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Complexities in Age Structured Predator-Prey System

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Abstract

Complex evolutionary behavior of an age-structured predator-prey three-dimensional system has been investigated analytically as well as numerically. Natural populations, whose generations are non-overlapping, can be described by model of difference equations that explains how the populations evolve in discrete time steps. In this paper stability criteria of fixed points of 3-dimentional discrete model are discussed. Bifurcation diagrams of this map are drawn by varying one parameter while fixing value of other parameters. To examine complexity of the map during evolution, certain chaotic measures such as calculations of Lyapunov characteristic exponents (LCEs), topological entropies and correlation dimension have been done and represented graphically.

Mathematics Subject Classification: 34H10, 34C23, 37L30, 37B40

Keywords: Chaos, Bifurcation, Lyapunov characteristic exponent (LCE), Topological entropy, Correlation dimension

1. Introduction

Chaotic evolutions in a dynamical system are subject to initial conditions as well as the set of parameter values incorporating the system. To analyse such fluctuations it is important to study through proper numerical simulation and (i) to visualize the bifurcation diagram of the system by varying one parameter while keeping others fixed, (ii) to obtain measurable quantities like LCEs and topological entropies and (iii) to obtain dimension of the chaotic attractor obtained while evolving. To control chaos and chaotic fluctuations in dynamical systems, recent articles reveal different methods with their merits, demerits and limitations. Complexity in a system happens somewhere between totally ordered (regular) and totally (chaos) random. Complexity of different type has been explained extendedly through some important articles [1- 5]. The measure suggested above as well as the dimension of the chaotic attractor, may also be termed as the measure of complexity.

While LCE measures the rate of divergence of orbits which originated nearby, the topological entropy measures the growth in randomness or number of typical orbits. The topological entropy concept is purely probabilistic while the LCE concept is primarily geometric [6]. However, both are powerful tools to measure complex nature of a dynamical system.

The objective of this work is to study further the complex dynamic behaviour of an age-structured three species predator-prey system and to find certain measures of complexities numerically such as LCEs, topological entropies and also correlation dimension of the chaotic attractor and to represent these graphically.

2. Age Structured Predator-Prey System

Evolutionary processes of age-structured population described in [3] and [7], through various mathematical formulation, provides great deal in understanding dynamical complexities in discrete time predator-prey system. Considerations are made here that the immature population neither feeds on prey nor reproduce and immature predators are raised by their parents. Also, the rate of their attack at prey be ignored and the reproductive rate can also be ignored. With these assumption a class of predator-prey models with age-structure for predator be suggested as, [3],

$$\begin{aligned} x_{n+1} &= x_n \,\mathrm{Exp}[r\,(1 - x_n - \beta z_n)] \\ y_{n+1} &= x_n\,(1 - \mathrm{Exp}\,[-\beta\, r\, z_n]) \qquad (1) \\ z_{n+1} &= b\, y_n + s\, z_n\,, \end{aligned}$$

Where, x_n denotes the density of prey population at generation n and y_n and z_n are, respectively, the densities of immature and mature predator at generation n. The parameters b, s, r and β, are responsible for regular and chaotic evolution of system (1). The parameter b, $0 < b \leq 1$, is the fraction of individuals in the immature class that survives to mature and s, $0 \leq s < 1$, denotes the fraction of mature individuals that are alive in the mature class after one generation and β is a non-negative parameter representing age dependent birth rate.

2.1. Fixed Points and Attractors

For the discrete system (1), the fixed points are the steady state solutions and which may be stable or unstable. They are also called as stationary points or equilibrium points. An orbit originating nearby a stable fixed point remain stable but that initiating nearby an unstable fixed point remain unstable and may also be chaotic.

For parameter values b = 0.95, s = 0.2, β = 1.3 and r = 2.9, the map (1) has two fixed points (1, 0, 0) and (0, 0, 0) out of which (1, 0, 0) is unstable fixed point. So, any orbit with initial point taken nearby this unstable fixed point would be unstable and may be chaotic. Taking an initial point (1.1, 0.1, 0.1) time series and an attractor for the map in the zx-plane are obtained and shown in Figure 1. These figures show that the system is chaotic and the figure on the right side is a chaotic attractor.

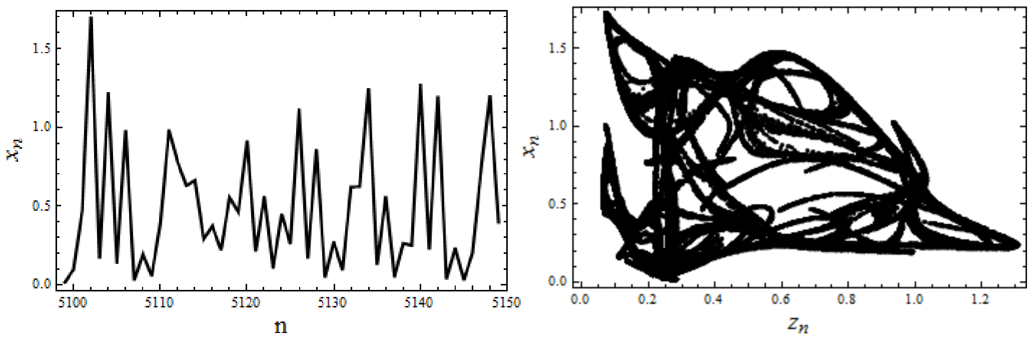

Figure 1.Left figure is a time series plot and the right figure is a phase plot in the zx-plane for map (1) for parameters b = 0.95,s = 0.2,β = 1.3 and r = 2.9.

2.2. Bifurcation Analysis

The qualitative change in dynamics of a system is called bifurcations, and the parameter value at which they occur are called bifurcation points. Chaos and bifurcation are two prominent phenomena of many nonlinear systems and they have a strong connection. Many observations in a system can be made by looking at its bifurcation diagram.

A traditional approach to gain preliminary insight into the properties of a dynamical system is to carry out a one-dimensional bifurcation analysis. One-dimensional bifurcation diagrams provide information about the dependence of the dynamics on a certain parameter. The analysis is expected to reveal the type of attractor, to which the dynamics will ultimately settle down after passing the initial transient phase and within which the trajectory will then remain forever.

For the map (1) we have drawn in Figure 2, the bifurcation diagrams by varying β from 0.5 to 1.5 and keeping rest of the parameters r = 2.9, b = 0.95 and s = 0.2 fixed with initial condition (1.1, 0.1, 0.1) which is very close to the unstable fixed point (1, 0, 0). Looking into these bifurcation diagrams, one can say that evolutionary dynamics the system (1) is highly complex. The figure on the right side, for small range of parameter β, indicates the inclusion of very complex pattern and existence of multiple attractors.

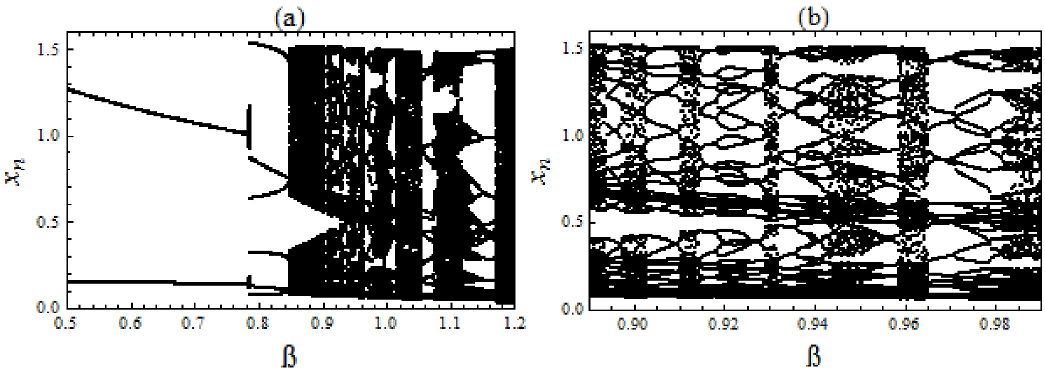

Figure 2(a) Bifurcation diagrams of map (1) when r = 2.9, b = 0.95, s = 0.2 and 0.5≤ β ≤ 1.5 and (b) its magnification in 0.89≤ β ≤ 0.99.

3. Measure of Chaos

By the measure of chaos usually we mean measuring LCEs, topological entropy and correlation dimension of chaotic orbit.

3.1 Lyapunov Characteristic Exponents

The motion of the attractor exhibits sensitive dependence on initial conditions. This means that the two trajectories starting together with nearby locations will rapidly diverge from each other and therefore have totally different futures. The practical implication is that long-term prediction becomes impossible in a system where small uncertainties are amplified enormously fast. Lyapunov characteristic exponents [6-13], is very effective tool for identification of regular and chaotic motions since this measures the degree of sensitivity to initial condition in a system.

Plot of Lyapunov number and LCEs have been drawn for regular and chaotic motions of the considered map (1). Both are drawn at r = 1.5, b = 0.95, s = 0.2 and β = 1.45 with initial point (1.1, 0.1, 0.1) given in Figure 3. Clearly this is the case of regularity of the system as at these points LCE is coming to be negative and Lyapunov number is less than 1.

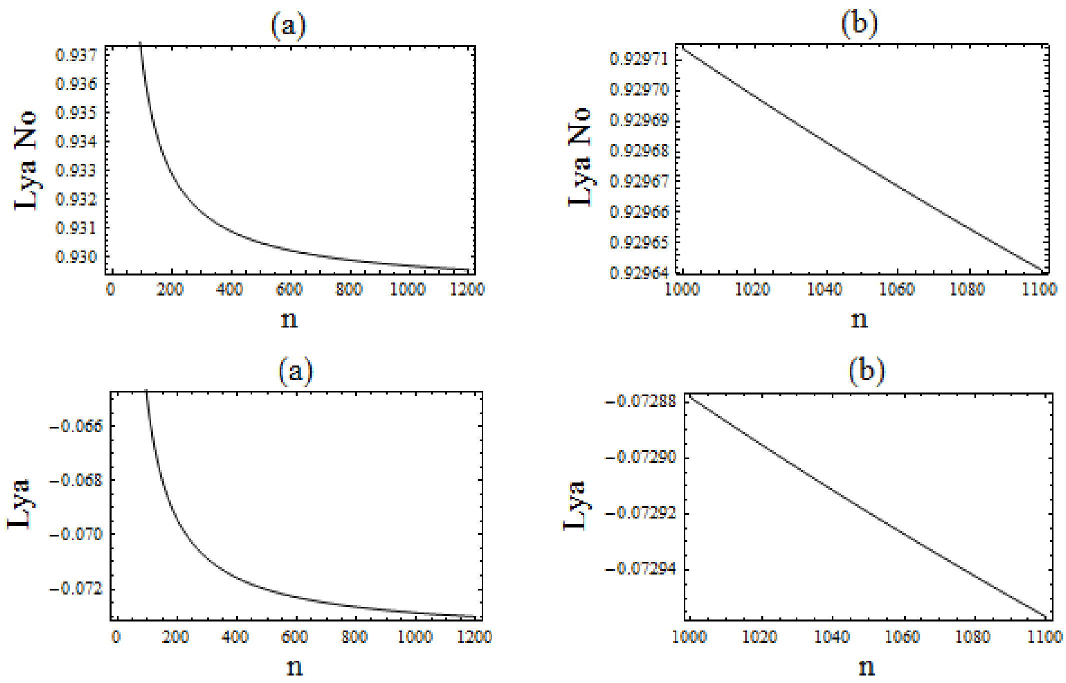

Figure 3(a): First row: plots of Lyapunov numbers with initial point (1.1, 0.1, 0.1) and $r = 1.5$, $b = 0.95,\ s = 0.2,\ \beta = 1.45$; Second row: plots of LCE with initial point (1.1, 0.1, 0.1) and $r = 1.5,\ b = 0.95,\ s = 0.2,\ \beta = 1.45$.

By increasing r, from 1.5 to 2.1 and keeping other parameters same, again we get a regular motion with the same initial point. However, plots of LCEs would look like, Figure 3 (b) below:

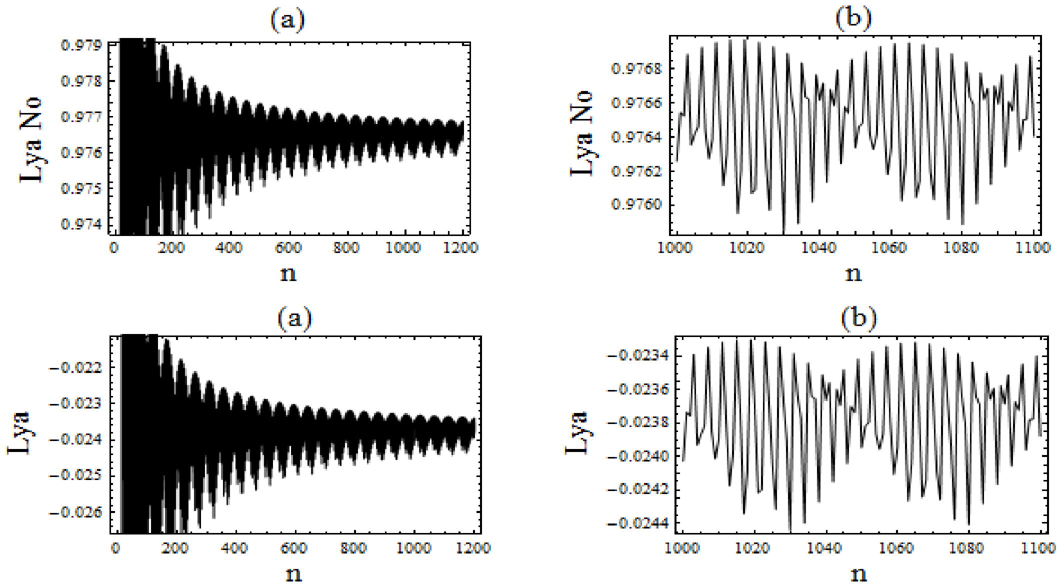

Figure 3(b): First row: plots of Lyapunov numbers with initial point (1.1, 0.1, 0.1), r = 2.1, b = 0.95, s = 0.2 and $\beta = 1.45$; Second row: plots of LCE with initial point (1.1, 0.1, 0.1) and$r = 2.1,$ $b = 0.95,\ s = 0.2,\ \beta = 1.45.$

Further increasing the value of $r\ \geq 2.75$ the system becomes fully chaotic. This can be seen with the help of plots of Lyapunov numbers and LCEs. Let us take $r = 2.75,\ b = 0.95,\ s = 0.2,\ \beta = 1.45$, with initial point $(1.1,\ 0.1,\ 0.1)$.Plots of Lyapunov numbers and LCEs with above said points are given by Figure 4.

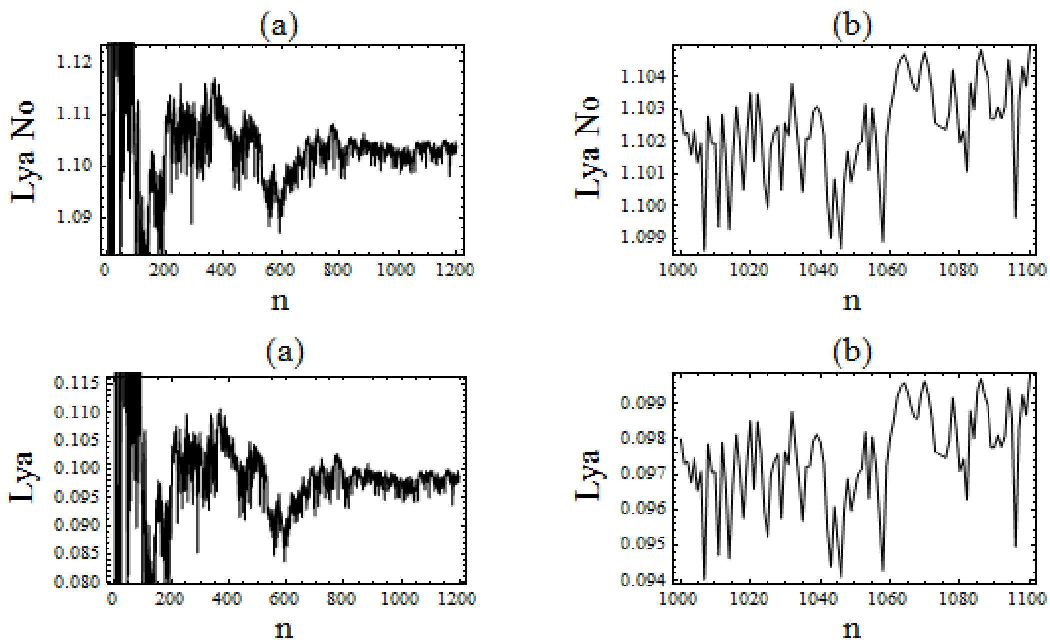

Figure 4. Chaotic case: Upper row plots are for Lyapunov numbers while those of lower are corresponding LCEs; parameters are: r = 2.9, b = 0.95, s = 0.2, β = 1.45 and initial condition is (1.1, 0.1, 0.1).

Clearly the system is showing chaos at this stage as value of Lyapunov number is greater than one and of LCE is greater than zero always.

3.2 Topological Entropy

One another good way to identify the complexity or chaotic nature of a dynamical system can be represented through its numerical measure of topological entropy [14] and [15]: the more complex a system is, the more topological entropy it will have. The topological entropy of the system is related to the growth rate of a material line. It measures the rate of increase in dynamical complexity as the system evolves with time. In chaotic systems, this is usually an exponential increasing growth rate.

Fixing values of parameters b, s, r as b = 0.95, s = 0.2, r = 2.9 and varying β, (i) $0.5 \le \beta \le 1.5$ and (ii) $0.5 \le \beta \le 1.8$, Plots of topological entropy have been obtained and shown in Figure 5. These indicate, approximately for $0.61 \le \beta \le 0.9$, topological entropy is zero and then increased highly. This is exactly what we observe in the bifurcation also; a regular evolution. The system moves chaotically after β exceeds value 0.9.

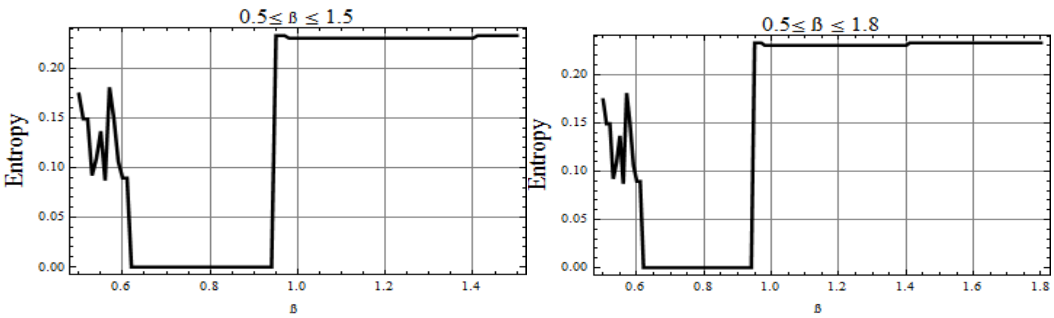

Figure 5. Two plots of topological entropy when b = 0.95, s = 0.2, r = 2.9.

Plot of topological entropies, when b = 0.95, s = 0.2, be obtained for four different values of r as 1.7, 1.2, 1, 0.7 and β lies in (0.3, 1.7) and these are represented in Figure 6. It is interesting to observe that the topological entropy decreases as value of r decreases.

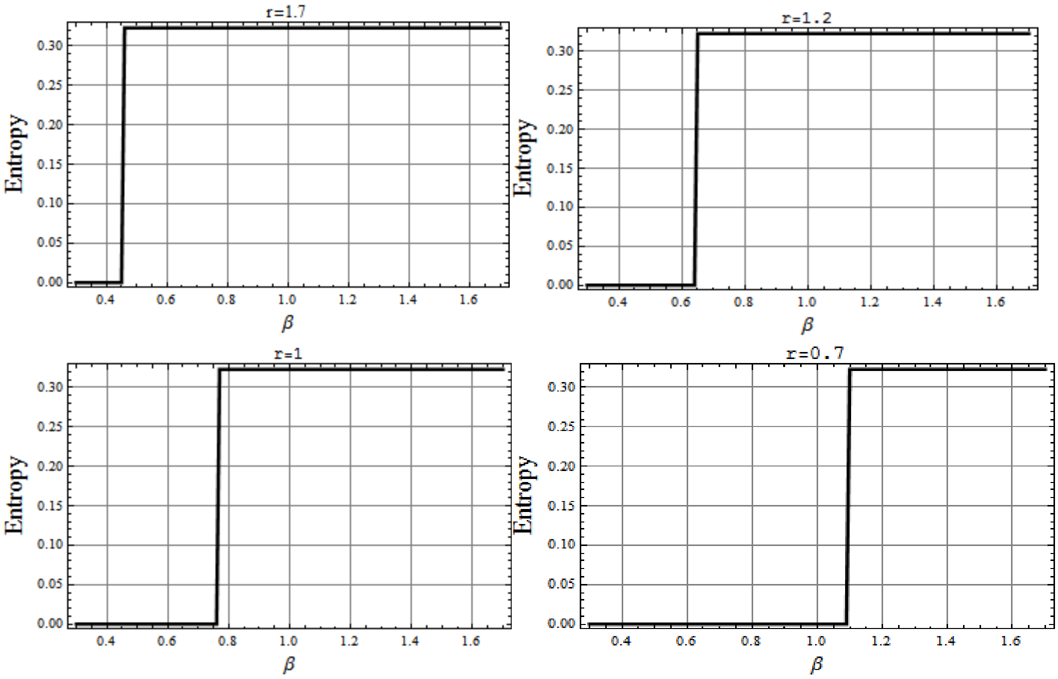

Figure 6. Topological entropy plots for different values of r for $0.3 \leq \beta \leq 1.7$.

Next, a 3-dimensional plot for topological entropy have been obtained where we have varied r and β; $0.5 \leq r \leq 1.8$ and $0.2 \leq \beta \leq 1.9$ for this problem and shown in Figure 7. Such plot can provide better understanding on complex behaviour of the system.

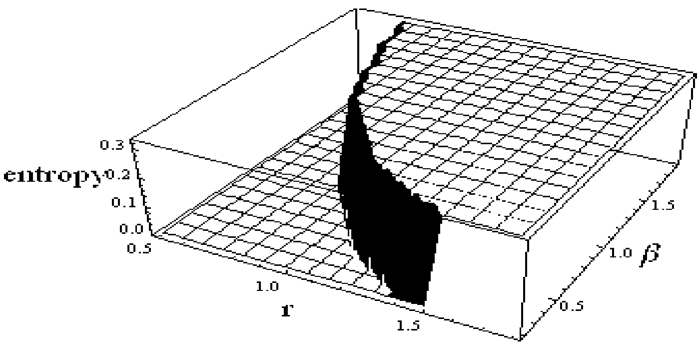

Figure 7.Topological entropy plot when $0.5 \leq r \leq 1.8$ and $0.2 \leq \beta \leq 1.9$.

3.3 Correlation Dimension

Correlation dimension [15-17] gives a measure of dimensionality of the chaotic attractor. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. To determine correlation dimension one has to use statistical method, [18].

Keeping parameters b = 0.95, s = 0.2,β = 1.45 fixed and changing r, two correlation curves have been drawn for (a) r = 2.9 (chaotic case) and (b) r = 1.0 (regular case), and with the initial condition $(x_0, y_0, z_0) = (1.1, 0.1, 0.1)$ as shown in Figure 8.

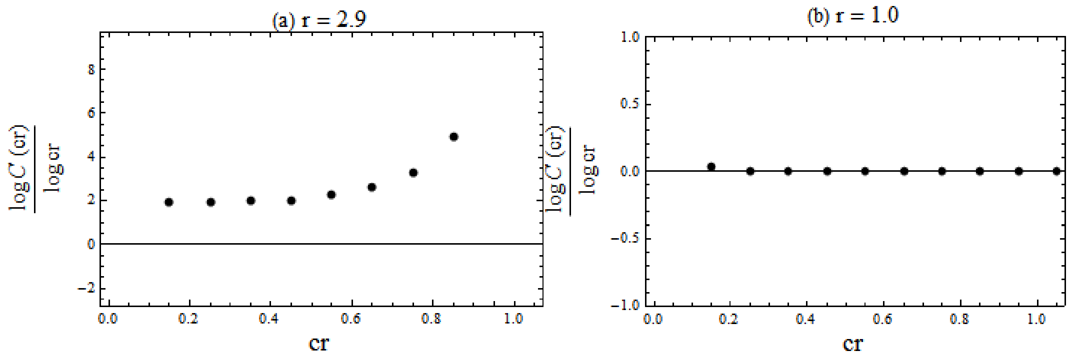

Figure 8. Plots of correlation curves with initial condition $(x_0, y_0, z_0) = (1.1, 0.1, 0.1)$ for (a) r = 2.9 and (b) r = 1.0. Other parameters are b = 0.95, s = 0.2, β = 1.45.

For case (a) if we use the least square linear fit to the data of correlation integral, we obtain the equation of the straight line fitting the data as

$$y = 2.9406 - 1.01737x \tag{2}$$

The y-intercept of straight line (2) is 2.9406. So, the correlation dimension of the chaotic attractor is 2.94 approximately. For the second case (b), it is clear from Figure 8, this is almost zero.

4. Discussion and Conclusion

Complex dynamics of age structured predator – prey system has been investigated and as complexity measures numerical studies have been performed to obtain LCEs, topological entropies and correlation dimension and represented these graphically. Plots of LCEs and topological entropies provide clear concepts of regular and chaotic evolution. A 3-dimensional plot of topological entropy, Figure 7, shows how complexity varies in the system.

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