Complexity for Conformal Field Theories in General Dimensions

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We study circuit complexity for conformal field theory states in an arbitrary number of dimensions. Our circuits start from a primary state and move along a unitary representation of the Lorentzian conformal group. Different choices of distance functions can be understood in terms of the geometry of coadjoint orbits of the conformal group. We explicitly relate our circuits to timelike geodesics in anti–de Sitter space and the complexity metric to distances between these geodesics. We extend our method to circuits in other symmetry groups using a group theoretic generalization of the notion of coherent states.

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Introduction.—The peculiarity of quantum systems is rooted in their entanglement pattern. Hence, there is increasing interest in studying measures characterizing entanglement in quantum states. The most famous of these measures is the entanglement entropy, which estimates the knowledge a given subsystem has about the full quantum state. In recent years, it became apparent that entanglement entropy is not enough to capture the full information about quantum correlations in a state. As a consequence, a new measure from quantum information became prominent in studies of quantum states. This measure, known as "quantum computational complexity" (QCC), estimates how hard it is to construct a given state from a set of elementary operations [1–3]. QCC is also of clear interest in recent efforts to construct quantum computers.

QCC has attracted a lot of attention in high energy theory due to its proposed relation to black holes [4,5]. This relation was explicitly formulated within the holographic (or AdS/CFT) correspondence [6]. It turns out that the growth of black hole interiors behaves in a very similar way to the growth of complexity during Hamiltonian evolution in quantum systems, see, e.g., [7–13]. These ideas suggest a promising avenue to address puzzles related to black hole spacetimes and their interior geometry.

However, the lack of a complete framework for studying QCC within quantum field theory (QFT) has been a stumbling block toward rigorously establishing the connection between black hole interiors and QCC. Significant progress was made for free and weakly coupled QFTs [14–20] and for strongly coupled two-dimensional conformal field theories (CFTs) [21–25]. Yet, no results exist at present for circuit complexity in CFTs in $d > 2$ and, further, its precise connection with holography has not been established in any dimension. The importance of studying complexity in $d > 2$ becomes evident when noting that holographic complexity behaves very differently in $d = 2$ and in $d > 2$, for example, when studying the complexity of formation of thermofield double states [26] or its sensitivity to defects [27,28]. The goal of this Letter is to bridge these gaps by studying complexity of CFTs in $d > 2$ and further by establishing a rigorous connection between complexity and geometry in holography.

We employ the symmetry generators to construct circuits in unitary representations of the Lorentzian conformal group and present explicit results for state-dependent distance functions along these circuits. Our circuits live in a phase space that is a coadjoint orbit of the conformal group and the various cost functions take the form of simple geometric notions on these orbits. Using symmetry generators to construct circuits restricts the circuits to move in the space of generalized coherent states. We use this fact to generalize our results to general symmetry groups. We illustrate our methods by focusing on circuits starting from a scalar primary state whose coadjoint orbit can be identified with the coset space $\mathrm{SO}(d,2)/[\mathrm{SO}(2)\times \mathrm{SO}(d)]$, but our techniques are also applicable to more general spinning states. We derive bounds on the complexity and its rate of change.

We explicitly relate our unitary circuits to timelike geodesics in anti–de Sitter spacetimes. We find that the line element in the complexity metric admits a very simple interpretation as the average of the minimal and maximal squared distances between two nearby geodesics. This provides a novel bulk description for complexity, which is rigorously derived from the CFT and opens new possibilities for testing the holographic complexity proposals.

This Letter is organized as follows: In Sec. II, we introduce the relevant complexity distance functions. In Sec. III, we present the result for the complexity of CFT states in general dimensions. In Secs. IV and V, we connect our results to the notions of coadjoint orbits and generalized coherent states. In Sec. VI, we connect our results to holography. We conclude in Sec. VII with a summary and outlook.

Preliminaries.—Explicitly, QCC is defined as the minimal number of gates required to reach a desired "target" state, starting from a (typically simpler) "reference" state. For several applications, it is advantageous to focus on continuous notions of complexity rather than a discrete gate counting. Such ideas were put forward by Nielsen [29–31] who translated the problem of studying minimal gate complexity to that of studying geodesics on the space of unitary transformations. In a very similar way, we can study notions of continuous complexity using geodesics through the space of quantum states.

Continuous complexity is defined using a cost function $\mathcal{F}(\sigma)$, with circuit parameter σ. The complexity is the minimal cost among all possible trajectories moving from the reference state to the target state: $\mathcal{C} \equiv \min \int d\sigma \mathcal{F}(\sigma)$. Past attempts to study state complexity in CFTs (e.g., [21]) focused on two cost functions: the $\mathcal{F}_1$ cost function and the Fubini-Study (FS) norm defined as

$$\mathcal{F}_1(\sigma) d\sigma = |\langle \psi | \partial_\sigma \psi \rangle | d\sigma = |\langle \psi_R | U^\dagger dU | \psi_R \rangle|, \tag{1a}$$

$$\mathcal{F}_{\mathrm{FS}}(\sigma) d\sigma = \sqrt{\langle \psi_R | dU^\dagger dU | \psi_R \rangle - |\langle \psi_R | U^\dagger dU | \psi_R \rangle|^2}, \tag{1b}$$

where $|\psi(\sigma)\rangle \equiv U(\sigma)|\psi_R\rangle$ are the states along the unitary circuit, $|\psi_R\rangle$ is the reference state, and $ds^2_{\mathrm{FS}} = \mathcal{F}^2_{\mathrm{FS}}(\sigma) d\sigma^2$ is the well known FS metric. Our analysis in the next section demonstrates that the $\mathcal{F}_1$ cost function assigns zero cost to certain gates and has, therefore, disadvantages as a complexity measure.

The FS metric along straight-line trajectories $e^{itH}|\psi_R\rangle$ is proportional to the variance $\Delta E = \sqrt{\langle H^2 \rangle - |\langle H \rangle|^2}$. We can interpret H as the Hamiltonian and t as the time. This variance was shown in [32] to bound the time required to reach an orthogonal state $\tau_{\mathrm{orth}} \geq \pi\hbar/(2\Delta E)$ on compact spaces. Inspired by these bounds on orthogonality time, Lloyd conjectured a bound on the rate of computation [33] (see also [8]). Unlike [32], our state manifold is noncompact and our states never become orthogonal. Nonetheless, we will derive bounds on the complexity and its rate of change by other means. Deriving bounds on the state overlap in our setup is an interesting question for future study.

Complexity in general dimensions.—Consider the Euclidean conformal algebra in $d \geq 2$ with D, P_μ, K_μ, and $L_{\mu\nu}$ as the Euclidean conformal generators (used to construct unitary representations of the Lorentzian conformal group; see Supplemental Material, Sec. A [34]) satisfying

$$D^\dagger = D, \qquad K^\dagger_\mu = P_\mu, \qquad L^\dagger_{\mu\nu} = -L_{\mu\nu}, \tag{2}$$

in radial quantization.

As the reference state, we consider a scalar primary state $|\psi_R\rangle = |\Delta\rangle$ satisfying $D|\Delta\rangle = \Delta|\Delta\rangle$ and $K_\mu|\Delta\rangle = L_{\mu\nu}|\Delta\rangle = 0$ and focus on circuits generated by the unitary

$$U(\sigma) \equiv e^{i\alpha(\sigma)\cdot P} e^{i\gamma_D(\sigma) D} \left(\prod_{\mu<\nu} e^{i\lambda_{\mu\nu}(\sigma) L_{\mu\nu}} \right) e^{i\beta(\sigma)\cdot K}, \tag{3}$$

with σ as a circuit parameter and $\alpha_\mu, \beta_\mu, \gamma_D$, and $\lambda_{\mu\nu}$ *a priori* complex parameters, further constrained by the restriction that $U(\sigma)$ be unitary. The circuits take the form $|\alpha(\sigma)\rangle \equiv U(\sigma)|\Delta\rangle \equiv \mathcal{N}(\sigma) e^{i\alpha(\sigma)\cdot P}|\Delta\rangle$, where $\mathcal{N}(\sigma) \equiv \exp[i\gamma_D(\sigma)\Delta]$ is a normalization factor and $\gamma_D(\sigma) \equiv \gamma^{\mathrm{Re}}_D(\sigma) + i\gamma^{\mathrm{Im}}_D(\sigma)$, with Re and Im indicating the real and imaginary part. Unitarity of $U(\sigma)$ implies $\gamma^{\mathrm{Im}}_D(\sigma) = -\frac{1}{2}\log A(\alpha, \alpha^*)$ (see Supplemental Material, Sec. B [34]), where

$$A(\alpha, \alpha^*) \equiv 1 - 2\alpha \cdot \alpha^* + \alpha^2 \alpha^{*2} > 0, \tag{4}$$

and requiring a positive spectrum for the Hamiltonian D along the circuit implies $\alpha^* \cdot \alpha < 1$ (equivalently $\alpha^2 \alpha^{*2} < 1$).

Substituting $|\alpha(\sigma)\rangle$ into the cost functions (1a) and (1b) and using the expectation values of $\{P_\mu, K_\mu, K_\mu P_\nu\}$ (see [34], Sec. B), we find for the $\mathcal{F}_1$ cost function

$$\frac{\mathcal{F}_1}{\Delta} = \left| \frac{\dot{\alpha} \cdot \alpha^* - \dot{\alpha}^* \cdot \alpha + \alpha^2 (\dot{\alpha}^* \cdot \alpha^*) - \alpha^{*2} (\dot{\alpha} \cdot \alpha)}{A(\alpha, \alpha^*)} + i\dot{\gamma}^{\mathrm{Re}}_D \right|, \tag{5}$$

while for the FS metric we obtain

$$\frac{ds^2_{\mathrm{FS}}}{d\sigma^2} = 2\Delta \left[\frac{\dot{\alpha} \cdot \dot{\alpha}^* - 2|\dot{\alpha} \cdot \alpha|^2}{A(\alpha, \alpha^*)} + 2\frac{|\dot{\alpha} \cdot \alpha^* - \alpha^{*2} \alpha \cdot \dot{\alpha}|^2}{A(\alpha, \alpha^*)^2} \right]. \tag{6}$$

The FS metric (6) is a positive-definite Einstein-Kähler metric on the complex manifold of states with d complex coordinates α bounded inside the domain (4). It satisfies $ds^2_{\mathrm{FS}} = \partial_\alpha \partial_{\alpha^*} K(\alpha, \alpha^*) d\alpha d\alpha^*$, where the associated Kähler potential is defined as $K(\alpha, \alpha^*) = -\Delta \log A(\alpha, \alpha^*)$. Denoting collectively the indices of α and α^* by capital Latin letters, one finds that $R_{AB} = -(2d/\Delta) g_{AB}$ and $R = -(4d^2/\Delta)$ and that all sectional curvatures are negative. This means that geodesics will deviate from each other.

In fact, (6) is a natural metric on the following quotient space of the conformal group:

$$\mathcal{M} = \frac{\mathrm{SO}(d,2)}{\mathrm{SO}(2) \times \mathrm{SO}(d)}, \tag{7}$$

which can also be identified with the space of timelike geodesics in AdS_{d+1} [46,47], see Sec. VI. This is similar to the relation between the metric on kinematic space and spacelike geodesics in AdS_{d+1}, [48–51] where the relevant orbit is $\mathrm{SO}(d,2)/\mathrm{SO}(1,1)\times\mathrm{SO}(1,d-1)$ [52]. While some of the above observations are well known in the context of geometry of Lie groups [53,54], here they find a novel role in the context of circuit complexity.

Since the coset space (7) is a negatively curved symmetric space, its geodesics (using the FS-metric) passing through $|\psi_R\rangle$ take the form [55]

$$|\psi(\sigma)\rangle = \exp\left[i\sigma(\tilde{\alpha}_\mu P^\mu + \tilde{\alpha}^*{}_\mu K^\mu)\right]|\psi_R\rangle, \qquad (8)$$

and do not reconnect; i.e., (7) has no conjugate points [54]. Here, we parametrized our geodesics in terms of the straight-line trajectory parameter $\tilde{\alpha}$ rather than α. Explicitly, in terms of the α parametrization, the complexity of a target state $|\alpha(\sigma=1)\rangle \equiv |\alpha_T\rangle$ is

$$\begin{aligned}\mathcal{C}[\tilde{\alpha}] &= \sqrt{2\Delta\tilde{\alpha}^*\cdot\tilde{\alpha}},\\ 2\tilde{\alpha}\cdot\tilde{\alpha}^* &= [(\tanh^{-1}\Omega_T^S)^2 + (\tanh^{-1}\Omega_T^A)^2],\end{aligned} \qquad (9)$$

where $\Omega_T^\pm \equiv \Omega_T^S \pm \Omega_T^A \equiv \sqrt{2\alpha_T\cdot\alpha_T^* \pm 2|\alpha_T^2|}$ (see Supplemental Material, Secs. C and D [34]). Earlier, we chose to parametrize the states with $\alpha(\sigma)$ rather than $\tilde{\alpha}$ since this facilitates the evaluation of correlation functions in the state and therefore provides its more natural characterization. We will see later that the relation to holography is also done using the parameter α. The complexity (9) can be bounded by employing the inequalities around (4)

$$\frac{\Delta}{E_T+\Delta}\sqrt{(E_T-\Delta)} \le \mathcal{C}[\alpha_T] \le \sqrt{E_T-\Delta}, \qquad (10)$$

where $E_T \equiv \langle\alpha_T|D|\alpha_T\rangle = \Delta(1-\alpha_T^2\alpha_T^{*2})/A(\alpha_T,\alpha_T^*)$ is the energy of the target state in radial quantization (see [34], Sec. E).

A substantial difference between the $\mathcal{F}_1$ cost function and the FS metric is that the former depends on γ_D^{Re}, which induces an overall phase in the states through which our circuits pass. In fact, the $\mathcal{F}_1$ cost function (5) without absolute values vanishes on shell except for its part associated with the overall phase γ_D^{Re} and is simply proportional to the Berry gauge field, cf. [22,56,57].

We close by observing that the FS distance along time evolved states $e^{i\tau D}|\alpha_0\rangle$ satisfies a Lloyd-like bound [33]

$$\frac{ds_{\mathrm{FS}}}{d\tau} \le \frac{E}{\sqrt{\Delta}} \le \sqrt{\frac{2}{d-2}}E, \qquad (11)$$

where $E \equiv \langle\alpha_0|D|\alpha_0\rangle$ is the energy, $|\alpha_0\rangle$ is an arbitrary initial state, and we used the unitarity bound $\Delta \ge d/2-1$ [58].

We compare our results to the existing literature for $d=2$ CFTs in the Supplemental Material [34], Sec. F. In that case, holomorphic factorization allows us to also treat spinning states ([34], Sec. G).

Geometric action and coadjoint orbits.—Our results for the cost functions (5) and (6) can be understood in terms of the geometry of coadjoint orbits, see, e.g., [59,60]. A similar connection was pointed out in two dimensions in [21,22].

Let us start by briefly describing the coadjoint orbit method in representation theory. Consider a Lie group G with Lie algebra $\mathfrak{g}$, a dual space $\mathfrak{g}^*$ consisting of linear maps on $\mathfrak{g}$, and a pairing $\langle\cdot,\cdot\rangle$ between the Lie algebra and dual space. For matrix groups, the adjoint action of $u\in G$ on $X\in\mathfrak{g}$ is defined as $\mathrm{Ad}_u(X) = uXu^{-1}$. At the level of the algebra, the adjoint action is simply the commutator $\mathrm{ad}_Y(X) = [Y,X]$, where $X,Y\in\mathfrak{g}$. The Maurer-Cartan (MC) form on the full group is $\Theta \equiv u^{-1}du$, where $u\in G$ and it satisfies $d\Theta = -\Theta\wedge\Theta$.

The coadjoint action on the dual space is defined implicitly by

$$\langle \mathrm{Ad}_u^*\xi, X\rangle = \langle\xi, \mathrm{Ad}_{u^{-1}}X\rangle, \qquad \xi\in\mathfrak{g}^*, \qquad X\in\mathfrak{g}, \quad u\in G, \qquad (12)$$

from which one can build the coadjoint orbit $\mathcal{O}_\lambda \equiv \{\mathrm{Ad}_u^*\lambda | u\in G\} \subset \mathfrak{g}^*$ of a given dual algebra element $\lambda\in\mathfrak{g}^*$. $\mathcal{O}_\lambda$ can be identified with the coset space G/H_λ, where the subgroup $H_\lambda = \mathrm{Stab}(\lambda) \equiv \{u\in G|\mathrm{Ad}_u^*\lambda = \lambda\}$ is the stabilizer and the corresponding algebra is $\mathfrak{h}_\lambda \equiv \mathrm{stab}(\lambda)$.

Each coadjoint orbit corresponds to a symplectic manifold with a local presymplectic form $\mathcal{A}_\lambda$ and the Kirillov-Kostant symplectic form ω_λ defined as

$$\mathcal{A}_\lambda = \langle\lambda,\Theta\rangle, \qquad \omega_\lambda = \langle\lambda, d\Theta\rangle. \qquad (13)$$

The geometric action associated with the coadjoint orbit is $S_\lambda = \int\mathcal{A}_\lambda$ [61,62].

The symplectic form ω_λ is compatible with a complex structure J_λ satisfying $J_\lambda^2 = -1$ if $\omega_\lambda(J_\lambda x, J_\lambda y) = \omega_\lambda(x,y)$. In this case it is possible to define a Kähler metric $ds^2_{G/H_\lambda}(x,y) = \omega_\lambda(x, J_\lambda y)$ on the coadjoint orbit $\mathcal{O}_\lambda$.

In the Supplemental Material [34], Sec. H, we apply the above definitions in the fundamental (matrix) representation of the conformal algebra $\mathrm{so}(d,2)$ with representative λ taken to be proportional to the dilatation matrix with stabilizer group $\mathfrak{h}_\lambda = \mathrm{so}(2)\times\mathrm{so}(d)$ and orbit corresponding to the quotient space G/H_λ from Eq. (7). This yields an agreement with Eqs. (5) and (6), i.e.,

$$\mathcal{F}_1 d\sigma = |\mathcal{A}_\lambda|, \qquad ds^2_{\mathrm{FS}} = ds^2_{G/H_\lambda}. \qquad (14)$$

As alluded to above, $\mathcal{A}_\lambda$ can also be interpreted as a Berry gauge field, and the Berry curvature is simply the symplectic form ω_λ. Circuits starting from spinning primary states in $d > 2$ amount to a different choice of representative to match with the relevant reduced stabilizer group.

Coherent state generalization.—The equivalence of the FS metric and the $\mathcal{F}_1$ cost function with their geometric counterparts on the coadjoint orbit is also valid within infinite-dimensional Hilbert spaces obtained via geometric quantization of the orbits of arbitrary Lie groups [21,25,63]. This can be understood using a group theoretical generalization of the notion of coherent states, see, e.g., [64–67]. The existence of these states is intrinsically connected to the representation theory of the symmetry in question. In this section, we explain how the coadjoint orbit perspective leads to the complexity functionals of (5) and (6) for general Lie groups.

As before, we consider some real Lie group G with Lie algebra $\mathfrak{g}$. The corresponding complex algebra admits a decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{n}_+ + \mathfrak{h}_\mathbb{C} + \mathfrak{n}_-$ with a real structure (a dagger) that maps $\mathfrak{h}_\mathbb{C}$ to itself and $\mathfrak{n}_+$ to $\mathfrak{n}_-$. For a detailed account of this decomposition, see [34], Sec. I. The generators of the real Lie algebra are anti-Hermitian. We denote the real subalgebra of $\mathfrak{h}_\mathbb{C}$ by $\mathfrak{h}$ and its associated Lie group by H. We also assume that $[\mathfrak{n}_+, \mathfrak{n}_+] \subset \mathfrak{n}_+$ and similar for $\mathfrak{n}_-$ and that $[\mathfrak{h}_\mathbb{C}, \mathfrak{n}_\pm] \subset \mathfrak{n}_\pm$. We take a basis of raising operators E_α for $\mathfrak{n}_+$ and lowering operators $E_{-\alpha}$ for $\mathfrak{n}_-$ with $E_\alpha^\dagger = E_{-\alpha}$ and a basis h_i for $\mathfrak{h}$.

We consider a unitary highest weight representation generated by a one-dimensional base state $|\psi_R\rangle$ satisfying $\mathcal{D}(E_\alpha)|\psi_R\rangle = 0$ and $\mathcal{D}(h_i)|\psi_R\rangle = \chi_i|\psi_R\rangle$ with χ_i constants and where $\mathcal{D}$ is the representation on the Hilbert space. In other words, the base state is invariant up to a phase under the action of the stabilizer subgroup $H \subset G$. This includes the possibility of spinning highest weight representations, cf. [68–71], in which case the stabilizer subgroup will be smaller compared to the spinless case.

We act on our base state with a unitary transformation $U = \exp[\sum_\alpha(\lambda_\alpha E_\alpha - \lambda_\alpha^* E_{-\alpha}) + \sum_i x_i h_i]$ in order to produce generalized coherent states

$$|u\rangle \equiv U|\psi_R\rangle = \mathcal{N}_H(z, z^*, x) \exp z_\alpha \mathcal{D}(E_{-\alpha})|\psi_R\rangle, \quad (15)$$

with $\mathcal{N}_H$ a normalization factor (including possibly an overall phase), x are real coordinates on the stabilizer, and z and z^* are holomorphic coordinates on the orbit. The relation between the coordinates that appear in U and the coordinates z can be quite complicated, in general. Of course, multiplying U from the right by an element of H does not modify $|u\rangle$ (up to an overall phase) and therefore U can be thought of as an element of $\mathcal{D}(G/H)$.

Generalized coherent states can be understood in terms of coadjoint orbits. Consider the dual element

$$\lambda(\mathcal{O}) = i\mathrm{Tr}[|\psi_R\rangle\langle\psi_R|\mathcal{D}(\mathcal{O})], \quad (16)$$

where the trace is taken in the infinite-dimensional representation space. The coadjoint action (12) on λ is simply $\langle \mathrm{Ad}_U^*\lambda, \mathcal{O}\rangle = i\mathrm{Tr}[|\psi_R\rangle\langle\psi_R|U^{-1}\mathcal{D}(\mathcal{O})U] = i\langle u|\mathcal{D}(\mathcal{O})|u\rangle$, which indeed remains unmodified by the stabilizing elements $U \in \mathcal{D}(H)$. Thus, we can view λ as a representative that selects the orbit G/H.

The MC form of the unitary U in Eq. (15) can be decomposed as $\Theta \equiv U^\dagger dU \equiv \Theta^- + \Theta^{(H)} + \Theta^+$ with $\Theta_\pm \in \mathfrak{n}_\pm$, $\Theta^{(H)} \in h_\mathbb{C}$. When acting with it on the base state, we obtain

$$\Theta|\psi_R\rangle = U^{-1}\left[\frac{d\mathcal{N}_H}{\mathcal{N}_H}U + \mathcal{N}_H d(e^{z_\alpha \mathcal{D}(E_{-\alpha})})\right]|\psi_R\rangle. \quad (17)$$

So $\Theta_-|\psi\rangle = (U^{-1}\mathcal{N}_H d e^{z_\alpha \mathcal{D}(E_{-\alpha})})_-|\psi\rangle$ and this only depends on dz_α and not on dz_α^*. Therefore, $\Theta^-|\psi_R\rangle = \Theta_\mu^- dz^\mu|\psi_R\rangle$ and by conjugation $\langle\psi_R|\Theta^+ = \langle\psi_R|\Theta_\mu^+ dz^{*\mu}$. Notice also that $\Theta^\dagger = -\Theta$ and therefore the FS metric (1b) becomes $ds^2_{\mathrm{FS}} = -\langle\psi_R|\Theta_\mu^+\Theta_\nu^-|\psi_R\rangle dz^{*\mu}dz^\nu$.

The metric has a manifest complex structure J compatible with the dagger, which maps z to $-iz$ and z^* to iz^*. Together, the metric and the complex structure define a closed 2-form according to $\omega(X, Y) = -g(X, JY)$, i.e.,

$$\begin{aligned}\omega &= -i\langle\psi_R|\Theta_\mu^+\Theta_\nu^-|\psi_R\rangle dz^{*\mu} \wedge dz^\nu \\ &= -i\langle\psi_R|\Theta \wedge \Theta|\psi_R\rangle = i\langle\psi_R|d\Theta|\psi_R\rangle. \end{aligned} \quad (18)$$

We recognize this as the Kirillov-Kostant symplectic form (13) through the representative (16).

Finally, the geometric action of the coadjoint orbit associated with the representative (16) relates to the $\mathcal{F}_1$ cost function (1a)

$$\mathcal{F}_1 d\sigma = |\langle\psi_R|U^\dagger dU|\psi_R\rangle| = |\langle\lambda, \Theta\rangle| = |\mathcal{A}_\lambda|. \quad (19)$$

For the specific case of the conformal algebra considered in Sec. III, we can take as base states the scalar primary states, $|\psi_R\rangle = |\Delta\rangle$. The stabilizing subalgebra is $\mathfrak{h} = \mathrm{so}(2)\times \mathrm{so}(d)$, generated by D and $L_{\mu\nu}$. The raising operators $\mathfrak{n}_+ = \{K_\mu\}$ annihilate highest weight states and the lowering operators are their conjugates $\mathfrak{n}_- = \{P_\mu\}$. Together these parametrize the coset (7).

Holography.—The symplectic geometry we found equally describes the space of timelike geodesics in anti–de Sitter (AdS) space, and this allows us to rigorously derive a bulk description of complexity. Explicitly, our circuits (3) starting from a scalar primary are mapped to the following particle trajectory in embedding coordinates in AdS_{d+1} of curvature radius R (following the conventions used in [72]):

$$\begin{aligned} X_0 &= r(t)\cos(t/R), \qquad X_{0'} = r(t)\sin(t/R), \\ X_\mu &= \frac{E_0 r(t)}{EA(\alpha, \alpha^*)}[\alpha_\mu B^*(t, \alpha^*) + \alpha_\mu^* B(t, \alpha)], \end{aligned} \quad (20)$$

where

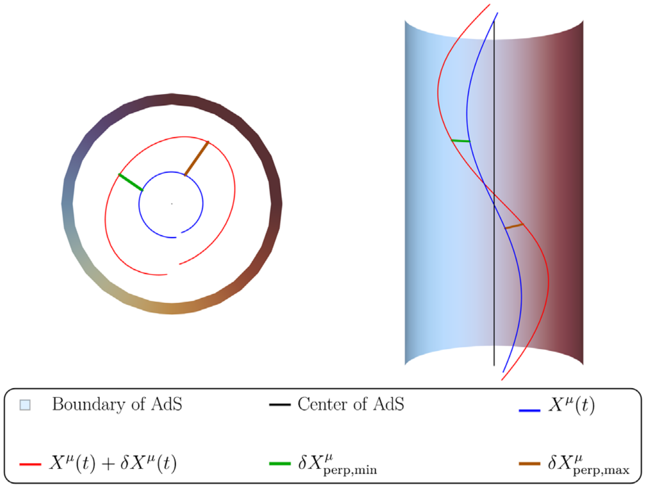

FIG. 1. Illustration of two nearby timelike geodesics in AdS_3 (blue, red) corresponding to two boundary circuits and the minimal (green) and maximal (brown) perpendicular distance between them. The infinitesimal variation was exaggerated to improve the visualization.

$$r(t) = \frac{RE}{E_0}\frac{\sqrt{A(\alpha,\alpha^*)}}{|B(t,\alpha)|}, \qquad E = E_0\frac{(1-\alpha^2\alpha^{*2})}{A(\alpha,\alpha^*)},$$
$$B(t,\alpha) = e^{it/R}\alpha^2 - e^{-it/R}. \tag{21}$$

Here, α parametrizes the phase space of the geodesics, and $A(\alpha,\alpha^*) > 0$ and $\alpha^2\alpha^{*2} < 1$. E is the energy of the massive particle, which is minimal at rest and equal to $E_0 = mR[1 + O(1/mR)]$, with m as the mass of the particle. The phase space is identical to that of the CFT_d with the identification $\Delta = E_0$. Time evolution $e^{i\tau D}|\alpha\rangle = |\alpha e^{i\tau}\rangle$ amounts to translating the geodesic in time in AdS_{d+1} and fixed radius geodesics correspond to $\alpha^2 = 0$. The complexity (9) is expressed in terms of the energy E and the angular momentum J of the massive particle through $\Omega_T^{S/A} = \sqrt{(E \pm J - \Delta/E \pm J + \Delta)}$ (see Supplemental Material, Sec. J [34]). For a circuit of circular geodesics starting at the origin and ending at a radius $r_T = R^2/\delta$ close to the boundary, the complexity diverges as $\mathcal{C}[\delta] \sim \sqrt{\Delta}\log[2R/\delta]$.

The FS metric over the space of circuits receives a surprisingly simple interpretation in terms of the maximal and minimal perpendicular distance between two infinitesimally nearby geodesics (as illustrated in Fig. 1; see [34], Sec. J)

$$ds^2_{\mathrm{FS}} = \frac{\Delta}{2R^2}(\delta X^2_{\mathrm{perp,min}} + \delta X^2_{\mathrm{perp,max}}). \tag{22}$$

Summary and outlook.—We studied the circuit complexity of trajectories associated with unitary representations of the conformal group in general dimensions. We considered primary states as reference states. Boundary states that are disentangled [73] could be an interesting alternative. Our gates, consisting of global conformal transformations, are nonlocal, similar to the gates relevant for holographic complexity [74]. We explained how our results can be understood using the geometry of coadjoint orbits. We presented general proofs relating the FS metric and $\mathcal{F}_1$ cost function to a coadjoint orbit metric and geometric action in the context of generalized coherent states. These proofs are also applicable to circuits starting from spinning primaries and to other symmetry groups.

Our complexity geometry does not provide a notion of distance between any two states in the CFT Hilbert space. It is an important question for the future to describe the complexity for circuits moving across different conformal families. Furthermore, considering more general states formed by nonlocal insertions could reveal the role of operator product expansion coefficients in studying complexity.

Considering the complexity of mixed states in CFT, e.g., thermal states or subregions of the vacuum, is another important question. For example, coherent states can be used as a starting point for the ensemble approach to mixed state complexity [75]. It is also interesting to explore the complexity of states with a conformal timelike defect or boundary and compare to holography [27,28,76].

The path-integral approach to complexity [77–82] involves the two-dimensional Liouville action and central charge. Hence, it relates to circuits going beyond the global conformal group. It is therefore compelling to study the $d > 2$ complexity of circuits constructed from general smearings of the stress tensor and tie the result to a higher-dimensional Liouville action [77,83].

Our complexity geometry is highly symmetric. It is interesting to break some of the symmetry by adding penalty factors—effectively favoring certain directions through the manifold of conformal unitaries. Our $\mathcal{F}_1$ cost function (1) (also considered in [77]) vanishes along certain nontrivial trajectories and differs from the $\mathcal{F}_1$ norm used when studying the complexity of Gaussian states (e.g., [14]). The difference is reminiscent of exchanging the order of the absolute value in the complexity definition and the sum over circuit generators. We intend to compare these two different definitions in the future.

We rigorously derived a bulk description of our circuits as trajectories between timelike geodesics in AdS space. We could connect this picture to the holographic complexity proposals, for instance, by exploring the influence of massive particles on the action. It is valuable to study generalizations of our circuit-geodesic duality in other spacetimes and for more than one (possibly spinning) particle. Further, it is important to explore the relation of our bulk picture to the phase space of Euclidean sources [84–87] and hence possibly to the complexity = volume proposal (see also [88]). Another compelling possibility is to connect our bulk picture to a parallel transport problem of timelike geodesics, similar to what was done for spacelike geodesics [89,90] in the context of kinematic space [48–51].

Complexity provides us with a new measure of entanglement in CFTs and it is interesting to probe its potential in diagnosing phase transitions. Some inspiration can be drawn from [91–95]. We hope to come back to this question in the future.

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