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Complexity Analysis of a Linear Complementarity Algorithm Based on a Lyapunov Function[1]

by

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Abstract

We consider a path following algorithm for solving linear complementarity problems with positive semi-definite matrices. This algorithm can start from any interior solution and attain a linear rate of convergence. Moreover, if the starting solution is appropriately chosen, this algorithm achieves a complexity of $O(\sqrt{m}L)$ iterations, where m is the number of variables and L is the size of the problem encoding in binary. We present a simple complexity analysis for this algorithm, which is based on a new Lyapunov function for measuring the nearness to optimality. This Lyapunov function has itself interesting properties that can be used in a line search to accelerate convergence. We also develop an inexact line search procedure in which the line search stepsize is obtainable in a closed form. Finally, we extended this algorithm to handle directly variables which are unconstrained in sign and whose corresponding matrix is positive definite. The rate of convergence of this extended algorithm is shown to be independent of the number of such variables.

Key Words: linear complementarity, Karmarkar's method, Newton iteration, path following.
Abbreviated Title: Linear Complementarity Algorithm

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1. Introduction

Let Q be an $m \times m$ matrix, c be an m-vector, A be an $n \times m$ matrix, and b be an n-vector. Consider the following problem, known as the linear complementarity problem (abbreviated by LCP), of finding an $(x,u) \in \Re^m \times \Re^n$ satisfying

$$x \geq 0, \qquad Qx + c - A^T u \geq 0, \qquad \langle x, Qx + c - A^T u \rangle = 0, \tag{1.1a}$$

$$Ax = b. \tag{1.1b}$$

[Here $\Re^m$ ($\Re^n$) denotes the m-dimensional (n-dimensional) Euclidean space and $\langle \cdot,\cdot \rangle$ is the usual Euclidean inner product.] This problem has important applications in linear and convex quadratic programs, bimatrix games, and some other areas of engineering (see [1], [3], [17], [20], [21]). In our notation, all vectors are column vectors and superscript T denotes the transpose. "Log" will denote the natural log and $\|\cdot\|_1$, $\|\cdot\|$ will denote, respectively the L_1-norm and the L_2-norm. For any $x \in \Re^m$, we will denote by x_j the j-th component of x and by X the $m \times m$ diagonal matrix whose j-th diagonal entry is x_j.

We make the following standing assumptions about (LCP):

Assumption A:

(a) Q is positive semi-definite.

(b) There exists an $(x,u) \in \Re^m \times \Re^n$ satisfying (1.1a)-(1.1b) with all inequalities strictly satisfied.

(c) A has full row rank.

Assumption A (b) is quite standard for interior point methods. Assumption A (c) is made to simplify the analysis and can be removed without affecting either the algorithm or the convergence results.

Let

$$\Xi = \{ x \in (0,\infty)^m \mid Ax = b \}$$

(Ξ is nonempty by Assumption A (b)) and, for each $w \in (0,\infty)^m$, let $g_w:(0,\infty)^m \to \Re$ denote the function

$$g_w(x) = Qx + c - X^{-1}w, \qquad \forall\, x > 0. \tag{1.2}$$

Consider the problem of finding an $(x,u) \in \Xi \times \Re^n$ satisfying the following system of nonlinear equations:

$$g_w(x) - A^T u = 0. \qquad (P_w)$$

It is easily seen that a solution (x,u) to this problem satisfies $x > 0$, $Qx + c - A^T u > 0$, $Ax = b$ and $X(Qx + c - A^T u) = w$; hence (x,u) is an approximate solution of (LCP), with an error of $O(\|w\|)$. Our approach to solving (LCP) will be based on solving (approximately) equations of the form (P_w) over $\Xi \times \Re^n$, with w tending to the zero vector.

We remark that in the case where Q is symmetric, the problem (LCP) reduces to the convex quadratic program

$$\begin{array}{ll} \text{Minimize} & \langle x,Qx\rangle/2 + \langle c,x\rangle \\ \text{subject to} & Ax = b,\ x \geq 0. \end{array} \qquad (QP)$$

In this case, an $(x,u) \in \Xi \times \Re^n$ satisfies (P_w) if and only if it is an optimal primal dual solution pair of the convex program $\min\{ \langle x,Qx\rangle/2 + \langle c,x\rangle - \sum_j w_j \log(x_j) \mid Ax = b \}$.

The first polynomial-time algorithm for solving (LCP) was given by Kojima, Mizuno and Yoshise [12], based on path following. Subsequently, Kojima, Megiddo, Yoshise [13] and Kojima, Mizuno, Ye [14] developed polynomial-time algorithms for solving (LCP), using a different notion of potential reduction. [Some of these papers treated only the special case of (LCP) where A = 0, b = 0. Although (LCP) can be transformed to this special case by adding artificial variables, the nonempty interior assumption (i.e. Assumption A (b)) would no longer hold.] For the special case of convex quadratic programs (QP), the first polynomial-time algorithm was given by Kozlov, Tarasov and Khachiyan [15] based on the ellipsoid method [23], [27]. This was followed by a number of algorithms of the interior point variety (see [9], [12], [18], [19], [26]). In this paper we consider a polynomial-time algorithm for solving (LCP) that is motivated by Karmarkar's method [11] and its interpretation as a projected Newton method based on the logarithmic barrier function [8]. Our approach, which is similar to that taken in [9], [12], [18], [19], [26] is to solve approximately a sequence of nonlinear equations $\{(P_{w^t})\}$ over $\Xi \times \Re^n$, where $\{w^t\}$ is a geometrically decreasing sequence of positive vectors. Each (P_{w^t}) is solved (approximately) by taking a Newton step for $(P_{w^{t-1}})$ starting at the previous solution. This algorithm scales using only primal solutions and, in this respect, it is closely related to the algorithms of [9], [26]. However, apart from the fact that it solves the more general linear complementarity problem, it differs from the latter in that it does not restrict w^t to be scalar multiples of e. This difference is significant since, as we shall see, it permits this algorithm to start with any interior solution and attain a linear rate of convergence. Moreover, the complexity proof, which is based on a certain Lyapunov function that measures the violation of complementary slackness in (1.1a), is simpler and reveals more of the algorithmic structure than existing proofs. The Lyapunov function has itself interesting properties that can be used in a line search procedure to accelerate convergence. For the special case where Q is a diagonal matrix, this line search is particularly simple. For general Q, we propose an inexact version of this line search that gives the stepsize in a closed form. Finally, we extend this algorithm to handle directly variables which are unconstrained in sign and whose corresponding matrix is positive definite. We show that the rate of convergence of this extended algorithm is independent of the number of such variables.

This paper proceeds as follows: in §2 we show that, for some fixed $\alpha \in (0,1)$, an approximate solution of $(P_{\alpha w})$ in $\Xi \times \Re^n$ can be obtained by taking a Newton step for (P_w) starting at an approximate solution of (P_w) in $\Xi \times \Re^n$. Based on this observation, in §3 and §4 we present our algorithm and analyze its complexity. In §5 we discuss the initialization of our algorithm. In §6 we consider extensions.

2. Technical Preliminaries

Consider an $\bar{w} \in (0,\infty)^m$ and an $(\bar{x},\bar{u}) \in \Xi \times \Re^n$. Consider applying a Newton step for $(P_{\bar{w}})$ at $(\bar{x},\bar{u})$ subject to the constraint (1.1b), and let $(x,u) \in \Re^m \times \Re^n$ be the vector generated by this step. Then (x,u) satisfies

$$
\begin{aligned}
g_{\bar{w}}(\bar{x}) + \nabla g_{\bar{w}}(\bar{x})(x-\bar{x}) - A^T u &= 0, \\
Ax &= b,
\end{aligned}
$$

or equivalently (cf. (1.2)),

$$Q\bar{x} + c - \bar{X}^{-1}\bar{w} + (Q + \bar{X}^{-2}\bar{W})z - A^T u = 0, \tag{2.1a}$$

$$Az = 0, \tag{2.1b}$$

where

$$z \quad = x - \bar{x}. \tag{2.2}$$

Let $d = \bar{X}^{-1}z$ and $\bar{r} = \bar{w} - \bar{X}(Q\bar{x} + c - A^T\bar{u})$. Then Eqs. (2.1a)-(2.1b) can be rewritten as

$$\begin{aligned}(\bar{W} + \bar{X}Q\bar{X})d - (A\bar{X})^T(u - \bar{u}) &= \bar{r},\\ A\bar{X}d &= 0.\end{aligned}$$

Solving for d gives

$$d \quad = \Theta^{-1}\bar{r} - \Theta^{-1}\bar{X}A^T[A\bar{X}\Theta^{-1}\bar{X}A^T]^{-1}A\bar{X}\Theta^{-1}\bar{r},$$

where $\Theta = \bar{W} + \bar{X}Q\bar{X}$. [Note that Θ is positive definite.] Straightforward calculation finds that

$$\langle d, \Theta d\rangle \quad = \langle \bar{r}, \Theta^{-1}\bar{r}\rangle - \langle A\bar{X}\Theta^{-1}\bar{r}, [A\bar{X}\Theta^{-1}\bar{X}A^T]^{-1}A\bar{X}\Theta^{-1}\bar{r}\rangle.$$

Since $[A\bar{X}\Theta^{-1}\bar{X}A^T]^{-1}$ is positive definite, this implies $\langle d, \Theta d\rangle \le \langle \bar{r}, \Theta^{-1}\bar{r}\rangle$ or, equivalently,

$$\|\bar{\Gamma}^{1/2}d\| \quad \le \|\bar{\Gamma}^{-1/2}\bar{r}\|, \tag{2.3}$$

where $\bar{\Gamma} = \bar{W} + \bar{X}\tilde{Q}\bar{X}$ and $\tilde{Q}$ denotes the symmetric part of Q, i.e. $\tilde{Q} = (Q + Q^T)/2$. Since (cf. (2.2)) $x = \bar{x} + \bar{X}d$, we also have $X = \bar{X} + D\bar{X}$ and therefore

$$\begin{aligned}\bar{w} - X(Qx + c - A^Tu) &= \bar{w} - \bar{X}(Qx + c - A^Tu) - D\bar{X}(Qx + c - A^Tu)\\ &= D\bar{w} - D\bar{X}(Qx + c - A^Tu)\\ &= D[\bar{w} - \bar{X}(Qx + c - A^Tu)]\\ &= D^2\bar{w},\end{aligned}$$

where the second and the third equality follows from (2.1a) and (2.2). This in turn implies

$$\begin{aligned}\|\bar{w} - X(Qx + c - A^Tu)\| &= \|D^2\bar{w}\|\\ &\le \|D^2\bar{w}\|_1\\ &= \langle d, \bar{W}d\rangle\\ &\le \langle d, \bar{\Gamma}d\rangle\\ &\le \|\bar{\Gamma}^{-1/2}\bar{r}\|,\end{aligned} \tag{2.4}$$

where the first inequality follows from properties of the L_1-norm and the L_2-norm, the second inequality follows from the fact $\langle d,\bar{\Gamma}d\rangle = \langle d,\bar{W}d\rangle + \langle \bar{X}d,\tilde{Q}\bar{X}d\rangle$ (also using the positive semi-definite property of $\tilde{Q}$), and the third inequality follows from (2.3).

Consider any $\beta\in(0,1)$ and any scalar α satisfying

$$(\beta^2+\|\bar{w}\|/\bar{\theta})/(\beta+\|\bar{w}\|/\bar{\theta}) \le \alpha < 1, \tag{2.5}$$

where $\bar{\theta} = \min_j\{\bar{w}_j\}$. Let $w = \alpha\bar{w}$, $\theta = \alpha\bar{\theta}$, $r = w - X(Qx + c - A^T u)$, and $\Gamma = W + X\tilde{Q}X$. Then

$$\begin{aligned}
\|\Gamma^{-1/2}r\|/\sqrt{\theta} &\le \|r\|/\theta \\
&= \|\alpha\bar{w} - X(Qx + c - A^T u)\|/(\alpha\bar{\theta}) \\
&\le \|\bar{w} - X(Qx + c - A^T u)\|/(\alpha\bar{\theta}) + (1-\alpha)\|\bar{w}\|/(\alpha\bar{\theta}) \\
&\le \|\bar{\Gamma}^{-1/2}\bar{r}\|^2/(\alpha\bar{\theta}) + (1/\alpha-1)\|\bar{w}\|/\bar{\theta},
\end{aligned}$$

where the first inequality follows from the fact that the eigenvalues of Γ are bounded from below by θ, the second inequality follows from the triangle inequality, and the third inequality follows from (2.4). Hence, by (2.5),

$$\|\bar{\Gamma}^{-1/2}\bar{r}\|/\sqrt{\bar{\theta}} \le \beta \quad\Rightarrow\quad \|\Gamma^{-1/2}r\|/\sqrt{\theta} \le \beta. \tag{2.6}$$

From (2.3) we would also have $\|d\| \le \|\bar{\Gamma}^{1/2}d\|/\sqrt{\bar{\theta}} \le \beta < 1$. Hence $e + d > 0$, where e is the vector in $\Re^m$ all of whose components are 1's, and (cf. (2.2)) $x > 0$. Also, by (2.1b) and (2.2), $Ax = A(\bar{x} + z) = b$. Furthermore, from (2.1a) and (2.2) we have that $Qx + c - A^T u = \bar{W}\bar{X}^{-1}(e - d)$. Since (cf. $\|d\| < 1$) $e - d > 0$, this implies that

$$0 < Qx + c - A^T u = \bar{W}X^{-1}(I + D)(e - d) \le \bar{W}X^{-1}e, \tag{2.7}$$

where the equality follows from the fact $\bar{X}^{-1} = X^{-1}(I + D)$ and the second inequality follows from the observation that $(I + D)(e - d) = e - Dd$. [Note that in the case where Q is symmetric, (2.7) implies that u is dual feasible for (QP). This is because if the dual cost of u, i.e. $\langle b,u\rangle + \min_{\zeta\ge 0}\{\langle\zeta,Q\zeta\rangle/2 + \langle c-A^T u,\zeta\rangle\}$, is not finite, then there exists $y\in[0,\infty)^m$ such that $Qy = 0$ and $\langle c-A^T u,y\rangle < 0$. Multiplying by y gives $0 < \langle Qx+c-A^T u,y\rangle = \langle c-A^T u,y\rangle < 0$, a contradiction.]

For any vector $w\in(0,\infty)^m$, let $\rho_w:\Xi\times\Re^n\to[0,\infty)$ denote the Lyapunov function

$$\rho_w(x,u) = \|(W+X\tilde{Q}X)^{-1/2}(w - X(Qx+c-A^Tu))\|/\sqrt{\min_j\{w_j\}}, \quad \forall\, x\in\Xi,\ \forall\, u\in\Re^n,$$

and let $\mu(w) = \|w\|/\min_j\{w_j\}$. [The function $\rho_w(x,u)$ measures the amount by which the complementary slackness condition $\langle x, Qx + c - A^Tu\rangle = 0$ in (1.1a) is violated. It also has some nice properties which we will discuss in §6.] We have then just proved the following important lemma (cf. (2.5)-(2.7)):

Lemma 1 For any $\beta\in(0,1)$, any $\bar{w}\in(0,\infty)^m$, and any $(\bar{x},\bar{u})\in\Xi\times\Re^n$ such that $\rho_{\bar{w}}(\bar{x},\bar{u}) \le \beta$, it holds that

$$(x,u)\in\Xi\times\Re^n, \quad \rho_{\alpha\bar{w}}(x,u) \le \beta,$$
$$0 < Qx + c - A^Tu \le X^{-1}\bar{w},$$

where $\alpha = (\beta^2+\mu(\bar{w}))/(\beta+\mu(\bar{w}))$ and (x,u) is defined as in (2.1a)-(2.1b), (2.2).

3. The Homotopy Algorithm

Choose $\beta\in(0,1)$, $\omega\in(0,\infty)^m$, and let $\alpha = (\beta^2+\mu(\omega))/(\beta+\mu(\omega))$. Lemma 1 and (2.1a)-(2.1b), (2.2) motivate the following algorithm for solving (LCP), parameterized by a scalar $\delta\in(0,1)$:

Homotopy Algorithm

<u>Step 0</u>: Choose any $(x^1,u^1)\in\Xi\times\Re^n$ such that $\rho_\omega(x^1,u^1) \le \beta$. Let $w^1 = \omega$.

<u>Step t</u>: Compute (z^{t+1},u^{t+1}) to be a solution of

$$\begin{bmatrix} Q+(X^t)^{-2}W^t & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} (X^t)^{-1}w^t - Qx^t - c \\ 0 \end{bmatrix}.$$

Set $x^{t+1} = x^t + z^{t+1}$, $w^{t+1} = \alpha w^t$.

If $\|w^{t+1}\| \le \delta\|\omega\|$, terminate.

[Note that, for convex quadratic programs, u^t is dual feasible for all t (cf. Lemma 1).] We gave the above algorithm the name "homotopy" [7] (or "path following") because it solves (approximately) a sequence of problems that approaches (LCP). This algorithm is closely related to one of Goldfarb and Liu [9]. In particular, if Q is symmetric and ω is a scalar multiple of e, then this algorithm reduces to the Goldfarb-Liu algorithm with $\gamma = 1$ and α reduces to the quantity $\bar{\sigma}$ given in Lemma 3.3 of [9]. However, in contrast to the complexity proof in [9], which is based on showing $\|(X^t)^{-1}z^{t+1}\| \leq \beta$ for all t, our complexity proof, as we shall see in §4, is based on showing $\rho_{w^t}(x^t,u^t) \leq \beta$ for all t. This latter approach simplifies the analysis and, as we shall see in §6, allows line search to be introduced into the algorithm to accelerate convergence.

4. Convergence Analysis

By Lemma 1 and the fact that $\mu(\cdot)$ is invariant under scalar multiplication, the homotopy algorithm generates, in at most $\log(\delta)/\log(\alpha)$ steps, an $(x,u) \in \Xi \times \Re^n$ satisfying

$$0 < Qx + c - A^T u \leq \delta X^{-1}\omega.$$

Since log is a concave function and its slope at 1 is 1, we have that $\log(1-\theta) \leq -\theta$, for any $\theta \in (0,1)$. Therefore

$$\begin{aligned} \log(\alpha) &= \log(1-\beta(1-\beta)/(\beta+\mu(\omega))) \\ &\leq -\beta(1-\beta)/(\beta+\mu(\omega)). \end{aligned}$$

Hence we have just proved following:

Lemma 2 For any $\beta \in (0,1)$, any $\omega \in (0,\infty)^m$ and any $\delta \in (0,1)$, the homotopy algorithm generates, in at most $-\log(\delta)(\beta+\mu(\omega))/\beta(1-\beta)$ steps, an $(x,u) \in \Xi \times \Re^n$ satisfying $0 < X(Qx + c - A^T u) \leq \delta\omega$.

Hence if we fix β and choose ω such that $\|\omega\| \leq 2^{\lambda L}$ and $\mu(\omega) = O(\sqrt{m})$, where L denotes the size of the problem encoding in binary (defined as in [9], [12]-[14], [18], [19], [26]) and λ is a positive scalar, then the homotopy algorithm with $\delta = 2^{-2\lambda L}$ would terminate in $O(\sqrt{m}L)$ steps with an approximate solution of (LCP) whose error is less than $2^{-\lambda L}$. For λ sufficiently large (independent of L), a solution for (LCP) can be recovered by using, say, the techniques described in [12]. Since the amount of computation per step is at most $O(m^3)$ arithmetic operations (not counting Step 0), the homotopy algorithm has a complexity of $O(m^{3.5}L)$ arithmetic operations. [We assume for the moment that ω is chosen such that Step 0 can be done very "fast". See §5 for justification.].

5. Algorithm Initialization

In this section we show that Step 0 of the homotopy algorithm (i.e. to find an $\omega \in (0,\infty)^m$ and an $(x,u) \in \Xi \times \Re^n$ satisfying $\rho_\omega(x,u) \le \beta$) can be done very "fast". [In fact, we can typically choose ω such that $\mu(\omega) = \sqrt{m}$ (i.e. ω is a scalar multiple of e) and $\|\omega\| \le 2^{O(L)}$.]

Suppose that the matrix A in (LCP) has all 1's in its last row, i.e.,

$$A = \begin{bmatrix} A' \\ e^T \end{bmatrix},$$

where A' is some $(n-1)\times m$ matrix, and $Ae = b$. [We can transform (LCP) into this form by using techniques similar to that used in [11, §5]. Also see [9, §4] and [12, §6].] Let $\eta = \|Qe+c\|/\beta$ and $u = (0,\ldots,0,-\eta)^T$. Then $(e,u) \in \Xi \times \Re^n$ and

$$\begin{aligned} \rho_{\eta e}(e,u) &= \|(\eta I + Q)^{-1/2}(\eta e - Qe - c + A^T u)\|/\sqrt{\eta} \\ &= \|(\eta I + Q)^{-1/2}(-Qe - c)\|/\sqrt{\eta} \\ &\le \|Qe + c\|/\eta \\ &= \beta, \end{aligned}$$

where the inequality follows from the fact that the eigenvalues of $\eta I + Q$ are bounded from below by η. Alternatively, we can solve (P_ω) over $\Xi \times \Re^n$, whose solution (x,u) can be seen to satisfy $\rho_\omega(x,u) = 0$. [It can be shown, by modifying the argument used in [12, Appendix A], that Assumptions A (a)-(b) implies the existence of such a solution.] If Ξ is bounded, we can solve the problem

$$\begin{aligned} &\text{Maximize } \textstyle\sum_j w_j \log(x_j) \\ &\text{subject to } \; Ax = b, \end{aligned} \tag{5.1}$$

for some $w \in (0,\infty)^m$. The optimal primal dual solution pair (x,p) of (5.1) can be seen to satisfy $w = -XA^Tp$ and $x \in \Xi$. Hence, for $\eta = \|XQx + Xc\|/\beta$, we have $\rho_{\eta w}(x,\eta p) \le \beta$. Polynomial-time algorithms for solving (5.1) are described in [4], [5], [24] and [25]. [Note that neither (P_ω) nor (5.1) needs to be solved exactly.]

However, each of the initialization procedures described above either is difficult to implement or requires that Ξ be bounded. We show below that the homotopy algorithm can start with any interior solution by choosing ω appropriately. Suppose that we have an $(x,u) \in \Xi^m \times \Re^n$ such that $Qx+c-A^Tu > 0$. Then it can be seen that $\rho_v(x,u) = 0$, where $v = X(Qx+c-A^Tu)$. By Lemma 1, the homotopy algorithm with $(x^1,u^1) = (x,u)$ and $\omega = v$ converges at a linear rate with a rate of convergence $(\beta^2+\mu(v))/(\beta+\mu(v))$. [Hence, if $\|v\| = 2^{O(L)}$ and $\mu(v)$ is a polynomial in L, then the homotopy algorithm can be terminated in a number of steps that is a polynomial in L. The quantity $\mu(v)$ in a sense measures how far the vector (x,u) is from satisfying $(P_{\eta e})$, where $\eta = \min_j\{v_j\}$.]

6. **Conclusion and Extensions**

In this paper we have proposed an algorithm for solving linear complementarity problems with positive semi-definite matrices and have provided a short proof of its complexity. This algorithm solves a sequence of approximations to the original problem where the accuracy of the approximation is improving at a geometric rate. This algorithm has a complexity (in terms of the number of steps) that is comparable to existing interior point methods (cf. [9]-[14], [18], [19], [26]) and, for convex quadratic programs, maintains both primal and dual feasibility.

There are many directions in which our results can be extended. For example, we can accelerate the rate of convergence of the homotopy algorithm by setting $w^t = \varepsilon^t w^{t-1}$, where ε^t is the smallest positive ε for which $\rho_{\varepsilon w^{t-1}}(x^t,u^t) \le \beta$. [More generally we can set $w^t = \text{argmin}\{\ \|w\| \mid \rho_w(x^t,u^t) \le \beta,\ \mu(w) \le \mu(\omega)\ \}$.] This minimization is difficult in general, but if Q is diagonal, it can be verified that the quantity $\varepsilon\rho_{\varepsilon w}(x,u)^2 = \sum_j (\varepsilon w_j - v_j)^2/(\varepsilon w_j + \theta_j)$ is convex in ε (the second derivative is non-negative), where v_j and w_j denote the j-th component of, respectively, $v = X^t(Qx^t + c - A^Tu^t)$ and $w = w^{t-1}$, and θ_j denotes the j-th diagonal entry of X^tQX^t. Hence in this case the above minimization reduces to finding a solution ε of the equation

$$\sum_j (\varepsilon w_j - v_j)^2/(\varepsilon w_j + \theta_j) \;=\; \varepsilon\beta^2 \min_j\{w_j\}. \tag{6.1}$$

Because the lefthand side is convex in ε, such a solution can be found using classical line search techniques (see [2], [16]). For linear programs (i.e. $Q = 0$), (6.1) further reduces to the scalar quadratic equation

$$\sum_j (\varepsilon w_j - v_j)^2/w_j \;=\; \varepsilon^2\beta^2 \min_j\{w_j\}, \tag{6.2}$$

whose solution is unique and is obtainable in a <u>closed form</u> (see Ye [26, §5] for a similar line search procedure). Alternatively, we can solve the simpler scalar quadratic equation

$$\|\varepsilon w - v\| \;=\; \varepsilon\beta \min_j\{w_j\}, \tag{6.3}$$

whose solution is more conservative than that given by (6.2), but has the advantage that it is usable even for general Q. [This follows from the fact that $\rho_{\varepsilon w}(x,u) \le \|\varepsilon w - v\|/(\varepsilon \min_j\{w_j\})$ for all $\varepsilon > 0$ and (see proof of Eq. (2.6)) $\|\alpha w - v\| \le \alpha\beta \min_j\{w_j\}$).] Hence the solution of (6.3) is at least as good as α and, for $\beta = 1/\sqrt{2}$ and $\mu(w) = \sqrt{m}$, can be shown by a more refined analysis to be less than $1 - 0.5/\sqrt{m}$. [The latter, to the best of our knowledge, gives the best theoretical rate of convergence amongst existing interior point algorithms.]

We can also choose β to minimize α (this gives $\alpha = 2\sqrt{\mu(\omega)^2+\mu(\omega)} - 2\mu(\omega)$). Also, Freund [6] noted that, at the t-th step, one can apply the Newton step for $(P_{\alpha w^t})$ instead of for (P_{w^t}). The resulting analysis is slightly different, but achieves the same complexity. Another direction of extension is to handle directly variables that are unconstrained in sign and whose corresponding matrix is positive <u>definite</u>. Consider the following generalization of (LCP) of finding an $(x,y,u) \in \Re^m \times \Re^{m'} \times \Re^n$ satisfying

$$x \ge 0, \qquad Qx + c - A^T u \ge 0, \qquad \langle x, Qx + c - A^T u\rangle = 0, \tag{6.4a}$$

$$Ax + By = b, \quad Hy + h - B^T u = 0, \tag{6.4b}$$

where H is a $m' \times m'$ positive definite matrix, B is an $n \times m'$ matrix and h is a m'-vector. Then it can be seen that Lemma 1 still holds with (x,y,u) given as a solution of

$$g_{\bar{w}}(\bar{x}) + \nabla g_{\bar{w}}(\bar{x})(x - \bar{x}) - A^T u \;=\; 0, \qquad Hy + h - B^T u \;=\; 0,$$

$$Ax + By \;=\; b.$$

This immediately suggests an extension of the homotopy algorithm for solving (6.4a)-(6.4b) that maintains $\rho_{w^t}(x^t,u^t) \leq \beta$ at all steps t, where $w^t = (\alpha)^t\omega$ and α, β, ω, ρ_w are defined as in the homotopy algorithm. This algorithm has a rate of convergence of $(\beta^2+\mu(\omega))/(\beta+\mu(\omega))$, which is <u>independent</u> of m′. [Of course, the line search procedures described earlier are also applicable here.]

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