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Complexity and behind the horizon cut off

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Abstract: Motivated by $T\overline{T}$ deformation of a conformal field theory we compute holographic complexity for a black brane solution with a cutoff using "complexity=action" proposal. In order to have a late time behavior consistent with Lloyd's bound one is forced to have a cutoff behind the horizon whose value is fixed by the boundary cutoff. Using this result we compute holographic complexity for two dimensional AdS solutions where we get expected late times linear growth. It is in contrast with the naively computation which is done without assuming the cutoff where the complexity approaches a constant at the late time.

Keywords: AdS-CFT Correspondence, Black Holes, Conformal Field Theory

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1 Introduction

According to the "complexity=action" proposal (CA) the quantum computational complexity of a holographic state is given by the on-shell action evaluated on a bulk region known as the 'Wheeler-De Witt' (WDW) patch [1, 2]

$$\mathcal{C}(\Sigma) = \frac{I_{\rm WDW}}{\pi\hbar}. \tag{1.1}$$

Here the WDW patch is defined as the domain of dependence of any Cauchy surface in the bulk whose intersection with the asymptotic boundary is the time slice Σ.

An interesting feature of the complexity is that it grows linearly with time at the late time with slope given by Lloyd's bound [3] that is twice of the energy of the state. Holographic complexity for two-sided black holes has been calculated in [4] where it was shown that although at the late time the growth rate approaches a constant value that is twice of the mass of the black hole, the constant is approached from above, violating the Lloyd's bound [3].

Another recent interesting development in the literature of theoretical higher energy is to study a conformal theory deformed by an irrelevant operator such as the one which is quadratic in the stress energy tensor known as $T\overline{T}$ deformation. Although typically deforming a conformal field theory by an irrelevant operator would remove UV fixed point and makes it non-local at high energies, it was shown that for the mentioned deformation the resultant theory is still exactly solvable [6, 7].

To be concrete let us consider a two dimensional conformal field theory deformed by the corresponding operator as follows

$$I_{\rm QFT} = I_{\rm CFT} + \mu \int d^2x T\overline{T}. \tag{1.2}$$

There are some interesting features of the resultant quantum field theory. First of all it is UV complete. Moreover the spectrum of the deformed theory can be determined non-perturbatively and rather in a compact form. More precisely for a conformal field theory

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on a cylinder with the circumference L the energy level $E_n(\mu, L)$ for a state denoted by conformal dimensions $(\Delta_n, \bar{\Delta}_n)$ is given by [6, 7]

$$E_n(\mu, L) = \frac{2L}{\mu}\left(1 - \sqrt{1 - \frac{2\pi\mu}{L^2}\left(M_n + \frac{2\pi\mu}{L^2}J_n^2\right)}\ \right), \tag{1.3}$$

where $M_n = \Delta_n + \bar{\Delta}_n - \frac{c}{12}$, and $J_n = \Delta_n - \bar{\Delta}_n$.

In the context of AdS/CFT correspondence it was proposed that the above deformation has a holographic dual. The corresponding dual gravitational theory may be described by an AdS_3 metric with a finite radial cutoff [8]. The radial cutoff r_c is given in terms of the deformed parameter μ, by $r_c^2 = \frac{16\pi G}{\mu}$.

Using AdS/CFT correspondence the generalization of $T\overline{T}$ deformation to higher dimensional conformal field theories has also been studied in [9, 10]. Following [8] one would also expect that a $d+2$ dimensional AdS black brane solution with a radial cutoff could provide a holographic dual for a $d+1$ dimensional $T\overline{T}$ deformed conformal field theory. Given the corresponding geometry by

$$ds^2 = \frac{\ell^2}{r^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^{d} d\vec{x}^2\right), \qquad f(r) = 1 - \left(\frac{r}{r_h}\right)^{d+1}, \tag{1.4}$$

where r_h and ℓ are radius of horizon the AdS radius, respectively, the spectrum of energy of the deformed theory is [9, 10]

$$E = \frac{V_d \ell^d d}{8\pi G}\frac{1}{r_c^{d+1}}\left(1 - \sqrt{1 - \frac{r_c^{d+1}}{r_h^{d+1}}}\right), \tag{1.5}$$

with V_d being the volume of d-dimensional internal space of the metric parametrized by x_i, $i = 1, \cdots d$.

Motivated by $T\overline{T}$ deformation and its holographic dual in the present paper we would like to compute the complexity growth of a black brane at a finite cutoff using CA proposal. We observe that requiring to reach the Lloyd's bound at the late time enforces us to have a cutoff behind the horizon whose value is fixed by boundary cutoff. More precisely for black brane solutions denoting cutoff radius inside the horizon by r_0 one finds (at leading order)

$$r_0 r_c^2 = 2^{\frac{4}{d+1}} r_h^3. \tag{1.6}$$

To explore the significance of our result we will then study holographic complexity for AdS_2 vacuum solutions of certain two dimensional Maxwell-Dilaton gravities. One observes that if we naively compute the complexity without taking into account the behind horizon cutoff the rate of growth vanishes at the late time. On the other had if one assumes that the UV cutoff would set a cutoff behind the horizon given by (1.6) the complexity exhibits late time linear growth, as expected.

The paper is organized as follows. In the next section we will compute holographic complexity for back brane solutions in the present of a cutoff where we show how the inside cutoff would emerge. In section three we will study complexity for AdS_2 taking into account the enforced behind the horizon cutoff. The last section is devoted to conclusions.

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2 CA complexity for cutoff geometries

In this section we would like to compute holographic complexity for a black brane solution with a radial cutoff. To do so, following CA proposal we will need to compute on shell action on the WDW patch associated with a boundary state given at $\tau = t_L + t_R$. Here $t_L(t_R)$, is time coordinate of left (right) boundary on the eternal black brane (see figure 1).

To proceed we note that the action consists of several parts including bulk, boundary and joint points as follows [12–14]

$$\begin{aligned} I = &\frac{1}{16\pi G_N}\int d^{d+2}x\sqrt{-g}(R-2\Lambda) + \frac{1}{8\pi G_N}\int_{\Sigma_t^{d+1}} K_t\, d\Sigma_t \\ &\pm \frac{1}{8\pi G_N}\int_{\Sigma_s^{d+1}} K_s\, d\Sigma_s \pm \frac{1}{8\pi G_N}\int_{\Sigma_n^{d+1}} K_n\, dS d\lambda \pm \frac{1}{8\pi G_N}\int_{J^d} a\, dS\,. \end{aligned} \tag{2.1}$$

Here the timelike, spacelike, and null boundaries and also joint points are denoted by $\Sigma_t^{d+1}, \Sigma_s^{d+1}, \Sigma_n^{d+1}$ and J^d, respectively. The extrinsic curvature of the corresponding boundaries are given by K_t, K_s and K_n. The function a at the intersection of the boundaries is given by the logarithm of the inner product of the corresponding normal vectors and λ is the null coordinate defined on the null segments. The sign of different terms depends on the relative position of the boundaries and the bulk region of interest (see [14] for more details).

The null boundaries B_1 and B_2 of the future interior are

$$B_1:\ t = t_R + r^*(r_c) - r^*(r), \qquad B_2:\ t = -t_L - r^*(r_c) + r^*(r), \tag{2.2}$$

where $r^*(r)$ is the tortoise coordinate. The null vectors associated with these null boundaries are also given by

$$k_1 = \alpha\left(\partial_t + \frac{1}{f(r)}\partial_r\right), \qquad k_2 = \beta\left(\partial_t - \frac{1}{f(r)}\partial_r\right). \tag{2.3}$$

Here α and β are two free constant parameters appearing due to the ambiguity of the normalization of null vectors.

For the black brane solution without cutoff the corresponding complexity rate growth has been computed in [4]

$$\frac{d\mathcal{C}}{d\tau} = \frac{1}{\pi}\frac{d}{d\tau}I_{\rm WDW} = \frac{2E}{\pi}\left(1 + \frac{1}{2}\tilde{f}(r_m(\tau))\log|f(r_m(\tau))|\right). \tag{2.4}$$

Here

$$\tilde{f}(r_m(\tau)) = \left(\frac{r_h^{d+1}}{r_m^{d+1}(\tau)} - 1\right), \qquad E = \frac{V_d \ell^d}{16\pi G_N}\,\frac{d}{r_h^d}\,. \tag{2.5}$$

It is clear from this expression that the late time value is approaches from above leading to the Lloyd's bound violation. Of course as far as the late lime linear growth of complexity is concerned it is sufficient to compute on shell action over the intersection of the WDW patch with the future interior shown by dark blue color in the figure 1 [11]. When the cutoff is not set the on shell action evaluated in this patch is [11]

$$I_{\rm FI} = 2E\tau + \text{time independent term}. \tag{2.6}$$

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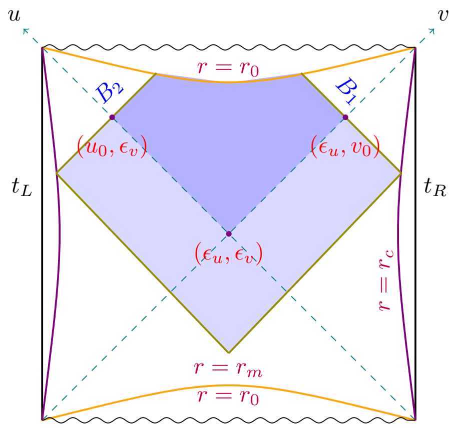

Figure 1. The intersection of WDW patch with the future interior of an eternal AdS black brane. The theory is defined at a radial finite cutoff r_c that fixes a cutoff behind the horizon denoted by r_0.

Now the aim is to compute on shell action over future interior for the case where we have a cutoff in the system. In this case, using the notation fixed above, the bulk part of the on shell action is

$$\begin{aligned} I_{\rm FI}^{\rm bulk} &= -\frac{V_d \ell^d}{4\pi G_N}(d+1)\int_{r_h}^{r_0} \frac{dr}{r^{d+2}}\left(\frac{\tau}{2} + r^*(r_c) - r^*(r)\right) \\ &= -\frac{V_d \ell^d}{8\pi G_N}\left(\frac{2}{d\, r_h^d} - \frac{2}{d\, r_0^d}\right) - \frac{V_d \ell^d}{8\pi G_N}\left(\frac{1}{r_h^{d+1}} - \frac{1}{r_0^{d+1}}\right)(\tau + \tau_c), \end{aligned} \tag{2.7}$$

where $\tau_c = 2(r^*(r_c) - r^*(r_0))$. Note that to find the last expression we have performed an integration by parts. Here r_0 is a cutoff near singularity which could have been sent to infinity for which all terms containing of different powers of $\frac{1}{r_0}$ in the above expression vanish. Of course as it will become clear later, in what follows we keep r_0 finite.

There are five boundaries four of which are null that have zero contribution if one uses the Affine parameter to parametrize the null directions. Therefore we are left with a space like boundary at future singularity whose contribution is given by

$$I_{\rm FI}^{\rm surf_1} = -\frac{1}{8\pi G_N}\int d^d x \int_{-t_L - r^*(r_c) + r^*(r)}^{t_R + r^*(r_c) - r^*(r)} dt\, \sqrt{h} K_s\Big|_{r=r_0}, \tag{2.8}$$

where K_s is the trace of extrinsic curvature of the boundary at $r = r_0$ and h is the determinant of the induced metric on it. To compute this term it is useful to note that for a constant r surface using the metric (1.4) one has

$$\sqrt{h}K = -\sqrt{g^{rr}}\partial_r \sqrt{h} = -\frac{1}{2}\frac{\ell^d}{r^d}\left(\partial_r f(r) - \frac{2(d+1)}{r} f(r)\right), \tag{2.9}$$

therefore the boundary term (2.8) reads

$$I_{\rm FI}^{\rm surf_1} = \frac{V_d \ell^d}{8\pi G_N}(d+1)\left(\frac{1}{2r_h^{d+1}} - \frac{1}{r_0^{d+1}}\right)(\tau + \tau_c). \tag{2.10}$$

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Note that there is also another boundary term to be evaluated at the surface cutoff behind the horizon that is given by

$$I_{\rm FI}^{\rm surf_2}=\frac{1}{8\pi G_N}\int d^dx\int_{-t_L-r^*(r_c)+r^*(r)}^{t_R+r^*(r_c)-r^*(r)}dt\,\sqrt{|h|}\,\,\frac{d}{\ell}\Big|_{r=r_0}=\frac{V_d\ell^d}{8\pi G_N}\frac{d}{r_0^{d+1}}\sqrt{\frac{r_0^{d+1}}{r_h^{d+1}}-1}\,\left(\tau+\tau_c\right). \tag{2.11}$$

There are also five joint points, two points at r_0 and three at the horizon $r=r_h$. Of course those at the horizon are not at the same point, though the coordinate system r cannot make any distinction between them. To label these points it is convenient to use the following coordinate system [15],

$$u=-e^{-\frac{1}{2}f'(r_h)(r^*(r)-t)},\qquad\qquad v=-e^{-\frac{1}{2}f'(r_h)(r^*(r)+t)}, \tag{2.12}$$

by which the points are located at (ϵ_u,v_0), (u_0,ϵ_v) and (ϵ_u,ϵ_v) as depicted in figure 1. Here in order to regularize quantities like $r^*(r)$ at $r=r_h$ we have put the horizon at $v=\epsilon_v$ and $u=\epsilon_u$ for small ϵ_v and ϵ_u. In what follows the radial coordinate associated with these three points are also labeled by r_{v_0},r_{u_0} and r_ϵ, respectively. Using this notation the contribution of joint points is [11]

$$\begin{aligned}I_{\rm FI}^{\rm joint}&=\frac{V_d\ell^d}{8\pi G_N}\left(\frac{\log\frac{\alpha\beta r_0^2}{\ell^2|f(r_0)|}}{r_0^d}+\frac{\log\frac{\alpha\beta r_\epsilon^2}{\ell^2|f(r_\epsilon)|}}{r_\epsilon^d}-\frac{\log\frac{\alpha\beta r_{u_0}^2}{\ell^2|f(r_{u_0})|}}{r_{u0}^d}-\frac{\log\frac{\alpha\beta r_{v_0}^2}{\ell^2|f(r_{v_0})|}}{r_{v0}^d}\right)\\&=-\frac{V_d\ell^d}{8\pi G_N}\left(\frac{\log|f(r_\epsilon)|-\log|f(r_{u_0})|-\log|f(r_{v_0})|}{r_h^d}+\frac{\log\frac{\alpha\beta r_h^2}{\ell^2}}{r_h^d}+\frac{\log\frac{\alpha\beta r_0^2}{\ell^2|f(r_0)|}}{r_0^d}\right).\end{aligned} \tag{2.13}$$

Here we have used the fact that $\{r_{u_m},r_{v_m},r_\epsilon\}\approx r_h$. On the other hand by making use of the fact that [15]

$$\log|f(r_{u,v})|=\log|uv|+c_0+\mathcal{O}(uv)\qquad\text{for }\ uv\to 0, \tag{2.14}$$

one arrives at

$$I_{\rm FI}^{\rm joint}=\frac{V_d\ell^d}{8\pi G_N}\left(\frac{\log|u_0v_0|+c_0}{r_h^d}-\frac{\log|f(r_0)|}{r_0^d}\right)-\frac{V_d\ell^d}{8\pi G_N}\left(\frac{\log\frac{\alpha\beta r_h^2}{\ell^2}}{r_h^d}-\frac{\log\frac{\alpha\beta r_0^2}{\ell^2}}{r_0^d}\right). \tag{2.15}$$

The only remaining part of the action to be considered is a term needed to remove the ambiguity associated with the normalization of null vectors [14, 16, 17]

$$I^{\rm amb}=\frac{1}{8\pi G_N}\int d\lambda d^dx\sqrt{\gamma}\Theta\log\frac{|\tilde{\ell}\Theta|}{d}, \tag{2.16}$$

where $\tilde{\ell}$ is an undetermined length scale and γ is the determinant of the induced metric on the joint point where two null segments intersect, and

$$\Theta=\frac{1}{\sqrt{\gamma}}\frac{\partial\sqrt{\gamma}}{\partial\lambda}. \tag{2.17}$$

In the present case the contribution of this term is (for more details see [11])

$$I_{\rm FI}^{\rm amb}=\frac{V_d\ell^d}{8\pi G_N}\left(\frac{\log\frac{\alpha\beta\tilde{\ell}^2 r_h^2}{\ell^4}}{r_h^d}-\frac{\log\frac{\alpha\beta\tilde{\ell}^2 r_0^2}{\ell^4}}{r_0^d}\right)+\frac{V_d\ell^d}{8\pi G_N}\left(\frac{2}{d\,r_h^d}-\frac{2}{d\,r_0^d}\right). \tag{2.18}$$

Taking all parts contributing to the on shell action into account one arrives at

$$\begin{aligned}I_{\rm FI}=\frac{V_d\ell^d}{8\pi G_N}\Bigg[&\left(\frac{d}{r_h^{d+1}}-\frac{d}{r_0^{d+1}}+\frac{d}{r_0^{d+1}}\sqrt{\frac{r_0^{d+1}}{r_h^{d+1}}-1}\right)(\tau+\tau_c)\\&+\frac{(d+1)r^*(r_0)+c_0r_h}{r_h^{d+1}}-\frac{\log|f(r_0)|}{r_0^d}+\left(\frac{1}{r_h^d}-\frac{1}{r_0^d}\right)\log\frac{\tilde{\ell}^2}{\ell^2}\Bigg],\end{aligned} \tag{2.19}$$

leading to the following rate of growth

$$\frac{dI_{\rm FI}}{d\tau}=\frac{V_d\ell^d d}{8\pi G_N}\left(\frac{1}{r_h^{d+1}}-\frac{1}{r_0^{d+1}}+\frac{1}{r_0^{d+1}}\sqrt{\frac{r_0^{d+1}}{r_h^{d+1}}-1}\right), \tag{2.20}$$

that is indeed the late time expression for the holographic complexity of the corresponding black brane solution [11]. It is worth noting that this expression does not depend on the UV cutoff r_c and it is also clear that for $r_0\to\infty$ the above expression reduces to (2.6).

Now the aim is to compare the above result with Lloyd's bound for the model under consideration that has a cutoff. In this case the bound should be read from the energy spectrum (1.5) that can be recast into the following inspiring form

$$E=\frac{V_d\ell^d d}{16\pi G}\frac{1}{r_h^{d+1}}+\frac{V_d\ell^d d}{16\pi G}\frac{1}{r_c^{d+1}}\left(1-\sqrt{1-\frac{r_c^{d+1}}{r_h^{d+1}}}\right)^2. \tag{2.21}$$

The first terms is energy associated with the black brane and the second terms arises from because of the cutoff. Therefore if one assumes that at the late time the growth rate of complexity saturates the Lloyd's bound, $2E$, one may conclude that

$$\frac{1}{r_0^{d+1}}\left(\sqrt{\frac{r_0^{d+1}}{r_h^{d+1}}-1}-1\right)=\frac{1}{r_c^{d+1}}\left(\sqrt{1-\frac{r_c^{d+1}}{r_h^{d+1}}}-1\right)^2, \tag{2.22}$$

which at leading order reduces to

$$r_0r_c^2=2^{\frac{4}{d+1}}r_h^3. \tag{2.23}$$

This means that the cutoff at the singularity should be given in terms of the UV cutoff at the boundary if the complexity is going to saturate at the late time to a value given by twice of the energy of the model. In other words this leads to a conclusion that as soon as we fixed the UV cutoff we are not allowed to consider another independent cutoff inside the horizon (let say near the singularity) and the UV cutoff will automatically regularize the modes inside the horizon. Therefore we were not actually allowed to send $r_0\to\infty$ from the first place. This is, indeed, the main result of the present paper.

To explore the importance of the above conclusion in what follows we will study holographic complexity for AdS_2 vacuum solutions of certain two dimensional gravities.

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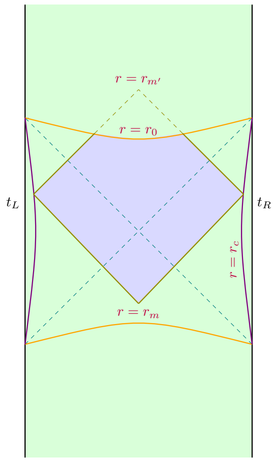

Figure 2. Penrose diagram of AdS_2 geometry. The green part is covered by AdS global coordinates, while the Rindler coordinates cover a portion shown in the figure. The actual WDW patch is shown by blue color.

3 Complexity for AdS_2 geometry

In this section we shall study holographic complexity for certain two dimensional Maxwell-Dilaton gravities that admit AdS vacuum solutions. The first model we will consider has the following action[1]

$$I = \frac{1}{8G} \int d^2x \sqrt{-g} \left(e^{\phi} \left(R + \frac{2}{\ell^2} \right) - F^2 \right). \tag{3.1}$$

Using the entropy function formalism [21] one can show that the above action admits constant dilaton AdS_2 vacuum solution as follows [22]

$$ds^2 = \ell^2 \left(-(r^2 - r_h^2)dt^2 + \frac{dr^2}{r^2 - r_h^2} \right), \qquad e^{\phi} = 4G^2Q^2\ell^2, \qquad F_{rt} = 2GQ\ell^2, \tag{3.2}$$

whose entropy is

$$S_{\rm BH} = 2\pi G Q^2 \ell^2, \tag{3.3}$$

that it is independent of r_h. Let us compute holographic complexity for a state given at $\tau = t_L + t_R$. The corresponding WDW patch is depicted in the figure 2.

One may naively compute on shell action in the WDW patch shown in the figure 2 with two joint points denoted by r_m and $r_{m'}$ (the later point is drown by dashed lines). Positions of the corresponding points are obtained from the following equations

$$\tau = -2(r^*(r_c) - r^*(r_{m'})) = 2(r^*(r_c) - r^*(r_m)), \tag{3.4}$$

where r_c is a UV cutoff.

[1]This is indeed one of the simplest example of two dimensional gravity having non-trivial vacuum. One could, as well, consider rather more complicated actions (see e.g. [18–20]).

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Following CA proposal the idea is to evaluate on shell action on the corresponding WDW with a UV cutoff but no, a priori, restriction on modes behind the horizon. This means that there is no cutoff behind the horizon and both corners denoted by m and m' should be taken into account. With this assumption the bulk part of the on shell action reads

$$\begin{aligned} I^{\text{bulk}} = GQ^2\ell^2 \bigg(& \int_{r_{m'}}^{r_h} dr\,(\tau + 2(r^*(r_c) - r^*(r))) + 2\int_{r_h}^{r_c} dr\, 2\,(r^*(r_c) - r^*(r)) \\ & + \int_{r_m}^{r_h} dr\,(-\tau + 2(r^*(r_c) - r^*(r))) \bigg) \\ = 2GQ^2\ell^2 \bigg(& (r_m - r_{m'})\frac{\tau}{2} + \int_{r_{m'}}^{r_c} dr(r^*(r_c) - r^*(r)) + \int_{r_m}^{r_c} dr\,(r^*(r_c) - r^*(r)) \bigg). \end{aligned} \tag{3.5}$$

By making use of an integration by parts one finds

$$I^{\text{bulk}} = GQ^2\ell^2 \Big(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_{m'})| \Big), \tag{3.6}$$

where $f(r) = r^2 - r_h^2$.

On the other hand using Affine parameter to parametrize the null direction one gets zero contribution from null boundaries. Therefore the only part one needs to further consider is the contribution of join points. Denoting the null vectors by

$$k_1 = \alpha\left(\partial_t - \frac{1}{f(r)}\partial_r\right), \qquad k_2 = \beta\left(\partial_t + \frac{1}{f(r)}\partial_r\right), \tag{3.7}$$

one gets

$$\begin{aligned} I^{\text{joint}} &= \frac{e^{\phi}}{4G}\left(\log\left|\frac{\alpha\beta}{\ell^2 f(r_m)}\right| + \log\left|\frac{\alpha\beta}{\ell^2 f(r_{m'})}\right| - 2\log\left|\frac{\alpha\beta}{\ell^2 f(r_c)}\right|\right) \\ &= GQ^2\ell^2 \Big(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_{m'})| \Big). \end{aligned} \tag{3.8}$$

Interestingly enough the free parameters α and β drop from the final result which means that there is no ambiguity associated with the normalization of null vectors. Therefore we do not need any further counter terms, except possibly the one that could cancel the most divergent term of the on shell action, $\log f(r_c)$. Of course since we are interested in the time derivative of the action this term does not play any role.

Taking all terms contributing to the on shell action one arrives at

$$I = I^{\text{bulk}} + I^{\text{joint}} = 2GQ^2\ell^2 \Big(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_{m'})| \Big), \tag{3.9}$$

whose time derivative is

$$\frac{dI}{d\tau} = 2GQ^2\ell^2(r_m - r_{m'}). \tag{3.10}$$

It is then notable that at the late time when $\{r_m, r_{m'}\} \to r_h$ the rate of growth vanishes, leading to a constant late time complexity that is counter intuitive. Indeed we would expect to get linear growth at the late time.

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Of course in light of our result in the previous section this conclusion is, indeed, misleading. In fact, as we have already demonstrated in the previous section, setting a UV cutoff at the boundary would enforce us to have a cutoff inside the horizon that prevents us to have access to all regions on WDW located behind the horizon.

In other words, as soon as we set the UV cutoff, r_c, at the boundary we will also have to consider a cutoff behind the horizon given by $r_0 \sim \frac{r_h^3}{r_c^2}$ at leading order. Actually having this cutoff will remove the joint point $r_{m'}$ from the WDW patch and instead we would have a space like boundary at $r = r_0$. Therefore one should redo our computations for on shell action for a new WDW patch that has no joint point m', as shown with blue color in the figure 2.

To proceed let us again start with the bulk action. In this case one gets

$$I_{\rm bulk} = GQ^2\ell^2\bigg(\int_{r_0}^{r_h} dr\,(\tau + 2(r^*(r_c) - r^*(r))) + 2\int_{r_h}^{r_c} dr\,2\,(r^*(r_c) - r^*(r)) + \int_{r_m}^{r_h} dr\,(-\tau + 2(r^*(r_c) - r^*(r)))\bigg), \tag{3.11}$$

that can be recast to the following form after making use of an integration by parts

$$I_{\rm bulk} = GQ^2\ell^2\bigg(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_0)| - r_0\,(\tau + 2(r^*(r_c) - r^*(r_0)))\bigg). \tag{3.12}$$

The boundary contributions associated with null boundaries are still zero when Affine parametrization is used. Of course in the present case we have a apace like boundary whose contribution is

$$I_{\rm surf} = -\frac{1}{4G}\int dt e^{\phi}\sqrt{-h}\left(K_s - \frac{1}{\ell}\right)\bigg|_{r_0} = GQ^2\ell^2(r_0 + r_h)\,(\tau + 2(r^*(r_c) - r^*(r_0)))\,. \tag{3.13}$$

As for joint points we have

$$\begin{aligned} I_{\rm joint} &= \frac{e^{\phi}}{4G}\left(\log\left|\frac{\alpha\beta}{\ell^2 f(r_m)}\right| + \log\left|\frac{\alpha}{\ell\sqrt{f(r_0)}}\right| + \log\left|\frac{\beta}{\ell\sqrt{f(r_0)}}\right| - 2\log\left|\frac{\alpha\beta}{\ell^2 f(r_c)}\right|\right) \\ &= GQ^2\ell^2\bigg(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_0)|\bigg). \end{aligned} \tag{3.14}$$

Now putting all terms together one arrives at

$$I = 2GQ^2\ell^2\bigg(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_0)|\bigg) + GQ^2\ell^2 r_h\,(\tau + 2(r^*(r_c) - r^*(r_0)))\,. \tag{3.15}$$

It is the easy to show

$$\frac{dI}{dt} = GQ^2\ell^2(r_h + 2r_m), \tag{3.16}$$

which approaches a constant at the late time

$$\frac{dI}{dt} = 3GQ^2\ell^2 r_h = 3\left(2\pi GQ^2\ell^2\right)\left(\frac{r_h}{2\pi}\right) = 3S_{\rm BH}T\,. \tag{3.17}$$

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Here T is the Hawking temperature associated with the geometry. This is in agreement with what is expected; namely one has late time linear growth with slop given by entropy times temperature. Of course the actual numerical factor does not look universal.

To further explore the above picture better it is also constructive to consider another two dimensional model admitting AdS_2 vacuum solutions as follows

$$S_2 = \frac{1}{8G}\int d^2x\sqrt{-g}\, e^{\phi}\left(R + \frac{2}{\ell^2} - \frac{\ell^2}{4}e^{2\phi}F^2\right). \tag{3.18}$$

Using the entropy function formalism [21] one can show that the above action admits the AdS_2 vacuum solution as follows [22][2]

$$ds^2 = \frac{\ell^2}{4}\left(-(r^2-r_h^2)dt^2 + \frac{dr^2}{r^2-r_h^2}\right), \qquad e^{\phi} = \sqrt{4GQ\ell^2}, \qquad F_{tr} = \sqrt{\frac{1}{16GQ\ell^2}} \tag{3.19}$$

with the entropy,

$$S_{\rm BH} = 2\pi\sqrt{\frac{Q\ell^2}{4G}}. \tag{3.20}$$

Now the aim is to compute the holographic complexity for this model. Of course the procedure is the same as that we considered in the previous case and the only difference is the numerical factors. More precisely for the bulk term one finds

$$I_{\rm bulk} = -\frac{\ell}{4}\sqrt{\frac{Q}{G}}\bigg(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_0)| - r_0\left(\tau + 2(r^*(r_c) - r^*(r_0))\right)\bigg). \tag{3.21}$$

As for joint points one gets

$$I_{\rm joint} = \frac{\ell}{2}\sqrt{\frac{Q}{G}}\bigg(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_0)|\bigg), \tag{3.22}$$

while for the surface term one has

$$I_{\rm surf} = \frac{\ell}{2}\sqrt{\frac{Q}{G}}\,(r_0 + r_h)\left(\tau + 2(r^*(r_c) - r^*(r_0))\right). \tag{3.23}$$

Therefore the total action is found

$$I = \frac{\ell}{4}\sqrt{\frac{Q}{G}}\bigg(2\log|f(r_c)| - \log|f(r_m)| - \log|f(r_0)|\bigg) + \frac{\ell}{2}\sqrt{\frac{Q}{G}}\,r_h\left(\tau + 2(r^*(r_c) - r^*(r_0))\right), \tag{3.24}$$

resulting to the following rate of growth for the on shell action

$$\frac{dI}{dt} = \frac{\ell}{4}\sqrt{\frac{Q}{G}}\,(r_m + 2r_h), \tag{3.25}$$

which approaches a constant at late time

$$\frac{dI}{dt} = 3\frac{\ell}{4}\sqrt{\frac{Q}{G}}\,r_h = \frac{3}{2}S_{\rm BH}T. \tag{3.26}$$

Note that the same as previous one, had not we considered the inside surface cutoff, the complexity growth would have been zero at the late time.

[2] See [23] for non-constant dilation solution of the model.

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4 Conclusions

In this paper we have studied holographic complexity for an AdS black brane geometry with a radial cutoff using CA proposal. Within this explicit example we have found that as soon as one sets a UV cutoff at the boundary the model enforces us to have a cutoff behind the horizon whose value is fixed by the UV cutoff. Indeed in the present case one has

$$\frac{1}{r_0^{d+1}}\left(\sqrt{\frac{r_0^{d+1}}{r_h^{d+1}}-1}-1\right)=\frac{1}{r_c^{d+1}}\left(\sqrt{1-\frac{r_c^{d+1}}{r_h^{d+1}}}-1\right)^2. \tag{4.1}$$

It is worth mentioning that in order to get a consistent result fulfilling the Lloyd's bound it was crucial to consider the contribution of certain counter term on the cutoff surface behind the horizon.

In this paper we have only considered uncharged black hole with flat boundary. It would be interesting to find an expression for behind the horizon cutoff in terms of the UV cutoff for a general charged black hole. In general the cutoff r_0 is a function of UV cutoff; $r_0 = r_0(r_h, r_c)$, though it might not have such a simple expression as above. Actually this relation should be intuitively understood from the fact that the energy is a charge defined at the boundary while the late time behavior of complexity is evaluated from the action behind the horizon. Indeed this is an interesting feature of complexity that could probe an object behind the horizon that quantities we are more familiar with (such an correlation functions) cannot do.

If our result works for a generic black hole, it means that near singularity modes may be regularized through a UV cutoff. It is, however, important to note that our conclusion will not affect the results people have found so far in the literature, though it might shed light on some new problems such as how to deal with Riemann tensor squared.

Actually in order to explore the importance of our result we have studied holographic complexity for AdS$_2$ vacuum solutions in certain two different Maxwell-Dilaton gravities. We have found that the complexity is finite at late times if one does not consider the cutoff enforced by the UV cutoff, that seems counter intuitive. Indeed one would expect that the complexity exhibits linear growth at the late time. On the other hand if one considers behind the horizon cutoff fixed by the UV cutoff, indeed one gets the corresponding linear growth.

Two dimensional AdS solutions we have considered were supported by a constant Dilaton, though it would be interesting to consider the case where the Dilaton is not constant. This might be more interesting as it could provide a holographic dual for SYK model [24–26] (see for example [27–29]).

Note added. While we were in the final stage of our work, the paper [30] appeared in the arXiv where the complexity of two dimensional gravity has also been studied. In this paper the authors resolved the undesired late time behavior by adding a new charge to the model. This in fact could be naturally accommodated if one considers the model as a dimensionally reduced four dimensional RN black hole.

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