML case because, in that case, the metric coefficients are independent of the spatial coordinates. In early developments of the PML concept, extensions of the Cartesian PML to curvilinear co-ordinates were suggested in which the metric coefficients were left unchanged, leading to only an approximate PML. This is discussed in more detail in Reference [29].

The equations for the curvilinear PML are then written as usual Fourier domain Maxwell's equations in an orthogonal curvilinear coordinate system but with the metric coefficients given by Equation (16). Because the vector calculus spatial operators depend on a metric, the complex space PML implementation amounts to a modification of these operators. For example, the gradient operator in the complex space for general orthogonal curvilinear co-ordinates, under the analytic continuation on $w(u,v,w)\to(u,v,\tilde{w})$ is changed to

$$\nabla=\hat{\mathbf{u}}\frac{1}{h_u}\frac{\partial}{\partial u}+\hat{\mathbf{v}}\frac{1}{h_v}\frac{\partial}{\partial v}+\hat{\mathbf{w}}\frac{1}{h_w}\frac{\partial}{\partial w}\to\tilde{\nabla}=\hat{\mathbf{u}}\frac{1}{\tilde{h}_u}\frac{\partial}{\partial u}+\hat{\mathbf{v}}\frac{1}{\tilde{h}_v}\frac{\partial}{\partial v}+\hat{\mathbf{w}}\frac{1}{\tilde{h}_w}\frac{\partial}{\partial w} \tag{17}$$

where we see that the modified operator $\tilde{\nabla}$ is just the usual gradient operator for (u,v,w) defined by the complex metric tensor given by Equation (16). Similar changes result for the curl and div operators. Note that, over the parallel surfaces which comprise the curvilinear PML, the unit vectors $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ are functions of u and v only and, therefore, the analytic continuation on w has no effect on them.

A brief digression should be made at this point about the analyticity of the resultant PML field solutions on the complex ω plane. An important characteristic of the eigenfunctions of Maxwell's equations for a passive medium is that they are free of singularities in the upper-half of the complex ω plane. This implies bounded (dynamically stable) solutions. For a PML defined over a surface with $r_u, r_v > 0$ (concave or planar termination when viewed from inside the computational domain), it can be shown that the singularities are translated downwards in the complex ω plane [30,31], so that the eigenfunctions after the analytic continuation are kept free of singularities over that domain. However, for a PML defined over a surface with r_u or r_v negative (e.g. convex terminations when viewed from inside the computational domain), the singularities are translated upwards in the complex ω plane, so that the resultant analytically continued eigenfunctions will eventually possess singularities in the upper-half ω plane [30,31]. As a result, dynamically unstable time-domain solutions are produced, which, in principle, precludes the application of the PML to such geometries. A more detailed analysis of this phenomenon is presented in References [30,31].

The Cartesian, cylindrical, and spherical PML's are special cases of the general orthogonal curvilinear case, followed (possibly) by a successive application of the analytic continuation in orthogonal directions, if needed to achieve absorption in corner regions. For example, in cylindrical co-ordinates, if we choose $u = \rho, v = \phi, w = z$, the metric tensor is given by $h_\rho = 1, h_\phi = \rho$, and $h_z = 1$, or

$$\bar{\mathbf{G}}(\rho) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

If we enforce an analytic continuation along ρ, i.e. $\rho \to \tilde{\rho}$, the end result is equivalent to changing the metric tensor to

$$\bar{\mathbf{G}}(\rho) \to \tilde{\bar{\mathbf{G}}}(\rho) = \bar{\boldsymbol{\sigma}}(\rho) \cdot \bar{\mathbf{G}}(\rho) \cdot \bar{\boldsymbol{\sigma}}(\rho) \tag{19}$$

with

$$\bar{\boldsymbol{\sigma}}(\rho) = \begin{bmatrix} s_p & 0 & 0 \\ 0 & \tilde{\rho}/\rho & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

In cylindrical co-ordinates, since the z co-ordinate is everywhere orthogonal to ρ, we may also simultaneously enforce an analytic continuation on $z, z \to \tilde{z}$, so that the above matrix $\bar{\boldsymbol{\sigma}}(\rho)$ is changed to

$$\bar{\boldsymbol{\sigma}}(\rho, z) = \begin{bmatrix} s_p & 0 & 0 \\ 0 & \tilde{\rho}/\rho & 0 \\ 0 & 0 & s_z \end{bmatrix} \tag{21}$$

A digression about the use of a *local* co-ordinate system for the definition of the conformal PML is in order. For arbitrary mesh termination surfaces, the use of global co-ordinate systems is not always adequate because it is restricted to cases where termination surfaces are equivalent by constant co-ordinate surfaces on the global co-ordinate system. This is also true for other ABCs [44, 45]. Moreover, the perfect matching condition does not require a global co-ordinate system to be established because the perfect matched condition is a local (boundary) condition on the fields (see next section).

The use of a local co-ordinate system (u, v, w) with the metric coefficients given by $h_u = (r_u + w)/r_u$, $h_v = (r_v + w)/r_v$, and $h_w = 1$, produces a system parallel surfaces (constant w) to the mesh termination, as illustrated in Figure 1. The conformal PML reduces to the Cartesian, cylindrical, and spherical PMLs for the case of planar, cylindrical, and spherical terminations, because, in these co-ordinate systems, surfaces of constant co-ordinate define parallel surfaces themselves. However, this is not the case in other global co-ordinate systems. For instance, in a elliptical system, surfaces of constant co-ordinate define elliptical surfaces which are not parallel to each other. The conformal PML defined using a local co-ordinate system attached to each point of an elliptical mesh termination defines parallel surfaces to the elliptical termination, not ellipses themselves, as illustrated in Figure 1. In other words, the conformal PML given by Equation (15) when defined over an elliptical surface is not equivalent to enforcing analytical continuation on the (global) radial co-ordinate of the elliptical system. This can also be established from the fact, that in elliptical co-ordinates, the metric coefficients do not correspond to the ones given by Equation (14) with $h_u = (r_u + w)/r_u$, $h_v = (r_v + w)/r_v$, and $h_w = 1$.

2.4. Perfect matching condition via complex-space

For completness, we present here a simple proof of the perfect matching condition for electromagnetic wave phenomena, using the complex space approach. The proof is based on the continuity of the fields everywhere.

We start by considering an open domain $\Omega \subset R^3$, where electromagnetic fields **E**, **H**, **D**, and **B** are present.

Lemma 1

Suppose $s_w(w)$ is a bounded, piecewise continuous function in some interval $I_w = \{w \in R \,|\, 0 \leqslant w \leqslant w_f\}$. Then, the complex spatial co-ordinate $\tilde{w}(w)$ is a C^n function of the real co-ordinate w, for any $w \in I_w$, where $n \geqslant 0$.

Proof

The proof is trivial if we write the expression for the complex space co-ordinate given by Equation (9):

$$\tilde{w} = \int_0^w s_w(w')\,\mathrm{d}w' \tag{9}$$

with $w \in I_w$. From the hypothesis on $s_w(w)$, the integral above exists, and $\tilde{w}$ is a C^n function in I_w, $n \geqslant 0$. In particular, if $s_w(w)$ have discontinuities (in numerical implementations, $s_w(w)$ is usually chosen to be continuous to minimize spurious reflections due to discretization, but in the continuum theory, continuity on $s_w(w)$ is not required), then $\tilde{w}(w)$ is a C^0 function in I_w.

Note that the important condition here is to have the complex stretching co-ordinate s_w being a function of *w only*. This precludes the application to non-separable geometries in a global co-ordinate system.

Lemma 2

Introduce an orthogonal and (possibly) curvilinear co-ordinate system (u, v, w) in Ω, such that, for any point $P \in \Omega$, we have $w \in I_w$. If the electromagnetic fields **E**, **H**, **D**, **B** are continuous functions of the co-ordinates (u, v, w) for any $P \in \Omega$, then the resultant fields under the analytic continuation $w \rightarrow \tilde{w}$, $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{B}}$, are continuous functions of (u, v, w) for any $P \in \Omega$.

Proof

Under the analytic continuation on the co-ordinate w, we define $\tilde{\psi}(u, v, w) = \psi(u, v, \tilde{w})$ where $\psi(u, v, w)$ is any electromagnetic field component. By hypothesis, $\psi(u, v, w)$ is continuous in (u, v, w) for any $P \in \Omega$. Since $\tilde{w}(w)$ is continuous in I_w by Lemma 1, then, by the continuity of compositions, the composition $\tilde{\psi}_i(u, v, w) = \tilde{\psi}_i(u, v, \tilde{w}(w))$ is continuous in terms of the variables (u, v, w) for any $P \in \Omega$.

Theorem

The analytic continuation $w \rightarrow \tilde{w}$ preserves the electromagnetic boundary conditions on the transformation **E**, **H**, **D**, **B** $\rightarrow$ $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{B}}$ through interfaces S_{w_0}, defined by the constraints $w = w_0$ (surfaces), for all $w_0 \in I_w$.

Proof

The source-free boundary conditions for electromagnetic fields at each point (u, v) of the interface S_{w_0} are given by

$$\mathbf{n} \times [\mathbf{E}(u, v, w^+) - \mathbf{E}(u, v, w^-)] = 0 \tag{22a}$$

$$\mathbf{n} \times [\mathbf{H}(u, v, w^+) - \mathbf{H}(u, v, w^-)] = 0 \tag{22b}$$

$$\mathbf{n} \cdot [\mathbf{D}(u, v, w^+) - \mathbf{D}(u, v, w^-)] = 0 \tag{22c}$$

$$\mathbf{n} \cdot [\mathbf{B}(u, v, w^+) - \mathbf{B}(u, v, w^-)] = 0 \tag{22d}$$

where $w^{\pm} = \lim_{\delta \to 0} (w_0 \pm \delta)$, $\mathbf{n} = (\partial \mathbf{r}/\partial w)/|\partial \mathbf{r}/\partial w|$, is the outward unit normal, and **r** is the position vector. Under the analytic continuation $w \rightarrow \tilde{w}$, the above become

$$\mathbf{n} \times [\mathbf{E}(u, v, \tilde{w}^+) - \mathbf{E}(u, v, \tilde{w}^-)] = \mathbf{F}_1(u, v) \tag{23a}$$

$$\mathbf{n} \times [\mathbf{H}(u, v, \tilde{w}^+) - \mathbf{H}(u, v, \tilde{w}^-)] = \mathbf{F}_2(u, v) \tag{23b}$$

$$\mathbf{n} \cdot [\mathbf{D}(u, v, \tilde{w}^+) - \mathbf{D}(u, v, \tilde{w}^-)] = \varphi_1(u, v) \tag{23c}$$

$$\mathbf{n} \cdot [\mathbf{B}(u, v, \tilde{w}^+) - \mathbf{B}(u, v, \tilde{w}^-)] = \varphi_2(u, v) \tag{23d}$$

where $\mathbf{F}_i(u, v)$, and $\varphi_i(u, v)$, $i = 1, 2$, are functions with support on S_{w_0}. Electromagnetic boundary conditions are preserved iff $\mathbf{F}_i(u, v) = 0$ and $\varphi_i(u, v) = 0$, $i = 1, 2$, for all points (u, v) in S_{w_0}. From Lemma 2, we write

$$\mathbf{E}(u, v, \tilde{w}^-) = \tilde{\mathbf{E}}(u, v, w^-) = \tilde{\mathbf{E}}(u, v, w^+) = \mathbf{E}(u, v, \tilde{w}^+) \tag{24}$$

and similarly for the other fields, for any $w_0 \in I_w$. Therefore, substituting Equation (24) into Equations (23), and similarly for the other fields, we arrive at $\mathbf{F}_i(u, v) = 0$ and $\varphi_i(u, v) = 0, i = 1, 2$, for all points (u, v) in S_{w_0}, so that boundary conditions are preserved through the interface S_{w_0} for any $w_0 \in I_w$. □

The proof is very simple and it is independent of the constitutive parameters on Ω. There is no need to determine reflection coefficients or matched artificial conductivities. The perfect matching essentially follows because $w \to \tilde{w}$ preserves continuity of the co-ordinate space and therefore, the continuity of the fields after mapping. Note also that no mention is made about the particular geometry of the surface defined by $w = w_0$. The perfect matching is obtained for any smooth surface, although, as discussed in Section 2.3, the resulting effect on the fields (i.e. damping or not) induced by $w \to \tilde{w}$ strongly depends on the geometry of the surface termination.

2.5. Analytic continuation as a general ABC for linear wave phenomena

Because the analytic continuation acts on the co-ordinate space only, being transparent to the particular constitutive equations of the medium, it is equally applicable to Maxwell's equations in media with dispersive and/or (bi-)anisotropic behavior [27]. To derive a PML in such a media, one may simply assume the same constitutive parameters everywhere and enforce the analytic continuation on the PML region.

Furthermore, the analytic continuation concept is also applicable to other kinds of linear wave phenomena, to produce PML ABCs in those cases, such as scalar PML [32], dispersive-media PML [33], paraxial PML [34], and elastic PML [35]. As in the case of Maxwell's equations, the general effect of this analytic continuation is to continuously map outward propagating modes (eigenfunctions) into exponentially decaying modes.

3. THE MAXWELLIAN PML AND THE METRIC INVARIANCE OF MAXWELL'S EQUATIONS

In this section, we discuss the fundamental relationship between the PML discussed in the previous section, derived through analytic continuation of Maxwell's equations and the Maxwellian PML, where Maxwell's equations are retained and the PML is represented as an artificial media with properly chosen constitutive parameters [6,7,10–15,20–22,27]. The discussion encompasses both the Cartesian PML and curvilinear PML, and is carried out in connection with symmetries of Maxwell's equations.

The Maxwellian PML was first derived to match isotropic, dispersion-less media in Cartesian co-ordinates [6]. The basic difference from the PML derived through the complex space approach is that the form of Maxwell's equations in the PML is preserved. Instead of modifying the spatial operators on the equations, the Maxwellian PML modifies only the constitutive parameters. Among the advantages of a Maxwellian PML is that it provides easier interfacing with the finite element method. Furthermore, the strongly well-posed property of the Maxwell system (symmetric hyperbolic system) is retained (in contrast to the Berenger split-field PML system, which is only weakly well posed) [22,36].

In later developments of the PML theory, it was shown that a Maxwellian PML also exists for general orthogonal curvilinear coordinates [21], as well as for more general linear media (i.e. with dispersive and (bi-)anisotropic behavior, possibly simultaneously) [27]. This includes all cases in which the PML was derived using the complex space approach. The derivation of the Maxwellian PML in such cases was done systematically, starting from the complex-space PML formulation and applying properly chosen field transformations on the electromagnetic fields [27]. These field transformations are given by

$$\mathbf{E}^{\mathrm{a}} = \bar{\boldsymbol{\sigma}} \cdot \mathbf{E}^{\mathrm{c}} \tag{25a}$$

$$\mathbf{H}^{\mathrm{a}} = \bar{\boldsymbol{\sigma}} \cdot \mathbf{H}^{\mathrm{c}} \tag{25b}$$

$$\mathbf{D}^{\mathrm{a}} = (\det \bar{\boldsymbol{\sigma}})\, \bar{\boldsymbol{\sigma}}^{-1} \cdot \mathbf{D}^{\mathrm{c}} \tag{25c}$$

$$\mathbf{B}^{\mathrm{a}} = (\det \bar{\boldsymbol{\sigma}})\, \bar{\boldsymbol{\sigma}}^{-1} \cdot \mathbf{B}^{\mathrm{c}} \tag{25d}$$

Where the superscript 'a' refers to Maxwellian PML fields and the superscript 'c' refers to complex space PML fields. Because the only (possibly) discontinuous element in $\bar{\boldsymbol{\sigma}}$ corresponds to the normal component (see Equation (16)), the above transformations for **E** and **H** preserve continuity of the tangential components. Similarly, the above transformations for **D** and **B** preserve continuity of the normal components. Because of this, Equations (25) preserve the electromagnetic boundary conditions (and, therefore, perfect matching conditions) given by Equations (22), independently of the underlying constitutive parameters, $\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\zeta}}$, and $\bar{\boldsymbol{\xi}}$, considered. In contrast to the complex space PML, however, where all field components are continuous over PML interfaces (as seen in Section 2.4), this is not necessarily true anymore for the normal components of **E** and **H**, and for the tangential components of **B** and **D**.

The Maxwellian PML is characterized through the insertion of artificial complex material constitutive tensors for the permeability and permittivity tensors within the PML layer. For the isotropic, dispersionless media case, these tensors were first derived for Cartesian co-ordinates in Reference [6], and are given in curvilinear co-ordinates [21] by $\bar{\boldsymbol{\varepsilon}} = \varepsilon \bar{\boldsymbol{\Lambda}}$ and $\bar{\boldsymbol{\mu}} = \mu \bar{\boldsymbol{\Lambda}}$, with

$$\bar{\boldsymbol{\Lambda}} = (\det \bar{\boldsymbol{\sigma}}) \bar{\boldsymbol{\sigma}}^{-1} \cdot \bar{\boldsymbol{\sigma}}^{-1} \tag{26}$$

represented in terms of the basis of unit vectors $\{\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}\}$, and with $\bar{\boldsymbol{\sigma}}$ defined in Section 2. We note that, since the matrices $\bar{\boldsymbol{\sigma}}$ depend on the local radii of curvature, these equations essentially state that the local constitutive parameters depend on the local termination surface, which reflects an interesting interplay between physics (constitutive parameters) and geometry (termination surface).

For a general, bianisotropic linear media characterized by the constitutive parameters $\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\zeta}}$, and $\bar{\boldsymbol{\xi}}$, the corresponding Maxwellian PML, derived in Reference [27], is characterized by constitutive tensors given by

$$\bar{\boldsymbol{\varepsilon}}_{\mathrm{PML}} = (\det \bar{\boldsymbol{\sigma}})\, \bar{\boldsymbol{\sigma}}^{-1} \cdot \bar{\boldsymbol{\varepsilon}} \cdot \bar{\boldsymbol{\sigma}}^{-1} \tag{27a}$$

$$\bar{\boldsymbol{\mu}}_{\mathrm{PML}} = (\det \bar{\boldsymbol{\sigma}})\, \bar{\boldsymbol{\sigma}}^{-1} \cdot \bar{\boldsymbol{\mu}} \cdot \bar{\boldsymbol{\sigma}}^{-1} \tag{27b}$$

$$\bar{\boldsymbol{\zeta}}_{\mathrm{PML}} = (\det \bar{\boldsymbol{\sigma}})\, \bar{\boldsymbol{\sigma}}^{-1} \cdot \bar{\boldsymbol{\zeta}} \cdot \bar{\boldsymbol{\sigma}}^{-1} \tag{27c}$$

$$\bar{\boldsymbol{\xi}}_{\mathrm{PML}} = (\det \bar{\boldsymbol{\sigma}})\, \bar{\boldsymbol{\sigma}}^{-1} \cdot \bar{\boldsymbol{\xi}} \cdot \bar{\boldsymbol{\sigma}}^{-1} \tag{27d}$$

The fact that it is possible to derive Maxwellian PMLs for the different co-ordinate systems (Cartesian, cylindrical, spherical), for general orthogonal curvilinear co-ordinates, and also for the general media case, is too conspicuous to be considered a mere coincidence. Instead, it should be treated as an indication of some inherent property of Maxwell's equations.

Maxwell's equations, like most equations of physics, are surprisingly rich in symmetries. Indeed, from a modern viewpoint, some symmetries form a principle from which field theories can be derived. Most symmetries can be easily traced in the vector calculus language. Others, however, are not adequately appreciated using this language. Our symmetry of interest here is an essentially geometric one: the metric invariance of Maxwell's equations, first discovered by Weyl and Cartan, and rediscovered a number of times [46,47]. The importance of this symmetry in our context arises because we have shown that the PML can be viewed as a complexification of the metric of space. In the vector language, the metric invariance is not obviated because the metric and topological structures of Maxwell's equations are intertwined. To uncover this invariance, a language which decomposes these two structures should be used. Such language already exists, but is hardly used (basically due to the lack of exposure in the engineering literature) by the electromagnetics community: the language of differential forms (or, simply, forms).

The use of forms as a concise and elegant language for electromagnetics has been pioneered by Deschamps [47]. Applications can be found, e.g. in [37,48–53]. In terms of forms, the source-free Maxwell's equations are written as

$$\mathrm{d}E = \mathrm{i}\omega B \tag{28a}$$

$$\mathrm{d}H = -\mathrm{i}\omega D \tag{28b}$$

$$\mathrm{d}D = 0 \tag{28c}$$

$$\mathrm{d}B = 0 \tag{28d}$$

where E and H are electric and magnetic field intensity 1-forms, D and B are electric and magnetic flux density 2-forms. The operator d is the usual exterior derivative, which plays the role of the curl and div operators of vector calculus. The exterior derivative is a purely topological operator. Maxwell's equations in the above form are manifestly invariant under diffeomorphisms, which is in contrast to the corresponding vector calculus equations.

Constitutive relations relate the 1-forms E, H to the 2-forms D, B and are given in terms of the so-called Hodge operators $\star_{\mathrm{e}}$ and $\star_{\mathrm{h}}$ as $D = \star_{\mathrm{e}} E$ and $B = \star_{\mathrm{h}} H$ [48–50]. The Hodge operators establish an isomorphism between 1- and 2-forms, and contain all information about the metric of space. The PML is obtained through the maps $\star_{\mathrm{e}} \rightarrow \tilde{\star}_{\mathrm{e}}$ and $\star_{\mathrm{h}} \rightarrow \tilde{\star}_{\mathrm{h}}$, via the complexification of the metric given by Equation (15). The resultant forms inside the PML, $\tilde{E}, \tilde{D}, \tilde{H}$, and $\tilde{B}$ therefore obey modified Hodge relations $\tilde{D} = \tilde{\star}_{\mathrm{e}}\tilde{E}$, and $\tilde{B} = \tilde{\star}_{\mathrm{h}}\tilde{H}$, but the form of Equations (28) is preserved.

The different PML formulations in the vector language arises from how to map the (unique) form fields to corresponding vector fields. This mapping is also an isomorphism governed by the metric of space [47–50]. If the original, real metric is chosen, then the Maxwellian PML formulation is recovered. Alternatively, if the complex metric is chosen, then the complex-space PML formulation is recovered. This is discussed in more detail in Reference [37]. The forms viewpoint also reveals that if other metrics are chosen to govern the form-to-vector isomorphism, other PML formulations in the Fourier domain are possible [37]. The choice of metrics should be consistent in preserving the perfect matching conditions and reducing to the real metric in the interior (physical) domain. In this context, the complex space PML and Maxwellian PML are special cases for these choices.

4. CONCLUSIONS

We have presented a review and summary of new developments on the complex space approach to the PML. We have discussed the PML as an analytic continuation of Maxwell's equations such that the original propagating eigenfunctions are mapped into exponentially decaying eigenfunctions. The extensions of the PML for general orthogonal curvilinear coordinates (conformal PML) and for general media (dispersive and/or (bi-)anisotropic) are reviewed. A geometrical viewpoint of the PML as a change on the metric of space is also discussed. Using this new interpretation, the relationship between the various PML formulations (complex space and Maxwellian) is clarified in view of the metric invariance of Maxwell's equations.

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