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Complex T-Spherical Fuzzy Relations With Their Applications in Economic Relationships and International Trades

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ABSTRACT Uncertainty is the unavoidable part of the life. In almost all circumstances, we regularly find ourselves in a state of uncertainty. Several reasons can lead to uncertainty, such as randomness, vagueness and rough knowledge. Fuzzy set (FS) theory deals with these kinds of information. Many generalizations had been made in the theory of FSs, such as intuitionistic FSs (IFSs), q-rung orthopair FSs (qROFSs), complex qROFSs (CqROFSs), spherical FSs (SFSs), T-spherical FSs (TSFSs), and complex TSFSs (CTSFSs). Among these generalizations of FSs, the CTSFSs are the most dominant generalization of the FSs. Although fuzzy relations (FRs), IF relations (IFRs) and complex FRs (CFRs) were defined in the literature, the concepts of relations have not yet been introduced in the CTSFSs. This paper unveils the novel concept of CTSF relations (CTSFRs), which provides the extensive generalizations of FRs. The proposed CTSFRs can give many generalized types of FRs, such as IFRs, CFRs, Pythagorean FRs, qROFRs, SFRs and TSFRs, etc. Additionally, some useful properties and results are obtained for CTSFRs. Moreover, a couple of applications demonstrate the usefulness of the proposed concepts. These CTSFRs can be used to depict the time-related interdependence of global market. Thus, we apply these CTSFRs to analyze the interdependence of the international trades among countries and compare the financial factors affecting business markets. Furthermore, the economic relationships with respect to time lag can be modeled by using the CTSFSs and the CTSFRs. Finally, a comparative analysis illuminates the supremacy of the proposed way in contrast with the existing ones.

INDEX TERMS Fuzzy sets (FSs), T-spherical FSs (TSFSs), complex TSFSs (CTSFSs), CTSF relations (CTSFRs), Cartesian product in CTSFSs, CTSF composite relation, financial and economic relationships.

I. INTRODUCTION

Nearly in all circumstances, we find ourselves in some sort of uncertain situation. Uncertainty has many causes, ranging from almost complete absence of awareness or belief to just falling short of conviction. Probability has been used to measure and cope the uncertainty of randomness, such as outcomes of rolling dice or tossing a coin. Uncertainty also occurs due to vagueness or imprecise information. In 1965, Zadeh [1] proposed fuzzy sets (FSs) and fuzzy logics which can deal with the uncertainty of vagueness. Unlike probability theory, fuzzy theory processed the data and information instead of demanding data element to be declared as member or non-member of the set, FSs introduced the concept of partial membership. An FS assigns the membership grades ranging between 0 and 1, to each object in the set. FSs soon caught the attention of researchers of the time. Atanassov [2] came up with a new idea of intuitionistic FSs (IFSs) and modified the FSs by adding the non-membership grade. The values of membership, non-membership grades and their sum belong to the unit interval. The advantage of an IFS over an FS is that it discusses the level of satisfaction as well as the dissatisfaction in terms of membership and non-membership grades. Yager [3] sensed the restrictions in the IFSs so he devised the concept of Pythagorean FSs (PyFSs) by modifying the constraints in IFSs. According to the innovative constraints, the sum of the squares of membership and non-membership grades must be in the unit interval

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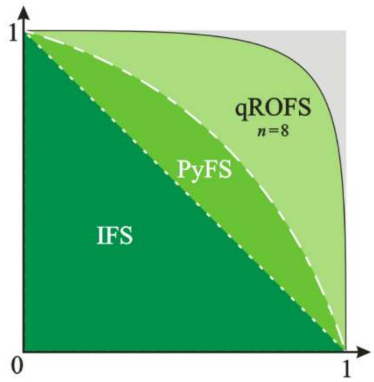

FIGURE 1. Evaluation of ranges of IFS, PyFS and qROFS.

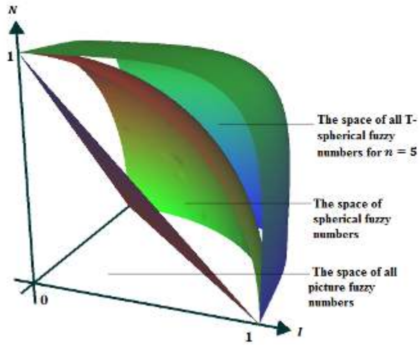

FIGURE 2. Evaluation of ranges of PFS, SFS and TSFS.

which expands the codomain. Furthermore, Yager [4] again realized the limitations in the assignment of the grades to objects in PyFSs, consequently he presented the notion of q-rung orthopair FSs (qROFSs). These sets allow the professionals and researchers to assign the membership and non-membership grades freely by relaxing the restrictions in the previous set theories. In qROFSs the sum of n^{th} power of membership and non-membership grades must be in the unit interval, where n is a positive integer. Figure 1 depicts the comparison among the ranges of IFS, PyFS and qROFS. The largest area of the plane is covered under the curve of the qROFS for $n = 8$, that area further extends by increasing the value of n. So qROFS was considered to be the most reliable generalization of FSs. Garg [5] introduced the linguistic PFSs along with their applications in multiattribute decision-making process. Yang and Hussain [6] defined the fuzzy entropy for PFSs with their application to multicriterion decision-making. Moreover, Hussain and Yang [7] also defined the distance and similarity measures of PFSs with application to TOPSIS. Zhou *et al.* [8] discussed a new divergence measure of PFSS and applied it in the medical diagnosis. Yang *et al.* [9] proposed the belief and plausibility measure on IFSs with construction of belief-plausibility TOPSIS. Faizi *et al.* [10] used the characteristic objects method for multicriterian group decision-making problems through IFSs and Zeng *et al.* [11] proposed a fuzzy methodology, neutrosophic fuzzy set, for sustainable supplier selection based on fuzzy information.

Later on, the inclusion of abstinence grade lead to the concept of picture FS (PFS) which was developed by Cuong [12]. The membership, abstinence and non-membership grades come from the unit interval such that the sum of all three grades is contained within the fuzzy interval. Like IFSs, PFSs are also held back by the same restrictions. Ullah *et al.* [13] came up with the notions of spherical FS (SFS) by adjusting the constraints, so that the sum of the squares of membership, abstinence and non-membership grades lie in the unit interval. Moreover, Ullah *et al.* [13] also defined the T-spherical FSs (TSFSs) because of the limitations that PFSs and SFSs had. In TSFSs, the sum of membership, abstinence and non-membership grades when raised to the power n must be contained in the unit interval, where n is a natural number. TSFS are the generalizations of all previous set theories and thus the most powerful tool in the field. In PFSs, all the three grades must sum up to a number in unit interval, there is a great boundedness. Say, if membership, abstinence and non-membership grades are 0.3, 0.4, 0.5 respectively, then PFSs fail to work because they sum-up to 1.2. But SFSs can tackle such situations, the sum of squares of three grades of SFSs should lie in the unit interval. Again, SFSs are limited because they cannot handle a situation when membership, abstinence and non-membership grades are 0.8, 08, 0.9 respectively. In these kinds of situations, the TSFS is the most reliable tool as the grades can be raised to sufficiently high exponent so that the sum ends up in the unit interval. Figure 2 clearly portrays the grander codomain of the TSFSs as compared to that of PFSs and SFSs. Mahmood *et al.* [14] used SFSs in medical diagnosis and decision making problems. Gundogdu and Kahraman [15]–[17] discussed the SFSs and spherical fuzzy TOPSIS method, extension of WASPAS with SFSs and spherical fuzzy hierarchy process and its energy applications. Ashraf et al. [18] applied the concept of SFSs in multi-attribute decision making problems. Garg *et al.* [19] provided the algorithm for TSF multi attribute decision making based on aggregation operators. Ullah *et al.* [20] devised the correlation coefficients for TSFSs and applied them in clustering and multi attribute decision making and Wu *et al.* [21] proposed the divergence measure for TFS with application in pattern recognition.

Although, TSFSs are absolutely great in coping with many uncertainty problems, but they could really not tackle the problems with multi-dimensions. Henceforth, Ramot *et al.* [22] changed the membership grade of the FSs from a real number in unit interval to a complex number in the unit disc in complex plane, and initiated the concept of complex FSs (CFSs). The membership grade being a complex number is expressed in the polar form as $\alpha_{\mathbb{C}}(\mathfrak{X})\, e^{\beta_{\mathbb{C}}(\mathfrak{X})2\pi i}$, where $\alpha_{\mathbb{C}}(\mathfrak{X})$ and $\beta_{\mathbb{C}}(\mathfrak{X})$ are real numbers from the unit interval, representing two different entities. $\alpha_{\mathbb{C}}(\mathfrak{X})$ is called the amplitude term and $\beta_{\mathbb{C}}(\mathfrak{X})$ is called the phase term. The phase term has a major role as it refers to the altering phases or periodicity. The CFSs only talk about the supportive, satisfaction, truth or membership grade but do not inform about the dissatisfaction or non-membership grade. To overcome this scarcity in CFSs, Alkouri *et al.* 23] added the non-membership grade and put forward the concept of complex intuitionistic FSs (CIFSs). CIFSs restrict the sum of membership and non-membership grades to the unit disc in complex plane. Equivalently, the sum of amplitude terms of both the grades as well as the sum of phase terms of both the grades must lie in the real unit interval. Because of this restriction Ullah *et al.* [24] extended the codomain by altering the constraint such that the sum of the squares of the real parts of membership and non-membership grades should be in the unit interval. They named this new set as complex PyFS (CPyFS). The CPyFSs were further generalized to complex qROFSs (CqROFSs) by Liu *et al.* [25]. Like qROFSs, the limitations are milder in CqROFSs as compared to CIFSs and CPyFSs. The sum of modulus of complex valued membership and non-membership grades is restricted to the unit interval. Ullah *et al.* [26] evaluated the investment policy based on multiattribute decision making through interval valued TSF aggregation operators. Peng *et al.* [27] proposed the decision making method with a new score function via exponential and aggregation operators for qROFSs. Du *et al.* [28] proposed Minkowski-type distance for qROFSs. Bai *et al.* [29] and Liu *et al.* [30] used the qROF power Maclaurin symmetric mean operators. Wei *et al.* [31] considered the qROF Heronian mean operators in multiattribute decision making and Shu *et al.* [32] integrated the qROF continuous information. Keeping in view the importance of fuzzy sets, other generalizations of fuzzy sets had been introduced, such as rough sets [33], soft sets [34], bipolar-valued fuzzy sets [35] and bipolar soft sets [36]–[38].

Despite the fact that the CqROFSs are so strong and can model several problems with uncertainty, they still are unable to handle many problems because these sets only consider the membership and non-membership grades. So it opens up the gates for the inclusion of complex abstinence grade. Ali *et al.* [39] proposed the notion of complex TSFSs (CTSFSs) which are the generalization of complex SFSs (CSFSs) and complex PFSs (CPFSs). In CTSFSs the elements of the set are given the complex valued membership, abstinence and non-membership grades. The sum of modulus of the membership, abstinence and non-membership grades when raised to power n, must lie in the unit interval, where n is a natural number. For $n = 1$ the CTSFS turns into a CPFS and for $n = 2$ it becomes a CSFS.

The classical set theory discusses the relationships between different sets by using the notion of relations. Relations have many applications in mathematics, engineering, social sciences and several other disciplines. The classical relations (CRs) were defined by Klir [40] which only tell that whether the two events or objects have any relationship or not. Several types of relations are defined, such as inverse relation, reflexive relation, symmetric relation, transitive relation, composite relation, equivalence relation etc. The concept of relations has been also brought in the fuzzy set theory. Fuzzy relations (FRs) were introduced by Mendel [41]. The advantage of FRs over CRs is that the FRs state the strength or quality of the relationship in the form of membership grade. In addition to identify the level of weak and poor relationship, Burillo *et al.* [42] devised the IF relation (IFR). Since these relations with real valued membership and non-membership grades cannot be used to model multidimensional problems. Therefore, the complex FRs (CFRs) were presented by Ramot *et al.* [22]. Further, Ramot *et al.* [43] worked on the complex fuzzy logic. Hu *et al.* [44] studied the distances of CFSs and continuity of CF operations. Al-Qudah and Hassan [45] described the entropy and similarity measure of complex multi-fuzzy soft sets. Garg and Rani [46] achieved some results on information measures for CIFSs. Gulzar *et al.* [47] came up with a novel application of CIFSs in group theory. Quek and Selvachandran [48] talked about the algebraic structure of CIF soft sets associated with groups and subgroups. Zhou *et al.* [49] applied the complex cubic fuzzy aggregation operators in group decision making. Xiao [50]–[52] studied the complex mass functions and used them to predict the interference effects as well as discussed the distance for complex mass functions.

It is known that CTSFSs are the extensive type of all the generalizations of FSs. However, the concepts of fuzzy relations have not yet been introduced for the CTSFSs. The definition of fuzzy relations for the CTSFSs is important because it can be applied in various fields. Henceforth, this paper presents the novel concept of CTSF relations (CTSFRs), which provides the extensive generalizations of FRs. The benefit of TSFSs is that the researchers and professionals are free to choose any grade of membership, abstinence and non-membership, as long as anyone of the grades is not exactly equal to 1. Likewise, CTSFS is the set with the greatest codomain and has the ability to cope with almost all the problems that its predecessors could handle. Moreover, the types of CTSFRs are also proposed with examples such as CTSF inverse, reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive, composite, equivalence and order relations. Besides these types, the equivalence class has also been defined for the CTSFRs. Also, some results have been found regarding the CTSFRs and its types. Furthermore, these concepts are applied to real world scenarios. This paper proposes a model for the economic relationships that can be vital in identifying the key factors affecting the economy growth, unemployment and quality of life. In addition, an application is presented that determined the grades of direct and indirect effects of one country trades on the trades of other countries.

Some predefined concepts are discussed in Section II such as FSs, CFSs, IFSs, CIFSs, qROFSs, CqROFSs, PFSs, CPFSs, SFSs, CSFSs, TSFSs, the complex Cartesian products and relations that have already been studied. Section III contains the main definitions including CTSFSs and CTSFRs. Additionally, the types of CTSFRs are defined with examples. Section IV contains some theorems and their proofs. In section V, a couple of applications of CTSFRs are presented. The first application talks about the direct and indirect effects of trades i.e. import and exports of one country on the trade of other countries. In addition, the grades of positive and negative effects are determined with respect to some time lag. The second application is also very important that discusses the quality and grade of economic relationships. Section VI enlightens the novelty and dominance of this study and compares this work with the previous works when applied to the applications discussed. Section VII highlights some of the advantages of CTSFSs, CTSFRs and their types. Finally, Section VIII concludes this research work.

II. PRELIMINARIES

This section consists of the necessary definitions of FSs, IFSs, PyFSs, qROFSs, PFSs, SFSs, TSFSs, CFSs, CIFSs, CPyFSs, CqROFSs, CPFSs, CSFSs. Moreover, the Cartesian products of CFSs and CFRs are reviewed with example.

Definition 1 [22]: A set $\widetilde{\mathrm{A}}$ on a universal set Џ defined as $\widetilde{\mathrm{A}} = \left\{\text{Ӿ}, \text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}) \mid \text{Ӿ} \in \text{Џ}\right\}$ is called a complex fuzzy set (CFS) with the membership grade $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})$ that is defined as $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}\colon \text{Џ} \to \{\text{ʒ} \mid \text{ʒ} \in \mathbb{C},\ |\text{ʒ}| \leq 1\}$. $\mathbb{C}$ is the set of complex numbers and the complex number ʒ is of the form

$$\text{ʒ}(\text{Ӿ}) = \alpha_{\mathbb{C}}(\text{Ӿ})\, e^{\beta_{\mathbb{C}}(\text{Ӿ})2\pi i} \quad \text{where } \alpha_{\mathbb{C}}(\text{Ӿ}),\ \beta_{\mathbb{C}}(\text{Ӿ}) \in [0, 1]$$

and $\alpha_{\mathbb{C}}(\text{Ӿ})$ is called the amplitude term and $\beta_{\mathbb{C}}(\text{Ӿ})$ is called the phase term.

Definition 2 [22]: The Cartesian product of two CFSs $\widetilde{\mathrm{A}} = \left\{\text{Ӿ}_{í}, \text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}_{í}) \mid \text{Ӿ}_{í} \in \text{Џ}\right\}$ and $\widetilde{\mathrm{B}} = \left\{\text{Ӿ}_{ȷ}, \text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{B}}}(\text{Ӿ}_{ȷ}) \mid \text{Ӿ}_{ȷ} \in \text{Џ}\right\}$, $í, ȷ \in \mathbb{N}$ is denoted and defined as

$$\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}} = \left\{(\text{Ӿ}_{í}, \text{Ӿ}_{ȷ}), \text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}}(\text{Ӿ}_{í}, \text{Ӿ}_{ȷ}) \mid \text{Ӿ}_{í} \in \widetilde{\mathrm{A}}, \text{Ӿ}_{ȷ} \in \widetilde{\mathrm{B}}\right\}$$

where the mapping $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}} : \text{Џ} \to \{\text{ʒ} \mid \text{ʒ} \in \mathbb{C},\ |\text{ʒ}| \leq 1\}$ symbolizes the membership grade of the Cartesian product $\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}$ which is defined as

$$\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}}(\text{Ӿ}_{í}, \text{Ӿ}_{ȷ}) = \min\left\{\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}_{í}), \text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{B}}}(\text{Ӿ}_{ȷ})\right\}.$$

Further, the complex number ʒ for $\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}$ is of the form

$$\begin{aligned}\text{ʒ}(\text{Ӿ}_{í}, \text{Ӿ}_{ȷ}) &= \alpha^{\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}}(\text{Ӿ}_{í}, \text{Ӿ}_{ȷ})\, e^{\beta^{\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}}(\text{Ӿ}_{í}, \text{Ӿ}_{ȷ})2\pi i} \\ &= \min\left\{\alpha^{\widetilde{\mathrm{A}}}(\text{Ӿ}_{í}), \alpha^{\widetilde{\mathrm{B}}}(\text{Ӿ}_{ȷ})\right\} e^{\min\left\{\beta^{\widetilde{\mathrm{A}}}(\text{Ӿ}_{í}), \beta^{\widetilde{\mathrm{B}}}(\text{Ӿ}_{ȷ})\right\}2\pi i}\end{aligned}$$

where $\alpha^{\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}}(\text{Ӿ}_{í}, \text{Ӿ}_{ȷ}), \beta^{\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}}(\text{Ӿ}_{í}, \text{Ӿ}_{ȷ}) \in [0, 1]$.

Definition 3 [22]: A complex fuzzy relation (CFR) denoted by $\mathfrak{R}$ is any non-empty subset of $\widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}$, where $\widetilde{\mathrm{A}}$ and $\widetilde{\mathrm{B}}$ are CFSs.

Example 1: Suppose that $\widetilde{\mathrm{A}}$ is a CFS on Џ defined as

$$\widetilde{\mathrm{A}} = \left\{\left(\text{Ӿ}, 0.3e^{(0.1)2\pi i}\right), \left(\Upsilon, 0.5e^{(0.2)2\pi i}\right), \left(\text{ƶ}, 0.2e^{(0.5)2\pi i}\right)\right\}.$$

Then the Cartesian product $\widetilde{\mathrm{A}} \times \widetilde{\mathrm{A}}$ is defined as

$$\widetilde{\mathrm{A}} \times \widetilde{\mathrm{A}} = \left\{\begin{array}{c}\left((\text{Ӿ}, \text{Ӿ}), 0.3e^{(0.1)2\pi i}\right), \left((\text{Ӿ}, \Upsilon), 0.3e^{(0.1)2\pi i}\right), \\ \left((\text{Ӿ}, \text{ƶ}), 0.2e^{(0.1)2\pi i}\right), \left((\Upsilon, \text{Ӿ}), 0.3e^{(0.1)2\pi i}\right), \\ \left((\Upsilon, \Upsilon), 0.5e^{(0.2)2\pi i}\right), \left((\Upsilon, \text{ƶ}), 0.2e^{(0.2)2\pi i}\right), \\ \left((\text{ƶ}, \text{Ӿ}), 0.2e^{(0.1)2\pi i}\right), \left((\text{ƶ}, \Upsilon), 0.2e^{(0.2)2\pi i}\right), \\ \left((\text{ƶ}, \text{ƶ}), 0.2e^{(0.5)2\pi i}\right)\end{array}\right\}.$$

Since the subset of $\widetilde{\mathrm{A}} \times \widetilde{\mathrm{A}}$ is a CFR, $\mathfrak{R}$ is given as

$$\mathfrak{R} = \left\{\begin{array}{c}\left((\text{Ӿ}, \text{Ӿ}), 0.3e^{(0.1)2\pi i}\right), \left((\text{Ӿ}, \Upsilon), 0.3e^{(0.1)2\pi i}\right), \\ \left((\Upsilon, \Upsilon), 0.5e^{(0.2)2\pi i}\right), \left((\text{ƶ}, \Upsilon), 0.2e^{(0.2)2\pi i}\right)\end{array}\right\}.$$

Definition 4 [2]: A set $\widetilde{\mathrm{A}}$ on a universal set Џ defined as $\widetilde{\mathrm{A}} = \{\text{Ӿ}, \text{ɱ}(\text{Ӿ}), \text{ɳ}(\text{Ӿ}) \mid \text{Ӿ} \in \text{Џ}\}$ is called an intuitionistic fuzzy set (IFS), where $\text{ɱ}(\text{Ӿ})$, $\text{ɳ}(\text{Ӿ})$ are mappings such that $\text{ɱ}, \text{ɳ} : \text{Џ} \to [0, 1]$ symbolize the membership and non-membership grades of the IFS $\widetilde{\mathrm{A}}$, on condition that $0 \leq \text{ɱ}(\text{Ӿ}) + \text{ɳ}(\text{Ӿ}) \leq 1$.

Definition 5 [23]: A set $\widetilde{\mathrm{A}}$ on a universal set Џ defined as $\widetilde{\mathrm{A}} = \left\{\text{Ӿ}, \text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}), \text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}) \mid \text{Ӿ} \in \text{Џ}\right\}$ is called a complex intuitionistic fuzzy set (CIFS) with the membership and non-membership grades $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})$ and $\text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})$ respectively, defined as $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}\colon \text{Џ} \to \{\text{ʒ}_{\text{ɱ}} \mid \text{ʒ}_{\text{ɱ}} \in \mathbb{C},\ |\text{ʒ}_{\text{ɱ}}| \leq 1\}$ and $\text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}\colon \text{Џ} \to \{\text{ʒ}_{\text{ɳ}} \mid \text{ʒ}_{\text{ɳ}} \in \mathbb{C},\ |\text{ʒ}_{\text{ɳ}}| \leq 1\}$. $\mathbb{C}$ is the set of complex numbers and the complex numbers $\text{ʒ}_{\text{ɱ}}$ and $\text{ʒ}_{\text{ɳ}}$ are of the form $\text{ʒ}_{\text{ɱ}}(\text{Ӿ}) = \alpha_{\text{ɱ}}(\text{Ӿ})\, e^{\beta_{\text{ɱ}}(\text{Ӿ})2\pi i}$ and $\text{ʒ}_{\text{ɳ}}(\text{Ӿ}) = \alpha_{\text{ɳ}}(\text{Ӿ})\, e^{\beta_{\text{ɳ}}(\text{Ӿ})2\pi i}$ where $\alpha_{\text{ɱ}}(\text{Ӿ}), \alpha_{\text{ɳ}}(\text{Ӿ}), \beta_{\text{ɱ}}(\text{Ӿ}), \beta_{\text{ɳ}}(\text{Ӿ}) \in [0, 1]$, on condition that $0 \leq \left|\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})\right| + \left|\text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})\right| = |\text{ʒ}_{\text{ɱ}}| + |\text{ʒ}_{\text{ɳ}}| \leq 1$, or $0 \leq \alpha_{\text{ɱ}}(\text{Ӿ}) + \alpha_{\text{ɳ}}(\text{Ӿ}) \leq 1$ and $0 \leq \beta_{\text{ɱ}}(\text{Ӿ}) + \beta_{\text{ɳ}}(\text{Ӿ}) \leq 1$ and $\alpha_{\text{ɱ}}(\text{Ӿ}), \alpha_{\text{ɳ}}(\text{Ӿ})$ are called the amplitude terms and $\beta_{\text{ɱ}}(\text{Ӿ}), \beta_{\text{ɳ}}(\text{Ӿ})$ are called the phase terms.

Definition 6 [24]: A set $\widetilde{\mathrm{A}}$ on a universal set Џ defined as $\widetilde{\mathrm{A}} = \left\{\text{Ӿ}, \text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}), \text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}) \mid \text{Ӿ} \in \text{Џ}\right\}$ is called a complex Pythagorean fuzzy set (CPyFS) with the membership and non-membership grades $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})$ and $\text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})$ respectively, defined as $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}\colon \text{Џ} \to \{\text{ʒ}_{\text{ɱ}} \mid \text{ʒ}_{\text{ɱ}} \in \mathbb{C},\ |\text{ʒ}_{\text{ɱ}}| \leq 1\}$ and $\text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}\colon \text{Џ} \to \{\text{ʒ}_{\text{ɳ}} \mid \text{ʒ}_{\text{ɳ}} \in \mathbb{C},\ |\text{ʒ}_{\text{ɳ}}| \leq 1\}$. $\mathbb{C}$ is the set of complex numbers and the complex numbers $\text{ʒ}_{\text{ɱ}}$ and $\text{ʒ}_{\text{ɳ}}$ are of the form $\text{ʒ}_{\text{ɱ}}(\text{Ӿ}) = \alpha_{\text{ɱ}}(\text{Ӿ})\, e^{\beta_{\text{ɱ}}(\text{Ӿ})2\pi i}$ and $\text{ʒ}_{\text{ɳ}}(\text{Ӿ}) = \alpha_{\text{ɳ}}(\text{Ӿ})\, e^{\beta_{\text{ɳ}}(\text{Ӿ})2\pi i}$ where $\alpha_{\text{ɱ}}(\text{Ӿ}), \alpha_{\text{ɳ}}(\text{Ӿ}), \beta_{\text{ɱ}}(\text{Ӿ}), \beta_{\text{ɳ}}(\text{Ӿ}) \in [0, 1]$, on condition that $0 \leq \left|\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})\right|^2 + \left|\text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})\right|^2 = |\text{ʒ}_{\text{ɱ}}|^2 + |\text{ʒ}_{\text{ɳ}}|^2 \leq 1$, or $0 \leq (\alpha_{\text{ɱ}}(\text{Ӿ}))^2 + (\alpha_{\text{ɳ}}(\text{Ӿ}))^2 \leq 1$ and $0 \leq (\beta_{\text{ɱ}}(\text{Ӿ}))^2 + (\beta_{\text{ɳ}}(\text{Ӿ}))^2 \leq 1$ and $\alpha_{\text{ɱ}}(\text{Ӿ}), \alpha_{\text{ɳ}}(\text{Ӿ})$ are called the amplitude terms and $\beta_{\text{ɱ}}(\text{Ӿ}), \beta_{\text{ɳ}}(\text{Ӿ})$ are called the phase terms.

Definition 7 [4]: A set $\widetilde{\mathrm{A}}$ on a universal set Џ defined as $\widetilde{\mathrm{A}} = \{\text{Ӿ}, \text{ɱ}(\text{Ӿ}), \text{ɳ}(\text{Ӿ}) \mid \text{Ӿ} \in \text{Џ}\}$ is called q-rung orthopair fuzzy set (qROFS), where $\text{ɱ}(\text{Ӿ})$, $\text{ɳ}(\text{Ӿ})$ are mappings such that $\text{ɱ}, \text{ɳ} : \text{Џ} \to [0, 1]$ symbolize the membership and non-membership grades of the IFS $\widetilde{\mathrm{A}}$, on condition that $0 \leq (\text{ɱ}(\text{Ӿ}))^n + (\text{ɳ}(\text{Ӿ}))^n \leq 1$, with n a natural number.

Remark 1: For $n = 1$ and $n = 2$, the qROFS becomes an IFS and PyFS respectively.

Definition 8 [25]: A set $\widetilde{\mathrm{A}}$ on a universal set Џ defined as $\widetilde{\mathrm{A}} = \left\{\text{Ӿ}, \text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}), \text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ}) \mid \text{Ӿ} \in \text{Џ}\right\}$ is called a complex q-rung orthopair fuzzy set (CqROFS) with the membership and non-membership grades $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})$ and $\text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\text{Ӿ})$ respectively, defined as $\text{ɱ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}\colon \text{Џ} \to \{\text{ʒ}_{\text{ɱ}} \mid \text{ʒ}_{\text{ɱ}} \in \mathbb{C},\ |\text{ʒ}_{\text{ɱ}}| \leq 1\}$ and $\text{ɳ}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}\colon \text{Џ} \to \{\text{ʒ}_{\text{ɳ}} \mid \text{ʒ}_{\text{ɳ}} \in \mathbb{C},\ |\text{ʒ}_{\text{ɳ}}| \leq 1\}$. $\mathbb{C}$ is the set of complex numbers and the complex numbers $\text{ʒ}_{\text{ɱ}}$ and $\text{ʒ}_{\text{ɳ}}$ are of the form $\text{ʒ}_{\text{ɱ}}(\text{Ӿ}) = \alpha_{\text{ɱ}}(\text{Ӿ})\, e^{\beta_{\text{ɱ}}(\text{Ӿ})2\pi i}$ and $\text{ʒ}_{\text{ɳ}}(\text{Ӿ}) = \alpha_{\text{ɳ}}(\text{Ӿ})\, e^{\beta_{\text{ɳ}}(\text{Ӿ})2\pi i}$

where $\alpha_{\mathfrak{m}}(\mathfrak{x})$, $\alpha_{\mathfrak{n}}(\mathfrak{x})$, $\beta_{\mathfrak{m}}(\mathfrak{x})$, $\beta_{\mathfrak{n}}(\mathfrak{x}) \in [0,1]$, on condition that $0 \le \left|\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})\right|^{n} + \left|\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})\right|^{n} = |\mathfrak{z}_{\mathfrak{m}}|^{n} + |\mathfrak{z}_{\mathfrak{n}}|^{n} \le 1$, or $0 \le (\alpha_{\mathfrak{m}}(\mathfrak{x}))^{n} + (\alpha_{\mathfrak{n}}(\mathfrak{x}))^{n} \le 1$ and $0 \le (\beta_{\mathfrak{m}}(\mathfrak{x}))^{n} + (\beta_{\mathfrak{n}}(\mathfrak{x}))^{n} \le 1$, that n is a natural number, and $\alpha_{\mathfrak{m}}(\mathfrak{x})$, $\alpha_{\mathfrak{n}}(\mathfrak{x})$ are called the amplitude terms and $\beta_{\mathfrak{m}}(\mathfrak{x})$, $\beta_{\mathfrak{n}}(\mathfrak{x})$ are called the phase terms.

Definition 9 [12]: A set $\widetilde{A}$ on a universal set $\mathfrak{U}$ defined as $\widetilde{A} = \{\mathfrak{x}, \mathfrak{m}(\mathfrak{x}), \mathfrak{p}(\mathfrak{x}), \mathfrak{n}(\mathfrak{x}) \,|\, \mathfrak{x} \in \mathfrak{U}\}$ is called a picture fuzzy set (PFS), where $\mathfrak{m}(\mathfrak{x})$, $\mathfrak{p}(\mathfrak{x})$ and $\mathfrak{n}(\mathfrak{x})$ are mappings such that $\mathfrak{m}, \mathfrak{p}, \mathfrak{n} : \mathfrak{U} \to [0,1]$ symbolize the membership, abstinence and non-membership grades of the PFS $\widetilde{A}$, on condition that $0 \le \mathfrak{m}(\mathfrak{x}) + \mathfrak{p}(\mathfrak{x}) + \mathfrak{n}(\mathfrak{x}) \le 1$.

Definition 10 [39]: A set $\widetilde{A}$ on a universal set $\mathfrak{U}$ defined as $\widetilde{A} = \left\{\mathfrak{x}, \mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}), \mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}), \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}) \,|\, \mathfrak{x} \in \mathfrak{U}\right\}$ is called a complex picture fuzzy set (CPFS) with the membership, abstinence and non-membership grades $\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})$, $\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})$ and $\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})$ respectively, defined as $\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}: \mathfrak{U} \to \{\mathfrak{z}_{\mathfrak{m}} \,|\, \mathfrak{z}_{\mathfrak{m}} \in \mathbb{C}, |\mathfrak{z}_{\mathfrak{m}}| \le 1\}$, $\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}: \mathfrak{U} \to \{\mathfrak{z}_{\mathfrak{p}} \,|\, \mathfrak{z}_{\mathfrak{p}} \in \mathbb{C}, |\mathfrak{z}_{\mathfrak{p}}| \le 1\}$ and $\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}: \mathfrak{U} \to \{\mathfrak{z}_{\mathfrak{n}} \,|\, \mathfrak{z}_{\mathfrak{n}} \in \mathbb{C}, |\mathfrak{z}_{\mathfrak{n}}| \le 1\}$. $\mathbb{C}$ is the set of complex numbers and the complex numbers $\mathfrak{z}_{\mathfrak{m}}$, $\mathfrak{z}_{\mathfrak{p}}$ and $\mathfrak{z}_{\mathfrak{n}}$ are of the form $\mathfrak{z}_{\mathfrak{m}}(\mathfrak{x}) = \alpha_{\mathfrak{m}}(\mathfrak{x})\, e^{\beta_{\mathfrak{m}}(\mathfrak{x})2\pi i}$, $\mathfrak{z}_{\mathfrak{p}}(\mathfrak{x}) = \alpha_{\mathfrak{p}}(\mathfrak{x})\, e^{\beta_{\mathfrak{p}}(\mathfrak{x})2\pi i}$ and $\mathfrak{z}_{\mathfrak{n}}(\mathfrak{x}) = \alpha_{\mathfrak{n}}(\mathfrak{x})\, e^{\beta_{\mathfrak{n}}(\mathfrak{x})2\pi i}$ where $\alpha_{\mathfrak{m}}(\mathfrak{x})$, $\alpha_{\mathfrak{p}}(\mathfrak{x})$, $\alpha_{\mathfrak{n}}(\mathfrak{x})$, $\beta_{\mathfrak{m}}(\mathfrak{x})$, $\beta_{\mathfrak{p}}(\mathfrak{x})$, $\beta_{\mathfrak{n}}(\mathfrak{x}) \in [0,1]$, on condition that $0 \le \left|\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})\right| + \left|\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})\right| + \left|\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})\right| = |\mathfrak{z}_{\mathfrak{m}}| + |\mathfrak{z}_{\mathfrak{p}}| + |\mathfrak{z}_{\mathfrak{n}}| \le 1$, or $0 \le \alpha_{\mathfrak{m}}(\mathfrak{x}) + \alpha_{\mathfrak{p}}(\mathfrak{x}) + \alpha_{\mathfrak{n}}(\mathfrak{x}) \le 1$ and $0 \le \beta_{\mathfrak{m}}(\mathfrak{x}) + \beta_{\mathfrak{p}}(\mathfrak{x}) + \beta_{\mathfrak{n}}(\mathfrak{x}) \le 1$. $\alpha_{\mathfrak{m}}(\mathfrak{x})$, $\alpha_{\mathfrak{p}}(\mathfrak{x})$, $\alpha_{\mathfrak{n}}(\mathfrak{x})$ are called the amplitude terms and $\beta_{\mathfrak{m}}(\mathfrak{x})$, $\beta_{\mathfrak{p}}(\mathfrak{x})$, $\beta_{\mathfrak{n}}(\mathfrak{x})$ are called the phase terms.

Definition 11 [13]: A set $\widetilde{A}$ on a universal set $\mathfrak{U}$ defined as $\widetilde{A} = \{\mathfrak{x}, \mathfrak{m}(\mathfrak{x}), \mathfrak{p}(\mathfrak{x}), \mathfrak{n}(\mathfrak{x}) \,|\, \mathfrak{x} \in \mathfrak{U}\}$ is called a spherical fuzzy set (SFS), where $\mathfrak{m}(\mathfrak{x})$, $\mathfrak{p}(\mathfrak{x})$ and $\mathfrak{n}(\mathfrak{x})$ are mappings such that $\mathfrak{m}, \mathfrak{p}, \mathfrak{n} : \mathfrak{U} \to [0,1]$, symbolize the membership, abstinence and non-membership grades of the PFS $\widetilde{A}$, on condition that $0 \le (\mathfrak{m}(\mathfrak{x}))^2 + (\mathfrak{p}(\mathfrak{x}))^2 + (\mathfrak{n}(\mathfrak{x}))^2 \le 1$.

Definition 12 [39]: A set $\widetilde{A}$ on a universal set $\mathfrak{U}$ defined as $\widetilde{A} = \left\{\mathfrak{x}, \mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}), \mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}), \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}) \,|\, \mathfrak{x} \in \mathfrak{U}\right\}$ is called a complex spherical fuzzy set (CSFS) with the membership, abstinence and non-membership grades $\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})$, $\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})$ and $\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})$ respectively, defined as $\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}: \mathfrak{U} \to \{\mathfrak{z}_{\mathfrak{m}} \,|\, \mathfrak{z}_{\mathfrak{m}} \in \mathbb{C}, |\mathfrak{z}_{\mathfrak{m}}| \le 1\}$, $\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}: \mathfrak{U} \to \{\mathfrak{z}_{\mathfrak{p}} \,|\, \mathfrak{z}_{\mathfrak{p}} \in \mathbb{C}, |\mathfrak{z}_{\mathfrak{p}}| \le 1\}$ and $\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}: \mathfrak{U} \to \{\mathfrak{z}_{\mathfrak{n}} \,|\, \mathfrak{z}_{\mathfrak{n}} \in \mathbb{C}, |\mathfrak{z}_{\mathfrak{n}}| \le 1\}$. $\mathbb{C}$ is the set of complex numbers and the complex numbers $\mathfrak{z}_{\mathfrak{m}}$, $\mathfrak{z}_{\mathfrak{p}}$ and $\mathfrak{z}_{\mathfrak{n}}$ are of the form

$$\mathfrak{z}_{\mathfrak{m}}(\mathfrak{x}) = \alpha_{\mathfrak{m}}(\mathfrak{x})\, e^{\beta_{\mathfrak{m}}(\mathfrak{x})2\pi i}, \mathfrak{z}_{\mathfrak{p}}(\mathfrak{x}) = \alpha_{\mathfrak{p}}(\mathfrak{x})\, e^{\beta_{\mathfrak{p}}(\mathfrak{x})2\pi i}$$
$$\text{and } \mathfrak{z}_{\mathfrak{n}}(\mathfrak{x}) = \alpha_{\mathfrak{n}}(\mathfrak{x})\, e^{\beta_{\mathfrak{n}}(\mathfrak{x})2\pi i}$$

where $\alpha_{\mathfrak{m}}(\mathfrak{x})$, $\alpha_{\mathfrak{p}}(\mathfrak{x})$, $\alpha_{\mathfrak{n}}(\mathfrak{x})$, $\beta_{\mathfrak{m}}(\mathfrak{x})$, $\beta_{\mathfrak{p}}(\mathfrak{x})$, $\beta_{\mathfrak{n}}(\mathfrak{x}) \in [0,1]$, on condition that

$$0 \le \left|\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})\right|^2 + \left|\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})\right|^2 + \left|\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x})\right|^2$$
$$= |\mathfrak{z}_{\mathfrak{m}}|^2 + |\mathfrak{z}_{\mathfrak{p}}|^2 + |\mathfrak{z}_{\mathfrak{n}}|^2 \le 1, \text{ or } 0 \le (\alpha_{\mathfrak{m}}(\mathfrak{x}))^2$$
$$+(\alpha_{\mathfrak{p}}(\mathfrak{x}))^2 + (\alpha_{\mathfrak{n}}(\mathfrak{x}))^2 \le 1 \text{ and } 0 \le (\beta_{\mathfrak{m}}(\mathfrak{x}))^2$$
$$+(\beta_{\mathfrak{p}}(\mathfrak{x}))^2 + (\beta_{\mathfrak{n}}(\mathfrak{x}))^2 \le 1.$$

$\alpha_{\mathfrak{m}}(\mathfrak{x})$, $\alpha_{\mathfrak{p}}(\mathfrak{x})$, $\alpha_{\mathfrak{n}}(\mathfrak{x})$ are called the amplitude terms and $\beta_{\mathfrak{m}}(\mathfrak{x})$, $\beta_{\mathfrak{p}}(\mathfrak{x})$, $\beta_{\mathfrak{n}}(\mathfrak{x})$ are called the phase terms.

Definition 13 [13]: A set $\widetilde{A}$ on a universal set $\mathfrak{U}$ defined as $\widetilde{A} = \{\mathfrak{x}, \mathfrak{m}(\mathfrak{x}), \mathfrak{p}(\mathfrak{x}), \mathfrak{n}(\mathfrak{x}) \,|\, \mathfrak{x} \in \mathfrak{U}\}$ is called a T-spherical fuzzy set (TSFS), where $\mathfrak{m}(\mathfrak{x})$, $\mathfrak{p}(\mathfrak{x})$ and $\mathfrak{n}(\mathfrak{x})$ are mappings such that $\mathfrak{m}, \mathfrak{p}, \mathfrak{n} : \mathfrak{U} \to [0,1]$ symbolize the membership, abstinence and non-membership grades of the PFS $\widetilde{A}$, on condition that

$$0 \le (\mathfrak{m}(\mathfrak{x}))^n + (\mathfrak{p}(\mathfrak{x}))^n + (\mathfrak{n}(\mathfrak{x}))^n \le 1, \; n \in \mathbb{N}.$$

III. COMPLEX T-SPHERICAL FUZZY RELATIONS AND THEIR TYPES

The objective of this section is to introduce novel concepts of CTSFSs, Cartesian products in CqROFSs, CTSFSs, CqROFRs, CPFRs, CSFRs and CTSFRs. Also, the types of CTSFRs are delineated. These notions are supported with the clear examples.

Definition 14: The Cartesian product of two CqROFSs $\widetilde{A} = \left\{\mathfrak{x}_{\mathfrak{i}}, \mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\mathfrak{i}}), \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\mathfrak{i}}) \,|\, \mathfrak{x}_{\mathfrak{i}} \in \mathfrak{U}\right\}$ and $\widetilde{B} = \left\{\mathfrak{x}_{\mathfrak{j}}, \mathfrak{m}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\mathfrak{j}}), \mathfrak{n}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\mathfrak{j}}) \,|\, \mathfrak{x}_{\mathfrak{j}} \in \mathfrak{U}\right\}$, $\mathfrak{i}, \mathfrak{j} \in \mathbb{N}$ is denoted and defined as

$$\widetilde{A} \times \widetilde{B} = \left\{(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}}), \mathfrak{m}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}}), \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}}) \,|\, \mathfrak{x}_{\mathfrak{i}} \in \widetilde{A}, \mathfrak{x}_{\mathfrak{j}} \in \widetilde{B}\right\}$$

where the mappings $\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}} : \mathfrak{U} \to \{\mathfrak{z}_{\mathfrak{m}} \,|\, \mathfrak{z}_{\mathfrak{m}} \in \mathbb{C}, |\mathfrak{z}_{\mathfrak{m}}| \le 1\}$ and $\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}} : \mathfrak{U} \to \{\mathfrak{z}_{\mathfrak{n}} \,|\, \mathfrak{z}_{\mathfrak{n}} \in \mathbb{C}, |\mathfrak{z}_{\mathfrak{n}}| \le 1\}$ symbolize the membership and non-membership grades of the Cartesian product $\widetilde{A} \times \widetilde{B}$ which are defined as

$$\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}}) = \min\left\{\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\mathfrak{i}}), \mathfrak{m}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\mathfrak{j}})\right\}$$
$$\text{and } \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}}) = \max\left\{\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\mathfrak{i}}), \mathfrak{n}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\mathfrak{j}})\right\}$$

Further, the complex numbers $\mathfrak{z}_{\mathfrak{m}}$ and $\mathfrak{z}_{\mathfrak{n}}$ for $\widetilde{A} \times \widetilde{B}$ are of the form

$$\mathfrak{z}_{\mathfrak{m}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}}) = \alpha_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}})\, e^{\beta_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}})2\pi i}$$
$$= \min\left\{\alpha_{\mathfrak{m}}^{\widetilde{A}}(\mathfrak{x}_{\mathfrak{i}}), \alpha_{\mathfrak{m}}^{\widetilde{B}}(\mathfrak{x}_{\mathfrak{j}})\right\} e^{\min\left\{\beta_{\mathfrak{m}}^{\widetilde{A}}(\mathfrak{x}_{\mathfrak{i}}), \beta_{\mathfrak{m}}^{\widetilde{B}}(\mathfrak{x}_{\mathfrak{j}})\right\}2\pi i}$$

and

$$\mathfrak{z}_{\mathfrak{n}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}}) = \alpha_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}})\, e^{\beta_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}})2\pi i}$$
$$= \max\left\{\alpha_{\mathfrak{n}}^{\widetilde{A}}(\mathfrak{x}_{\mathfrak{i}}), \alpha_{\mathfrak{n}}^{\widetilde{B}}(\mathfrak{x}_{\mathfrak{j}})\right\} e^{\max\left\{\beta_{\mathfrak{n}}^{\widetilde{A}}(\mathfrak{x}_{\mathfrak{i}}), \beta_{\mathfrak{n}}^{\widetilde{B}}(\mathfrak{x}_{\mathfrak{j}})\right\}2\pi i}$$

on condition that

$$0 \le \left(\alpha_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}})\right)^n + \left(\alpha_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}})\right)^n \le 1$$
$$\text{and } 0 \le \left(\beta_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}})\right)^n + \left(\beta_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}(\mathfrak{x}_{\mathfrak{i}}, \mathfrak{x}_{\mathfrak{j}})\right)^n \le 1$$

where n is a natural number and

$$\alpha_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\alpha_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\beta_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),$$
$$\beta_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\in[0,1].$$

Definition 15: A complex q-rung orthopair fuzzy relation (CqROFR) denoted by $\mathfrak{R}$ is any non-empty subset of $\widetilde{A}\times\widetilde{B}$, where $\widetilde{A}$ and $\widetilde{B}$ are CqROFSs.

Remark 2: For $n=1$ and $n=2$, the CqROFRs convert to CIFRs and CPyFRs, respectively. In the same way, Cartesian products are defined.

Example 2: Suppose that $\widetilde{A}$ is a CqROFS for $n=3$ on $\mathfrak{U}$ defined as

$$\widetilde{A}=\left\{\begin{array}{c}\left(\mathfrak{X},0.9e^{(0.1)2\pi i},0.5e^{(0.7)2\pi i}\right),\\ \left(\Upsilon,0.8e^{(0.2)2\pi i},0.4e^{(0.5)2\pi i}\right),\\ \left(\mathfrak{z},0.6e^{(0.5)2\pi i},0.3e^{(0.9)2\pi i}\right)\end{array}\right\}.$$

Then the Cartesian product $\widetilde{A}\times\widetilde{A}$ is defined as

$$\widetilde{A}\times\widetilde{A}=\left\{\begin{array}{c}\left((\mathfrak{X},\mathfrak{X}),0.9e^{(0.1)2\pi i},0.5e^{(0.7)2\pi i}\right),\\ \left((\mathfrak{X},\Upsilon),0.8e^{(0.1)2\pi i},0.5e^{(0.7)2\pi i}\right),\\ \left((\mathfrak{X},\mathfrak{z}),0.6e^{(0.1)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{X}),0.8e^{(0.1)2\pi i},0.5e^{(0.7)2\pi i}\right),\\ \left((\Upsilon,\Upsilon),0.8e^{(0.2)2\pi i},0.4e^{(0.5)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{z}),0.6e^{(0.2)2\pi i},0.4e^{(0.9)2\pi i}\right),\\ \left((\mathfrak{z},\mathfrak{X}),0.6e^{(0.1)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left((\mathfrak{z},\Upsilon),0.6e^{(0.2)2\pi i},0.4e^{(0.9)2\pi i}\right),\\ \left((\mathfrak{z},\mathfrak{z}),0.6e^{(0.5)2\pi i},0.3e^{(0.9)2\pi i}\right)\end{array}\right\}$$

Since the subset of $\widetilde{A}\times\widetilde{A}$ is a CIFR, $\mathfrak{R}$ is given as

$$\mathfrak{R}=\left\{\begin{array}{c}\left((\mathfrak{X},\Upsilon),0.8e^{(0.1)2\pi i},0.5e^{(0.7)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{z}),0.6e^{(0.2)2\pi i},0.4e^{(0.9)2\pi i}\right),\\ \left((\mathfrak{z},\mathfrak{z}),0.6e^{(0.5)2\pi i},0.3e^{(0.9)2\pi i}\right)\end{array}\right\}$$

Definition 16: A set $\widetilde{A}$ on a universal set $\mathfrak{U}$ defined as $\widetilde{A}=\left\{\mathfrak{X},\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X})\,|\,\mathfrak{X}\in\mathfrak{U}\right\}$ is called a complex T-spherical fuzzy set (CTSFS) with the membership, abstinence and non-membership grades $\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X})$, $\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X})$ and $\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X})$ respectively, defined as

$$\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}:\mathfrak{U}\rightarrow\left\{\mathfrak{z}_{\mathfrak{m}}\,|\,\mathfrak{z}_{\mathfrak{m}}\in\mathbb{C},\ \left|\mathfrak{z}_{\mathfrak{m}}\right|\leq1\right\},$$
$$\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}:\mathfrak{U}\rightarrow\left\{\mathfrak{z}_{\mathfrak{p}}\,|\,\mathfrak{z}_{\mathfrak{p}}\in\mathbb{C},\ \left|\mathfrak{z}_{\mathfrak{p}}\right|\leq1\right\}$$

and $\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}:\mathfrak{U}\rightarrow\left\{\mathfrak{z}_{\mathfrak{n}}\,|\,\mathfrak{z}_{\mathfrak{n}}\in\mathbb{C},\ \left|\mathfrak{z}_{\mathfrak{n}}\right|\leq1\right\}$. $\mathbb{C}$ is the set of complex numbers and the complex numbers $\mathfrak{z}_{\mathfrak{m}}$, $\mathfrak{z}_{\mathfrak{p}}$ and $\mathfrak{z}_{\mathfrak{n}}$ are of the form

$$\mathfrak{z}_{\mathfrak{m}}(\mathfrak{X})=\alpha_{\mathfrak{m}}(\mathfrak{X})\,e^{\beta_{\mathfrak{m}}(\mathfrak{X})2\pi i},\ \mathfrak{z}_{\mathfrak{p}}(\mathfrak{X})=\alpha_{\mathfrak{p}}(\mathfrak{X})\,e^{\beta_{\mathfrak{p}}(\mathfrak{X})2\pi i}$$
$$\text{and } \mathfrak{z}_{\mathfrak{n}}(\mathfrak{X})=\alpha_{\mathfrak{n}}(\mathfrak{X})\,e^{\beta_{\mathfrak{n}}(\mathfrak{X})2\pi i},$$

where $\alpha_{\mathfrak{m}}(\mathfrak{X}),\ \alpha_{\mathfrak{p}}(\mathfrak{X}),\alpha_{\mathfrak{n}}(\mathfrak{X}),\beta_{\mathfrak{m}}(\mathfrak{X}),\beta_{\mathfrak{p}}(\mathfrak{X}),\beta_{\mathfrak{n}}(\mathfrak{X})\in[0,1]$, on condition that

$$0\leq\left|\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X})\right|^n+\left|\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X})\right|^n+\left|\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{X})\right|^n$$
$$=\left|\mathfrak{z}_{\mathfrak{m}}\right|^n+\left|\mathfrak{z}_{\mathfrak{p}}\right|^n+\left|\mathfrak{z}_{\mathfrak{n}}\right|^n\leq1,\ n\in\mathbb{N},$$

$$\text{or } 0\leq\left(\alpha_{\mathfrak{m}}(\mathfrak{X})\right)^n+\left(\alpha_{\mathfrak{p}}(\mathfrak{X})\right)^n+\left(\alpha_{\mathfrak{n}}(\mathfrak{X})\right)^n\leq1 \text{ and } 0$$
$$\leq\left(\beta_{\mathfrak{m}}(\mathfrak{X})\right)^n+\left(\beta_{\mathfrak{p}}(\mathfrak{X})\right)^n+\left(\beta_{\mathfrak{n}}(\mathfrak{X})\right)^n\leq1 \text{ for } n\in\mathbb{N}.$$

$\alpha_{\mathfrak{m}}(\mathfrak{X}),\alpha_{\mathfrak{p}}(\mathfrak{X}),\ \alpha_{\mathfrak{n}}(\mathfrak{X})$ are called the amplitude terms and $\beta_{\mathfrak{m}}(\mathfrak{X}),\beta_{\mathfrak{p}}(\mathfrak{X}),\beta_{\mathfrak{n}}(\mathfrak{X})$ are called the phase terms.

Definition 17: The Cartesian product of two CTSFSs $\widetilde{A}=\left\{\mathfrak{x}_{\acute{\imath}},\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}})\,|\,\mathfrak{x}_{\acute{\imath}}\in\mathfrak{U}\right\}$ and $\widetilde{B}=\left\{\mathfrak{x}_{\jmath},\mathfrak{m}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\jmath}),\mathfrak{p}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\jmath}),\mathfrak{n}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\,|\,\mathfrak{x}_{\jmath}\in\mathfrak{U}\right\}$, $\acute{\imath},\jmath\in\mathbb{N}$ is denoted and defined as

$$\widetilde{A}\times\widetilde{B}=\left\{\begin{array}{c}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\\ \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)|\mathfrak{x}_{\acute{\imath}}\in\widetilde{A},\mathfrak{x}_{\jmath}\in\widetilde{B}\end{array}\right\}$$

where the mappings $\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}:\mathfrak{U}\rightarrow\left\{\mathfrak{z}_{\mathfrak{m}}\,|\,\mathfrak{z}_{\mathfrak{m}}\in\mathbb{C},\ \left|\mathfrak{z}_{\mathfrak{m}}\right|\leq1\right\}$, $\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}:\mathfrak{U}\rightarrow\left\{\mathfrak{z}_{\mathfrak{p}}\,|\,\mathfrak{z}_{\mathfrak{p}}\in\mathbb{C},\ \left|\mathfrak{z}_{\mathfrak{p}}\right|\leq1\right\}$ and $\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}:\mathfrak{U}\rightarrow\left\{\mathfrak{z}_{\mathfrak{n}}\,|\,\mathfrak{z}_{\mathfrak{n}}\in\mathbb{C},\ \left|\mathfrak{z}_{\mathfrak{n}}\right|\leq1\right\}$ symbolize the membership, abstinence and non-membership grades of the Cartesian product $\widetilde{A}\times\widetilde{B}$ which are defined as

$$\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)=\min\left\{\mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\mathfrak{m}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\},$$
$$\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)=\min\left\{\mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\mathfrak{p}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\}$$
$$\text{and } \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)=\max\left\{\mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\mathfrak{n}_{\mathbb{C}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\}.$$

Further, the complex numbers $\mathfrak{z}_{\mathfrak{m}}$, $\mathfrak{z}_{\mathfrak{p}}$ and $\mathfrak{z}_{\mathfrak{n}}$ for $\widetilde{A}\times\widetilde{B}$ are of the form

$$\begin{aligned}&\mathfrak{z}_{\mathfrak{m}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\\&\quad=\alpha_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)e^{\beta_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)2\pi i}\\&\quad=\min\left\{\alpha_{\mathfrak{m}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\alpha_{\mathfrak{m}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\}e^{\min\left\{\beta_{\mathfrak{m}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\beta_{\mathfrak{m}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\}2\pi i},\\&\quad\mathfrak{z}_{\mathfrak{p}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\\&\quad=\alpha_{\mathfrak{p}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)e^{\beta_{\mathfrak{p}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)2\pi i}\\&\quad=\min\left\{\alpha_{\mathfrak{p}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\alpha_{\mathfrak{p}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\}e^{\min\left\{\beta_{\mathfrak{p}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\beta_{\mathfrak{p}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\}2\pi i},\\&\quad\text{and } \mathfrak{z}_{\mathfrak{n}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\\&\quad=\alpha_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)e^{\beta_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)2\pi i}\\&\quad=\max\left\{\alpha_{\mathfrak{n}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\alpha_{\mathfrak{n}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\}e^{\max\left\{\beta_{\mathfrak{n}}^{\widetilde{A}}(\mathfrak{x}_{\acute{\imath}}),\beta_{\mathfrak{n}}^{\widetilde{B}}(\mathfrak{x}_{\jmath})\right\}2\pi i}\end{aligned}$$

on condition that

$$\begin{aligned}&0\leq\left(\alpha_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\right)^n+\left(\alpha_{\mathfrak{p}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\right)^n+\left(\alpha_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\right)^n\\&\quad\leq1 \text{ and } 0\leq\left(\beta_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\right)^n+\left(\beta_{\mathfrak{p}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\right)^n\\&\quad+\left(\beta_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\right)^n\leq1 \text{ for } n\in\mathbb{N},\end{aligned}$$

where $\alpha_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\alpha_{\mathfrak{p}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\alpha_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\beta_{\mathfrak{m}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\beta_{\mathfrak{p}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right),\beta_{\mathfrak{n}}^{\widetilde{A}\times\widetilde{B}}\left(\mathfrak{x}_{\acute{\imath}},\mathfrak{x}_{\jmath}\right)\in[0,1]$.

Definition 18: A complex T-spherical fuzzy relation (CTSFR) denoted by $\mathfrak{R}$ is any non-empty subset of $\widetilde{A}\times\widetilde{B}$, where $\widetilde{A}$ and $\widetilde{B}$ are CTSFSs.

Remark 3: Like TSFS which is the generalization of PFS and SFS, the CTSFS is the generalization of CPFS and CSFS for $n = 1$ and $n = 2$, respectively. In the same way, Cartesian products and the relations are defined, i.e. CTSFR converts to CPFR for $n = 1$ and it converts to CSPR for $n = 2$.

Example 3: Suppose that $\widetilde{A}$ is a CPFS on $\mathfrak{U}$ defined as

$$\widetilde{A} = \left\{ \begin{array}{l} \left(\chi, 0.3e^{(0.1)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.4e^{(0.2)2\pi i}\right), \\ \left(\Upsilon, 0.2e^{(0.4)2\pi i}, 0.4e^{(0.3)2\pi i}, 0.1e^{(0.3)2\pi i}\right) \end{array} \right\}.$$

Then the Cartesian product $\widetilde{A} \times \widetilde{A}$ is defined as

$$\widetilde{A} \times \widetilde{A} = \left\{ \begin{array}{l} \left((\chi, \chi), 0.3e^{(0.1)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.4e^{(0.2)2\pi i}\right), \\ \left((\chi, \Upsilon), 0.2e^{(0.1)2\pi i}, 0.2e^{(0.3)2\pi i}, 0.4e^{(0.3)2\pi i}\right), \\ \left((\Upsilon, \chi), 0.2e^{(0.1)2\pi i}, 0.2e^{(0.3)2\pi i}, 0.4e^{(0.3)2\pi i}\right), \\ \left((\Upsilon, \Upsilon), 0.2e^{(0.4)2\pi i}, 0.4e^{(0.3)2\pi i}, 0.1e^{(0.3)2\pi i}\right) \end{array} \right\}.$$

Since, the subset of $\widetilde{A} \times \widetilde{A}$ is a CPFR, $\mathfrak{R}$ is given as

$$\mathfrak{R} = \left\{ \begin{array}{l} \left((\chi, \chi), 0.3e^{(0.1)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.4e^{(0.2)2\pi i}\right), \\ \left((\chi, \Upsilon), 0.2e^{(0.1)2\pi i}, 0.2e^{(0.3)2\pi i}, 0.4e^{(0.3)2\pi i}\right) \end{array} \right\}.$$

Example 4: Suppose that $\widetilde{A}$ is a CSFS on $\mathfrak{U}$ defined as

$$\widetilde{A} = \left\{ \begin{array}{l} \left(\chi, 0.5e^{(0.4)2\pi i}, 0.6e^{(0.7)2\pi i}, 0.4e^{(0.3)2\pi i}\right), \\ \left(\Upsilon, 0.2e^{(0.6)2\pi i}, 0.4e^{(0.6)2\pi i}, 0.8e^{(0.3)2\pi i}\right) \end{array} \right\}.$$

Then the Cartesian product $\widetilde{A} \times \widetilde{A}$ is defined as

$$\widetilde{A} \times \widetilde{A} = \left\{ \begin{array}{l} \left((\chi, \chi), 0.5e^{(0.4)2\pi i}, 0.6e^{(0.7)2\pi i}, 0.4e^{(0.3)2\pi i}\right), \\ \left((\chi, \Upsilon), 0.2e^{(0.4)2\pi i}, 0.4e^{(0.6)2\pi i}, 0.8e^{(0.3)2\pi i}\right), \\ \left((\Upsilon, \chi), 0.2e^{(0.4)2\pi i}, 0.4e^{(0.6)2\pi i}, 0.8e^{(0.3)2\pi i}\right), \\ \left((\Upsilon, \Upsilon), 0.2e^{(0.6)2\pi i}, 0.4e^{(0.6)2\pi i}, 0.8e^{(0.3)2\pi i}\right) \end{array} \right\}$$

Since the subset of $\widetilde{A} \times \widetilde{A}$ is a CSFR, $\mathfrak{R}$ is given as

$$\mathfrak{R} = \left\{ \begin{array}{l} \left((\chi, \Upsilon), 0.2e^{(0.4)2\pi i}, 0.4e^{(0.6)2\pi i}, 0.8e^{(0.3)2\pi i}\right), \\ \left((\Upsilon, \chi), 0.2e^{(0.4)2\pi i}, 0.4e^{(0.6)2\pi i}, 0.8e^{(0.3)2\pi i}\right) \end{array} \right\}.$$

Example 5: Suppose that $\widetilde{A}$ is a CTSFS for $n = 4$ on $\mathfrak{U}$ defined as

$$\widetilde{A} = \left\{ \begin{array}{l} \left(\chi, 0.5e^{(0.6)2\pi i}, 0.6e^{(0.7)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left(\Upsilon, 0.5e^{(0.6)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right) \end{array} \right\}.$$

Then the Cartesian product $\widetilde{A} \times \widetilde{A}$ is defined as

$$\widetilde{A} \times \widetilde{A} = \left\{ \begin{array}{l} \left((\chi, \chi), 0.5e^{(0.6)2\pi i}, 0.6e^{(0.7)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((\chi, \Upsilon), 0.5e^{(0.6)2\pi i}, 0.6e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\Upsilon, \chi), 0.5e^{(0.6)2\pi i}, 0.6e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\Upsilon, \Upsilon), 0.5e^{(0.6)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right) \end{array} \right\}$$

Since the subset of $\widetilde{A} \times \widetilde{A}$ is a CSFR, $\mathfrak{R}$ is given as

$$\mathfrak{R} = \left\{ \begin{array}{l} \left((\chi, \chi), 0.5e^{(0.6)2\pi i}, 0.6e^{(0.7)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((\Upsilon, \Upsilon), 0.5e^{(0.6)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right) \end{array} \right\}.$$

Definition 19. Let $\widetilde{A}$ be a CTSFS on $\mathfrak{U}$ and $\mathfrak{R} = \left\{\left(\chi_{\acute{\imath}}, \chi_{\jmath}\right), \mathfrak{m}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{A}}\left(\chi_{\acute{\imath}}, \chi_{\jmath}\right), \mathfrak{p}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{A}}\left(\chi_{\acute{\imath}}, \chi_{\jmath}\right), \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{A}}\left(\chi_{\acute{\imath}}, \chi_{\jmath}\right) \mid \left(\chi_{\acute{\imath}}, \chi_{\jmath}\right) \in \mathfrak{R}\right\}$ be a CTSFR on $\widetilde{A}$. Then the inverse CTSFR of $\mathfrak{R}$ is denoted and defined as

$$\mathfrak{R}^{-1} = \left\{\left(\chi_{\jmath}, \chi_{\acute{\imath}}\right), \mathfrak{m}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{A}}\left(\chi_{\jmath}, \chi_{\acute{\imath}}\right), \mathfrak{p}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{A}}\left(\chi_{\jmath}, \chi_{\acute{\imath}}\right), \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}\times\widetilde{A}}\left(\chi_{\jmath}, \chi_{\acute{\imath}}\right) \mid \left(\chi_{\acute{\imath}}, \chi_{\jmath}\right) \in \mathfrak{R}\right\}$$

Example 6: Let

$$\mathfrak{R} = \left\{ \begin{array}{l} \left((\chi, \Upsilon), 0.3e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\Upsilon, z), 0.3e^{(0.5)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((z, \chi), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \end{array} \right\}$$

be a CTSFR on a CTSFS $\widetilde{A}$, for $n = 7$.

$$\widetilde{A} = \left\{ \begin{array}{l} \left(\chi, 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left(\Upsilon, 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right), \\ \left(z, 0.8e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \end{array} \right\}, \text{ then}$$

$$\mathfrak{R}^{-1} = \left\{ \begin{array}{l} \left((\chi, z), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right), \\ \left((\Upsilon, \chi), 0.3e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((z, \Upsilon), 0.3e^{(0.5)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right) \end{array} \right\}$$

is a CTSF inverse relation of $\mathfrak{R}$ on $\widetilde{A}$.

Definition 20: Let $\widetilde{A}$ be a CTSFS on $\mathfrak{U}$. Then a CTSF reflexive relation $\mathfrak{R}$ is defined as $\left((\chi, \chi), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\chi, \chi), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\chi, \chi), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\chi, \chi)\right) \in \mathfrak{R}, \forall \left(\chi, \mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\chi), \mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\chi), \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\chi)\right) \in \widetilde{A}$.

Example 7: Suppose that $\widetilde{A}$ is a CTSFS for $n = 7$ on $\mathfrak{U}$ defined as

$$\widetilde{A} = \left\{ \begin{array}{l} \left(\chi, 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left(\Upsilon, 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right), \\ \left(z, 0.8e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \end{array} \right\}.$$

Then the Cartesian product $\widetilde{A} \times \widetilde{A}$ is defined as

$$\widetilde{A} \times \widetilde{A} = \left\{ \begin{array}{l} \left((\chi, \chi), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left((\chi, \Upsilon), 0.3e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\chi, z), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right), \\ \left((\Upsilon, \chi), 0.3e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\Upsilon, \Upsilon), 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right), \\ \left((\Upsilon, z), 0.3e^{(0.5)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((z, \chi), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right), \\ \left((z, \Upsilon), 0.3e^{(0.5)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((z, z), 0.8e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \end{array} \right\}$$

The subset $\mathfrak{R}$ of $\widetilde{A} \times \widetilde{A}$ is a CTSF reflexive relation, given below

$$\mathfrak{R} = \left\{ \begin{array}{l} \left((\chi, \chi), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left((\Upsilon, \Upsilon), 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right), \\ \left((z, z), 0.8e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \end{array} \right\}$$

Definition 21: Let $\widetilde{A}$ be a CTSFS on $\mathfrak{U}$. Then a CTSF irreflexive relation $\mathfrak{R}$ is defined as

$$\left((\chi, \chi), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\chi, \chi), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\chi, \chi), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\chi, \chi)\right) \notin \mathfrak{R}, \quad \forall \left(\chi, \mathfrak{m}_{\mathbb{C}}^{\widetilde{A}}(\chi), \mathfrak{p}_{\mathbb{C}}^{\widetilde{A}}(\chi), \mathfrak{n}_{\mathbb{C}}^{\widetilde{A}}(\chi)\right) \in \widetilde{A}$$

Example 8:

$$\mathfrak{R} = \left\{ \begin{array}{l} \left((\chi, z), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right), \\ \left((\Upsilon, z), 0.3e^{(0.5)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right) \end{array} \right\}$$

is a CTSF irreflexive relation on a CTSFS $\widetilde{A}$, for $n = 7$ with

$$\widetilde{A} = \left\{ \begin{array}{l} \left(\chi, 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left(\Upsilon, 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right), \\ \left(z, 0.8e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \end{array} \right\}$$

Definition 22: Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$. Then a CTSF symmetric relation $\mathfrak{R}$ is a set such that

$$\text{If}\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right)\in\mathfrak{R}$$
$$\Longrightarrow\left((\Upsilon,\mathfrak{X}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X})\right)\in\mathfrak{R}$$

Example 9:

$$\mathfrak{R}=\left\{\begin{array}{l}\left((\mathfrak{X},\mathfrak{X}),0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left((\mathfrak{X},\Upsilon),0.3e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.9)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{X}),0.3e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.9)2\pi i}\right),\\ \left((\mathfrak{z},\mathfrak{z}),0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

is a CTSF symmetric relation on a CTSFS $\widetilde{\mathrm{A}}$, for $n=7$ with

$$\widetilde{\mathrm{A}}=\left\{\begin{array}{l}\left(\mathfrak{X},0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left(\Upsilon,0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left(\mathfrak{z},0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

Definition 23: Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$. Then a CTSF asymmetric relation $\mathfrak{R}$ is a set such that

$$\text{If}\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right)\in\mathfrak{R}$$
$$\Longrightarrow\left((\Upsilon,\mathfrak{X}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X})\right)\notin\mathfrak{R}.$$

Example 10:

$$\mathfrak{R}=\left\{\begin{array}{l}\left((\mathfrak{X},\Upsilon),0.3e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.9)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{z}),0.3e^{(0.5)2\pi i},0.7e^{(0.5)2\pi i},0.8e^{(0.9)2\pi i}\right)\end{array}\right\}$$

is a CTSF asymmetric relation on a CTSFS $\widetilde{\mathrm{A}}$, for $n=7$ with

$$\widetilde{\mathrm{A}}=\left\{\begin{array}{l}\left(\mathfrak{X},0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left(\Upsilon,0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left(\mathfrak{z},0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

Definition 24. Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$. Then a CTSF anti-symmetric relation $\mathfrak{R}$ is a set such that

$$\text{If}\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right)\in\mathfrak{R}$$
$$\text{and}\left((\Upsilon,\mathfrak{X}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X})\right)\in\mathfrak{R}$$
$$\Longrightarrow\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right)$$
$$=\left((\Upsilon,\mathfrak{X}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X})\right)$$

Example 11:

$$\mathfrak{R}=\left\{\begin{array}{l}\left((\mathfrak{X},\mathfrak{z}),0.5e^{(0.2)2\pi i},0.2e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right),\\ \left((\Upsilon,\Upsilon),0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{z}),0.3e^{(0.5)2\pi i},0.7e^{(0.5)2\pi i},0.8e^{(0.9)2\pi i}\right)\end{array}\right\}$$

is a CTSF anti-symmetric relation on a CTSFS $\widetilde{\mathrm{A}}$, for $n=7$ with

$$\widetilde{\mathrm{A}}=\left\{\begin{array}{l}\left(\mathfrak{X},0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left(\Upsilon,0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left(\mathfrak{z},0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

Definition 25: Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$, then a CTSF transitive relation $\mathfrak{R}$ is a set such that

$$\text{If}\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right),$$
$$\left((\Upsilon,\mathfrak{z}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{z})\right)\in\mathfrak{R}$$
$$\Longrightarrow\left((\mathfrak{X},\mathfrak{z}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right)\in\mathfrak{R}$$

Example 12:

$$\mathfrak{R}=\left\{\begin{array}{l}\left((\mathfrak{X},\mathfrak{X}),0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left((\mathfrak{X},\Upsilon),0.3e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.9)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{X}),0.3e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.9)2\pi i}\right)\end{array}\right\}$$

is a CTSF transitive relation on a CTSFS $\widetilde{\mathrm{A}}$, for $n=7$. with

$$\widetilde{\mathrm{A}}=\left\{\begin{array}{l}\left(\mathfrak{X},0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left(\Upsilon,0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left(\mathfrak{z},0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

Definition 26: A CTSF equivalence relation $\mathfrak{R}$ is a relation that holds the properties of the following relations:

i. A CTSF reflexive relation;
ii. A CTSF symmetric relation;
iii. A CTSF transitive relation.

Example 13:

$$\mathfrak{R}=\left\{\begin{array}{l}\left((\mathfrak{X},\mathfrak{X}),0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left((\mathfrak{X},\Upsilon),0.3e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.9)2\pi i}\right),\\ \left((\Upsilon,\Upsilon),0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{X}),0.3e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.9)2\pi i}\right),\\ \left((\mathfrak{z},\mathfrak{z}),0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

is a CTSF equivalence relation on a CTSFS $\widetilde{\mathrm{A}}$, for $n=7$ with

$$\widetilde{\mathrm{A}}=\left\{\begin{array}{l}\left(\mathfrak{X},0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left(\Upsilon,0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left(\mathfrak{z},0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

Definition 27: A CTSF order relation $\mathfrak{R}$ is a relation that holds the properties of the following relations:

i. A CTSF reflexive relation;
ii. A CTSF anti-symmetric relation;
iii. A CTSF transitive relation.

Example 14:

$$\mathfrak{R}=\left\{\begin{array}{l}\left((\mathfrak{X},\mathfrak{X}),0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left((\Upsilon,\Upsilon),0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left((\Upsilon,\mathfrak{z}),0.3e^{(0.5)2\pi i},0.7e^{(0.5)2\pi i},0.8e^{(0.9)2\pi i}\right),\\ \left((\mathfrak{z},\mathfrak{z}),0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

is a CTSF order relation on a CTSFS $\widetilde{\mathrm{A}}$, for $n=7$ with

$$\widetilde{\mathrm{A}}=\left\{\begin{array}{l}\left(\mathfrak{X},0.5e^{(0.2)2\pi i},0.2e^{(0.7)2\pi i},0.8e^{(0.4)2\pi i}\right),\\ \left(\Upsilon,0.3e^{(0.6)2\pi i},0.7e^{(0.8)2\pi i},0.5e^{(0.9)2\pi i}\right),\\ \left(\mathfrak{z},0.8e^{(0.5)2\pi i},0.9e^{(0.5)2\pi i},0.8e^{(0.7)2\pi i}\right)\end{array}\right\}$$

Definition 28. Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$, and $\mathfrak{R}$ be any CTSFR on $\widetilde{\mathrm{A}}$. Then the relation $\mathfrak{R}\circ\mathfrak{R}$ is said to be a CTSF composite relation if $\forall\, \mathfrak{X}, \mathfrak{Y}, \mathfrak{z} \in \mathfrak{U}$

$$\left((\mathfrak{X},\mathfrak{Y}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y})\right), \left((\mathfrak{Y},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z})\right) \in \mathfrak{R}$$
$$\Longrightarrow \left((\mathfrak{X},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right) \in \mathfrak{R}\circ\mathfrak{R}$$

Example 15: Consider a CTSFR $\mathfrak{R}$,

$$\mathfrak{R} = \left\{ \begin{array}{l} \left((\mathfrak{X},\mathfrak{Y}), 0.3e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\mathfrak{Y},\mathfrak{X}), 0.3e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\mathfrak{Y},\mathfrak{z}), 0.3e^{(0.5)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\mathfrak{z},\mathfrak{Y}), 0.3e^{(0.5)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \end{array} \right\}$$

then,

$$\mathfrak{R}\circ\mathfrak{R} = \left\{ \begin{array}{l} \left((\mathfrak{X},\mathfrak{X}), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left((\mathfrak{X},\mathfrak{z}), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right), \\ \left((\mathfrak{Y},\mathfrak{Y}), 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right),, \\ \left((\mathfrak{z},\mathfrak{X}), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \\ \left((\mathfrak{z},\mathfrak{z}), 0.8e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \end{array} \right\}.$$

Definition 29: Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$, and $\mathfrak{R}$ be any CTSFR. Then the equivalence class of $\mathfrak{X}$ modulo $\mathfrak{R}$ is given as

$$\overbrace{\mathfrak{R}}^{\mathfrak{X}} = \left\{ \begin{array}{c} \left(\mathfrak{z}, \mathfrak{m}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\mathfrak{z})\right) : \\ \left((\mathfrak{z},\mathfrak{X}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X})\right) \in \mathfrak{R} \end{array} \right\}$$

for $\left(\mathfrak{X}, \mathfrak{m}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\mathfrak{X}), \mathfrak{p}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\mathfrak{X}), \mathfrak{n}_{\mathbb{C}}^{\widetilde{\mathrm{A}}}(\mathfrak{X})\right)$.

Example 16: Consider a CTSF equivalence relation $\mathfrak{R}$,

$$\mathfrak{R} = \left\{ \begin{array}{l} \left((\mathfrak{X},\mathfrak{X}), 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left((\mathfrak{X},\mathfrak{Y}), 0.3e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\mathfrak{Y},\mathfrak{Y}), 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right), \\ \left((\mathfrak{Y},\mathfrak{X}), 0.3e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.9)2\pi i}\right), \\ \left((\mathfrak{z},\mathfrak{z}), 0.8e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right) \end{array} \right\}.$$

Then,

$$\overbrace{\mathfrak{R}}^{\mathfrak{X}} = \left\{ \begin{array}{l} \left(\mathfrak{X}, 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left(\mathfrak{Y}, 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right) \end{array} \right\},$$

$$\overbrace{\mathfrak{R}}^{\mathfrak{Y}} = \left\{ \begin{array}{l} \left(\mathfrak{X}, 0.5e^{(0.2)2\pi i}, 0.2e^{(0.7)2\pi i}, 0.8e^{(0.4)2\pi i}\right), \\ \left(\mathfrak{Y}, 0.3e^{(0.6)2\pi i}, 0.7e^{(0.8)2\pi i}, 0.5e^{(0.9)2\pi i}\right) \end{array} \right\},$$

and

$$\overbrace{\mathfrak{R}}^{\mathfrak{z}} = \left\{\left(\mathfrak{z}, 0.8e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}, 0.8e^{(0.7)2\pi i}\right)\right\}.$$

IV. MAIN RESULTS

This section presents some results of CTSF symmetric relations, CTSF transitive, CTSF equivalence relation and CTSF composite relation.

Theorem 1: Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$, and $\mathfrak{R}$ be a CTSFR on $\widetilde{\mathrm{A}}$. Then $\mathfrak{R}$ is a CTSF symmetric relation on $\widetilde{\mathrm{A}}$ *iff* $\mathfrak{R} = \mathfrak{R}^{-1}$

Proof. First suppose that $\mathfrak{R}$ is a CTSF symmetric relation on $\widetilde{\mathrm{A}}$,

$$\Longrightarrow \left((\mathfrak{X},\mathfrak{Y}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y})\right) \in \mathfrak{R}$$
$$\Longleftrightarrow \left((\mathfrak{Y},\mathfrak{X}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X})\right) \in \mathfrak{R}$$

Also, $\left((\mathfrak{Y},\mathfrak{X}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X})\right) \in \mathfrak{R}^{-1} \Longrightarrow \mathfrak{R} = \mathfrak{R}^{-1}$ Now, let $\mathfrak{R} = \mathfrak{R}^{-1}$, then for $\left((\mathfrak{X},\mathfrak{Y}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y})\right) \in \mathfrak{R}$,

$$\left((\mathfrak{X},\mathfrak{Y}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y})\right) \in \mathfrak{R}$$
$$\Longleftrightarrow \left((\mathfrak{Y},\mathfrak{X}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X})\right) \in \mathfrak{R}^{-1}$$
$$\Longleftrightarrow \left((\mathfrak{Y},\mathfrak{X}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{X})\right) \in \mathfrak{R} \Longrightarrow \mathfrak{R}$$

is a CTSF symmetric relation on $\widetilde{\mathrm{A}}$.

Theorem 2: Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$ and $\mathfrak{R}$ be a CTSFR on $\widetilde{\mathrm{A}}$. Then $\mathfrak{R}$ is a CTSF transitive relation on $\widetilde{\mathrm{A}}$ *iff* $\mathfrak{R}\circ\mathfrak{R} \subseteq \mathfrak{R}$.

Proof: First, suppose that $\left((\mathfrak{X},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right) \in \mathfrak{R}\circ\mathfrak{R}$ and $\mathfrak{R}$ is a CTSF transitive relation then $\exists$ an element $\mathfrak{Y} \in \mathfrak{U} \ni$

$$\left((\mathfrak{X},\mathfrak{Y}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y})\right) \in \mathfrak{R}$$
$$\text{and } \left((\mathfrak{Y},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z})\right) \in \mathfrak{R}$$
$$\Longrightarrow \left((\mathfrak{X},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right) \in \mathfrak{R}$$
$$\Longrightarrow \mathfrak{R}\circ\mathfrak{R} \subseteq \mathfrak{R}$$

Conversely suppose that $\mathfrak{R}\circ\mathfrak{R} \subseteq \mathfrak{R}$ then by definition of CTSF composite relation,

$$\left((\mathfrak{X},\mathfrak{Y}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y})\right) \in \mathfrak{R}$$
$$\text{and } \left((\mathfrak{Y},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z})\right) \in \mathfrak{R}$$
$$\Longrightarrow \left((\mathfrak{X},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right) \in \mathfrak{R}\circ\mathfrak{R}$$

But $\mathfrak{R}\circ\mathfrak{R} \subseteq \mathfrak{R} \Longrightarrow \left((\mathfrak{X},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right) \in \mathfrak{R}$. Thus $\mathfrak{R}$ is a CTSF transitive relation.

Theorem 3: Let $\widetilde{\mathrm{A}}$ be a CTSFS on $\mathfrak{U}$ and $\mathfrak{R}$ be a CTSF equivalence relation on $\widetilde{\mathrm{A}}$. Then $\mathfrak{R}\circ\mathfrak{R} = \mathfrak{R}$.

Proof: Since $\mathfrak{R}$ is a CTSF equivalence relation on $\widetilde{\mathrm{A}}$, for $\left((\mathfrak{X},\mathfrak{Y}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{Y})\right) \in \mathfrak{R}$, we have that $\left((\mathfrak{Y},\mathfrak{z}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{Y},\mathfrak{z})\right) \in \mathfrak{R}$ by using symmetry.

Also, $\left((\mathfrak{X},\mathfrak{X}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{X}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{X}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{X})\right) \in \mathfrak{R}$ by using transitivity. But, according to the definition of CTSF composite relation,

$$\left((\mathfrak{X},\mathfrak{X}), \mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{X}), \mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{X}), \mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{X})\right) \in \mathfrak{R}\circ\mathfrak{R}$$
$$\Longrightarrow \mathfrak{R} \subseteq \mathfrak{R}\circ\mathfrak{R} \qquad (1)$$

Conversely, suppose that $\left((\mathfrak{X},\mathfrak{z}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right)\in\mathfrak{R}\circ\mathfrak{R}$. Then $\exists\ \Upsilon\in\mathfrak{U}\ni\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right)\in\mathfrak{R}$ and $\left((\Upsilon,\mathfrak{z}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{z})\right)\in\mathfrak{R}\Longrightarrow\left((\mathfrak{X},\mathfrak{z}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right)$. Because $\mathfrak{R}$ is CTSF equivalence relation and thus CTSF transitive relation, i.e.

$$\mathfrak{R}\circ\mathfrak{R}\subseteq\mathfrak{R} \tag{2}$$

Equations (1) and (2) imply that $\mathfrak{R}\circ\mathfrak{R}=\mathfrak{R}$.

Theorem 4: Let $\widetilde{A}$ be a CTSFS on $\mathfrak{U}$ and $\mathfrak{R}$ be a CTSF equivalence relation on $\widetilde{A}$. Then $\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right)\in\mathfrak{R}$, *iff* $\overbrace{\mathfrak{R}}^{\mathfrak{X}}=\overbrace{\mathfrak{R}}^{\Upsilon}$.

Proof: Since $\mathfrak{R}$ is a CTSF equivalence relation. Let $\overbrace{\mathfrak{R}}^{\mathfrak{X}}=\overbrace{\mathfrak{R}}^{\Upsilon}$, then for any $\mathfrak{z}\in\mathfrak{U}$, $\left(\mathfrak{z},\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z})\right)\in\overbrace{\mathfrak{R}}^{\mathfrak{X}}\Longrightarrow\left((\mathfrak{z},\mathfrak{X}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X})\right)\in\mathfrak{R}$. By symmetry, we have

$$\left((\mathfrak{X},\mathfrak{z}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\mathfrak{z})\right)\in\mathfrak{R} \tag{3}$$

and

$$\left(\mathfrak{z},\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z})\right)\in\overbrace{\mathfrak{R}}^{\Upsilon}$$

$$\Longrightarrow\left((\mathfrak{z},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon)\right)\in\mathfrak{R} \tag{4}$$

Equation (3) and (4) imply through transitivity that

$$\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right)\in\mathfrak{R}.$$

Conversely, suppose that

$$\left((\mathfrak{X},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{X},\Upsilon)\right)\in\mathfrak{R} \tag{5}$$

and

$$\left(\mathfrak{z},\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z})\right)\in\overbrace{\mathfrak{R}}^{\mathfrak{X}}$$

$$\Longrightarrow\left((\mathfrak{z},\mathfrak{X}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X})\right)\in\mathfrak{R} \tag{6}$$

Equations (5) and (6) imply through transitivity that

$$\left((\mathfrak{z},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon)\right)\in\mathfrak{R}$$

$$\Longrightarrow\left(\mathfrak{z},\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z})\right)\in\overbrace{\mathfrak{R}}^{\Upsilon}$$

$$\Longrightarrow\overbrace{\mathfrak{R}}^{\mathfrak{X}}\subseteq\overbrace{\mathfrak{R}}^{\Upsilon} \tag{7}$$

In the same way, suppose that

$$\left((\Upsilon,\mathfrak{X}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\Upsilon,\mathfrak{X})\right)\in\mathfrak{R} \tag{8}$$

and

$$\left(\mathfrak{z},\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z})\right)\in\overbrace{\mathfrak{R}}^{\Upsilon}$$

$$\Longrightarrow\left((\mathfrak{z},\Upsilon),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\Upsilon)\right)\in\mathfrak{R} \tag{9}$$

Equations (8) and (9) imply through transitivity that

$$\left((\mathfrak{z},\mathfrak{X}),\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z},\mathfrak{X})\right)\in\mathfrak{R}$$

$$\Longrightarrow\left(\mathfrak{z},\mathfrak{m}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{p}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z}),\mathfrak{n}_{\mathbb{C}}^{\mathfrak{R}}(\mathfrak{z})\right)\in\overbrace{\mathfrak{R}}^{\mathfrak{X}}.$$

Then, we have

$$\overbrace{\mathfrak{R}}^{\Upsilon}\subseteq\overbrace{\mathfrak{R}}^{\mathfrak{X}} \tag{10}$$

Equations (7) and (10) imply that $\overbrace{\mathfrak{R}}^{\mathfrak{X}}=\overbrace{\mathfrak{R}}^{\Upsilon}$.

V. APPLICATIONS

This section presents a couple of applications of CTSFSs, CTSFRs and the types of CTSFRs. In subsection A, the interdependence of the international trades among countries is discussed. Subsection B proposes a useful application that talks over the economic relationships. It also compares the financial factors affecting the business markets and presents the effects of fundamental elements of economic structure on each other.

A. INTERNATIONAL TRADE INTERDEPENDENCE

The term international trade refers to the give-and-take of goods and exchange of services among the countries. This global exchange of goods exposes the countries to the products that are unavailable in the local markets, or it provides the same products at lower prices. So, the international trade lets countries to grow their markets and helps them to access those goods and services that are unobtainable in the country. This makes the market more competitive. Eventually, due to competitive pricing, the products are offered on cheaper prices to the consumers. The global market has significant effects on the domestic markets. Hence, the international trades are interdependent. For example, consider an American mobile company based in China. If the labor cost rises in China due to some political changes or some other reasons, the mobile company has to pay more to its workers and employees. Ultimately the prices of smartphones in the local markets of USA will rise as well.

There are several tools in the fuzzy set theory for modeling such topics. The objective is to discuss the influences of one parameter over the other through three grades. These grades would represent the positive effects, no effects or neutral effects and the negative effects. Keeping these targets in the mind, the options of applicable tools become clearer. Numerous tools from FRs to CqROFRs are disregarded because of their limitations. Moreover, we would also like to compare these parameters with respect to some time periods. Therefore, the choices of tools with three grades, such as PFRs, SFRs and TSFRs, are also neglected, because they discuss one dimensional relations. However, CPFRs, CSFRs and CTSFRs are characterized by three grades and are capable of modeling problems with periodic nature. Among these three tools, the CTSFRs are the strongest and the most efficient. Henceforth, this trading structure has been modeled using the CTSFSs and CTSFRs. This application presents the interdependence of the international trades among countries; i.e. the positive effects, negative effects and no effects of the trades of one country over the others. The membership grades indicate the positive effects, the non-membership grades indicate the negative effects and the abstinence grades indicate no effects at all. To clearly understand the idea, the following example is presented in which the aforementioned concepts are applied to the supposed situations.

Consider the sets $A = \{China, \ Germany, \ USA\}$ and $B = \{China, \ Japan, Russia\}$ of the countries that directly trade with each other. By direct trade we mean that the countries in a set such as China, Germany and USA directly trade goods and services with each other, while the countries in different sets have no direct trades. But they are still somehow related to each other through indirect trades. For convenience, let us assign variables to each of the country; China (C), USA (U), Germany (G), Russia (R) and Japan (J). Now, constructing the associated CTSFSs $\widetilde{\mathrm{A}}$ and $\widetilde{\mathrm{B}}$ by assigning the membership, abstinence and non-membership grades to the countries in sets A and B as

$$\widetilde{\mathrm{A}} = \left\{ \begin{array}{c} \left(C, 0.8e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left(G, 0.6e^{(0.5)2\pi i}, 0.7e^{(0.75)2\pi i}, 0.4e^{(0.75)2\pi i}\right), \\ \left(U, 0.9e^{(0.25)2\pi i}, 0.1e^{(0.75)2\pi i}, 0.6e^{(0.25)2\pi i}\right) \end{array} \right\}$$

and

$$\widetilde{\mathrm{B}} = \left\{ \begin{array}{c} \left(C, 0.8e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left(J, 0.5e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}, 0.2e^{(0.25)2\pi i}\right), \\ \left(R, 0.9e^{(0.75)2\pi i}, 0.4e^{(0.25)2\pi i}, 0.5e^{(0.5)2\pi i}\right) \end{array} \right\}$$

Now, to study the relationships among the countries of set $\widetilde{\mathrm{A}}$, the Cartesian product is found as

$$\begin{aligned} &\mathfrak{R}_1 \\ &= \widetilde{\mathrm{A}} \times \widetilde{\mathrm{A}} \\ &= \left\{ \begin{array}{c} \left((C, C), 0.8e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((C, G), 0.6e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.75)2\pi i}\right), \\ \left((C, U), 0.8e^{(0.25)2\pi i}, 0.1e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((G, C), 0.6e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.75)2\pi i}\right), \\ \left((G, G), 0.6e^{(0.5)2\pi i}, 0.7e^{(0.75)2\pi i}, 0.4e^{(0.75)2\pi i}\right), \\ \left((G, U), 0.6e^{(0.25)2\pi i}, 0.1e^{(0.75)2\pi i}, 0.6e^{(0.75)2\pi i}\right), \\ \left((U, C), 0.8e^{(0.25)2\pi i}, 0.1e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((U, G), 0.6e^{(0.25)2\pi i}, 0.1e^{(0.75)2\pi i}, 0.6e^{(0.75)2\pi i}\right), \\ \left((U, U), 0.9e^{(0.25)2\pi i}, 0.1e^{(0.75)2\pi i}, 0.6e^{(0.25)2\pi i}\right) \end{array} \right\} \end{aligned} \tag{11}$$

and the relationships among the countries of set $\widetilde{\mathrm{B}}$ are explored by the following Cartesian product,

$$\begin{aligned} &\mathfrak{R}_2 \\ &= \widetilde{\mathrm{B}} \times \widetilde{\mathrm{B}} \\ &= \left\{ \begin{array}{c} \left((C, C), 0.8e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((C, J), 0.5e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((C, R), 0.8e^{(0.25)2\pi i}, 0.2e^{(0.25)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((J, C), 0.5e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((J, J), 0.5e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}, 0.2e^{(0.25)2\pi i}\right), \\ \left((J, R), 0.5e^{(0.5)2\pi i}, 0.4e^{(0.25)2\pi i}, 0.5e^{(0.5)2\pi i}\right), \\ \left((R, C), 0.8e^{(0.25)2\pi i}, 0.2e^{(0.25)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((R, J), 0.5e^{(0.5)2\pi i}, 0.4e^{(0.25)2\pi i}, 0.5e^{(0.5)2\pi i}\right), \\ \left((R, R), 0.9e^{(0.75)2\pi i}, 0.4e^{(0.25)2\pi i}, 0.5e^{(0.5)2\pi i}\right) \end{array} \right\} \end{aligned} \tag{12}$$

In $\mathfrak{R}_1$ and $\mathfrak{R}_2$, each element describes the quality and quantity of effects of one country over the other. For instance, $\left((G, U), 0.6e^{(0.25)2\pi i}, 0.1e^{(0.75)2\pi i}, 0.6e^{(0.75)2\pi i}\right)$ from $\mathfrak{R}_1$ tells that the grade of positive influence of the trades of Germany on the trades of USA is 0.6 with respect to half a year, because the exponent shows the time delay, so $(0.25)\ 2 = 0.5$ years. Moreover, the grade of negative influence is also 0.6, but with respect to one and a half year and the grade of no influence is 0.1 with respect to one and a half year. Similarly, $\left((C, R), 0.8e^{(0.25)2\pi i}, 0.2e^{(0.25)2\pi i}, 0.8e^{(0.5)2\pi i}\right)$ from $\mathfrak{R}_2$ tells that the grade of positive influence of trades of China over trades of Russia is 0.8 with respect to half a year, grade of negative influence is 0.8 with respect to a year and grade of no influence is 0.2 with respect to half a year.

Since, in the above relations, the effects of trades among the countries having direct trades are determined. What if there is no direct trade between two countries, then how could one find out the indirect trade effects in such cases. For instance, it is observed that USA and Russia are from different sets, so they do not have any direct trades, but both of them have trades with China. Henceforth, they can exchange goods and services in some way through China. For determining the relationships between USA and Russia, we find $\mathfrak{R}_3 = \widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}}$. As, in the current example USA and Russia do not trade directly, but they are somehow related to each other indirectly through China. That means the trades of USA influence the trades of China, which in turn influence the trades of Russia. Hence, there are effects in an indirect manner. Let us find out the CTSFR $\mathfrak{R}_3$. It is

$$\begin{aligned} &\mathfrak{R}_3 \\ &= \widetilde{\mathrm{A}} \times \widetilde{\mathrm{B}} \\ &= \left\{ \begin{array}{c} \left((C, C), 0.8e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((C, J), 0.5e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((C, R), 0.8e^{(0.25)2\pi i}, 0.2e^{(0.25)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((G, C), 0.6e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.8e^{(0.75)2\pi i}\right), \\ \left((G, J), 0.5e^{(0.5)2\pi i}, 0.7e^{(0.5)2\pi i}, 0.4e^{(0.75)2\pi i}\right), \\ \left((G, R), 0.6e^{(0.5)2\pi i}, 0.4e^{(0.25)2\pi i}, 0.5e^{(0.75)2\pi i}\right), \\ \left((U, C), 0.8e^{(0.25)2\pi i}, 0.1e^{(0.5)2\pi i}, 0.8e^{(0.5)2\pi i}\right), \\ \left((U, J), 0.5e^{(0.25)2\pi i}, 0.1e^{(0.5)2\pi i}, 0.6e^{(0.25)2\pi i}\right), \\ \left((U, R), 0.9e^{(0.25)2\pi i}, 0.1e^{(0.25)2\pi i}, 0.6e^{(0.5)2\pi i}\right) \end{array} \right\} \end{aligned} \tag{13}$$

FIGURE 3. Flowchart of the process followed in the example.

So now, $\mathfrak{R}_3$ provides an opportunity to study the indirect effects of trades of one country over the trades of others. The event $\left((U, R), 0.9e^{(0.25)2\pi i}, 0.1e^{(0.25)2\pi i}, 0.6e^{(0.5)2\pi i}\right)$ in (3) explains that the positive influence of trade of USA on trade of Russia is of grade 0.9 with respect to half a year, grade of negative influence is 0.6 with respect to a year and the grade of no effects is 0.1 with respect to half a year. To find out the effects of Russia on USA a CTSF inverse relation is used.

Figure 3 illustrates the algorithm and the process followed in the application. The flow chart explains the stepwise progression in order to find the relationships between the countries. First of all, pull together the countries that are needed be inspected. Then make the sub collections of the original set on the basis of direct relationships i.e. the sub collections contain the countries that have direct relations among them. To learn the interdependence and the level of effectiveness of the trades of one country over the others. After that, the Cartesian products are found to get all the possible relations among the listed countries. Next, by applying the types of CTSFRs, the desired indirect relationships are established. Finally, the information is interpreted by reading off the relations. The ordered pair in a Cartesian product tells the effects of the trades of first country on the trades of the second country. The amplitude term refers to the level of effectiveness and the phase term refers to the time lag.

B. ECONOMIC RELATIONSHIPS

The prosperity of a nation highly depends on its economy. Economics is one of the social sciences that deals with the study of interaction of people with the value or wealth. It discusses the way in which goods and services are produced, distributed and consumed. Economic development is the main factor that pushes up the economic growth in the economy. The economic growth is very important because it creates higher salary jobs and plays the fundamental role in the improvement of the quality of life. For instance, the economic developers assist the jobs providing companies to expand their businesses by connecting them to other companies and partners. Eventually it opens up the gates for other industries to jump in. As a result, the economy is diversified which means that many industries are running. The diverse economy helps the businesses to grow and generates higher tax revenues. Ultimately there are more and more job opportunities, and the quality of life improves. In addition, the knowledge of economics also helps to settle a wonderful, successful and an efficient business. Some basic financial and economic relationships are discussed below.

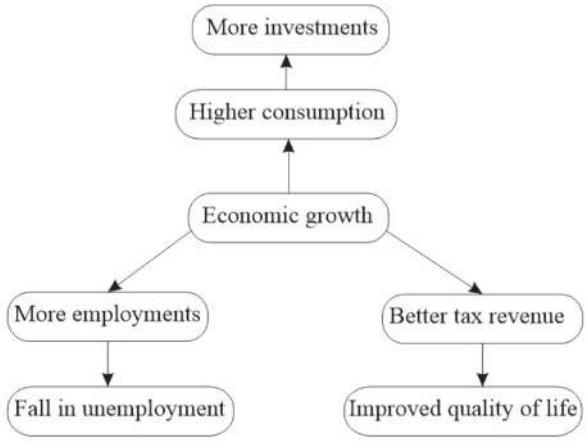

FIGURE 4. Effects of economic growth on various factors.

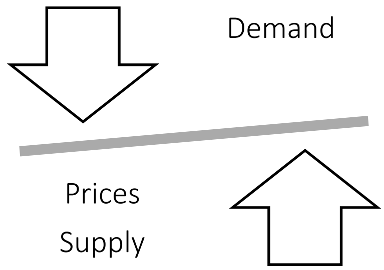

FIGURE 5. Relationship of price with supply and demand.

1) PRICE, SUPPLY AND DEMAND
The prices, supply and demand of goods and services are directly related to each other. Almost always the sales of products and services drop down whenever a business increases the prices of its products or services because buyers prefer cheaper products. As the prices go higher, the less people will be able to afford the products. So the demand for the product drops. On the other hand, when something is being sold at higher prices, an increased supply will generate greater revenue. Figure 5 illustrates the relation of price with demand and supply of the goods and services. It is an economic principle that prices fall whenever the supply of a good or service surpasses the demand for that good or service. The prices tend to rise whenever the demand exceeds the supply.
2) INTEREST RATES, INVESTMENT AND MONEY SUPPLY
The investment helps to grow the businesses as well as the economy of a country. The financings or investments highly depend on the interest rates. Since, an investor always looks for a higher return, so he prefers the industry with a higher interest rates. Thus,

the higher interest rates invite and attract the investors to put in their money in the business. Consequently, the business grows bigger. The prices of certain product are usually determined by the quantity of that product in the market. When the quantity of money in an economic structure increases, the value of money decreases and vice versa. Therefore, the interest rates are lower in the economic structures with greater money supply and vice versa. This fact leads to an important relation among the supply of money, interest rates and the investment. The interest rates are inversely proportional to the amount of money available. Henceforth, the printing of currency notes has bad impacts on the economy.

3) ECONOMIC GROWTH AND UNEMPLOYMENT
Since the amount of money determines the interest rates and the interest rates determine the investments made. Although investments play a major role in determining the economy, gross domestic product (GDP) is also of great concern. GDP is the amount spent on the consumption, investment, exports and government services. For instance, if the spending on the services and products is not enough then it makes sense to stop the production of new products. On the other hand, if the expenditure is more, then there is a need of more production. More production will need more workers which will create job opportunities. So there is a key link between GDP and the unemployment. The economic growth is the measure of the change of GDP from one year to the other. For better understanding, a visual summary of economic relationships is given in Figure 4.

CTSFRs are used to study these financial and economic relationships. Modeling this problem using the notion of CTSFSs and CTSFRs will not only help to determine the influence of one factor on the other, but also the grades of supportive effects, destructive effects and even the grade of no effects with respect to the time lag can be investigated. The beauty of CTSFRs is that they empower to find out the indirect effects between some events.

Let P, S and D symbolize the prices, supply and demand respectively. Investment, interest rates, money supply and unemployment are denoted by I, IR, M and U, respectively. So, the set of all factors is given in (14)

$$\widehat{\mathrm{F}} = \left\{ \begin{array}{c} \left(P, 0.9e^{(0.5)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.2e^{(0.75)2\pi i}\right), \\ \left(S, 0.7e^{(0.5)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.3e^{(0.5)2\pi i}\right), \\ \left(D, 0.4e^{(0.75)2\pi i}, 0.1e^{(0.5)2\pi i}, 0.8e^{(0.25)2\pi i}\right), \\ \left(I, 0.9e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.3e^{(0.5)2\pi i}\right), \\ \left(IR, 0.8e^{(0.25)2\pi i}, 0.3e^{(0.75)2\pi i}, 0.4e^{(0.75)2\pi i}\right), \\ \left(M, 0.2e^{(0.5)2\pi i}, 0.3e^{(0.5)2\pi i}, 0.8e^{(0.25)2\pi i}\right), \\ \left(GDP, 0.8e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.1e^{(0.5)2\pi i}\right), \\ \left(U, 0.2e^{(0.75)2\pi i}, 0.4e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}\right) \end{array} \right\} \tag{14}$$

But, these factors need to be grouped on the basis of direct relationships. Thus, the following three sets $\widetilde{\mathrm{A}}_1$, $\widetilde{\mathrm{A}}_2$ and $\widetilde{\mathrm{A}}_3$ are constructed whose members are directly related to one another, i.e., they have direct impacts on their fellow members.

$$\widetilde{\mathrm{A}}_1 = \left\{ \begin{array}{c} \left(P, 0.9e^{(0.5)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.2e^{(0.75)2\pi i}\right), \\ \left(S, 0.7e^{(0.5)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.3e^{(0.5)2\pi i}\right), \\ \left(D, 0.4e^{(0.75)2\pi i}, 0.1e^{(0.5)2\pi i}, 0.8e^{(0.25)2\pi i}\right) \end{array} \right\},$$

$$\widetilde{\mathrm{A}}_2 = \left\{ \begin{array}{c} \left(I, 0.9e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.3e^{(0.5)2\pi i}\right), \\ \left(IR, 0.8e^{(0.25)2\pi i}, 0.3e^{(0.75)2\pi i}, 0.4e^{(0.75)2\pi i}\right), \\ \left(M, 0.2e^{(0.5)2\pi i}, 0.3e^{(0.5)2\pi i}, 0.8e^{(0.25)2\pi i}\right) \end{array} \right\},$$

$$\widetilde{\mathrm{A}}_3 = \left\{ \begin{array}{c} \left(GDP, 0.8e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.1e^{(0.5)2\pi i}\right), \\ \left(U, 0.2e^{(0.75)2\pi i}, 0.4e^{(0.5)2\pi i}, 0.9e^{(0.5)2\pi i}\right) \end{array} \right\}$$

The effects of factors lying in the same set can be found in the similar fashion as in the previous application, i.e. using simple Cartesian product of set on itself. To study the relationships between the elements of set $\widetilde{\mathrm{A}}_1$ and $\widetilde{\mathrm{A}}_2$, a CTSFR $\mathfrak{R}_{\widetilde{\mathrm{A}}_1\times\widetilde{\mathrm{A}}_2}$ is found which is the subset of the Cartesian product $\widetilde{\mathrm{A}}_1 \times \widetilde{\mathrm{A}}_2$.

$$\mathfrak{R}_{\widetilde{\mathrm{A}}_1\times\widetilde{\mathrm{A}}_2} = \left\{ \begin{array}{c} \left((P,\ I), 0.9e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.3e^{(0.75)2\pi i}\right), \\ \left((P,\ M), 0.9e^{(0.5)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.2e^{(0.75)2\pi i}\right), \\ \left((S,\ I), 0.9e^{(0.5)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.2e^{(0.75)2\pi i}\right), \\ \left((S,\ IR), 0.9e^{(0.5)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.2e^{(0.75)2\pi i}\right), \\ \left((D,\ I), 0.9e^{(0.5)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.2e^{(0.75)2\pi i}\right) \end{array} \right\} \tag{15}$$

The CTSFR $\mathfrak{R}_{\widetilde{\mathrm{A}}_1\times\widetilde{\mathrm{A}}_2}$ contains only the events that need to be inspected, and so the remaining events of the Cartesian product are excluded. In equation (15), the event $\left((P,\ I), 0.9e^{(0.25)2\pi i}, 0.2e^{(0.5)2\pi i}, 0.3e^{(0.75)2\pi i}\right)$ provides the information about the effects of prices of goods and services on the investment. The membership grade 0.9 indicates that the better prices highly support the investment and that too in short amount of time, since the phase term in the exponent expresses the time lag to be half a year. The abstinence grade $0.2e^{(0.5)2\pi i}$ identifies that with respect to one year, there is a very little chances of investment being not effected by the prices. In other words, the prices enormously affect the investment. Likewise, the non-membership grade $0.3e^{(0.75)2\pi i}$ discloses that the better prices scantly discourage the investment. Its adverse influence on investment is as low as 0.3 with respect to one and half a year. Hence, its degree of destructing the investment is very near to the ground and these effects are sluggish too.

In the same way, another CTSFR $\mathfrak{R}_{\widetilde{\mathrm{A}}_2\times\widetilde{\mathrm{A}}_3}$ is found to study the relationship among the elements of sets $\widetilde{\mathrm{A}}_2$ and $\widetilde{\mathrm{A}}_3$. $\mathfrak{R}_{\widetilde{\mathrm{A}}_2\times\widetilde{\mathrm{A}}_3}$ is a subset of the Cartesian product $\widetilde{\mathrm{A}}_2 \times \widetilde{\mathrm{A}}_3$, that is given in (16).

$$\mathfrak{R}_{\widetilde{A}_2\times\widetilde{A}_3} = \left\{ \begin{array}{c} \left((I,\ GDP),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right), \\ \left((I,\ U),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right), \\ \left((IR,\ GDP),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right), \\ \left((M,\ GDP),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right) \end{array} \right\} \tag{16}$$

Each event in the above relation describes the helpful effects, discouraging effects as well as the neutral effects of first variable on the second variable. For example, $\left((I,\ GDP),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right)$ describes the effects of investment (I) on the gross domestic product (GDP). The membership $0.9e^{(0.5)2\pi i}$, abstinence $0.2e^{(0.5)2\pi i}$ and the non-membership grades $0.2e^{(0.75)2\pi i}$ show that the higher investment prominently helps GDP in short period of time. The overall effects of investment on GDP are higher as translated by the grade of abstinence. Furthermore, in rare case of negative effects, GDP declines very slowly due to investments.

The CTSF composite relation $\mathfrak{R}$ is found to find the chain relationship among the elements of set $\widetilde{A}_1$ and $\widetilde{A}_3$. Previously, the effects of prices on the investments and the effects of investment on GDP have been determined. Using this chain, the composite relation helps in relating prices to GDP. Some of the indirect relationships are given the following composite relation set $\mathfrak{R}$.

$$\mathfrak{R} = \left(\mathfrak{R}_{\widetilde{A}_1\times\widetilde{A}_2}\right)\circ\left(\mathfrak{R}_{\widetilde{A}_2\times\widetilde{A}_3}\right) = \left\{ \begin{array}{c} \left((P,\ GDP),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right), \\ \left((P,\ U),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right), \\ \left((S,\ GDP),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right), \\ \left((D,\ GDP),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right) \end{array} \right\} \tag{17}$$

As mentioned, the event $\left((P,\ GDP),0.9e^{(0.5)2\pi i},0.2e^{(0.5)2\pi i},0.2e^{(0.75)2\pi i}\right)$ provides the information about the effects of prices on GDP. In this particular case, it learns that the prices play a vital role in the growth of economy. Prices benefit the GDP up to 0.9 degree with respect to a year and detriment GDP as low as 0.2 degree with respect to one and a half year. The abstinence grade $0.2e^{(0.5)2\pi i}$ imparts that chances of prices not effecting the GDP are 0.2 with respect to a year, which means that the prices greatly affect the GDP.

VI. COMPARITIVE ANALYSIS

In this section, a comparison among the proposed methods and existing methods is carried out. The CTSFSs and CTSFRs stand preeminent above all concepts and methods that are meant to handle the fuzziness. Obviously, these sets discuss three different grades i.e. membership grade, abstinence grade and non-membership grade. However, FSs, IFSs, PyFSs, qROFSs, CFSs, CIFSs, CPyFSs and CqROFSs fail to express this situation because they are only characterized by single and dual grades. In the proposed applications, we talked about the supportive effects, neutral effects and discouraging effects of one factor on the other which were depicted by membership, abstinence and non-membership grades respectively. In order to discuss the overall strength of the effects, it is essential to consider the abstinence grade. Moreover, PFSs, SFSs and TSFSs are capable of stating all three grades, but they flop when it comes to multidimensional problems. Whereas CPFSs, CSFSs and CTSFSs are made to model the multidimensional problems. Henceforth, Tables 1 and 2 compare CTSFSs with CSFSs and CPFSs respectively when applied to the applications discussed above.

Since CPFSs and CSFSs are equipped with complex valued membership grades, abstinence grades and non-membership grades, so it's convenient to test them on the presented applications. Table 1 clearly shows that CPFSs completely fail to deal with the problem, as the sum of the grades do not lie within its constraints. Although CSFSs have a wider range than CPFSs, they barely passed only on one occasion, table 2.

TABLE 1. CPFSs dealing with the application.

Element	CPFS	Result	Status
$\left(C,0.8e^{(0.25)2\pi i},0.2e^{(0.5)2\pi i},0.8e^{(0.5)2\pi i}\right)$	$(0.8+0.2+0.8)e^{(0.25+0.5+0.5)2\pi i}$	$1.8e^{(1.25)2\pi i}$	Fail
$\left(G,0.6e^{(0.5)2\pi i},0.7e^{(0.75)2\pi i},0.4e^{(0.75)2\pi i}\right)$	$(0.6+0.7+0.4)e^{(0.5+0.75+0.75)2\pi i}$	$1.7e^{(2.0)2\pi i}$	Fail
$\left(U,0.9e^{(0.25)2\pi i},0.1e^{(0.75)2\pi i},0.6e^{(0.25)2\pi i}\right)$	$(0.9+0.1+0.6)e^{(0.25+0.75+0.25)2\pi i}$	$1.6e^{(1.25)2\pi i}$	Fail
$\left(J,0.5e^{(0.5)2\pi i},0.8e^{(0.5)2\pi i},0.2e^{(0.25)2\pi i}\right)$	$(0.5+0.8+0.2)e^{(0.5+0.5+0.25)2\pi i}$	$1.5e^{(1.25)2\pi i}$	Fail
$\left(R,0.9e^{(0.75)2\pi i},0.4e^{(0.25)2\pi i},0.5e^{(0.5)2\pi i}\right)$	$(0.9+0.4+0.5)e^{(0.75+0.25+0.5)2\pi i}$	$1.8e^{(1.5)2\pi i}$	Fail

TABLE 2. CPFSs dealing with the application.

Element	CSFS	Result	Status
$\left(C,0.8e^{(0.25)2\pi i},0.2e^{(0.5)2\pi i},0.8e^{(0.5)2\pi i}\right)$	$(0.8^2+0.2^2+0.8^2)e^{(0.25^2+0.5^2+0.5^2)2\pi i}$	$1.32e^{(0.56)2\pi i}$	Fail
$\left(G,0.6e^{(0.5)2\pi i},0.7e^{(0.75)2\pi i},0.4e^{(0.75)2\pi i}\right)$	$(0.6^2+0.7^2+0.4^2)e^{(0.5^2+0.75^2+0.75^2)2\pi i}$	$1.01e^{(1.37)2\pi i}$	Fail
$\left(U,0.9e^{(0.25)2\pi i},0.1e^{(0.75)2\pi i},0.6e^{(0.25)2\pi i}\right)$	$(0.9^2+0.1^2+0.6^2)e^{(0.25^2+0.75^2+0.25^2)2\pi}$	$1.18e^{(0.56)2\pi i}$	Fail
$\left(J,0.5e^{(0.5)2\pi i},0.8e^{(0.5)2\pi i},0.2e^{(0.25)2\pi i}\right)$	$(0.5^2+0.8^2+0.2^2)e^{(0.5^2+0.5^2+0.25^2)2\pi i}$	$0.93e^{(0.56)2\pi i}$	Pass
$\left(R,0.9e^{(0.75)2\pi i},0.4e^{(0.25)2\pi i},0.5e^{(0.5)2\pi i}\right)$	$(0.9^2+0.4^2+0.5^2)e^{(0.75^2+0.25^2+0.5^2)2\pi i}$	$1.22e^{(0.88)2\pi i}$	Fail

Besides that, CSFSs failed to cope with the problem because the sum of squares of real and imaginary parts do not belong to unit interval. Henceforth, considering the dominance of CTSFSs, we used the notion with broader range. CTSFSs facilitates the professionals to show their discernment without restrictions.

Moreover, the CTSFSs are the ultimate generalization of FSs. By substituting $n = 1$ in a CTSFS, it turns to be a CPFS, when $n = 2$, it turns to be a CSFS. A CTSFS becomes a CqROFS by putting the abstinence degree equal to zero. Similarly, it generalizes the CIFS, CPyFS and CFS. For zero phase term, i.e. the zero imaginary part, it makes the CTSFS to be a TSFS, which also generalizes the SFS, PFS, qROFS, PyFS, IFS and FS. Thus, CTSFSs cover all the previous methods and techniques that actually demonstrate the superiority of the notion of CTSFSs.

VII. ADVANTAGES OF CTSFSs AND CTSFRs

This section intends to highlight some of the advantages of the notions of CTSFSs, CTSFRs and the types of CTSFRs.

- The concept of CTSFS generalizes all the existing structures in the fuzzy set theory, i.e. FSs, CFSs, IFSs, CIFSs, PyFSs, CPyFSs, qROFSs, CqROFSs, PFSs, CPFSs, SFSs, CSFSs and TSFSs. It means that the structure of CTSFSs can deal with the data provided in any of the existing environments.
- The CTSFRs generalize all the relations presented in the literature, such as FRs, CFRs, IFRs, CIFRs, PyFRs, CPyFRs, qROFRs, CqROFRs, PFRs, CPFRs, SFRs, CSFRs and TSFRs.
- The relations defined on the CTSFSs such as CTSF equivalence relation, CTSF order relation and CTSF composite relations also generalize the types of relations in the other structures.

VIII. CONCLUSION

There are several theories, such as fuzzy sets (FSs), intuitionistic FSs (IFSs), q-rung orthopair FSs (qROFSs), complex qROFSs (CqROFSs), spherical FSs (SFSs), and T-spherical FSs (TSFSs) in the literature that cope with the problems of imprecise information, but there are limitations to those concepts. This study presented the novel concepts of complex TSFSs (CTSFSs), Cartesian products in CTSFSs and CTSFS relations (CTSFRs). The proposed CTSFRs are the supreme tools that are capable of handling a wide range of vagueness problems with periodicity. In contrast with the available methods, CTSFSs assign complex valued membership, abstinence, non-membership grades to objects from the unit disc in the complex plane, with the least restrictions. Furthermore, we introduced the types of CTSFRs and provided their examples. Moreover, the concepts of CTSFRs and its types were used to identify the degree of positive, negative and neutral effects of financial exchanges of a country on that of the other countries. Likewise, to depict the capabilities of CTSFRs, the economic relationships are modeled using the novel techniques. Lastly, a comparative analysis was carried out to spot off the proposed work. Since fuzzy relations with composition and transitivity can be used in clustering, we will improve the proposed method with the CTSFRs composition and transitivity properties in applications of clustering and pattern recognition in our future works. Furthermore, we will also extend the proposed concepts to fuzzy uncertain environments by defining the aggregation operators, in order to use them in the assessment of express service quality with entropy weight.

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