Complexity of conservative Constraint Satisfaction Problems

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In a constraint satisfaction problem (CSP) the aim is to find an assignment of values to a given set of variables, subject to specified constraints. The CSP is known to be NP-complete in general. However, certain restrictions on the form of the allowed constraints can lead to problems solvable in polynomial time. Such restrictions are usually imposed by specifying a constraint language, that is, a set of relations that are allowed to be used as constraints. A principal research direction aims to distinguish those constraint languages that give rise to tractable CSPs from those that do not.

We achieve this goal for the important version of the CSP, in which the set of values for each individual variable can be restricted arbitrarily. Restrictions of this type can be studied by considering those constraint languages which contain all possible unary constraints; we call such languages *conservative*. We completely characterize conservative constraint languages that give rise to polynomial time solvable CSP classes. In particular, this result allows us to obtain a complete description of those (directed) graphs H for which the LIST H-COLOURING problem is solvable in polynomial time. The result, the solving algorithm and the proofs heavily use the algebraic approach to CSP developed in [Jeavons et al. 1997; Jeavons 1998; Bulatov et al. 2005; Bulatov and Jeavons 2003; 2001b].

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1. INTRODUCTION

In a Constraint Satisfaction Problem (CSP) the aim is to find an assignment of values to a given set of variables, subject to constraints on the values that can be assigned simultaneously to certain specified subsets of variables. A CSP can also be expressed as the problem of deciding whether a given conjunctive formula has a

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model, or as the problem of deciding whether there exists a homomorphism between two relational structures.

The general CSP is NP-complete [Montanari 1974]. However, many practical and theoretical problems can be expressed in terms of CSP using constraints of a certain restricted form. Such restricted CSPs often are tractable (i.e., decidable in polynomial time). Constraints are usually specified by relations, and the constraint satisfaction problem can therefore be restricted by specifying a *constraint language*, that is, a set of relations that are allowed to be used as constraints. In the homomorphism form of the CSP this is equivalent to restricting the possible form of the target structure (or the *template*). The research problem of distinguishing those constraint languages that give rise to tractable problems from those which do not is acknowledged to be very important and has attracted much attention (see e.g., [Feder and Vardi 1998; Creignou et al. 2001; Bulatov 2002a]). We shall call this problem the *complexity classification problem.*

Boolean constraint satisfaction problem, in which two values are available, is equivalent to the propositional Generalized Satisfiability problem, and provides one of the most important particular cases of the CSP. In this case the complexity classification problem mentioned above is completely solved for the standard CSP [Schaefer 1978; Kolaitis and Vardi 2000a], and for various related problems (see [Creignou et al. 2001; Bulatov et al. 2003] for a survey and further references). In [Bulatov 2002a; 2006b], this problem was solved for CSPs over a 3-element set of values.

The analogous problem in the case of a bigger domain is believed to be very hard and to require more advanced approaches. Several such approaches based on logic, algebra, graph theory and databases have been developed [Hell and Nešetřil 1990; Jeavons et al. 1997; Feder and Vardi 1998; Jeavons 1998; Gottlob et al. 1999; Bulatov et al. 2005; Kolaitis and Vardi 2000a; Bulatov and Jeavons 2003; 2001b; Bulatov 2002a; 2006b; Dalmau 2002]. In spite of substantial progress achieved during the last decade, the problem has withstood all attacks.

Remarkably, all studied CSP classes turn out to be either tractable or NP-complete. Dichotomy results of this type are of particular interest in the study of CSP, because, on the one hand, they determine the precise complexity of a problem, and on the other hand, the a priori existence of a dichotomy result cannot be taken for granted. In [Feder and Vardi 1998], it was conjectured that a dichotomy theorem holds for the complexity of an arbitrary restricted CSP, and in [Bulatov et al. 2005], a possible criteria for tractable problems has been suggested.

In this paper we study the complexity classification problem for the important and widely used variant of the CSP, in which the set of values for each individual variable can be restricted arbitrarily. As is easily seen, the availability of this type of restrictions is equivalent to including into a constraint language all possible unary relations, or including into the target structure all possible unary predicates. A constraint language or a relational structure satisfying this condition is said to be *conservative.*

Constraint satisfaction problems related to some particular types of conservative constraint languages have already been studied. In [Cohen et al. 1994], a dichotomy theorem has been proved for such languages containing all permutation relations;

and in [Bulatov and Jeavons 2000], conservative constraint languages related to binary operations have been studied. In [Dalmau 2000], the result of [Cohen et al. 1994] was generalized to the class of constraint languages containing all permutation relations without assumptions on the unary relations. In this paper we generalize the results of [Cohen et al. 1994; Bulatov and Jeavons 2000].

We completely characterize conservative constraint languages that give rise to tractable constraint satisfaction problems, and provide a polynomial time algorithm (which appears to be quite non-trivial) solving the problem in the tractable cases. In particular, this algorithm allows one to solve numerous problems that cannot be solved by known algorithms. We also prove that in all other cases the constraint satisfaction problem is NP-complete.

It is somewhat unexpected that, despite the fact that our result is applicable to CSPs over sets of arbitrary size, the form of the criterion stated is quite similar to that found by Schaefer (see [Schaefer 1978]) for the GENERALIZED SATISFIABILITY problem.

We apply the main result to the H-COLOURING problem, an intensively studied combinatorial problem that can be naturally formulated as a CSP (see e.g. [Hell and Nešetřil 1990; Dyer and Greenhill 2000; Galuccio et al. 2000]. In the H-COLOURING problem the question is whether there exists a homomorphism of a given graph G to the fixed graph H. This problem is equivalent to the CSP for a constraint language consisting of one binary relation. The conservative version of H-COLOURING problem is called LIST H-COLOURING [Kratochvil and Tuza 1994; Feder and Hell 1998; Feder et al. 1999; 2003]. In this problem the homomorphism sought is subject to restrictions on possible images of vertices of the input graph. Therefore, LIST H-COLOURING is a particular case of the conservative CSP, and we obtain a complete description of those directed graphs H for which LIST H-COLOURING is solvable in polynomial time as a direct consequence of our result. However, we were unable to reformulate this description in graph theoretic terms.

Our result, the solving algorithm and proofs heavily use the algebraic approach to the CSP developed in [Jeavons et al. 1997; Jeavons 1998; Bulatov et al. 2005; Bulatov and Jeavons 2001b; 2003]. This method relies upon the fact that one can extract much information about the complexity of a restricted constraint satisfaction problem from knowing certain operations, called polymorphisms, related to the constraint language.

The paper is organized as follows. In Section 2.1, we give basic definitions and examples, and also pose the research problems we solve. Section 2.2 contains a short introduction to invariance properties of constraints. In Sections 2.3 and 2.4, we formulate the main result and apply it to the LIST H-COLOURING problem. Then, in Section 2.5, we recall a more general framework for CSP, *multi-sorted CSP*, and algebraic methods of studying CSPs of this form. This provides the technique used in this paper and also allows us to state our results in a more general form. In Section 2.6, we recall the notion of problems of bounded width and of bounded relational width and characterize those conservative CSPs that have bounded relational width and those conservative homomorphism problems that have bounded width. We outline the proof in Section 2.7. Sections 3–6 are devoted to considering particular types of conservative CSPs (for details, see Section 2.7) and auxiliary results. Finally, in Section 7 we describe a solving algorithm for tractable cases of the conservative CSP.

Due to space restrictions, several proofs are moved to the Electronic Appendix.

2. DEFINITIONS AND THE RESULT

2.1 Constraint Satisfaction Problem

Let A be a finite set. An *n-ary relation* on A is a set of n-tuples of elements from A; we use $\mathbf{R}_A$ to denote the set of all finitary relations on A. The components of an n-tuple $\mathbf{a}$ are denoted by $\mathbf{a}[1], \ldots, \mathbf{a}[n]$. A *constraint language* is a subset of $\mathbf{R}_A$, and may be finite or infinite.

Definition 2.1 *(Constraint Satisfaction Problem).* The *constraint satisfaction problem over a constraint language* $\Gamma \subseteq \mathbf{R}_A$, denoted CSP($\Gamma$), is defined to be the decision problem with instance $\mathcal{P} = (V; A; \mathcal{C})$, where

- V is a finite set of *variables*,
- A is a *set of values* (sometimes called a *domain*), and
- $\mathcal{C}$ is a set of *constraints*, in which each constraint $C \in \mathcal{C}$ is a pair $\langle s, R\rangle$ with s a list of variables of length m_C, called the *constraint scope*, and R an m_C-ary relation on A, belonging to Γ, called the *constraint relation.*

The question is whether there exists a *solution* to $\mathcal{P}$, that is, a mapping $\varphi\colon V \to A$ such that, for each constraint $\langle s, R\rangle$, $s = (v_1, \ldots, v_n)$, in $\mathcal{C}$, the image of the constraint scope $(\varphi(v_1), \ldots, \varphi(v_n))$ is a member of the constraint relation.

The size of a problem instance is defined to be the length of a reasonable encoding of a list of variables, and, for each constraint, of the constraint scope and of all the tuples in the constraint relation.

It was observed in [Feder and Vardi 1998] that the constraint satisfaction problem can be equivalently reformulated in the form of the homomorphism problem. Let $\mathfrak{H}$ be a class of relational structures. In the *uniform homomorphism problem* (HOM($\mathfrak{H}$)) *associated with* $\mathfrak{H}$, the question is, given a structure $\mathcal{H} \in \mathfrak{H}$ and a structure $\mathcal{G}$ over the same vocabulary as $\mathcal{H}$, whether there exists a homomorphism from $\mathcal{G}$ to $\mathcal{H}$. If $\mathfrak{H}$ consists of a single structure $\mathcal{H}$, then we write HOM($\mathcal{H}$) instead of HOM($\{\mathcal{H}\}$). We refer to such a problem as a *non-uniform homomorphism problem*, because the inputs are just source structures.

The intuition behind the mentioned equivalence between the constraint satisfaction problem and the homomorphism problem is that the source structure in the latter represents the variables and the constraint scopes, while the target structure represents the domain of values and the constraint relations. Moreover, the homomorphisms between the structures are precisely the solutions to the constraint satisfaction problem.

If a constraint language $\Gamma = \{R_1, \ldots, R_n\}$ on a set H is finite, then CSP(Γ) can be identified with the non-uniform problem HOM($\mathcal{H}$), where $\mathcal{H} = (H; R_1, \ldots, R_n)$. If Γ is infinite, then CSP(Γ) is equivalent to the uniform problem HOM($\mathfrak{H}$), where $\mathfrak{H}$ is the class of structures $\mathcal{H} = (H; R_1^{\mathcal{H}}, \ldots, R_n^{\mathcal{H}})$, $R_1^{\mathcal{H}}, \ldots, R_n^{\mathcal{H}} \in \Gamma$. (Since from now on we use the homomorphism setting of the CSP mostly in the non-uniform case, it will not cause a confusion to use the same notation for relation symbols and particular relations.) Non-uniform problems have been widely studied, see, e.g. [Feder and Vardi 1998; Jeavons et al. 1998; Jeavons 1998; Kolaitis and Vardi 2000b; 2000a; Kolaitis 2003]. Since the technique we use in this paper makes it more natural to deal with infinite constrain languages, we state our main results within the framework of the constraint satisfaction problem and, correspondingly, uniform homomorphism problems. However, some results can be stated in a stronger form for non-uniform homomorphism problems. We therefore use this formalism as well.

Example 2.2 *(*H-COLOURING*)*. Let H be a (directed) graph. In the H-COLOURING problem we are asked whether there is a homomorphism from a given graph G to H. So, the H-COLOURING problem is just the problem HOM(H).

Every (directed) graph $H = (V(H), E(H))$ corresponds to a binary relation R_H: $(a, b) \in R_H$ if and only if (a, b) is an edge of H. Thus every instance $G = (V(G); E(G))$ of the H-COLOURING problem corresponds to the instance $\big(V(G); V(H); \{\langle(a, b), R_H\rangle \mid (a, b) \in E(G)\}\big)$ of CSP($\{R_H\}$).

We shall be concerned with distinguishing between those constraint languages that give rise to *tractable* problems, and those which do not.

A relational structure $\mathcal{H}$ is said to be *tractable* if HOM($\mathcal{H}$) is tractable. It is said to be *NP-complete* if HOM($\mathcal{H}$) is NP-complete. A class of finite structures $\mathfrak{H}$ is said to be *tractable* if every $\mathcal{H} \in \mathfrak{H}$ is tractable. It is said to be *NP-complete* if there is an NP-complete $\mathcal{H} \in \mathfrak{H}$. Analogously, a constraint language Γ is said to be *tractable*, if CSP(Γ') is tractable for each finite subset $\Gamma' \subseteq \Gamma$. It is said to be *NP-complete*, if CSP(Γ') is NP-complete for some finite subset $\Gamma' \subseteq \Gamma$.

Note that algorithms solving CSP(Γ') for different finite subsets of a tractable constraint language Γ can be quite different. If there exists a uniform polynomial time algorithm for CSP(Γ), then we say that Γ is *globally tractable*. Clearly, every finite tractable constraint language is globally tractable. For classes of relational structures, global tractability is equivalent to the tractability of the corresponding uniform problem. In all known cases every tractable constraint language is also globally tractable.

Several different definitions of the CSP have appeared in literature; some of them are equivalent to that given above, and some are not. In this paper we concentrate on one of the standard variants of the CSP (see, e.g., [Dechter 2003]) which we will call the *conservative CSP*.

The *conservative constraint satisfaction problem over a constraint language* $\Gamma \subseteq \mathbf{R}_A$, denoted c-CSP($\Gamma$), is defined to be the decision problem with instance $\mathcal{P} = (V; A; \mathcal{L}; \mathcal{C})$, where $V, A, \mathcal{C}$ are defined as for the usual CSP, and $\mathcal{L}$ is a list of subsets L_v $(v \in V)$ from A. The question is whether there exists a solution φ to $\mathcal{P}$ defined as in Definition 2.1 and such that $\varphi(v) \in L_v$ for $v \in V$. For a relational structure $\mathcal{H}$, the LIST HOMOMORPHISM problem LHOM($\mathcal{H}$) is defined in a similar way.

Example 2.3 *(*LIST-H-COLOURING*)*. Let H be a (directed) graph. In the LIST H-COLORING problem we are given a graph G and, for each vertex v of G, a set L_v of vertices of H. The question is whether there is a homomorphism φ from G to H such that $\varphi(v) \in L_v$ for every vertex v of G. Clearly, LIST H-COLOURING can be represented in the form of the conservative CSP in a way similar to that in Example 2.2.

Notice that, for any constraint language $\Gamma \subseteq \mathbf{R}_A$, the problem c-CSP($\Gamma$) is equivalent to CSP($\Gamma \cup 2^A$), where 2^A denotes the set of all subsets from A. A constraint language Γ with $2^A \subseteq \Gamma$ will be called *conservative*. Analogously, the List Homomorphism problem LHOM($\mathcal{H}$) is equivalent to HOM($\mathcal{H}^+$), where $\mathcal{H}^+$ is obtained from $\mathcal{H}$ by adding unary relations R_B, for all $B \subseteq H$ (where H is the universe of $\mathcal{H}$) such that $R_B^{\mathcal{H}^+} = B$. A relational structure that contains all possible unary relations will also be called *conservative*. Therefore, we may get rid of conservative constraint satisfaction problems, and, instead, consider problems of the form CSP(Γ) [HOM($\mathcal{H}$)] for conservative constraint languages Γ [conservative relational structures $\mathcal{H}$].

The main research problem studied in this paper is.

Problem 2.4 (Complexity Classification problem). *Characterize tractable [globally tractable] and NP-complete conservative constraint languages.*

In the important particular case of Boolean CSP (i.e., Generalized Satisfiability problem), and in the case of the CSP over a 3-element domain, this problem has been completely solved [Schaefer 1978; Kolaitis and Vardi 2000a; Bulatov 2002a] for arbitrary constraint languages, not only conservative. Remarkably, in both cases a *dichotomy* has been proved: every problem of the form CSP(Γ) [HOM($\mathcal{H}$)] is either tractable or NP-complete. In [Feder and Vardi 1998], it was conjectured that this is the case for arbitrary constraint languages [relational structures].

Problem 2.5 (Dichotomy problem). *Is it true that every conservative constraint language is either tractable or NP-complete?*

2.2 Polymorphisms and tractability

The algebraic approach to the study of CSP uses, in addition to relations, arbitrary *operations* on the set of values. The set of all finitary operations on A (that is mappings $f: A^n \to A$, $n \in \mathbb{N}$) will be denoted by $\mathbf{O}_A$.

Any operation on A can be extended in a standard way to an operation on tuples over A, as follows. For any n-ary operation $f \in \mathbf{O}_A$, and any collection of tuples $\mathbf{a}_1, \ldots, \mathbf{a}_n \in A^m$, set

$$f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = (f(\mathbf{a}_1[1], \ldots, \mathbf{a}_n[1]), \ldots, f(\mathbf{a}_1[m], \ldots, \mathbf{a}_n[m])).$$

For any m-ary relation $R \in \mathbf{R}_A$, and any n-ary operation $f \in \mathbf{O}_A$, if $f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \in R$ for all choices of $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$, then R is said to be *invariant* under f, and f is called a *polymorphism* of R. We will often use the fact that a superposition of polymorphisms is a polymorphism, that is if $f(x_1, \ldots, x_n)$ and $g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_)$ are polymorphisms then

$$h(x_1, \ldots, x_k) = f(g_1(x_1, \ldots, x_k), \ldots, g_n(x_1, \ldots, x_k))$$

is also a polymorphism, and that if $f(x_1, \ldots, x_n)$ is a polymorphism of R then so is $f(x_{i_1}, \ldots, x_{i_n})$, where $x_{i_1}, \ldots, x_{i_n} \in \{x_1, \ldots, x_n\}$ (see, e.g. [Denecke and Wismath 2002]).

The set of all relations which are invariant under each operation from some set $C \subseteq \mathbf{O}_A$ is denoted by $\mathsf{Inv}(C)$. The set of all operations that are polymorphisms of every relation from some set $\Gamma \subseteq \mathbf{R}_A$ is denoted by $\mathsf{Pol}(\Gamma)$. Analogously, for a relational structure $\mathcal{H}$, an n-ary operation on the same universe is said to be a polymorphism of $\mathcal{H}$ if it is a polymorphism of every relation of $\mathcal{H}$. The set of all polymorphisms of $\mathcal{H}$ is denoted by $\mathsf{Pol}(\mathcal{H})$.

The operators Inv and Pol form a Galois correspondence between $\mathbf{R}_A$ and $\mathbf{O}_A$ (see Proposition 1.1.14 of [Pöschel and Kalužnin 1979]). A basic introduction to this correspondence can be found in [Pippenger 1997], and a comprehensive study in [Pöschel and Kalužnin 1979] and [Denecke and Wismath 2002]. The following result describes a connection between polymorphisms and the complexity of CSP.

PROPOSITION 2.6 [JEAVONS 1998]. *Let* Γ_1 *and* Γ_2 *be constraint languages over a finite set such that* Γ_2 *is finite. If* $\mathsf{Pol}(\Gamma_1) \subseteq \mathsf{Pol}(\Gamma_2)$ *then* $\mathrm{CSP}(\Gamma_2)$ *is polynomial time reducible to* $\mathrm{CSP}(\Gamma_1)$.

The proposition amounts to say that the complexity of $\mathrm{CSP}(\Gamma)$, for Γ finite, is completely determined by the polymorphisms of Γ. Moreover, this is true for all infinite constraint languages that have been studied.

A number of results on the complexity of constraint satisfaction problems have been obtained using this approach (e.g., [Jeavons et al. 1997; Jeavons 1998; Bulatov et al. 2005; Bulatov and Jeavons 2000; 2001b; Bulatov et al. 2001; Bulatov 2002a; 2002b; Dalmau 2002]. In particular, it was proved that certain types of operations yield tractability.

PROPOSITION 2.7. *If one of the following operations is a polymorphism of a constraint language* Γ *over a finite set* A*, then* $\mathrm{CSP}(\Gamma)$ *is tractable:*

- *a* semilattice *operation, that is a binary operation* f *satisfying the conditions: (a)* $f(a,a) = a$ *(idempotency); (b)* $f(a,b) = f(b,a)$ *(commutativity); (c)* $f(f(a,b),c) = f(a,f(b,c))$ *(associativity), for all* $a,b,c \in A$*;*
- *a* conservative commutative *operation, that is a binary operation* f *such that* $f(a,b) = f(b,a)$ *and* $f(a,b) \in \{a,b\}$*;*
- *a* majority *operation, that is a ternary operation* g *such that* $g(a,a,b) = g(a,b,a) = g(b,a,a) = a$*, for all* $a,b \in A$*.*
- *a* Mal'tsev *operation, that is a ternary operation* h *such that* $h(a,a,b) = h(b,a,a) = b$*, for all* $a,b \in A$*.*

Moreover, Schaefer's Dichotomy Theorem [Schaefer 1978], when appropriately restated, easily follows from Proposition 2.6, 2.7 and well known algebraic results [Post 1941] (see [Jeavons 1998]).

PROPOSITION 2.8 (SCHAEFER'S DICHOTOMY THEOREM). *For any* $\Gamma \subseteq \mathbf{R}_{\{0,1\}}$, $\mathrm{CSP}(\Gamma)$ *is tractable if* $\mathsf{Pol}(\Gamma)$ *contains one of the following:*

– *the constant* 0 *or constant* 1 *operations;*

– *the conjunction or disjunction operations (which are semilattice);*

– *the majority operation* $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$*;*

– *the Mal'tsev operation* $x - y + z \pmod 2$ *(it is also called the* affine *or* minority *operation).*

In all other cases $\mathrm{CSP}(\Gamma)$ *is NP-complete.*

Schaefer's Dichotomy Theorem completely solves Problems 2.4, 2.5 for Boolean constraint languages.

2.3 A dichotomy theorem for conservative constraint languages

First, we show that the NP-completeness part of our dichotomy theorem can be easily deduced from the existing results.

It has been observed [Jeavons et al. 1997; Jeavons et al. 1998; Bulatov et al. 2005] that there are two benchmark NP-complete problems that explain the NP-completeness of all known NP-complete CSPs. These are the problem CSP($\{R\}$),

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

in the Boolean case (the tuples are written vertically), which is equivalent to the Not-All-Equal SAT [Schaefer 1978], and CSP($\{\neq_A\}$), which is equivalent to the $|A|$-Colourability problem, in the non-Boolean case, where $\neq_A$ denotes the disequality relation $\{(a, b) \in A^2 \mid a \neq b\}$.

An operation f is a polymorphism of one of those relations if and only if f is an *essentially unary surjective operation*, that is $f(x_1, \ldots, x_n) = g(x_i)$ where g is a bijection [Jeavons et al. 1997; Jeavons et al. 1998]. Therefore, by Proposition 2.6, if every polymorphism of a constraint language Γ is an essentially unary surjective operation, then CSP(Γ) is NP-complete.

Note that every (n-ary) polymorphism f of a conservative constraint language satisfies the condition: $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$, for all $a_1, \ldots, a_n \in A$. Such an operation is said to be *conservative*. For example, the operations listed in Proposition 2.8, excluding the constant ones, are conservative. By $f|_B$ we denote the restriction of an operation f onto a set B. The following result from [Bulatov et al. 2005] provides a necessary condition for the tractability of CSP(Γ).

Proposition 2.9. *Let Γ be a constraint language on a finite set A, and suppose that $B \subseteq A$, $|B| \geq 2$, is a subset which, when treated as a unary relation, belongs to Γ. If for all $f \in \mathsf{Pol}(\Gamma)$, $f|_B$ is an essentially unary surjective operation, then* CSP(Γ) *is NP-complete.*

If Γ is a conservative language, then every $B \subseteq A$ belongs to Γ. Therefore, if CSP(Γ) is tractable then, for any 2-element subset $B \subseteq A$ (we assume $B = \{0, 1\}$), there exists a polymorphism f^B of Γ such that $f^B|_B$ is not an essentially unary surjective operation. Well-known properties of Boolean operations [Post 1941], imply that f^B can be chosen such that $f^B|_B$ is either a semilattice (that is conjunction or disjunction) operation, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, or the Mal'tsev operation $x - y + z \pmod 2$. Note that the constant operations are not in this list since Γ is conservative. The main result of this paper states that this property is also sufficient for the tractability of CSP(Γ).

Theorem 2.10. *A conservative constraint language Γ is tractable if and only if, for any 2-element subset $B \subseteq A$ (we assume $B = \{0, 1\}$), there exists an operation $f \in \mathsf{Pol}(\Gamma)$ such that $f|_B$ is either a semilattice operation $x \vee y$ or $x \wedge y$, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, or the Mal'tsev operation $x - y + z$* $\pmod 2$. *In this case Γ is also globally tractable. Otherwise Γ is NP-complete.*

In [Bulatov et al. 2005; Bulatov and Jeavons 2001b] a possible criterion has been conjectured characterizing tractable constraint language over a finite set. It can be shown that, the criterion stated in Theorem 2.10 is equivalent to that from [Bulatov et al. 2005; Bulatov and Jeavons 2001b] in the particular case of conservative constraint languages.

Theorem 2.10 completely solves the complexity classification problem (Problem 2.4), and gives a positive solution to the dichotomy problem (Problem 2.5). It also provides a polynomial time algorithm to recognize tractable conservative constraint satisfaction problems. Indeed, to verify the condition stated in the theorem for a finite conservative constraint language Γ [a relational structure $\mathcal{H}$], one just has to calculate all binary and ternary polymorphisms of Γ [respectively $\mathcal{H}$]. Their number is bounded by a constant when the size of A is fixed; and then, for each 2-element subset $B \subseteq A$, check all the computed polymorphisms, looking for an operation f such that $f|_B$ is of the specified type. In Section 7 we provide a polynomial time algorithm solving CSP(Γ) for a conservative Γ whenever it is globally tractable. For the homomorphism problem this implies the following

COROLLARY 2.11. *The uniform problem* HOM($\mathfrak{T}_A$)*, where* $\mathfrak{T}_A$ *denotes the class of all conservative relational structures that have universe* A *and satisfy the conditions of Theorem 2.10, is polynomial time solvable.*

In fact, Theorem 2.10 can be stated for a wider class of constraint languages than conservative languages. Suppose that Γ is a constraint language on A containing all 2- and 3-element unary relations (subsets of A). If the conditions of Theorem 2.10 do not hold then CSP($\{R\}$) is reducible to CSP(Γ) and, hence, the latter problem is NP-complete. Otherwise, let C denote the set of all binary and ternary polymorphisms of Γ. Then the conditions of Theorem 2.10 hold for the language $\mathsf{Inv}(C) \supseteq \Gamma$. However, if a binary or ternary operation is a polymorphism of all 3-element unary relations then it is also a polymorphisms of all unary relations. Therefore, $\mathsf{Inv}(C)$ is conservative and, by Theorem 2.10, is tractable.

A constraint language [relational structures] containing all at most k-element unary relations will be called *k-conservative.*

COROLLARY 2.12. *If* Γ *is a 3-conservative constraint language then* Γ *is globally tractable if and only if the conditions of Theorem 2.10 holds. Otherwise it is NP-complete.*

Corollary 2.12 has been observed independently in [Feder and Hell 2003].

2.4 LIST H-COLOURING problem

We apply Theorem 2.10 to characterize those digraphs H for which the LIST H-COLOURING problem is tractable. Such a characterization has been obtained in [Feder et al. 1999; 2003] for undirected graphs. For an undirected graph H, the LIST H-COLOURING is tractable if and only if H is *bi-arc*. Remarkably, bi-arc graphs can be described in terms of polymorphisms: a graph H is bi-arc if and only if R_H is symmetric and has a majority polymorphism [Feder et al. 2003; Brewster et al. 2008]. To our best knowledge, no progress has been made so far in studying the LIST H-COLOURING for directed graphs. Theorem 2.10 allows us to obtain a complete classification of the complexity of the LIST H-COLOURING problem in this case.

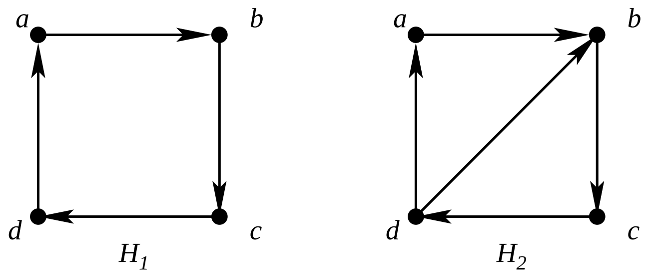


Figure 2.1

COROLLARY 2.13. *The* LIST H-COLOURING *problem for a digraph H is tractable if and only if R_H satisfies the conditions of Theorem 2.10. Otherwise it is NP-complete.*

Example 2.14. Consider the LIST H-COLOURING problems for the graphs shown in Fig. 2.1. It is not hard to see that $\mathsf{Pol}(\{R_{H_1}\})$ contains the operation of *dual discriminator*

$$g(x,y,z) = \begin{cases} x, & \text{if } x = y, \\ z, & \text{otherwise} \end{cases}$$

(it follows, for example, from results of [Szendrei 1986] and the fact that R_{H_1} is the graph of a mapping), which is a majority operation on on all pairs. By Corollary 2.13, the LIST H_1-COLOURING is tractable. Observe that $\mathsf{Pol}(\{R_{H_1}\})$ also contains a binary operation f, such that the restriction of f onto 2-element sets $\{a,b\},\{b,c\},\{c,d\},\{d,a\}$ is a semilattice operation. For example, one can define f by the equalities: $f(a,b) = b$, $f(b,c) = c$, $f(c,d) = d$, $f(d,a) = a$, and $f(b,d) = b$, $f(d,b) = d$, $f(a,c) = a$, $f(c,a) = c$. As we shall see later, this yields a certain 'type' of those 2-element sets.

In Section 3.1 we show that the LIST H_2-COLOURING is NP-complete.

The problem of describing the class of digraphs specified in Corollary 2.13 in graph-theoretic terms remains open.

2.5 Multi-sorted constraints satisfaction problem

We prove our results in a form more general than that in Theorem 2.10, namely, for the multi-sorted constraint satisfaction problem. In this generalized form, every variable is allowed to have its own domain. We follow the approach developed in [Bulatov and Jeavons 2001a; 2003].

In multi-sorted constraint satisfaction problems we allow *multi-sorted relations*. For any collection of sets $\mathcal{A} = \{A_i \mid i \in I\}$, and any list of indices $(i_1, i_2, \ldots, i_m) \in I^m$, a subset R of $A_{i_1} \times A_{i_2} \times \cdots \times A_{i_m}$, together with the list $(i_1, i_2, \ldots, i_m)$, will be called a *multi-sorted relation* over $\mathcal{A}$ with arity m and *signature* $(i_1, i_2, \ldots, i_m)$. For any such relation R, the signature of R will be denoted $\sigma(R)$. Any set of multi-sorted relations over $\mathcal{A}$ is called a *multi-sorted constraint language* over $\mathcal{A}$.

Definition 2.15 *(Multi-sorted CSP).* Let Γ be a multi-sorted constraint language over a collection of sets $\mathcal{A} = \{A_i \mid i \in I\}$. The multi-sorted constraint satisfaction

problem MCSP(Γ) is defined to be the decision problem with instance $(V; \mathcal{A}; \delta; \mathcal{C})$, where V is a set of variables; δ is a mapping from V to I, called the *domain function*; $\mathcal{C}$ is a set of constraints where each constraint $C \in \mathcal{C}$ is a pair $\langle s, R\rangle$, such that $s = (v_1, \dots, v_m)$ is a tuple of variables of length m, called the constraint scope; R is an m-ary relation over $\mathcal{A}$ with signature $(\delta(v_1), \dots, \delta(v_m))$, called the constraint relation.
The question is whether there exists a solution, i.e. a function φ, from V to $\bigcup_{A\in\mathcal{A}} A$, such that, for each variable $v \in V$, $\varphi(v) \in A_{\delta(v)}$, and for each constraint $\langle s, R\rangle \in \mathcal{C}$, with $s = (v_1, \dots, v_m)$, the tuple $(\varphi(v_1), \dots, \varphi(v_m))$ belongs to R.

Tractable and NP-complete multi-sorted constraint languages are defined in the same way as usual ones.

To extend the algebraic approach to the multi-sorted case, we need to define a suitable extension of the notion of a polymorphism. It is not hard to see that we cannot simply separate out different domains and consider polymorphisms on each domain separately; we must ensure that all of the domains are treated in a co-ordinated way. In the following definition, this is achieved by defining different interpretations for the same operation applied to different sets.

Let $\mathcal{A}$ be a collection of sets. An n-ary *multi-sorted operation* t on $\mathcal{A}$ is defined by a collection of *interpretations* $\{t^A \mid A \in \mathcal{A}\}$, where each t^A is an n-ary operation on the corresponding set A.

The multi-sorted operation t is said to be a *polymorphism* of an m-ary multi-sorted relation R over $\mathcal{A}$ with signature $(\delta(1), \dots, \delta(m))$ or R is said to be *invariant* with respect to t if, for any $(a_{11}, \dots, a_{m1}), \dots, (a_{1n}, \dots, a_{mn}) \in R$, we have

$$t\left(\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1}\end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn}\end{pmatrix}\right) = \begin{pmatrix} t^{A_{\delta(1)}}(a_{11}, \dots, a_{1n}) \\ \vdots \\ t^{A_{\delta(m)}}(a_{m1}, \dots, a_{mn})\end{pmatrix} \in R.$$

For a multi-sorted constraint language Γ, the set of all those multi-sorted operations which are polymorphisms of every relation in Γ is denoted $\mathsf{MPol}(\Gamma)$.

A multi-sorted constraint language Γ over $\mathcal{A}$ is said to be *conservative* [*k-conservative*] if for any $A \in \mathcal{A}$ and any $B \subseteq A$ [any $B \subseteq A$ with $|B| \le k$], $B \in \Gamma$ (as a relation over A). By technical reasons we shall also assume that $B \in \mathcal{A}$. In Sections 3–7 we prove the following theorem. Note that Theorem 2.10 is a particular case of this theorem when $\mathcal{A}$ is the collection of all subsets of a finite set.

THEOREM 2.16. *A multi-sorted 3-conservative constraint language Γ over a collection of sets $\mathcal{A}$ is tractable if and only if for any $A \in \mathcal{A}$ and any 2-element subset $B \subseteq A$ (we assume $B = \{0, 1\}$), there exists a multi-sorted operation $f \in \mathsf{MPol}(\Gamma)$ such that $f^A|_B$ is either a semilattice operation $x \vee y$ or $x \wedge y$, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$, or the Mal'tsev operation $x - y + z \pmod 2$. Otherwise* CSP(Γ) *is NP-complete. Moreover, if the size of the members of $\mathcal{A}$ is bounded, then Γ is tractable if and only if it is globally tractable.*

A weaker version of Theorem 2.16 (involving only finite constraint languages and providing no criterion for tractable cases) has been independently derived from Theorem 2.10 in [Feder and Hell 2003].

2.6 Partial solutions and bounded relational width

We recall a property of a subclass of the CSP that provides a polynomial time solving algorithm.

We use $\underline{n}$ to denote the set $\{1, \ldots, n\}$. For an n-ary (multi-sorted) relation R, $\mathbf{a} \in R$, and $J = \{i_1, \ldots, i_k\} \subseteq \underline{n}$, by $\mathrm{pr}_J \mathbf{a}$ we denote the tuple $(\mathbf{a}[i_1], \ldots, \mathbf{a}[i_k])$, and by $\mathrm{pr}_J R$ the set $\{\mathrm{pr}_J \mathbf{b} \mid \mathbf{b} \in R\}$. We sometimes will also write $\mathrm{pr}_{i_1,\ldots,i_k} R$ instead of $\mathrm{pr}_J R$. Clearly, $R \subseteq \mathrm{pr}_1 R \times \ldots \times \mathrm{pr}_n R$; we say that R is a *subdirect product* of $A_1, \ldots, A_n$ if $\mathrm{pr}_i R = A_i$ for any i.

It will often be convenient for us to consider relations whose coordinate positions are indexed by elements of a certain arbitrary set, not necessarily by natural numbers. For example, the coordinate positions of constraint relations will be supposed to be indexed by variables. Note that this might cause troubles as in general a variable may occur in a constraint scope more than once. However, it is not hard to see that any CSP can be transformed to an equivalent one without repetitions of variables in constraint scope. Indeed, to remove such a repetition from a constraint $C = \langle s = (v, v, \ldots), R\rangle$ we throw out from R all tuples $\mathbf{a}$ with $\mathbf{a}[1] \neq \mathbf{a}[2]$ obtaining a relation R' and replace C with $\langle s' = (v, \ldots), \mathrm{pr}_{s'} R'\rangle$. For CSPs without repetitions of variables it is safe to index coordinate positions with variables.

Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be an instance of the multi-sorted CSP, and $W \subseteq V$. By $\mathcal{P}_W$ we denote the *problem restricted on* W, that is, the problem instance defined as $(W; \mathcal{A}; \delta|_W; \mathcal{C}')$ where, for every $\langle s, R\rangle \in \mathcal{C}$, there is $\langle s', R'\rangle \in \mathcal{C}'$ with $s' = s \cap W$[1], and $R' = \mathrm{pr}_{s'} R$. Every solution to $\mathcal{P}_W$ is said to be a *partial solution* to $\mathcal{P}$ on W. Let us denote the set of all partial solutions on W by $\mathcal{S}_W$. Notice that $\mathcal{S}_W$ can be viewed as a $|W|$-ary relation.

The problem $\mathcal{P}$ is said to be *k-consistent* if, for any subsets $W \subseteq V$ containing $k-1$ elements and any subset U with $W \subseteq U \subseteq V$ containing k elements, every partial solution on W can be extended to a partial solution on U. The problem is called *k-minimal* if, for any k-element subset W of V, there is a constraint $\langle s, R\rangle \in \mathcal{C}$ such that $W \subseteq s$, and for any $\langle s, R\rangle \in \mathcal{C}$, we have $\mathrm{pr}_{s\cap W} R = \mathrm{pr}_{s\cap W} \mathcal{S}_W$.

Any problem instance $\mathcal{P}$ can be transformed to an equivalent k-consistent or k-minimal instance $\mathcal{P}'$. To do this we employ one of the standard constraint propagation algorithms in the former case and, in the latter case, the algorithm k-MINIMALITY, see Fig. 2.2. A class $\mathbf{K}$ of constraint satisfaction problems is said to be of *width k* if any problem instance $\mathcal{P}$ from $\mathbf{K}$ has a solution if and only if the k-consistent problem associated with $\mathcal{P}$ contains no constraint with empty constraint relation. If $\mathbf{K}$ is of width k for a certain k, then $\mathbf{K}$ is said to be of *bounded width*. Analogously, a class $\mathbf{K}$ of constraint satisfaction problems is said to be of *relational width k* if any problem instance $\mathcal{P}$ from $\mathbf{K}$ has a solution if and only if the k-minimal problem associated with $\mathcal{P}$ contains no constraint with empty constraint relation. If $\mathbf{K}$ is of relational width k for a certain k, then $\mathbf{K}$ is said to be of *bounded relational width*. Any class of bounded width or bounded relational width is tractable, because, assuming k fixed, establishing k-consistency and k-minimality takes polynomial time.

[1] This is a slight abuse of notation; formally speaking, the tuple s' is obtained from s by omitting those entries that are not contained in W.

Input. A problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.
Output. A k-minimal problem instance $\mathcal{P}' = (V; \mathcal{A}; \delta; \mathcal{C}')$ equivalent to $\mathcal{P}$.

Step 1 **set** $\mathcal{P}' := (V; \mathcal{A}; \delta; \mathcal{C}')$ where $\mathcal{C}' = \mathcal{C} \cup \{\langle W, \prod_{v \in W} A_{\delta(v)}\rangle \mid W \subseteq V,\ |W| = k\}$; and $\mathcal{P}'' := \mathcal{P}'$
Step 2 **do**
Step 3 **set** $\mathcal{P}' := \mathcal{P}''$
Step 4 **for each** $W \subseteq V$ with $|W| = k$ **do**
Step 5 **solve** the restricted problem $\mathcal{P}''_W$; let $\mathcal{S}''_W$ be the set of its solutions
Step 6 **for each** constraint $C = \langle s, R\rangle \in \mathcal{C}''$, **replace** C with $\langle s, R'\rangle$ where $R' = \{\mathbf{a} \in R \mid \mathrm{pr}_{s\cap W}\mathbf{a} \in \mathrm{pr}_{s\cap W}\mathcal{S}''_W\}$
Step 7 **until** $\mathcal{P}'' = \mathcal{P}'$
Step 8 **output** $\mathcal{P}'$

Fig. 2.2. Algorithm k-minimality

It is not hard to see that every k-minimal problem is also k-consistent. Therefore, every problem of width k has also relational width k; hence, every problem of bounded width is of bounded relational width. The converse, that is that every problem of bounded relational width has bounded width, is true in the important case of problems of the form CSP(Γ), where Γ is finite, or, equivalently, problems of the form HOM($\mathcal{H}$).

In the case of conservative constraint languages it is possible to characterize those languages that give rise to a problem of bounded relational width. We prove such a characterization only in the case when the number of allowed domains is finite (although a constraint language can be infinite). In spite of this, we believe that this result still holds in the general case. However, proving it in the general case would require a Galois theory for multi-sorted relations and operations, which is beyond the scope of the present paper.

Theorem 2.17. *Let Γ be a multi-sorted 3-conservative constraint language over a finite collection of sets $\mathcal{A}$. Then* MCSP(Γ) *has relational width 3 if and only if for any 2-element subset $B \subseteq A \in \mathcal{A}$ (we assume $B = \{0, 1\}$), there exists an operation $f \in \mathsf{MPol}(\Gamma)$ such that $f^A|_B$ is either a semilattice operation, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. Otherwise* MCSP($\Gamma$) *is not of bounded relational width.*

Proof. If Γ satisfies the conditions specified in the theorem then MCSP(Γ) is of relational width 3 by Lemma 7.2 (see Section 7).

Conversely, as MCSP(Γ) is tractable, it satisfies the conditions of Theorem 2.16. Suppose that there exists a 2-element $B \subseteq A \in \mathcal{A}$ (we denote the elements of B by 0,1) such that $f^A|_B$ is a semilattice or majority operation for no $f \in \mathsf{MPol}(\Gamma)$. It was shown in [Bulatov and Jeavons 2003] that if $\mathcal{A}$ is finite then MCSP(Γ) is polynomial time equivalent to CSP($\chi(\Gamma)$) where $\chi(\Gamma)$ is a one-sorted constraint language on $C = A_1 \times \cdots \times A_n$, where $\{A_1, \ldots, A_n\} = \mathcal{A}$ (we assume $A = A_1$). Moreover, it easily follows from the proof of Proposition 1 [Bulatov and Jeavons 2003] that MCSP(Γ) has relational width k for some k if and only if CSP($\chi(\Gamma)$) does. Let us fix some elements $a_i \in A_i$, $i \in \{2, \ldots, n\}$, and set $0' = (0, a_2, \ldots, a_n), 1' = (1, a_2, \ldots, a_n) \in C$. Then by the construction of $\chi(\Gamma)$, and the results of [Post 1941] and Proposition 2.6 (see also [Jeavons 1998]) the 4-ary relation R on $\{0', 1'\}$ defined by $R = \{(a, b, c, d) \in \{0', 1'\}^4 \mid a + b = c + d\}$, where $+$ denotes addition modulo 2, belongs to $\mathsf{InvPol}(\chi(\Gamma))$. The problem of solving systems of linear equations over the 2-element field is equivalent to CSP(R). However, this problem is not of bounded relational width; hence, by Lemma 3.1 of [Larose and Zadori 2007] the same holds for CSP(R) and therefore for CSP($\chi(\Gamma)$) and MCSP(Γ).[2] $\square$

COROLLARY 2.18. *Let $\mathcal{H}$ be a finite 3-conservative relational structure. Then* HOM($\mathcal{H}$) *has bounded width if and only if for any 2-element subset $B \subseteq \mathcal{H}$ (we assume $B = \{0, 1\}$), there exists an operation $f \in \mathsf{Pol}(\mathcal{H})$ such that $f|_B$ is either a semilattice operation, or the majority operation $(x \vee y) \wedge (y \vee z) \wedge (z \vee x)$. Otherwise* HOM($\mathcal{H}$) *is not of bounded width.*

2.7 Outline of the proof of Theorem 2.16

In this section we explain the main ideas behind the algorithm solving conservative CSPs and outline the auxiliary results needed to justify the algorithm.

Graph of a relational structure. First of all we associate with every 3-conservative relational structure $\mathcal{H}$ a complete graph whose vertices are the elements of the structure and the edges are coloured in 3 colours. The colour of an edge (a, b) reflects the local structure of $\mathcal{H}$, namely, the existence of polymorphisms whose action on $\{a, b\}$ is of some particular type. Thus, a red edge indicates that there is a polymorphism acting on $\{a, b\}$ as a semilattice operation, a yellow edge corresponds to majority polymorphism, and a blue edge corresponds to affine or minority polymorphisms. (For precise definitions see Section 3.1.)

Then we define several concepts of connectedness and strong connectedness in such graphs, based on the existence of paths consisting of edges of some particular colours. The most important types of connectedness will be red-connectedness (two vertices are red-connected if there is a "red" path connecting then), red-yellow-connectedness (there is a path consisting of red and yellow edges), and red-blue-connectedness. We also define, in the usual way, partial orders on the strongly red-, red-yellow-, and red-blue-connected components of the graph. Vertices from the maximal components in the sense of these orders we call maximal, ry-maximal, and rb-maximal elements, respectively.

Outline of the algorithm. The main idea of the algorithm is to decompose a CSP into several smaller problems depending on the graph structure of the domains involved in the problem. Usually, the smaller problems will be obtained by restricting domains to connected components of certain kind or to the sets of maximal elements, and also by restricting the set of variables included into the subproblem. The choice of variables to include is usually based on the interplay between possible values of the variables. Roughly speaking, we include variable w along with variable v only if we can conclude that, for any two solutions of the original CSP, the values of v in these solutions belong to the same connected component if and only if the values of w belong to the same connected component.

[2] Note that Lemma 4.1 of [Larose and Zadori 2007] allows one to conclude that CSP($\chi(\Gamma)$) has not bounded width without using relation R. However, that lemma uses non-trivial algebraic terminology that we would like to avoid here.

The key parts of the proof will be, first, the Tightening Proposition 7.1, claiming that if some of the smaller problems have no solutions then the original problem can be tightened by removing some elements from the domains, and then the algorithm is to be applied to the new smaller problem; and second, the results showing that if all the smaller problems have solutions then a solution to the entire problem can be constructed from solutions of the restricted problems. In order to prove these two results we will need a long series of auxiliary results that helps to understand the relationship between the properties of relations of conservative structures and the edge-coloured graphs of those structures.

The mentioned two key results look differently in different cases. We distinguish the following four cases.

Single coloured and double connected graphs. The base case of our algorithm, when no further iterations required, includes the case when the edges of the graphs of all structures involved coloured the same colour. In this case the problem can be solved by the existing algorithms from previous papers (see the Single Colour Proposition 3.3). Another case in which no further recursion is needed is the case when the graphs of all structures are both strongly red-blue- and strongly red-yellow-connected (Section 4). In this case, we use an algorithm similar to that from [Bulatov 2006a]. Note although that there may be a complication in this case when some further preparations are required. This possibility will be considered in Section 5.5 (the Double Connected Tightening Lemma 5.16).

Multiple red-blue-components. If some domains of the problem are not red-blue-connected, then we use red-blue-connected components to split it into subproblems. Suppose that domain A_v for some variable v is not red-blue-connected. Then, for any red-blue-connected component B of A_v, we define $J(\mathrm{rb}, v, B)$ to be the set of variables w such that it can be concluded that, for any solutions φ, ψ, we have $\varphi(v), \psi(v) \in B$ if and only if $\varphi(w), \psi(w)$ are in the same red-blue-connected component of A_w. The restricted problem for v and B has the set of variables $J(\mathrm{rb}, v, B)$ and red-blue-connected components of A_w, $w \in J(\mathrm{rb}, v, B)$, as domains. Finally, we use the Rectangularity Proposition 5.4(2) that claims that if the original problem has a solution assembled from some solutions of the restricted problems then any solutions of those restricted problems can be used to build a solution to the entire problem, and the Red-Blue Decomposition Proposition 5.7 that states that the CSP in this case can be solved as if it had a majority polymorphism, that is by considering the restrictions of the problem onto 2-element sets of variables.

Strongly red-blue connected domains. Due to the previous case we may assume that all the domains are red-blue-connected. Since the case when every domain is both red-blue- and red-yellow-connected is already considered, in the next case to consider, every red-connected component of every domain is strongly red-connected, but some domains are not red-yellow-connected. Using this property, we define, similar to the previous case, sets of the form $J(\mathrm{ry}, v, B)$ and the corresponding restricted problems, and we use the Rectangularity Proposition 5.4(1) to show that if the original problem has a solution assembled from some solutions of the restricted problems then any solutions of those restricted problems can be used to build a solution to the entire problem. Then we construct the *skeleton* problem leaving in the constraint relations only those tuples that are parts of some chosen solutions of the restricted problems. The key property of this new problem is that the domains of each variable in this problem has only blue edges and therefore can be solved by one the existing algorithms.

Isolating maximal components. As we shall see, red edges is the only type of edges that can be directed. This is why we consider the situation when there are vertices that are red-connected but not strongly red-connected as a separate case. And this is the most difficult part of the proof. The overall idea is again to restrict the problem using sets of variables of the form similar to $J(\mathrm{rb}, v, B), J(\mathrm{ry}, v, B)$, if one of such restricted problems has no solution then using the Tightening Proposition 7.1 we show that the domains of some variables can be reduced. Otherwise we distinguish the case when the set $A_v^{\max}$ of elements belonging to the maximal red-connected components of each domain A_v is red-yellow-connected. The All Minimal Lemma 6.20 shows that in this case the problem can be solved in the same way as in the case of multiple red-blue-components. Finally, if there are sets of the form $A_v^{\max}$ that are not red-yellow-connected, then the Maximal Rectangularity Proposition 6.11 shows that a property similar to that established in the Rectangularity Proposition 5.4(1) holds for $\max(A_v)$, and therefore the problem can be solved in a manner similar to that in the case of strongly connected domains.

Complexity and Soundness. We complete the proof by an analysis of the algorithm showing that it is correct (Proposition 7.3) and polynomial time (Proposition 7.4).

3. STRUCTURE OF RELATIONS FROM A CONSERVATIVE LANGUAGE

This section is mostly devoted to basic definitions and their properties. In Section 3.1, we formally introduce the edge-coloured graph associated with a relational structure that has already been mentioned in Section 2.7, prove that the colours of edges of this graph can be defined using only 3 polymorphisms of the structure (the Three Operations Proposition 3.1), and use this fact to show that the List H_2-Colouring problem, where H_2 is a digraph defined in Example 2.14, is NP-complete. Section 3.2 introduces various notions of connectedness in graphs of relational structures and also in graphs associated with multi-sorted relations. Finally, in Section 3.3, we study connected and strongly connected components of several types, partial orders on the sets of such components, and also the relationship between maximal components of the graph of a multi-sorted relation and maximal components of graphs of its components.

3.1 Red, yellow, and blue

Let Γ be a conservative constraint language on a collection of sets $\mathcal{A}$, satisfying the conditions of Theorem 2.16. Recall that, as Γ is conservative, we assume that along with every $A \in \mathcal{A}$ the collection $\mathcal{A}$ contains also every non-empty subset of A. For every $A \in \mathcal{A}$, we consider the graph $\mathcal{G}_\Gamma(A)$, an edge-coloured digraph with vertex set A. An edge (a, b) is present and is coloured *red* if there is $f_{a,b} \in \mathsf{MPol}(\Gamma)$ such that $f^A_{a,b}|_{\{a,b\}}$ is a semilattice operation with $f^A_{a,b}(a, b) = f^A_{a,b}(b, a) = f^A_{a,b}(b, b) = b$, $f^A_{a,b}(a, a) = a$. Edges $(a, b), (b, a)$ are present and are coloured *yellow* if neither (a, b)

nor (b,a) can be coloured red and there is $f^A_{a,b} \in \mathsf{MPol}(\Gamma)$ such that $f^A_{a,b}|_{\{a,b\}}$ is a majority operation. Edges $(a,b),(b,a)$ exist and are coloured *blue* if none of them can be coloured red or yellow, and there is $f_{a,b} \in \mathsf{MPol}(\Gamma)$ such that $f^A_{a,b}|_{\{a,b\}}$ is an affine operation. Thus, for each pair $a,b \in A$, either (a,b) or (b,a) is an edge of $\mathcal{G}_\Gamma(A)$; if (a,b) is a yellow or blue edge then (b,a) is also an edge of the same colour; while if (a,b) is red then the edge (b,a) may not exist. Since Γ is usually fixed, we shall use $\mathcal{G}(A)$ instead of $\mathcal{G}_\Gamma(A)$. First we show that all operations of the form $f_{a,b}$ can be considerably unified. Recall that an operation f on A is called a *projection* if there is $i \in \underline{n}$ such that $f(x_1,\dots,x_n) = x_i$ for any $x_1,\dots,x_n \in A$. It is called *idempotent* if $f(x,\dots,x) = x$ for any $x \in A$.

Proposition 3.1 (Three Operations Proposition). *There are polymorphisms $f(x,y), g(x,y,z), h(x,y,z) \in \mathsf{MPol}(\Gamma)$ such that, for every $A \in \mathcal{A}$ and every two-element subset $B \subseteq A$,*

—*$f^A|_B$ is a semilattice operation whenever B is red, and $f^A|_B(x,y) = x$ otherwise;*

—*$g^A|_B$ is a majority operation if B is yellow, $g^A|_B(x,y,z) = x$ if B is blue, and $g^A|_B(x,y,z) = f^A|_B(f^A|_B(x,y),z)$ if B is red;*

—*$h^A|_B$ is the affine operation if B is blue, $h^A|_B(x,y,z) = x$ if B is yellow, and $h^A|_B(x,y,z) = f^A|_B(f^A|_B(x,y),z)$ if B is red.*

There is also a polymorphism $p(x,y)$ such that $p^A|_B = f^A|_B$ if B is red, $p^A|_B(x,y) = y$ if B is yellow, and $p^A|_B(x,y) = x$ if B is blue.

Proof. The required operations are constructed from polymorphisms that are semilattice, majority, or affine operations on pairs of elements of sets from $\mathcal{A}$. Specifically, we first construct operation f by a chain of substitutions of polymorphisms whose restriction on some 2-element subset is a semilattice operation. Then g and h are constructed in a similar way. □

Example 3.2. We are now able to show that the problem List H_2-Colouring, where H_2 is the graph from Fig. 2.1, is NP-complete. If this problem were tractable then there would exist polymorphisms f,g,h of $R = R_{H_2}$ specified in the Three Operations Proposition 3.1. We show that the restrictions of these operations onto $B = \{b,d\}$ cannot be a semilattice, majority, or affine operation.

Notice first that if $f|_B$ is a semilattice operation with $f(b,d) = f(d,b) = d$ $[f(b,d) = f(d,b) = b]$ then so is $f|_{\{a,c\}}$ and $f(a,c) = f(c,a) = c$ $[f(a,c) = f(c,a) = a]$. Indeed, since f is a polymorphism of R and $f(b,d) = f(d,b) = d$, both

$$f\left(\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}c\\d\end{pmatrix}\right) \quad\text{and}\quad f\left(\begin{pmatrix}c\\d\end{pmatrix},\begin{pmatrix}a\\b\end{pmatrix}\right)$$

must be equal to (c,d), which means $f(a,c) = f(c,a) = c$. On the other hand, under the same assumption we must have

$$f\left(\begin{pmatrix}b\\c\end{pmatrix},\begin{pmatrix}d\\a\end{pmatrix}\right) = f\left(\begin{pmatrix}d\\a\end{pmatrix},\begin{pmatrix}b\\c\end{pmatrix}\right) = \begin{pmatrix}d\\a\end{pmatrix},$$

and $f(a,c) = f(c,a) = a$, a contradiction.

Analogously, if $g|_B$ is a majority operation then so are $g|_{\{b,c\}}, g|_{\{c,d\}}$. Indeed, if $g|_B$ is a majority operation then $g\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}\right) = \begin{pmatrix} d \\ g(c,b,b) \end{pmatrix} \in R_{H_2}$, which implies $g(c,b,b) \in \{a,b\}$. As g is conservative, $g(c,b,b) = b$. In a similar way one can check the remaining equalities required to show that $g|_{\{b,c\}}$ is a majority operation. A proof for the other restriction is similar.

Let us determine the possible values for $g(a,c,b)$. Since $g\left(\begin{pmatrix} d \\ a \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}\right) = \begin{pmatrix} d \\ g(a,c,b) \end{pmatrix} \in R_{H_2}$, we have $g(a,c,b) \in \{a,b\}$. If $g(a,c,b) = a$ then from $g\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}\right) = \begin{pmatrix} a \\ g(b,d,c) \end{pmatrix} \in R_{H_2}$ we get $g(b,d,c) = b$, from $g\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} d \\ a \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{pmatrix} b \\ g(c,a,d) \end{pmatrix} \in R_{H_2}$ that $g(c,a,d) = c$, from $g\left(\begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} d \\ a \end{pmatrix}\right) = \begin{pmatrix} c \\ g(d,b,a) \end{pmatrix} \in R_{H_2}$ that $g(d,b,a) = d$, and from $g\left(\begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}, \begin{pmatrix} d \\ a \end{pmatrix}\right) = \begin{pmatrix} g(c,d,d) \\ d \end{pmatrix} \in R_{H_2}$ that $g(c,d,d) = c$. The last equality gives a contradiction with the fact that $g|_{\{c,d\}}$ is a majority operation. The case $g(a,c,b) = b$ is similar, except that we use values $g(b,d,c), g(c,b,d), g(d,c,b)$, and $g(d,d,c)$. Thus an operation g with the required properties cannot exist.

Finally, in a similar way one can show that a polymorphism h such that $h|_B$ is a minority operation is also impossible.

Clearly, the constraint language $\mathsf{MInv}(\{f,g,h,p\}) \supseteq \Gamma$ satisfies the conditions of Theorem 2.16. Hence, the tractability of Γ will be proved if we show that $\mathsf{MInv}(\{f,g,h,p\})$ is tractable. So, we replace Γ with $\mathsf{MInv}(\{f,g,h,p\})$. Notice that $\mathcal{G}_{\mathsf{MInv}(\{f,g,h,p\})}(A)$, $A \in \mathcal{A}$, has no pair of mutually inverse red edges, because f uniquely determines the direction of a red edge.

To conclude this section we survey several particular cases when the colouring of $\mathcal{G}(A)$ is restricted.

Proposition 3.3 (Single Colour Proposition). *Let Γ be a conservative constraint language on a collection of sets $\mathcal{A}$. If one of the following conditions holds then Γ is tractable.*

(1) *All edges of $\mathcal{G}(A)$ for every $A \in \mathcal{A}$ are blue.*

(2) *All edges of $\mathcal{G}(A)$ for every $A \in \mathcal{A}$ are yellow.*

(3) *All edges of $\mathcal{G}(A)$ for every $A \in \mathcal{A}$ are red.*

In the first case, operation h from the Three Operations Proposition 3.1 is an affine operation on every two-element subset of each set from $\mathcal{A}$; hence it satisfies the conditions $h(x,y,y) = h(y,y,x) = x$. This means that h is a Mal'tsev operation. The tractability of Γ follows from [Bulatov 2002b; Bulatov and Dalmau 2006].

In the second case, g is a majority operation on every two-element subset of each set from $\mathcal{A}$; hence it satisfies the conditions $g(x,x,y) = g(x,y,x) = g(y,x,x) = x$. This means that g is a majority operation. Then the tractability of Γ follows from the results of [Jeavons et al. 1997; Jeavons et al. 1998; Bulatov and Jeavons 2003; 2001a].

Finally, in the third case, operation f is commutative and satisfies the condition $f(a,b) \in \{a,b\}$. Therefore, f is a commutative conservative binary operation. As is proved in [Bulatov and Jeavons 2000; Bulatov 2006a], Γ is tractable.

3.2 Connectedness

For an integer n, a partition I_1, I_2 of $\underline{n}$, and tuples $\mathbf{a}, \mathbf{b}$ whose components are indexed by elements of sets I_1, I_2 correspondingly, we use $(\mathbf{a}, \mathbf{b})$ to denote the n-ary tuple $\mathbf{c}$ such that $\mathbf{c}[i] = \mathbf{a}[i]$ if $i \in I_1$ and $\mathbf{c}[i] = \mathbf{b}[i]$ if $i \in I_2$. By $\langle \mathbf{c}, \mathbf{d} \rangle$ we denote the pair of tuples $\mathbf{c}, \mathbf{d}$ of the same arity.

We use several forms of connectedness in $\mathcal{G}(A)$, $A \in \mathcal{A}$, defined as the existence of a path (directed or not) consisting of edges of colours from some restricted set. Vertices a, b are said to be (*strongly*) *r-* [*ry-, rb-*]*connected* if there is a (directed) path from a to b consisting of red [red and yellow, red and blue] edges. We then define *r-* [*ry-, rb-*] *connected components*, and *strongly r-* [*ry-, rb-*]*connected components* in a natural way.

By a *strongly ry/rb-connected* component of $\mathcal{G}(A)$ we mean a maximal set of vertices that is both strongly ry- and strongly rb-connected. Observe that a strongly ry/rb-connected component is not necessarily a ry- (or rb-) connected component. Indeed, although such a component is strongly ry-connected, it may be not maximal strongly ry-connected set, as it is required for a strongly ry-connected component. Since any strongly ry/rb-connected component is a subset of some strongly ry- and some strongly rb-connected component, such a set can be defined as the first set, which is both strongly ry- and strongly rb-connected, in a sequence $B_1, C_1, B_2, C_2, \ldots$. Set B_1 to be a strongly ry-connected component of $\mathcal{G}(A)$ if $\mathcal{G}(A)$ is not strongly ry-connected, and A otherwise; set C_1 to be a strongly rb-connected component of $\mathcal{G}(A)$ if $\mathcal{G}(A)$ is strongly ry-connected, but not strongly rb-connected. Then set B_i to be a strongly ry-connected component of $\mathcal{G}(C_{i-1})$, and C_i to be a strongly rb-connected component of $\mathcal{G}(B_i)$.

It will be convenient for us to use 3 binary relations on each $A \in \mathcal{A}$. Let α, β, γ be reflexive binary relations defined as follows: α consists of red edges (and pairs of the form (a,a)), β consists of yellow edges (and the pairs (a,a)), and γ consists of blue ones (and the pairs (a,a)). These relations will mostly be used in the technical parts of the paper.

Every relation invariant with respect to f, g, h, that is from Γ, can be treated as an edge-coloured graph. Let R be a subdirect product of $A_1, \ldots, A_n \in \mathcal{A}$. The vertex set of the graph $\mathcal{G}(R)$ is R; a pair $\langle \mathbf{a}, \mathbf{b} \rangle$, $\mathbf{a}, \mathbf{b} \in R$ and $\mathbf{a} \neq \mathbf{b}$, is a red [yellow, blue] edge of $\mathcal{G}(R)$ if and only if, for each $i \in \underline{n}$, $(\mathbf{a}[i], \mathbf{b}[i])$ is a red [yellow, blue] edge of $\mathcal{G}(A_i)$ or $\mathbf{a}[i] = \mathbf{b}[i]$. (Strongly) connectedness in $\mathcal{G}(R)$ is defined in the same way as for $\mathcal{G}(A)$. We also introduce relations analogous to α, β, γ as follows: for $\mathbf{a}, \mathbf{b} \in R$, $\langle \mathbf{a}, \mathbf{b} \rangle \in \alpha$ [β, γ] iff $(\mathbf{a}[i], \mathbf{b}[i]) \in \alpha$ [β, γ], for all $i \in \underline{n}$. To simplify the notation we do not specify the set on which the relations α, β, γ are defined. Since this set is always clear from the context, this does not lead to a confusion.

If (a,b) is a red edge and $f(a,b) = b$, or if $a = b$ then we write $a \leq b$. For relations from Γ we use the same notation. We also use $\prec$ for the transitive closure of $\leq$.

We complete this section with several easy properties of operations f and p.

Lemma 3.4 (P-Lemma). *Let $R \in \Gamma$. Then for any $\mathbf{x}, \mathbf{y} \in R$*
(1) $\mathbf{x} \le f(\mathbf{x}, \mathbf{y})$;
(2) $\mathbf{x}$ is strongly rb-connected to $p(\mathbf{y}, \mathbf{x})$;
(3) $\mathbf{x}$ is strongly ry-connected to $p(\mathbf{x}, \mathbf{y})$.

Proof. (1) is straightforward.

(2) Let R be a subdirect product of $A_1, \ldots, A_n \in \mathcal{A}$. We set $\mathbf{a} = f(\mathbf{x}, \mathbf{y})$ and $\mathbf{b} = p(\mathbf{y}, \mathbf{a})$. By (1), $\mathbf{x} \le \mathbf{a}$, moreover, $\mathbf{a}[i] = \mathbf{x}[i]$ for all $i \in \underline{n}$ such that $\mathbf{x}[i] \not\le \mathbf{y}[i]$. Therefore, $\mathbf{b}[i] = \mathbf{a}[i]$ whenever $\mathbf{y}[i] \le \mathbf{x}[i] = \mathbf{a}[i]$, or $\mathbf{x}[i] \le \mathbf{y}[i] = \mathbf{a}[i]$, or $\langle \mathbf{y}[i], \mathbf{x}[i] \rangle \in \beta$. Hence $\langle \mathbf{a}, \mathbf{b} \rangle \in \gamma$, and $\mathbf{x}$ is strongly rb-connected to $\mathbf{b}$. Finally, noticing that $\mathbf{b} = p(\mathbf{y}, \mathbf{x})$ we get the result.

The proof of (3) is quite similar. □

A relation $R \subseteq R_1 \times \ldots \times R_n$, where for every $i \in \underline{n}$ the relation R_i is m_i-ary, such that $\mathrm{pr}_{I_i} R = R_i$, where $I_i = \{m_1 + \ldots + m_{i-1} + 1, \ldots, m_1 + \ldots + m_i\}$, is said to be a *subdirect product* of $R_1, \ldots, R_n$.

Lemma 3.5. *Let R be a subdirect product of strongly rb-connected relations R_1, R_2. If there exists an element $\mathbf{a} \in R_1$ such that $\{\mathbf{a}\} \times R_2 \subseteq R$, then $R = R_1 \times R_2$.*

Proof. We prove that $\{\mathbf{c}\} \times R_2 \subseteq R$ for every $\mathbf{c} \in R_1$. Let $C \subseteq R_1$ be the set of all $\mathbf{d} \in R_1$, for which the property holds. Since $\{\mathbf{a}\} \times R_2 \subseteq R$, $C \neq \varnothing$. To derive a contradiction, suppose that $C \neq R_1$. Since $\mathcal{G}(R_1)$ is strongly rb-connected, there are $\mathbf{d} \in C$, $\mathbf{c} \in R_1 - C$ such that $\mathbf{d} \le \mathbf{c}$ or $\langle \mathbf{d}, \mathbf{c} \rangle \in \gamma$. Set $B = \{\mathbf{b} \in R_2 \mid (\mathbf{c}, \mathbf{b}) \in R\}$. Again by the strongly rb-connectedness of $\mathcal{G}(R_2)$, there are $\mathbf{b} \in B$, $\mathbf{a} \in R_2 - B$ such that $\mathbf{b} \le \mathbf{a}$ or $\langle \mathbf{a}, \mathbf{b} \rangle \in \gamma$. If $\mathbf{d} \le \mathbf{c}$ or $\mathbf{b} \le \mathbf{a}$ then we get a contradiction, because

$$\begin{pmatrix}\mathbf{c}\\ \mathbf{a}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{d}\\ \mathbf{a}\end{pmatrix}, \begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix}\right) \in R \text{ if } \mathbf{d} \le \mathbf{c}, \quad \begin{pmatrix}\mathbf{c}\\ \mathbf{a}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix}, \begin{pmatrix}\mathbf{d}\\ \mathbf{a}\end{pmatrix}\right) \in R \text{ if } \mathbf{b} \le \mathbf{a}.$$

If $\langle \mathbf{d},\mathbf{c} \rangle, \langle \mathbf{b},\mathbf{a} \rangle \in \gamma$ then $\begin{pmatrix}\mathbf{c}\\ \mathbf{a}\end{pmatrix} = h\left(\begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix}, \begin{pmatrix}\mathbf{d}\\ \mathbf{b}\end{pmatrix}, \begin{pmatrix}\mathbf{d}\\ \mathbf{a}\end{pmatrix}\right) \in R$, a contradiction again. □

3.3 Orders, paths and maximal elements

In this subsection we introduce several concepts related to the natural partial order on the set of strongly connected components of a graph. We study basic properties of these concepts for graphs $\mathcal{G}(A)$, $\mathcal{G}(R)$, where A, R are the universe of a conservative relational structure and a subdirect product of such universes, respectively, which will be intensively used throughout the rest of the proof. Our main concern is the relationship between various maximality conditions for the graph $\mathcal{G}(R)$ and those for the graphs of factors of R.

Let $R \in \Gamma$. For strongly r- [ry-, rb] connected components B, C of $\mathcal{G}(R)$ we write $A \le B$ [$A \le_{\mathrm{ry}} B$, $A \le_{\mathrm{rb}} B$] if there are $\mathbf{b} \in B, \mathbf{c} \in C$ such that $\mathbf{b} \le \mathbf{c}$ [$\mathbf{b} \le \mathbf{c}$ or $\langle \mathbf{b}, \mathbf{c} \rangle \in \beta$, and $\mathbf{b} \le \mathbf{c}$ or $\langle \mathbf{b}, \mathbf{c} \rangle \in \alpha$]. Clearly, the family of strongly r- [ry-, rb-] connected component endowed with the transitive closure $\prec$ [$\prec_{\mathrm{ry}}$, $\prec_{\mathrm{rb}}$] of $\le$ [$\le_{\mathrm{ry}}$, $\le_{\mathrm{rb}}$] is a poset. We use $\le$ and $\prec$ instead of $\le_r$ and $\prec_r$, because in what follows $\le_{\mathrm{ry}}, \le_{\mathrm{rb}}, \prec_{\mathrm{ry}}, \prec_{\mathrm{rb}}$ are scarcely used, and we can simplify notation. The maximal elements of this poset will be called *r-maximal [ry-maximal, rb-maximal]* components. For every $\mathbf{a} \in R$ there is a r-maximal [ry-, rb-maximal] component C such that if $\mathbf{a}$ belongs to a strongly r- [ry-, rb-] connected component B then $B \prec C$

[respectively, $B \prec_y C$, $B \prec_b C$]. For $B \subseteq R$ we denote the *filter* $\{\mathbf{b} \in R \mid$ there is $\mathbf{a} \in B$ with $\mathbf{a} \prec \mathbf{b}\}$ by $\mathcal{F}(B)$. *Ry/rb-maximal* components of $\mathcal{G}(R)$ are defined in a similar way as strongly ry/rb-connected components of $\mathcal{G}(A)$, but instead of just any strongly ry- or rb-connected component to obtain the next member of the sequence we choose a maximal one. Formally, an *ry/rb-maximal* component of $\mathcal{G}(R)$ is the first set in a sequence $B_1, C_1, B_2, C_2, \ldots$ which is both strongly ry- and strongly rb-connected. Set B_1 to be an ry-maximal component of $\mathcal{G}(R)$ if $\mathcal{G}(R)$ is not strongly ry-connected and R_1 otherwise; set C_1 to be an rb-maximal component of $\mathcal{G}(R)$ if $\mathcal{G}(R)$ is strongly ry-connected, but not strongly rb-connected. Then set B_i to be an ry-maximal component of $\mathcal{G}(C_{i-1})$, and C_i to be an rb-maximal component of $\mathcal{G}(B_i)$.

Elements from an r-maximal [rb-maximal, ry-maximal] component of $\mathcal{G}(A)$, $A \in \mathcal{A}$, are called *r-maximal* [*rb-maximal, ry-maximal*] *elements*. An element $\mathbf{a}$ of $R \in \Gamma$ is called *r-maximal* [*rb-maximal, ry-maximal*] if it belongs to an r- [rb-, ry-]maximal component of $\mathcal{G}(R)$. In Section 5 we show that every element of R such that every its component is r-maximal [rb-maximal] is also r-maximal [rb-maximal].

A sequence $\mathbf{a}_1, \ldots, \mathbf{a}_k \in R$ is called an *r-path* if $\mathbf{a}_i \le \mathbf{a}_{i+1}$ for every i. It is called an *rb-path* if, for any i, either $\mathbf{a}_i \le \mathbf{a}_{i+1}$ or $\langle \mathbf{a}_i, \mathbf{a}_{i+1} \rangle \in \gamma$, and it is called an *ry-path* if, for any i, either $\mathbf{a}_i \le \mathbf{a}_{i+1}$ or $\langle \mathbf{a}_i, \mathbf{a}_{i+1} \rangle \in \beta$. First of all we show that if there is a path in a factor of a subdirect product, then it can be expanded to a path in the product.

LEMMA 3.6 (PATH EXPANSION LEMMA). *Let R be a subdirect product of relations R_1 and R_2, R_1 m-ary, and $(\mathbf{a}, \mathbf{b}) \in R$.*
(1) If $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \in R_1$ is an r-path, then there are $\mathbf{b} = \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k \in R_2$ such that $(\mathbf{a}_1, \mathbf{b}_1), \ldots, (\mathbf{a}_k, \mathbf{b}_k)$ is an r-path in R. Moreover, if $\mathbf{b}$ is r-maximal, then the $\mathbf{b}_i$ are also r-maximal and belong to the same r-maximal component.
(2) If $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \in R_1$ is an rb-path [ry-path], then there are $\mathbf{b} = \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{2k} \in R_2$ such that $(\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_1, \mathbf{b}_2), (\mathbf{a}_2, \mathbf{b}_3), \ldots, (\mathbf{a}_k, \mathbf{b}_{2k})$ is an rb-path [ry-path] in R. Moreover, if $\mathbf{b}$ is rb- [ry-] maximal, then the $\mathbf{b}_i$ are also rb- [ry-] maximal and belong to the same rb- [ry-]maximal component.

PROOF. (1) Let $\mathbf{c}_1, \ldots, \mathbf{c}_k \in R_2$ be such that $(\mathbf{a}_1, \mathbf{c}_1), \ldots, (\mathbf{a}_k, \mathbf{c}_k) \in R$, and $\mathbf{c}_1 = \mathbf{b}$. Then set $\mathbf{b}_1 = \mathbf{c}_1$, and $\begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_i\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_{i-1}\\ \mathbf{b}_{i-1}\end{pmatrix}, \begin{pmatrix}\mathbf{a}_i\\ \mathbf{c}_i\end{pmatrix}\right)$. By the P-Lemma 3.4(1), $(\mathbf{a}_{i-1}, \mathbf{b}_{i-1}) \le (\mathbf{a}_i, \mathbf{b}_i)$. Finally, if $\mathbf{b}$ is r-maximal, then, as $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathcal{F}(\mathbf{b})$, they all belong to the same r-maximal component.

(2) Let $\mathbf{c}_1, \ldots, \mathbf{c}_k \in R_2$ be such that $(\mathbf{a}_1, \mathbf{c}_1), \ldots, (\mathbf{a}_k, \mathbf{c}_k) \in R$, and $\mathbf{c}_1 = \mathbf{b}$. Then set $\mathbf{b}_1 = \mathbf{c}_1$. If $\mathbf{a}_i \le \mathbf{a}_{i+1}$ then we set $\begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_{2i}\end{pmatrix} = \begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_{2i-1}\end{pmatrix}$ and $\begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_{2i+1}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_{2i-1}\end{pmatrix}, \begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{c}_{i+1}\end{pmatrix}\right)$. If $\langle \mathbf{a}_i, \mathbf{a}_{i+1} \rangle \in \gamma$, then set $\begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_{2i}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_{i-1}\end{pmatrix}, \begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{c}_{i+1}\end{pmatrix}\right)$ and $\begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_{2i+1}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_i\\ \mathbf{b}_{2i}\end{pmatrix}, \begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{c}_{i+1}\end{pmatrix}\right)$. Making use of the P-Lemma 3.4(2), it is straightforward that the obtained tuples form an rb-path. Finally, if $\mathbf{b}$ is rb-maximal, then, as $\mathbf{b}_1, \ldots, \mathbf{b}_{2k}$ are strongly rb-connected with $\mathbf{b}$, they all belong to the same rb-maximal component.

For ry-paths the proof is quite similar. $\Box$

Our next goal is to prove that the components of an r-maximal tuple from a subdirect product are r-maximal elements. This goal will be achieved in the next two lemmas and a corollary. For a subdirect product $R_1, \dots, R_n$ we will use I_i to denote the set of coordinate positions of relation R_i.

LEMMA 3.7 (MAXIMALITY LEMMA). *Let R be a subdirect product of $R_1, \dots, R_n$.*
(1) For any $\mathbf{a} \in R$, there is $\mathbf{b} \in R$ such that $\mathbf{a} \prec \mathbf{b}$, tuple $\mathbf{b}$ is r-maximal in R, and, therefore, for any $j \in \underline{n}$, $\mathrm{pr}_{I_j}\mathbf{b}$ is r-maximal. Moreover, for any maximal $\mathbf{c} \in R_1$ with $\mathrm{pr}_{I_1}\mathbf{a} \prec \mathbf{c}$ $\mathbf{b}$ can be chosen such that $\mathrm{pr}_{I_1}\mathbf{b} = \mathrm{pr}_{I_1}\mathbf{c}$.
(2) For any $\mathbf{a} \in R$, there is $\mathbf{b} \in R$ such that $\mathbf{a} \prec_{\mathrm{rb}} \mathbf{b}$ [$\mathbf{a} \prec_{\mathrm{ry}} \mathbf{b}$] tuple $\mathbf{b}$ is rb- [ry-]maximal in R, and, therefore, for any $j \in \underline{n}$, $\mathrm{pr}_{I_j}\mathbf{b}$ is rb- [ry-] maximal. Moreover, for any rb- [ry-] maximal $\mathbf{c} \in R_1$ with $\mathrm{pr}_{I_1}\mathbf{a} \prec_{\mathrm{rb}} \mathbf{c}$ [$\mathrm{pr}_{I_1}\mathbf{a} \prec_{\mathrm{ry}} \mathbf{c}$]$\mathbf{b}$ can be chosen such that $\mathrm{pr}_{I_1}\mathbf{b} = \mathrm{pr}_{I_1}\mathbf{c}$.

PROOF. (1) We prove by induction on $j \le n$ that there is $\mathbf{b}_j$ such that $\mathbf{a} \prec \mathbf{b}_j$ and $\mathrm{pr}_{I_1}\mathbf{b}_j, \dots, \mathrm{pr}_{I_j}\mathbf{b}_j$ are r-maximal. In the base case for induction, $j = 0$, one may set $\mathbf{b}_0 = \mathbf{a}$.

Suppose that the claim is proved for $j-1$. Then there is a path $\mathrm{pr}_{I_j}\mathbf{b}_{j-1} = \mathbf{c}_1 \le \mathbf{c}_2 \le \dots \le \mathbf{c}_k$ such that $\mathbf{c}_k$ is r-maximal. By the Path Expansion Lemma 3.6(1), this path can be expanded to a path $\mathbf{b}_{j-1} = \mathbf{c}'_1 \le \mathbf{c}'_2 \le \dots \le \mathbf{c}'_k \in R$. Since, for any $i \le j-1$, the element $\mathrm{pr}_{I_i}\mathbf{b}_{j-1}$ is r-maximal and $\mathrm{pr}_{I_i}\mathbf{b}_{j-1} = \mathrm{pr}_{I_i}\mathbf{c}'_1 \le \mathrm{pr}_{I_i}\mathbf{c}'_2 \le \dots \le \mathrm{pr}_{I_i}\mathbf{c}'_k$, the element $\mathrm{pr}_{I_i}\mathbf{c}'_k$ is also r-maximal, and $\mathbf{b}_j$ can be chosen to be $\mathbf{c}'_k$. Finally, $\mathbf{b}_n$ is the required tuple.

If $\mathrm{pr}_{I_1}\mathbf{a}$ is maximal then $\mathrm{pr}_{I_1}\mathbf{a}$ and $\mathrm{pr}_{I_1}\mathbf{b}$ are in the same r-maximal component and therefore connected with an r-path $\mathrm{pr}_{I_1}\mathbf{b} = \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k = \mathrm{pr}_{I_1}\mathbf{a}$. Let $\mathbf{b} = \mathbf{b}'_1 \le \mathbf{b}'_2 \le \dots \le \mathbf{b}'_l$ be its expansion. Since $\mathrm{pr}_{I_j}\mathbf{c}$ is r-maximal and $\mathbf{c} \prec \mathbf{b}'_1, \dots, \mathbf{b}'_l$, the tuple $\mathbf{b}'_l$ is as required.

The proof for (2) is quite similar; we just use the Path Expansion Lemma 3.6(2) instead of part (1) of the same lemma. $\square$

The next lemma shows that any r-maximal element of a factor of a subdirect product can be expanded by r-maximal elements of the other factors.

LEMMA 3.8 (MAXIMAL EXPANSION LEMMA). *Let R be a subdirect product of $R_1, \dots, R_n$.*

(1) For any r-maximal element $\mathbf{d}$ from R_1, there is $\mathbf{a}$ such that $\mathrm{pr}_{I_1}\mathbf{a} = \mathbf{d}$ and, for any $j \in \underline{n}$, $\mathrm{pr}_{I_j}\mathbf{b}$ is an r-maximal element from R_j.
(2) For any rb- [ry-] maximal element $\mathbf{d}$ from R_1, there is an rb- [ry-] maximal $\mathbf{a}$ such that $\mathrm{pr}_{I_1}\mathbf{a} = \mathbf{d}$ and, for any $j \in \underline{n}$, $\mathrm{pr}_{I_j}\mathbf{b}$ is an rb- [ry-] maximal element of $\mathcal{G}(R_j)$.
(3) For any ry/rb-maximal element $\mathbf{d}$ from R_1 there exists $\mathbf{a} \in R$ such that $\mathrm{pr}_{I_1}\mathbf{a} = \mathbf{d}$ and, for any $j \in \underline{n}$, $\mathrm{pr}_{I_j}\mathbf{a}$ is ry/rb-maximal element of R_j.

PROOF. (1) and (2) follow straightforwardly from the Maximality Lemma 3.7(1),(2).

(3) To prove the result we construct a series of relations $R = R^0 \supseteq R^1 \supseteq \dots \supseteq R^s$ such that $\mathrm{pr}_{I_j}R^i$ is an ry-maximal component of $\mathcal{G}(\mathrm{pr}_{I_j}R^{i-1})$ if i is odd, and $\mathrm{pr}_{I_j}R^i$ is an rb-maximal component of $\mathcal{G}(\mathrm{pr}_{I_j}R^{i-1})$ if i is even. The relations will also be chosen such that $\mathbf{d} \in \mathrm{pr}_{I_1}R^i$. As $R^0 = R$ satisfies the conditions, let us

suppose that R^{i-1} is already defined. Then, if i is odd then, by the Maximality Lemma 3.7(2), there is $\mathbf{a} \in R^{i-1}$ such that $\mathrm{pr}_{I_1}\mathbf{a} = \mathbf{d}$ and $\mathrm{pr}_{I_j}\mathbf{a}$ belongs to an ry-maximal component B_j of $\mathcal{G}(\mathrm{pr}_{I_j} R^{i-1})$. Analogously, if i is even then, by the same result, there is $\mathbf{a} \in R^{i-1}$ such that $\mathrm{pr}_{I_1}\mathbf{a} = \mathbf{d}$ and $\mathrm{pr}_{I_j}\mathbf{a}$ belongs to an rb-maximal component C_j of $\mathcal{G}(\mathrm{pr}_{I_j} R^{i-1})$. We set

$$R^i = \begin{cases} \{\mathbf{b} \in R^{i-1} \mid \mathrm{pr}_{I_j}\mathbf{b} \in B_j, 2 \le j \le k\}, & \text{if } i \text{ is odd,} \\ \{\mathbf{b} \in R^{i-1} \mid \mathrm{pr}_{I_j}\mathbf{b} \in C_j, 2 \le j \le k\}, & \text{if } i \text{ is even} \end{cases}$$

By what was proved above $\mathrm{pr}_{I_j} R^i = B_j$ if i is odd and $\mathrm{pr}_{I_j} R^i = C_j$ if i is even.

For a certain s we get $R^s = R^{s+1}$. This means that, for any $j \ge 2$, $\mathcal{G}(\mathrm{pr}_{I_j} R^s)$ is both strongly ry- and strongly rb-connected. Moreover, $\mathrm{pr}_{I_j} R^s$ is an ry/rb-maximal component of $\mathcal{G}(\mathrm{pr}_{I_j} R)$. Since $\mathbf{d} \in \mathrm{pr}_{I_1} R^s$, the lemma is proved. $\square$

It is an easy exercise combining the Maximality Lemma 3.7 and the Maximal Expansion Lemma 3.8 to show that every maximal component of a subdirect product is a subdirect product of maximal components of the factors.

Corollary 3.9. *If R is a subdirect product of $R_1, \dots, R_n$, then every r- [rb-, ry-, ry/rb-] maximal component of $\mathcal{G}(R)$ is a subdirect product of r- [rb-, ry-, ry/rb-] maximal components of $\mathcal{G}(R_1), \dots, \mathcal{G}(R_n)$.*

Indeed, if, say, a tuple $\mathbf{a}$ is r-maximal, but $\mathrm{pr}_{I_1}\mathbf{a}$ is not, then there exists $\mathbf{b} \in R_1$ such that $\mathbf{a} \prec \mathbf{b}$ and $\mathrm{pr}_{I_1}\mathbf{b}$ is r-maximal. Clearly, $\mathbf{b} \not\prec \mathbf{a}$; a contradiction.

Unfortunately, Corollary 3.9 does not guarantee that any tuple of elements from maximal components also belongs to a maximal component of the subdirect product.

The last result of this section amounts to say that if one of the factors of a subdirect product is strongly connected, then any element from this factor can be expanded by an element from any maximal component of the second factor.

Lemma 3.10. *Let R be a subdirect product of relations R_1 and R_2.*
(1) If $\mathcal{G}(R_2)$ is strongly r-connected, then, for any r-maximal component B of $\mathcal{G}(R_1)$ and any $\mathbf{a} \in R_2$, there is $\mathbf{a}' \in B$ such that $(\mathbf{a}', \mathbf{a}) \in R$.
(2) If $\mathcal{G}(R_2)$ is strongly rb-connected, then, for any rb-maximal component B of $\mathcal{G}(R_1)$ and any $\mathbf{a} \in R_2$, there is $\mathbf{a}' \in B$ such that $(\mathbf{a}', \mathbf{a}) \in R$.
(3) If $\mathcal{G}(R_2)$ is strongly ry-connected, then, for any ry-maximal component B of $\mathcal{G}(R_1)$ and any $\mathbf{a} \in R_2$, there is $\mathbf{a}' \in B$ such that $(\mathbf{a}', \mathbf{a}) \in R$.
(4) If $\mathcal{G}(R_2)$ is strongly ry/rb-connected, then, for any ry/rb-maximal component B of $\mathcal{G}(R_1)$ and any $\mathbf{a} \in R_2$, there is $\mathbf{a}' \in B$ such that $(\mathbf{a}', \mathbf{a}) \in R$.

Proof. (1) Let $B \subseteq R_1$ be an r-maximal component and $D_B = \{\mathbf{a} \in R_2 \mid (\mathbf{a}', \mathbf{a}) \in R$ for a certain $\mathbf{a}' \in B\}$. Suppose the contrary, $D_B \neq R_2$. Since $\mathcal{G}(R_2)$ is strongly r-connected, there are $\mathbf{a} \in D_B$ and $\mathbf{b} \in R_2 - D_B$ such that $\mathbf{a} \le \mathbf{b}$. There also exist $\mathbf{d}, \mathbf{c} \in R_1$ with $(\mathbf{c}, \mathbf{b}), (\mathbf{d}, \mathbf{a}) \in R$ and $\mathbf{d} \in B$, $\mathbf{c} \in R_1 - B$. By the P-Lemma 3.4(1), we get $\begin{pmatrix}\mathbf{d}'\\ \mathbf{b}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{d}\\ \mathbf{a}\end{pmatrix}, \begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix}\right) \in R$ and $\mathbf{d}' \in B$. Hence $\mathbf{b} \in D_B$, a contradiction.

(2) Let $B \subseteq R_1$ be an rb-maximal component. Suppose that $D_B \neq R_2$. Since $\mathcal{G}(R_2)$ is strongly rb-connected, there are $\mathbf{a} \in D_B$ and $\mathbf{b} \in R_2 - D_B$ such that

$\mathbf{a} \le \mathbf{b}$ or $\langle \mathbf{a}, \mathbf{b}\rangle \in \gamma$. There also exist $\mathbf{d}, \mathbf{c} \in R_1$ with $(\mathbf{c}, \mathbf{b}), (\mathbf{d}, \mathbf{a}) \in R$ and $\mathbf{d} \in B$, $\mathbf{c} \in R_1 - B$. If $\mathbf{a} \le \mathbf{b}$, then we get $\begin{pmatrix}\mathbf{d}'\\ \mathbf{b}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{d}\\ \mathbf{a}\end{pmatrix}, \begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix}\right) \in R$, where $\mathbf{d}' = f(\mathbf{d}, \mathbf{c}) \in B$. Hence $\mathbf{b} \in D_B$, a contradiction. If $\langle \mathbf{a}, \mathbf{b}\rangle \in \gamma$ then $\begin{pmatrix}\mathbf{d}'\\ \mathbf{b}\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{c}\\ \mathbf{b}\end{pmatrix}, \begin{pmatrix}\mathbf{d}\\ \mathbf{a}\end{pmatrix}\right) \in R$, and, by the P-Lemma 3.4(2), $\mathbf{d}' = p(\mathbf{c}, \mathbf{d}) \in B$, a contradiction again.

For (3) the proof is quite similar.

(4) Let B be a strongly ry/rb-connected component resulted from the sequence $C_0 = B_0 = R_1, B_1, C_1, \dots, B_k, C_k = B$ where B_i is an ry-maximal component of $\mathcal{G}(C_{i-1})$, and C_i is a rb-maximal component of $\mathcal{G}(B_i)$. We prove by induction that for any $j \in \underline{k}$, $D_{B_j} = D_{C_j} = R_2$, where D_C for $C \subseteq R_1$ denotes the set $\{\mathbf{a} \in R' \mid (\mathbf{a}', \mathbf{a}) \in R \text{ for a certain } \mathbf{a}' \in C\}$. The base case for induction, $j = 0$, is obvious. Suppose that the required condition is proved for $j - 1$. Applying Lemma 3.10(2),(3) to $R \cap (C_{j-1} \times D_{C_{j-1}})$ and $R \cap (B_j \times D_{B_j})$, we get $D_{B_j} = R'$ and $D_{C_j} = R'$, as required. $\square$

4. DOUBLE-CONNECTED RELATIONS

In this section we consider subdirect products of sets from $\mathcal{A}$ that are both strongly ry-connected and strongly rb-connected. We show that such products satisfy some properties that can be used when solving corresponding constraint satisfaction problems. The main result of this section, the Hereditarily Double Connected Proposition 4.11, allows one to solve problems over hereditarily ry/rb-connected domains (for a definition see Section 4.3), and together with the Single Colour Proposition 3.3 constitutes the base case for our recursive algorithm. In some statements we will use weaker types of connectedness.

Some parts of this section are very close to certain parts of [Bulatov and Jeavons 2000; Bulatov 2006a]. A *subalgebra* of a relation R is any subset of R, which belongs to $\Gamma = \mathsf{MInv}(\{f, g, h\})$. If R is a unary relation, that is a set from $\mathcal{A}$, then every its subset is a subalgebra.

A *congruence* of an n-ary relation R is an equivalence relation on R invariant under f, g, h. Note that when talking about invariance properties we consider a congruence as a $2n$-ary relation. A congruence divides R up into subsets that are called *congruence classes*. It is also not hard to see that, since f, g, h are idempotent, every congruence class is a subalgebra. Indeed, if θ is a congruence of R and $\mathbf{a}, \mathbf{b}$ belong to the same class of θ then

$$f\left(\begin{pmatrix}\mathbf{a}\\ \mathbf{a}\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right) = \begin{pmatrix}\mathbf{a}\\ f(\mathbf{a}, \mathbf{b})\end{pmatrix} \in \theta,$$

that shows that $f(\mathbf{a}, \mathbf{b})$ belongs to the same class. In a similar way one can check that this class is also invariant under g and h. The following lemma is an easy observation.

LEMMA 4.1. *Let θ be a congruence of domain A.*
(1) If $\mathcal{G}(A)$ is (strongly) r- [ry-, rb-, ry/rb-] connected then so is $\mathcal{G}(A/_\theta)$.
(2) If B, C are different classes of θ and $b_1, b_2 \in B$, $c_1, c_2 \in C$ and $b_1 \le c_1$ [$c_1 \le b_1$, $\langle b_1, c_1\rangle \in \beta$, $\langle b_1, c_1\rangle \in \gamma$], then $b_2 \le c_2$ [$c_2 \le b_2$, $\langle b_2, c_2\rangle \in \beta$, $\langle b_2, c_2\rangle \in \gamma$].

Relation R is said to be *simple* if the equality relation and the total relation are the only congruences of R. The congruence class containing element $\mathbf{a}$ will be denoted by $\mathbf{a}^\theta$. A proper congruence of R maximal with respect to inclusion is called a *maximal congruence* of R. Notice that if R is a subdirect product of $A_1, \dots, A_n$, then a congruence can always be chosen in the following way. Take a congruence η of A_1 (if A_1 is simple then η is the equality relation) and set $\langle \mathbf{a}, \mathbf{b}\rangle \in \theta$ for $\mathbf{a}, \mathbf{b} \in R$ if and only if $\langle \mathbf{a}[1], \mathbf{b}[1]\rangle \in \eta$. Such a congruence is called a *projection congruence.* If η is chosen to be a maximal congruence of A_1, then θ is a maximal congruence of R called a *maximal projection congruence.* For a maximal projection congruence θ every θ-class corresponds to a class of η, and, thus, the set $R/_\theta$ of θ-classes can be viewed as the set $R_1/_\eta$ of classes of η. Therefore we may think of $R/_\theta$ as of a unary relation. We shall assume that such a relation belongs to $\mathcal{A}$.

Section 4.1, studies the structure of subdirect products of simple strongly rb-connected relations. The main result of that section, the Almost Trivial Proposition 4.7, claims that such products are almost trivial relations, that is direct products of graphs of bijective mappings. In order to prove this result we show that if such a subdirect product has no binary projections which are graphs of mappings then it is a direct product of its factors (the Simple Double Connected Lemma 4.2 for products of two relations, Lemma 4.5 for products of 3 relations, and Lemma 4.6 for the general case). In Section 4.2, we prove a similar result for a subdirect product, in which only one of the factors is a simple relation: Lemma 4.8 shows that such a relation is a direct product of its simple factor to the rest of the relation. This result is used in Section 4.3 to prove that CSPs over hereditarily ry/rb-connected domains (for a definition see Section 4.3) have relational width 3 (the Hereditarily Double Connected Proposition 4.11). Finally, in Section 5.5, we show these results help to handle CSPs whose domains are strongly ry/rb-connected, but not hereditarily ry/rb-connected.

4.1 Simple red-blue-connected relations

Our first goal is to show that a subdirect product of simple strongly r- [rb-, ry/rb-] connected relations is of a very restricted form. The graph of a mapping $\pi\colon A \to B$ is the binary relation $\{(a, \pi(a)) \mid a \in A\}$.

Let R be a subdirect product of $A_1, \dots, A_n \in \mathcal{A}$ and θ its projection congruence onto the first coordinate. If R is simple then θ is the equality relation, which means that there are $\pi_i\colon A_1 \to A_i$, $i \in \{2, \dots, n\}$, such that $R = \{(a, \pi_2(a), \dots, \pi_n(a)) \mid a \in A_1\}$. Therefore in a simple relation if, say, $\mathbf{a}[1] \le \mathbf{b}[1]$ then $\mathbf{a}[i] \le \mathbf{b}[i]$ for all coordinate positions i. Clearly the similar property holds for all other relations $=, \beta, \gamma$. Therefore, we always can treat a simple relation as a unary relation.

Lemma 4.2 (Simple Double Connected Lemma). *Let R be a subdirect product of simple domains R_1, R_2 such that $\mathcal{G}(R_1), \mathcal{G}(R_2)$ are strongly rb-connected. Then R is either the graph of a bijective mapping from R_1 to R_2, or $R_1 \times R_2$.*

Proof. Notice first, that if R is the graph of a mapping $\pi : R_1 \to R_2$, then the relation θ: $(\mathbf{a}, \mathbf{b}) \in \theta \iff \pi(\mathbf{a}) = \pi(\mathbf{b})$ is a congruence of R_1 and, since R_1 is simple, π is a bijection. The same holds if R is the graph of a mapping from R_2 onto R_1.

Suppose that R is neither $R_1 \times R_2$ nor the graph of a bijective mapping, and that $|R_1| + |R_2|$ is the smallest number such that there exists a subdirect product of simple strongly rb-connected relations with this property. We show that there is $\mathbf{b} \in R_1$ [or $\mathbf{b} \in R_2$] such that $\{\mathbf{b}\} \times R_2 \subseteq R$ [respectively, $R_1 \times \{\mathbf{b}\} \subseteq R$].

CLAIM 1. For any $D \subset R_1$ [$D \subset R_2$], there is $\mathbf{a} \in R_2$ [respectively, $\mathbf{a} \in R_1$] and $\mathbf{b} \in D$, $\mathbf{c} \in R_1 - D$ [respectively, $\mathbf{b} \in D$, $\mathbf{c} \in R_2 - D$] such that $(\mathbf{b}, \mathbf{a}), (\mathbf{c}, \mathbf{a}) \in R$ [respectively, $(\mathbf{a}, \mathbf{b}), (\mathbf{a}, \mathbf{c}) \in R$].

We consider the relation η on R_1 defined as follows: $(\mathbf{a}_1, \mathbf{a}_2) \in \eta$ if and only if there is $\mathbf{b}$ such that $(\mathbf{a}_1, \mathbf{b}), (\mathbf{a}_2, \mathbf{b}) \in R$. Obviously, η belongs to Γ and it is reflexive and symmetric. Since R is not the graph of a mapping, there are $(\mathbf{d}_1, \mathbf{e}), (\mathbf{d}_2, \mathbf{e}) \in R$ with $\mathbf{d}_1 \neq \mathbf{d}_2$, and η is not the equality relation. The transitive closure θ of η is an equivalence relation, and therefore it is a congruence of R_1. Since R_1 is simple, $\theta = R_1 \times R_1$.

Now, for any $D \subset R_1$ and any $\mathbf{a} \in R_1 - D$, $\mathbf{c} \in D$, we have $\langle \mathbf{a}, \mathbf{c} \rangle \in \theta$. Hence, there exists a sequence $\mathbf{a} = \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_k = \mathbf{c}$ such that $\langle \mathbf{b}_i, \mathbf{b}_{i+1} \rangle \in \eta$. For some i, $\mathbf{b}_i \in D$ and $\mathbf{b}_{i+1} \in R_1 - D$, and this pair satisfies the conditions of the claim.

For $\mathbf{a} \in R_1, \mathbf{b} \in R_2$ by $B_{\mathbf{a}}, C_{\mathbf{b}}$ we denote the sets $\{\mathbf{c} \mid (\mathbf{a}, \mathbf{c}) \in R\}$, $\{\mathbf{c} \mid (\mathbf{c}, \mathbf{b}) \in R\}$ respectively.

Take an arbitrary $\mathbf{a} \in R_1$, set $E_1 = \{\mathbf{a}\}$, and, for each $i > 0$, set

$$E_{i+1} = \begin{cases} \bigcup_{\mathbf{b} \in E_i} B_{\mathbf{b}} & \text{if } i \text{ is odd} \\ \bigcup_{\mathbf{b} \in E_i} C_{\mathbf{b}} & \text{if } i \text{ is even.} \end{cases}$$

By Claim 1, for each $i > 0$, $E_i \subset E_{i+2}$ unless $E_i = R_1$ or $E_i = R_2$. Therefore, for some $l > 1$, $E_l = R_1$ or $E_l = R_2$. Without loss of generality let $E_l = R_2$, and $E_{l-1} \neq R_1$, $E_{l-2} \neq R_2$.

By the mentioned property of simple relations, for each $i \in \underline{l}$, E_i is a subalgebra of R_1 or R_2.

Thus, E_{l-1} is a proper subalgebra of R_1 such that $\bigcup_{\mathbf{b} \in E_{l-1}} B_{\mathbf{b}} = R_2$.

Define a sequence $S_0, \dots, S_k$ of sets and a sequence of congruences $\theta_0, \dots, \theta_k$ where θ_i is a congruence of S_i through the following rules.

1) S_0 is a certain rb-maximal component of $\mathcal{G}(E_{l-1})$.

2) Suppose that S_i is already defined. Let θ_i be a maximal congruence of S_i or the identity relation if S_i is simple.

3) If S_i is a singleton, then $k = i$ and the process stops. Otherwise set S_{i+1} to be an rb-maximal component of a class of θ_i containing an element $\mathbf{b}$ with $|B_{\mathbf{b}}| > 1$ (as we shall prove later, such an element exists).

Further, set $S'_i = S_i/_{\theta_i}$ and

$$R^{(i)} = \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} = \mathbf{c}^{\theta_i} \text{ where } \mathbf{c} \in S_i,\ (\mathbf{c}, \mathbf{b}) \in R\} \subseteq S'_i \times R_2.$$

We prove that, for every i, (i) for any $\mathbf{b} \in R_2$ there exists $\mathbf{a} \in S_i$ such that $(\mathbf{a}, \mathbf{b}) \in R$, (ii) there exists $\mathbf{c} \in S_i$ such that $|B_{\mathbf{c}}| > 1$, and (iii) $R^{(i)} = S'_i \times R_2$.

If $i = 0$, then (i) holds by the choice of E_{l-1}, and Lemma 3.10. Furthermore, there is $\mathbf{a} \in E_{l-1}$ such that $|B_{\mathbf{a}}| > 1$. If $\mathbf{a}$ belongs to an rb-maximal component of $\mathcal{G}(E_{l-1})$ then (ii) can be satisfied. Otherwise let us suppose $R \cap (S_0 \times R_2)$ is the graph of a mapping $\pi\colon S_0 \to R_2$. Take $\mathbf{a} \in \mathbf{E}_{l-1} - S_0$ and $\mathbf{b}, \mathbf{c} \in R_2$ such that $(\mathbf{a}, \mathbf{b}), (\mathbf{a}, \mathbf{c}) \in R$. Since R_2 is strongly rb-connected we may assume that $\mathbf{b}, \mathbf{c}$ are such that $\mathbf{c} \leq \mathbf{b}$ or $\langle \mathbf{b}, \mathbf{c} \rangle \in \gamma$. Indeed, if $\mathbf{c} \neq f(\mathbf{c}, \mathbf{b})$ then $\mathbf{c} \leq f(\mathbf{c}, \mathbf{b})$ and $\mathbf{b}$

can be chosen to be $f(\mathbf{c},\mathbf{b})$. If $\mathbf{b} \neq f(\mathbf{b},\mathbf{c})$ then $\mathbf{b},\mathbf{c}$ can be chosen to be $f(\mathbf{b},\mathbf{c})$ and $\mathbf{b}$, respectively. In the remaining cases if $\langle \mathbf{b},\mathbf{c}\rangle \notin \gamma$ then $\langle \mathbf{b}, p(\mathbf{c},\mathbf{b})\rangle \in \gamma$, and again we can obtain a required pair choosing $\mathbf{c}$ to be $p(\mathbf{c},\mathbf{b})$. Let also $\pi^{-1}(\mathbf{c})$ denote any preimages of $\mathbf{c}$; then $(\pi^{-1}(\mathbf{c}),\mathbf{c}) \in R$. By the P-Lemma 3.4(2) the tuple $\mathbf{d} = p(\mathbf{a}, \pi^{-1}(\mathbf{c}))$ belongs to the same rb-maximal component as $\pi^{-1}(\mathbf{c})$. Hence,

$$\begin{pmatrix}\mathbf{d}\\ \mathbf{c}\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}\\ \mathbf{c}\end{pmatrix}, \begin{pmatrix}\pi^{-1}(\mathbf{c})\\ \mathbf{c}\end{pmatrix}\right) \in R, \quad \text{and} \quad \begin{pmatrix}\mathbf{d}\\ \mathbf{b}\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}, \begin{pmatrix}\pi^{-1}(\mathbf{c})\\ \mathbf{c}\end{pmatrix}\right) \in R.$$

Therefore, $R^{(0)}$ is not the graph of a mapping, and, since S'_0 is a simple relation and $|S'_0| + |R_2| < |R_1| + |R_2|$, we get $R^{(0)} = S'_0 \times R_2$.

Suppose that for $i-1$ the properties (i), (ii), (iii) hold. Then, for any $\mathbf{a}' \in S'_{i-1}$ we have $\{\mathbf{a}'\} \times R_2 \subseteq R^{(i-1)}$, that is, by Lemma 3.10(2), for every $\mathbf{b} \in R_2$ there exists $\mathbf{a} \in S_i$ such that $(\mathbf{a},\mathbf{b}) \in R$, that proves (i) for i. By (ii) for $i-1$ the θ_{i-1}-class D containing S_i contains an element $\mathbf{b}$ such that $|B_{\mathbf{b}}| > 1$. Arguing as in the previous paragraph $\mathbf{b}$ can be chosen to be in S_i. Finally, as S'_i is simple we have $R^{(i)} = S'_i \times R_2$.

We have proved $R^{(k)} = S'_k \times R_2$. Since S_k is a singleton, say, $S_k = \{\mathbf{b}\}$ this implies $S_k = S'_k$, that is $\{\mathbf{b}\} \times R_2 \subseteq R$.

To complete the proof we just apply Lemma 3.5. □

Observing that any strongly r- and ry/rb-connected graph is also strongly rb-connected we get the following

Corollary 4.3. *Let R be a subdirect product of simple relations R_1, R_2 such that $\mathcal{G}(R_1), \mathcal{G}(R_2)$ are strongly r-connected [strongly ry/rb-connected]. Then R is either the graph of a bijective mapping from R_1 to R_2, or $R_1 \times R_2$.*

In fact, the conditions of the Simple Double Connected Lemma 4.2 can be relaxed.

Corollary 4.4 (Semi-Simple Double Connected Corollary). *Let R be a subdirect product of relations R_1, R_2 where R_2 is simple and $\mathcal{G}(R_2)$ is strongly rb-connected [strongly r-, strongly ry/rb-connected] and R is not the graph of any mapping $\pi\colon R_1 \to R_2$. Then there exists $a \in R_1$ such that $\{a\} \times R_2 \subseteq R$.*

Proof. If R_1 is simple then we apply the Simple Double Connected Lemma 4.2. Otherwise the second part of the proof of Lemma 4.2 can be easily transformed. We use the same notation, S_i, θ_i, S'_i and $R^{(i)}$, but this time S_0 is an rb-maximal component of R_1. Note that we do not need the first part of the proof of Lemma 4.2. The only place to be changed is the proof of (ii). Again, suppose that $R \cap (S_0 \times R_2)$ is the graph of a mapping $\pi\colon S_0 \to R_2$. Let $\mathbf{a} \in R_1$ be such that $|B_{\mathbf{a}}| > 1$ and $\mathbf{b},\mathbf{c} \in B_{\mathbf{a}}$. Let also $\mathbf{b}' = \pi^{-1}(\mathbf{b}), \mathbf{c}' = \pi^{-1}(\mathbf{c})$, and let $\mathbf{d} \in R_2$ be such that $\mathbf{d} \le \mathbf{b}$ or $\langle \mathbf{b},\mathbf{d}\rangle \in \gamma$, and $\mathbf{d}' = \pi^{-1}(\mathbf{d})$. If $\mathbf{d} \neq \mathbf{c}$ we consider the tuple $\begin{pmatrix}\mathbf{e}'\\ \mathbf{b}\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}, \begin{pmatrix}\mathbf{d}'\\ \mathbf{d}\end{pmatrix}\right) \in R$. By the P-Lemma 3.4(2), $\mathbf{d}'$ is strongly rb-connected with $\mathbf{e}'$, therefore, $\mathbf{e}'$ belongs to the same rb-maximal component as $\mathbf{d}'$, and, hence, $\pi(\mathbf{e}') = \mathbf{b}$. Then $\begin{pmatrix}\mathbf{e}'\\ \mathbf{e}\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}\\ \mathbf{c}\end{pmatrix}, \begin{pmatrix}\mathbf{d}'\\ \mathbf{d}\end{pmatrix}\right) \in R$. As R_2 is simple, $\mathbf{e} \in \{\mathbf{c},\mathbf{d}\}$. However, $\mathbf{e} = \pi(\mathbf{e}') = \mathbf{b}$, a contradiction.

If $\mathbf{d} = \mathbf{c}$, and so $\mathbf{c} \le \mathbf{b}$ or $\langle \mathbf{c}, \mathbf{b} \rangle \in \gamma$, then

$$\begin{pmatrix} p(\mathbf{a}, \pi^{-1}(\mathbf{c}) \\ \mathbf{c} \end{pmatrix} = p\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} \pi^{-1}(\mathbf{c}) \\ \mathbf{c} \end{pmatrix}\right), \begin{pmatrix} p(\mathbf{a}, \pi^{-1}(\mathbf{c})) \\ \mathbf{b} \end{pmatrix} = p\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} \pi^{-1}(\mathbf{c}) \\ \mathbf{c} \end{pmatrix}\right) \in R,$$

a contradiction again. □

Our next step is to prove a similar property of ternary relations (subdirect products of three simple relations). As we shall use the results of this subsection in Section 5, we prove them in a slightly stronger form than it is necessary for the case of ry/rb-connected domains.

LEMMA 4.5. *Let R be a subdirect product of simple relations R_1, R_2, R_3 such that their graphs are strongly rb-connected and each of them contains a red or yellow edge. If $R_i \times R_j \subseteq \mathrm{pr}_{i,j} R$ for every $i, j \in \{1, 2, 3\}$, then $R = R_1 \times R_2 \times R_3$.*

PROOF. As all the relations are simple, we may assume them to be unary. Suppose without lost of generality that $|R_1| \le |R_2| \le |R_3|$. For $\mathbf{a} \in R_1$ set $R_{\mathbf{a}} = \{(\mathbf{b}_2, \mathbf{b}_3) \mid (\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3) \in R\}$. Notice that, for every $\mathbf{a} \in R_1$, $R_{\mathbf{a}}$ is a subalgebra of $\mathrm{pr}_{2,3} R$, and, since $\mathrm{pr}_{1,2} R = R_1 \times R_2$, $\mathrm{pr}_{1,3} R = R_1 \times R_3$, the relation $R_{\mathbf{a}}$ is a subdirect product of R_2, R_3. By the Simple Double Connected Lemma 4.2, $R_{\mathbf{a}}$ is either the graph of a bijective mapping or $R_2 \times R_3$.

Let us assume $R_{\mathbf{a}} = R_2 \times R_3$ for a certain $\mathbf{a} \in R_1$. Since $R_2 \times R_3$ and R_1 are strongly rb-connected, by Lemma 3.5, $R = R_1 \times R_2 \times R_3$.

Now suppose that, for every $\mathbf{a} \in R_1$, the set $R_{\mathbf{a}}$ is the graph of a bijective mapping $\pi_a: R_2 \to R_3$. This immediately implies $|R_2| = |R_3|$, let us denote this number by k, and as $\mathrm{pr}_{2,3} R = R_2 \times R_3$, there are at least k different relations of the form $R_{\mathbf{a}}$. Therefore, $|R_1| = k$ and $|R_{\mathbf{a}}| = k$ for any $\mathbf{a} \in R_1$. Moreover, $|\mathrm{pr}_{2,3} R| = k^2$, which means $R_{\mathbf{a}} \cap R_{\mathbf{a}'} = \varnothing$ whenever $\mathbf{a} \ne \mathbf{a}'$, $\mathbf{a}, \mathbf{a}' \in R_1$. The equivalence relation $\sim$ on $\mathrm{pr}_{2,3} R$ where $(\mathbf{a}, \mathbf{b}) \sim (\mathbf{c}, \mathbf{d})$ iff $(\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d}) \in R_{\mathbf{e}}$ for some $\mathbf{e} \in R_1$, is a congruence of $\mathrm{pr}_{2,3} R = R_2 \times R_3$.

Since there is a bijective mapping between R_2 and R_3, $R_2 \times R_3$ can be treated as a subalgebra of $R_2 \times R_2$. An element $\mathbf{a}$ of a relation S is said to be *absorbing* if whenever $t(x, y_1, \ldots, y_n)$ is an $(n+1)$-ary polymorphism of Γ such that t depends on x and $(\mathbf{b}_1, \ldots, \mathbf{b}_n) \in S^n$, then $t(\mathbf{a}, \mathbf{b}_1, \ldots, \mathbf{b}_n) = \mathbf{a}$. A congruence θ of S^2 is said to be *skew* if it is not a projection congruence, that is the kernel of no projection mapping of S^2 onto its factors. R_2 is a simple and all polymorphisms of Γ are idempotent, therefore, the results of [Kearnes 1996] can be rephrased in such a way that one of the following holds: (a) there is a finite ring K and an operation $+$ of an Abelian group on R_2 such that every polymorphism of Γ on R_2 can be represented as an operation of a module with group operation $+$ and ring K; (b) R_2 has an absorbing element; or (c) R_2^2 has no skew congruence. Case (a) is impossible, because R_2 contains a red or yellow edge and therefore has a 2-element subalgebra with a semilattice or majority term operation, but no module has such a subalgebra. If in case (b) $\mathbf{a}$ is an absorbing element, then $f(\mathbf{a}, \mathbf{b}) = \mathbf{a}$ for any $\mathbf{b} \in R_2$ that contradicts strongly rb-connectedness. Finally, case (c) is also impossible, because $\sim$ is a skew congruence. □

LEMMA 4.6. *Let R be a subdirect product of simple relations $R_1, \ldots, R_n$ such that their graphs are strongly rb-connected and each of them contains a red or yellow* *edge. If* $R_i \times R_j \subseteq \mathrm{pr}_{i,j} R$ *for every* $i, j \in \underline{n}$*, then* $R = R_1 \times \ldots \times R_n$*.*

PROOF. We prove the lemma by induction. The induction base $n = 2, 3$ is proved in Lemmas 4.2 and 4.5. Suppose that the lemma holds for each number less than n. Take $\mathbf{a} \in R_1$ and denote by $R_{\mathbf{a}}$ the set $\{(\mathbf{b}_2, \ldots, \mathbf{b}_n) \mid (\mathbf{a}, \mathbf{b}_2, \ldots, \mathbf{b}_n) \in R\}$. By Lemma 4.5, $R_1 \times R_i \times R_j \subseteq \mathrm{pr}_{1,i,j} R$ for every $2 \le i, j \le n$. Then $R_i \times R_j \subseteq \mathrm{pr}_{i,j} R_{\mathbf{a}}$ and by induction hypothesis $R_{\mathbf{a}} = R_2 \times \ldots \times R_n$. As this holds for any $\mathbf{a} \in R_1$, the lemma is proved. □

A relation $R \subseteq R_1 \times \ldots \times R_n$ is said to be *almost trivial* if there exists a partition of the set $\{1, \ldots, n\}$ with classes $I_1, \ldots, I_k$, such that $R = \mathrm{pr}_{I_1} R \times \ldots \times \mathrm{pr}_{I_k} R$ where $\mathrm{pr}_{I_j} R = \{(\mathbf{a}_{i_1}, \pi_{i_2}(\mathbf{a}_{i_1}), \ldots, \pi_{i_l}(\mathbf{a}_{i_1})) \mid \mathbf{a}_{i_1} \in R_{i_1}\}$, $I_j = \{i_1, \ldots, i_l\}$, for certain bijective mappings $\pi_{i_2} \colon R_{i_1} \to R_{i_2}, \ldots, \pi_{i_l} \colon R_{i_1} \to R_{i_l}$.

PROPOSITION 4.7 (ALMOST TRIVIAL PROPOSITION). *A subdirect product of simple relations such that their graphs are strongly rb-connected and each of them contains a red or yellow edge is an almost trivial relation.*

PROOF. Let R be a subdirect product of simple strongly rb-connected relations $R_1, \ldots, R_n$. We prove the proposition by induction on n. For $n = 1$ the result holds trivially.

We now prove the induction step. By Lemma 4.2, for any pair $i, j \in \underline{n}$ the projection $\mathrm{pr}_{i,j} R$ is either $R_i \times R_j$, or the graph of a bijective mapping. Assume that there exist i, j such that $\mathrm{pr}_{i,j} R$ is the graph of a mapping $\pi \colon R_i \to R_j$. By induction hypothesis $\mathrm{pr}_{\underline{n}-\{j\}} R$ is almost trivial, and therefore can be represented in the form $\mathrm{pr}_{\underline{n}-\{j\}} R = \mathrm{pr}_{I_1} R \times \ldots \times \mathrm{pr}_{I_k} R$ where $I_1 \cup \ldots \cup I_k = \underline{n} - \{j\}$. Suppose, for simplicity, that i is the coordinate position in I_1, that is,

$$\mathrm{pr}_{I_1} R = \{(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}, \mathbf{a}_i) \mid \mathbf{a}_{i_1} \in R_{i_1},\ \mathbf{a}_{i_s} = \pi_{s1}(\mathbf{a}_{i_1}) \text{ for } s \in \{2, \ldots, k\},\ \mathbf{a}_i = \pi_i(\mathbf{a}_{i_1})\}.$$

Then

$$\begin{aligned} \mathrm{pr}_{I_1 \cup \{j\}} R = \{ & (a_{i_1}, \ldots, a_{i_k}, a_i, a_j) \mid a_{i_1} \in R_{i_1},\ a_{i_s} = \pi_{s1}(a_{i_1}) \\ & \text{for } s \in \{2, \ldots, k\},\ a_i = \pi_i(a_{i_1}),\ a_j = \pi\pi_i(a_{i_1})\}, \end{aligned}$$

and we have $R = \mathrm{pr}_{I_1 \cup \{j\}} R \times \ldots \times \mathrm{pr}_{I_k} R$, as required.

Finally, if $\mathrm{pr}_{i,j} R = R_i \times R_j$ for all $i, j \in \underline{n}$, then the result follows by Lemma 4.6. □

4.2 General double-connected relations

In this section we partially generalize Proposition 4.7 by showing that a subdirect product in which only one of the factors is a simple relation is a direct product of its simple factor to the rest of the relation.

LEMMA 4.8 (DOUBLE CONNECTED RECTANGULARITY LEMMA). *Let R be a subdirect product of strongly ry/rb-connected relations $R_1, \ldots, R_n$ where R_1 is simple, $\mathcal{G}(\mathrm{pr}_{2,\ldots,n} R)$ is strongly ry/rb-connected, and $\mathrm{pr}_{1,i} R = R_1 \times R_i$ for $i \in \{2, \ldots, n\}$, Then $R = R_1 \times \mathrm{pr}_{2,\ldots,n} R$.*

PROOF. The case $n = 2$ is trivial. Consider the case $n = 3$. We proceed by induction on $|R_1| + |R_2| + |R_3|$. The trivial case $|R_1| + |R_2| + |R_3| = 3$ and the case, in which R_1, R_2, R_3 are simple, give the base case for induction. Note that in the latter case the result follows from Lemma 4.5. Take a maximal projection congruence θ of R_3, and a θ-class $C = \mathbf{d}^\theta$. Then set $R'_3 = R_3/_\theta$, and consider $R' \subseteq R_1 \times R_2 \times R_3/_\theta$, $R'' \subseteq R$ such that

$$\begin{aligned} R' &= \{(\mathbf{a}[1], \mathbf{a}[2], \mathbf{a}[3]/_\theta) \mid (\mathbf{a}[1], \mathbf{a}[2], \mathbf{a}[3]) \in R\} \\ R'' &= \{(\mathbf{a}[1], \mathbf{a}[2], \mathbf{a}[3]) \mid (\mathbf{a}[1], \mathbf{a}[2], \mathbf{a}[3]) \in R, \mathbf{a}[3] \in C\}. \end{aligned}$$

Set also R''' to be an ry/rb-maximal component of $\mathcal{G}(R'')$, and $\overline{C} = \mathrm{pr}_3 R'''$. Induction hypothesis implies that $R' = R_1 \times \mathrm{pr}_{2,3} R'$. Note that R'_3 is strongly ry/rb-connected. By Corollary 4.4 and Lemma 3.5, two cases are possible: $\mathrm{pr}_{2,3} R' = R_2 \times R_3/_\theta$, or there is a mapping $\pi : R_2 \to R'_3$ such that $\mathrm{pr}_{2,3} R' = \{(\mathbf{a}, \pi(\mathbf{a})) \mid \mathbf{a} \in R_2\}$.

CASE 1. $\mathrm{pr}_{2,3} R'$ is the graph of a mapping $\pi\colon R_2 \to R_3/_\theta$.

By Corollary 3.9, $\overline{B} = \mathrm{pr}_2 R'''$ is an ry/rb-maximal component of $\mathcal{G}(B)$, $B = \pi^{-1}(\mathbf{d}^\theta)$. Since for each $(\mathbf{a}, \mathbf{b}) \in R_1 \times B \subseteq \mathrm{pr}_{1,2} R$ there is $\mathbf{c} \in C$ with $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in R$, we have $\mathrm{pr}_{1,2} R'' = R_1 \times B$. Since $R_1 \times \overline{B}$ is a strongly ry/rb-connected component of $\mathcal{G}(R_1 \times B)$, by Lemma 3.10(4), $\mathrm{pr}_{1,2} R''' = R_1 \times \overline{B}$.

Since $|R_1| + |\overline{B}| + |\overline{C}| < |R_1| + |R_2| + |R_3|$, and $\mathcal{G}(\mathrm{pr}_{2,3} R''')$ is strongly ry/rb-connected, inductive hypothesis implies $R_1 \times \mathrm{pr}_{2,3} R''' \subseteq R'''$. In particular, there is $(\mathbf{a}, \mathbf{b}) \in \mathrm{pr}_{2,3} R''' \subseteq \mathrm{pr}_{2,3} R$ such that $R_1 \times \{(\mathbf{a}, \mathbf{b})\} \subseteq R$. To finish the proof we just apply Lemma 3.5.

CASE 2. $\mathrm{pr}_{2,3} R' = R_2 \times R_3/_\theta$.

Since $|R_1| + |R_2| + |R_3/_\theta| < |R_1| + |R_2| + |R_3|$, $R_3/_\theta$ is simple, and $\mathrm{pr}_{1,2} R = R_1 \times R_2$, by inductive hypothesis, $R' = R_1 \times R_2 \times R_3/_\theta$. Therefore, $\mathrm{pr}_{1,2} R'' = R_1 \times R_2$. By Lemma 3.10(4), since $\mathcal{G}(R_1 \times R_2)$ is strongly ry/rb-connected, $\mathrm{pr}_{1,2} R''' = R_1 \times R_2$. Then we argue as in Case 1.

Let now $n > 3$. Then $\mathrm{pr}_1 R \times \mathrm{pr}_{3,\dots,n} R \subseteq \mathrm{pr}_{1,3,\dots,n} R$. Denoting $\mathrm{pr}_{3,\dots,n} R$ by R' we have $R \subseteq R_1 \times R_2 \times R'$, and the conditions of the lemma hold for this subdirect product. Thus $R = R_1 \times \mathrm{pr}_{2,\dots,n} R$ as required. $\square$

The Double Connected Rectangularity Lemma 4.8 allows us to prove that a subdirect product of strongly ry/rb-connected domains is strongly ry/rb-connected.

PROPOSITION 4.9 (DOUBLE CONNECTEDNESS PROPOSITION). *Let R be a subdirect product of $A_1, \dots, A_n \in \mathcal{A}$ and $\mathcal{G}(A_i)$ is strongly ry/rb-connected for every i. Then $\mathcal{G}(R)$ is strongly ry/rb-connected.*

PROOF. We proceed by induction on n. The base case for induction, $n = 1$, is obvious, so suppose that the result is true for $n-1$, that is $\mathcal{G}(R')$, $R' = \mathrm{pr}_{\{1,\dots,n-1\}} R$, is strongly ry/rb-connected. Let us assume first that A_n is simple. Let us also denote

$$W = \{i \in \{1, \dots, n-1\} \mid \mathrm{pr}_{i,n} R \text{ is the graph of a mapping}\}.$$

If $W = \varnothing$ then the result follows from the Double Connected Rectangularity Lemma 4.8, so suppose that $W \neq \varnothing$. In this case, every tuple $\mathbf{a} \in R'$ has a unique extension $a \in A_n$ to a tuple $(\mathbf{a}, a) \in R$. If $\mathbf{a}, \mathbf{b} \in R'$ are such that $\mathbf{a} \le \mathbf{b}$ [or $\langle \mathbf{a}, \mathbf{b} \rangle \in \beta$, or $\langle \mathbf{a}, \mathbf{b} \rangle \in \gamma$] then so are their extensions $a, b \in A_n$. Indeed, if $\mathbf{a} \le \mathbf{b}$ and $a \not\le b$ then $\begin{pmatrix} \mathbf{b} \\ a \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a} \\ a \end{pmatrix}, \begin{pmatrix} \mathbf{b} \\ b \end{pmatrix}\right) \in R$, a contradiction. This implies the strongly ry/rb-connectedness of $\mathcal{G}(R)$.

Let us assume now that A_n is not simple, θ is a maximal congruence of A_n, and $\widetilde{R}$ denotes the relation $\{(a_1,\ldots,a_{n-1},a_n^\theta) \mid (a_1,\ldots,a_{n-1},a)\in R\}$. Also let W denote the set $\{i\in\{1,\ldots,n-1\} \mid \mathrm{pr}_{i,n}\widetilde{R}$ is the graph of a mapping $\pi_i\}$. Consider the relation $\widetilde{\widetilde{R}} = \{(a_1^{\theta_1},\ldots,a_{n-1}^{\theta_{n-1}},a_n^\theta) \mid (a_1,\ldots,a_{n-1},a_n)\in R\}$, where θ_i denotes the equality relation on A_i if $i\notin W$ and $\theta_i=\ker\pi_i$ otherwise. Since for any $i,j\in W\cup\{n\}$ the ith and jth components of any tuple from $\widetilde{\widetilde{R}}$ are related with a bijective mapping, $\mathcal{G}(\widetilde{\widetilde{R}})$ is strongly ry/rb-connected if and only if $\mathcal{G}(\mathrm{pr}_{\underline{n}-W}\widetilde{\widetilde{R}})$ is strongly ry/rb-connected. If $W\neq\varnothing$ then the graph of the latter relation is strongly ry/rb-connected by induction hypothesis, and if $W=\varnothing$ then it is so by what was proved in the case when A_n is simple. Without loss of generality let us assume that $W=\{k+1,\ldots,n-1\}$. The Double Connected Rectangularity Lemma 4.8 implies that $\widetilde{\widetilde{R}}=\mathrm{pr}_{\underline{k}}R\times\mathrm{pr}_{W\cup\{n\}}\widetilde{\widetilde{R}}$.

Thus, for any $\mathbf{a},\mathbf{b}\in R$, there is an rb-path [an ry-path] from $\mathbf{a}^\theta=(\mathbf{a}[1],\ldots,\mathbf{a}[k],(\mathbf{a}[k+1])^{\theta_{k+1}},\ldots,(\mathbf{a}[n-1])^{\theta_{n-1}},(\mathbf{a}[n])^\theta)$ to $\mathbf{b}^\theta=(\mathbf{b}[1],\ldots,\mathbf{b}[k],(\mathbf{b}[k+1])^{\theta_{k+1}},\ldots,(\mathbf{b}[n-1])^{\theta_{n-1}},(\mathbf{b}[n])^\theta)$. Note that since $|A_n/_\theta|>1$ we may assume that $(\mathbf{a}[n])^\theta\neq(\mathbf{b}[n])^\theta$. Indeed, if $(\mathbf{a}[n])^\theta=(\mathbf{b}[n])^\theta$ then choose $\mathbf{c}\in R$ such that $(\mathbf{a}[n])^\theta=(\mathbf{b}[n])^\theta\neq(\mathbf{c}[n])^\theta$ and prove that there is an rb-path [an ry-path] from $\mathbf{a}$ to $\mathbf{c}$ and from $\mathbf{c}$ to $\mathbf{b}$. Let $\mathrm{pr}_{\underline{k}}\mathbf{a}=\mathbf{c}_0,\mathbf{c}_1,\ldots,\mathbf{c}_l=\mathrm{pr}_{\underline{k}}\mathbf{b}$ be an ry-path [an rb-path]. The equality $\widetilde{\widetilde{R}}=\mathrm{pr}_{\underline{k}}R\times\mathrm{pr}_{W\cup\{n\}}\widetilde{\widetilde{R}}$ and the Path Expansion Lemma 3.6(2) imply that this path can be expanded to an ry-path [an rb-path] $\mathbf{a}=\mathbf{c}'_0,\mathbf{c}'_1,\ldots,\mathbf{c}'_{2l}$ such that $(\mathbf{c}'_0[i])^{\theta_i}=\ldots=(\mathbf{c}'_{2l}[i])^{\theta_i}$ for all $i\in W$ (and therefore $(\mathbf{c}'_0[n])^\theta=\ldots=(\mathbf{c}'_{2l}[n])^\theta$). Now, let $\mathrm{pr}_{W\cup\{n\}}\mathbf{c}'_{2l}=\mathbf{d}_0,\mathbf{d}_1,\ldots,\mathbf{d}_m=\mathrm{pr}_{W\cup\{n\}}\mathbf{b}$ be a sequence of tuples such that $\mathbf{d}_0^\theta,\ldots,\mathbf{d}_m^\theta$ is an ry-path [an rb-path] in $\mathcal{G}(\mathrm{pr}_{W\cup\{n\}}\widetilde{\widetilde{R}})$, elements $(\mathbf{d}_i[n])^\theta,(\mathbf{d}_{i+1}[n])^\theta$ are different for any $i\in\{0,\ldots,m-1\}$. Since $\widetilde{\widetilde{R}}=\mathrm{pr}_{\underline{k}}\times\mathrm{pr}_{W\cup\{n\}}\widetilde{\widetilde{R}}$, we have $(\mathrm{pr}_{\underline{k}}\mathbf{b},\mathbf{d}_i^\theta)\in\widetilde{\widetilde{R}}$ for all $i\in\underline{m}$. Therefore, these tuples can be chosen such that $\mathbf{d}'_i=(\mathrm{pr}_{\underline{k}}\mathbf{b},\mathbf{d}_i)=(\mathbf{c}_{2l},\mathbf{d}_i)\in R$ for $i\in\{0,\ldots,m\}$. It is not hard to see that, as $\mathrm{pr}_{j,n}\widetilde{R}$ is the graph of a mapping for any $j\in W$ and $(\mathbf{d}_i[n])^\theta\neq(\mathbf{d}_{i+1}[n])^\theta$, if $(\mathbf{d}_i[n])^\theta\le(\mathbf{d}_{i+1}[n])^\theta$ $[\langle(\mathbf{d}_i[n])^\theta,(\mathbf{d}_{i+1}[n])^\theta\rangle\in\beta$, $\langle(\mathbf{d}_i[n])^\theta,(\mathbf{d}_{i+1}[n])^\theta\rangle\in\gamma]$, then $\mathbf{d}_i[j]\le\mathbf{d}_{i+1}[j]$ $[\langle\mathbf{d}_i[j],\mathbf{d}_{i+1}[j]\rangle\in\beta$, $\langle\mathbf{d}_i[j],\mathbf{d}_{i+1}[j]\rangle\in\gamma]$ and $\mathbf{d}_i[n]\le\mathbf{d}_{i+1}[n]$ $[\langle\mathbf{d}_i[n],\mathbf{d}_{i+1}[n]\rangle\in\beta$, $\langle\mathbf{d}_i[n],\mathbf{d}_{i+1}[n]\rangle\in\gamma]$. Thus, $\mathbf{d}_0,\ldots,\mathbf{d}_m$ is an ry-path [an rb-path], and thus $\mathbf{c}'_0,\mathbf{c}'_1,\ldots,\mathbf{c}'_{2l},\mathbf{d}'_0,\mathbf{d}'_1,\ldots,\mathbf{d}'_m$ is an ry-path [an rb-path] from $\mathbf{a}$ to $\mathbf{b}$. The proposition is proved. $\Box$

4.3 Finding a solution to problems of bounded relational width

In this section we use the Double Connected Rectangularity Lemma 4.8 to show that a wide class of constraint satisfaction problems over strongly ry/rb-connected domains has relational width 3.

We say that $A\in\mathcal{A}$ is *hereditarily ry/rb-connected* if $\mathcal{G}(A)$ is strongly ry/rb-connected and either A is simple, or for any maximal congruence θ of A and any class B of θ, the set of r-maximal elements of $\mathcal{G}(B)$ is hereditarily ry/rb-connected.

Let $\mathcal{P}=(V;\mathcal{A};\delta;\mathcal{C})$ be a problem instance from MCSP(Γ) such that $A_{\delta(w)}=\mathcal{S}_w$ is hereditarily ry/rb-connected for all $w\in V$ (recall that $\mathcal{S}_w$ denotes the set of all partial solutions on w). Let also $v \in V$ and ξ be a maximal congruence of $\mathcal{S}_v$. For each $w \in V - \{v\}$, consider the sets $\mathcal{S}_v/_\xi = \{a^\xi \mid a \in \mathcal{S}_v\}$ and $\mathcal{S}_{v,w}/_\xi = \{(a^\xi, b) \mid (a,b) \in \mathcal{S}_{v,w}\}$. By the Semi-Simple Double Connected Corollary 4.4 and Lemma 3.5, $\mathcal{S}_{v,w}/_\xi$ is either the graph of a mapping $\pi_w : \mathcal{S}_w \to \mathcal{S}_v/_\xi$ or $\mathcal{S}_{v,w}/_\xi = \mathcal{S}_v/_\xi \times \mathcal{S}_w$. A variable w that satisfies the former condition will be called *connected* to v, ξ. Sometimes we shall use this term even if $\mathcal{G}(\mathcal{S}_v), \mathcal{G}(\mathcal{S}_w)$ are not strongly ry/rb-connected. In this case this simply means that $\mathcal{S}_{v,w}/_\xi$ is the graph of a mapping from $\mathcal{S}_w$ to $\mathcal{S}_v/_\xi$. For a class D of ξ and the set B of r-maximal elements of $\mathcal{G}(D)$, we define the problem $\mathcal{P}_{v,\xi,B}$ to be $(V; \mathcal{A}; \delta'; \mathcal{C}')$, where $A_{\delta'(w)}$ is $\mathcal{S}_w$ if $\mathcal{S}^\xi_{v,w} = \mathcal{S}^\xi_v \times \mathcal{S}_w$ or it is the set of r-maximal elements of $\mathcal{G}(\pi_w^{-1}(D))$ otherwise and, for every $C = \langle s, R\rangle \in \mathcal{C}$, there is $C' = \langle s, R'\rangle \in \mathcal{C}'$ such that $R' = \{\mathbf{a} \mid \mathbf{a} \in R, \mathbf{a}[w] \in A_{\delta'(w)} \text{ for } w \in s\}$.

Lemma 4.10. *If $\mathcal{P}$ is 3-minimal then $\mathcal{P}_{v,\xi,B}$ is also 3-minimal, and the set of its partial solutions $\mathcal{S}'_v$ on $\{v\}$ equals $A_{\delta'(v)}$ for any $v \in V$.*

Proof. For any constraint $C' \in \langle s, R'\rangle \in \mathcal{C}'$, we have $\mathrm{pr}_w R' = A_{\delta'(w)}$ for all $w \in s$. Indeed, if w is connected to v, ξ and $a \in A_{\delta'(w)}$, then for any $\mathbf{a} \in R$ with $\mathbf{a}[w] = a$ we have $\mathbf{a}[u] \in \pi_u^{-1}(D)$ due to 3-minimality of $\mathcal{P}$. By the Maximal Expansion Lemma 3.8(1), there is $\mathbf{b} \in R$ such that $\mathbf{b}[w] = a$ and $\mathbf{b}[u] \in A_{\delta'(u)}$ for variables u connected to v, ξ. If w is not connected to v, ξ and $a \in A_{\delta'(w)} = A_{\delta(w)}$, then there is $\mathbf{a} \in R$ with $\mathbf{a}[w] = a$ and $\mathbf{a}[u] \in \pi_u^{-1}(D)$ for variables u connected to v, ξ. Then we again use the Maximal Expansion Lemma 3.8(1). Therefore, $\mathcal{S}'_w = A_{\delta'(w)}$.

For $U = \{u_1, u_2, u_3\} \subseteq V$, set $\mathcal{T}_U = \mathcal{S}_U \cap (\mathcal{S}'_1 \times \mathcal{S}'_2 \times \mathcal{S}'_3)$, where $\mathcal{S}_U$ is the set of partial solutions of $\mathcal{P}$ on U and $S_i = A_{\delta'(u_i)}$. By the Maximal Expansion Lemma 3.8(3), $\mathcal{T}_U$ is non-empty. Clearly, for any $C' = \langle s, R'\rangle \in \mathcal{C}'$, we have $\mathrm{pr}_{s\cap U} R' \subseteq \mathrm{pr}_{s\cap U}\mathcal{T}_U$. Therefore, if we prove the reverse inclusion then we get both, the equality $\mathrm{pr}_{s\cap U} R' = \mathrm{pr}_{s\cap U}\mathcal{T}_U$, and the 3-minimality of $\mathcal{P}'$.

Let W denote the set consisting of v and all $w \in V$ such that $\mathcal{S}_{v,w}/_\xi$ is the graph of a mapping. Let $\mathcal{S}''_w$, for $w \in W$, denote the set $\pi_w^{-1}(B)$, and, for $w \notin W$, the set $\mathcal{S}_w$. Note that, for any $w_1, w_2 \in W$ and any $\mathbf{a} \in \mathcal{S}_{w_1,w_2}$, if $\mathbf{a}[w_1] \in \mathcal{S}''_{w_1}$ then $\mathbf{a}[w_2] \in \mathcal{S}''_{w_2}$. Indeed, let us consider $\mathbf{b} \in \mathcal{S}_{v,w_1,w_2}$ such that $\mathbf{b}[w_1] = \mathbf{a}[w_1]$ and $\mathbf{b}[w_2] = \mathbf{a}[w_2]$ (such $\mathbf{b}$ exists by the 3-minimality of $\mathcal{P}$). As $\mathbf{a}[w_1] \in \mathcal{S}''_{w_1}$, we have $\mathbf{a}[v] \in \mathcal{S}''_v$ and therefore $\mathbf{a}[w_2] \in \mathcal{S}''_{w_2}$.

We start with a weaker claim. Take $\mathbf{b} = (\mathbf{a}[u_1], \mathbf{a}[u_2], \mathbf{a}[u_3]) \in \mathcal{T}_U$, $\langle s, R\rangle \in \mathcal{C}$, and $\mathbf{a} \in R$ such that $\mathrm{pr}_{s\cap U}\mathbf{a} = \mathrm{pr}_{s\cap U}\mathbf{b}$. We show first that $\mathbf{a}$ can be chosen such that $\mathbf{b}[w] \in \mathcal{S}''_w$ for every $w \in s$. If $U \cap W \cap s \neq \varnothing$ then, for any $w \in s \cap W$, $\mathbf{a}[w] \in \mathcal{S}''_w$, and we are done. If $s \cap W = \varnothing$ then $R' = R$. Otherwise, notice that, for any $w \in W$, the kernel $\ker \pi_w$ is a maximal congruence of $A_{\delta(w)}$. Let us consider the relation

$$\widetilde{R} = \{\mathbf{c}' \mid \mathbf{c} \in R,\ \mathbf{c}'[w] = \mathbf{c}[w] \text{ for } w \in s - W \text{ and } \mathbf{c}'[w] = (\mathbf{c}[w])^{\ker \pi_w} \text{ for } w \in s \cap W\}.$$

Choose $w \in s \cap W$ and set $R'' = \mathrm{pr}_{(s-W)\cup\{w\}}\widetilde{R}$. Since $\mathrm{pr}_w R'' = A_{\delta(w)}/_{\ker \pi_w}$ is simple and $\mathcal{G}(\mathrm{pr}_{s-W}\widetilde{R})$ is strongly ry/rb-connected, by the Double Connected Rectangularity Lemma 4.8, $R'' = \mathrm{pr}_w R'' \times \mathrm{pr}_{s-W}\widetilde{R}$. This means that there is

$\mathbf{c} \in R$ such that $\mathrm{pr}_{s-W} R = \mathrm{pr}_{s-W}\mathbf{a}$ and $\mathbf{c}[w] \in \mathcal{S}''_w$. Therefore, $\mathbf{c}[u] \in \mathcal{S}''_u$ for any $u \in s \cap W$. Since $s \cap U \subseteq s - W$, we have $\mathrm{pr}_{s\cap U}\mathbf{c} = \mathrm{pr}_{s\cap U}\mathbf{b}$, as required.

Finally, we show that $\mathbf{a}$ can be chosen such that $\mathbf{a} \in R'$. Note that, by the Double Connectedness Proposition 4.9, R' is the set of r-maximal elements of the graph of the relation $R''' = R \cap (\mathcal{S}''_{w_1} \times \ldots \times \mathcal{S}''_{w_k})$, where $s = (w_1, \ldots, w_k)$, and $\mathrm{pr}_{s\cap U}\mathbf{b} \in \mathrm{pr}_{s\cap U} R'$. Since $\mathbf{a} \in R'''$, by the Maximal Expansion Lemma 3.8(3), there is $\mathbf{a}' \in R'$ such that $\mathrm{pr}_{s\cap U}\mathbf{b} = \mathrm{pr}_{s\cap U}\mathbf{a}'$ $\square$

Proposition 4.11 (Hereditarily Double Connected Proposition). *Let* $\mathbf{K}$ *be a class of 3-minimal constraint satisfaction problems from* MCSP(Γ) *in which all the domains are hereditarily ry/rb-connected. Then* $\mathbf{K}$ *is of relational width 3.*

Proof. Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a 3-minimal problem instance without empty constraints. We prove by induction on the number of elements in $\mathcal{S}_v$, $v \in V$, that $\mathcal{P}$ has a solution.

The base case for induction. If all $v \in V$, $|\mathcal{S}_v| = 1$, then every variable can be assigned the only element in its domain. If every $\mathcal{S}_v$ is simple, then by Proposition 4.7, every constraint relation is almost trivial. This means that, for any $v, w \in V$, the relation $\mathcal{S}_{v,w}$ is either the graph of a bijection, or $\mathcal{S}_v \times \mathcal{S}_w$. By the 3-minimality of $\mathcal{P}$, there is a partition $V_1, \ldots, V_k$ of V, a collection of representatives $v_i \in V_i$, $i \in \{1, \ldots, k\}$, and a collection of bijections $\pi_w : \mathcal{S}_{v_i} \to \mathcal{S}_w$, $w \in V_i$, such that an assignment φ is a solution of $\mathcal{P}$ if and only if, for any $w \in V$, $\varphi(w) = \pi_w(\varphi(v_i))$, where i is such that $w \in V_i$.

Induction step. Suppose that the theorem holds for all problem instances $\mathcal{P}' = (V; \mathcal{A}; \delta'; \mathcal{C}')$ where $|A_{\delta'(v)}| \le |\mathcal{S}_v|$ for $v \in V$ and at least one inequality is strict.

Let us assume that, for a certain $v \in V$, $\mathcal{S}_v$ is not simple and ξ is a maximal congruence of $\mathcal{S}_v$, which is not the equality relation. Take a class of ξ and the set B of r-maximal elements from this class. By Lemma 4.10, the problem $\mathcal{P}_{v,\xi,B}$ is 3-minimal. Since all the domains of $\mathcal{P}$ are hereditarily strongly ry/rb-connected, so are the domains of $\mathcal{P}_{v,\xi,B}$. Finally, $|\mathcal{S}'_v| < |\mathcal{S}_v|$, where $\mathcal{S}'_v$ is the set of partial solutions of $\mathcal{P}_{v,\xi,B}$ on $\{v\}$; therefore $\mathcal{P}_{v,\xi,B}$ has a solution by induction hypothesis. $\square$

For our main algorithm we also need to be able to find a solution to a problem instance from a class of relational width 3 (provided there exists one). As is easily seen, this can be done by employing algorithm 3-Width (see Fig. 4.1).

5. CONNECTEDNESS, RECTANGULARITY AND DECOMPOSITION

In this section we study problems such that the graphs of all their domains are strongly ry-connected or strongly rb-connected. First of all we concentrate on individual relations over such domains. In Section 5.1, we show that strong ry-[rb-] connectedness can be extended from domains to relations. The Connectedness Proposition 5.1 claims that a subdirect product of domains whose graphs are strongly connected is strongly connected. Then the main result of Section 5.2, the Rectangularity Proposition 5.4, shows that relations with strongly ry-connected but not strongly rb-connected graphs (as well as relations with strongly rb-connected but not strongly ry-connected graphs) have a representation close to direct product decomposition. This representation allows us to reduce problems over domains with

INPUT: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.
OUTPUT: A solution to $\mathcal{P}$ if it has one, $\varnothing$ otherwise.

Step 1. **invoke** 3-MINIMALITY($\mathcal{P}$)
Step 2. **if** the constraint relations of $\mathcal{P}$ are empty, **then output** $\varnothing$ **and stop**
Step 3. **if**, for all $v \in V$, $|\mathcal{S}_v| = 1$, **then output**(φ), where $\varphi(v) = a$, $\mathcal{S}_v = \{a\}$
Step 4. **if**, for all $v \in V$, $\mathcal{S}_v$ is simple, **then do**
Step 4.1. let $V_1, \dots, V_k$ be the partition of V, $v_1, \dots, v_k$ a collection of representatives of its classes, and $\pi_w: \mathcal{S}_{v_i} \to \mathcal{S}_w$ bijections, where $w \in V_i$, such that $(a, b) \in \mathcal{S}_{v,w}$ if and only if $a \in \mathcal{S}_v$, $b \in \mathcal{S}_w$ and v, w are from different classes of the partition, or $a = \pi_v(c)$, $b = \pi_w(c)$ for some $c \in \mathcal{S}_{v_i}$, $v, w \in V_i$
Step 4.2. **for** $i = 1$ **to** k **do**
Step 4.2.1. **set** $\varphi(v_i) = a$, $a \in \mathcal{S}_{v_i}$ is any
Step 4.2.2. **for each** $w \in V_i - \{v_i\}$ **set** $\varphi(w) = \pi_w(a)$
endfor
Step 4.3. **output** φ
Step 5. **else** choose a variable $v \in V$ such that $|\mathcal{S}_v| > 1$, a maximal congruence ξ of $\mathcal{S}_v$ and a class B of ξ; **return** 3-WIDTH($\mathcal{P}_{v,\xi,B}$)

Fig. 4.1. Algorithm 3-WIDTH

strongly connected graphs to smaller problems corresponding to the factors of the mentioned decomposition, and to various types of 'skeleton' problems (Sections 5.3 and 5.4).

5.1 Connectedness for relations

The main result of this section is the following

PROPOSITION 5.1 (CONNECTEDNESS PROPOSITION). *Let R be a subdirect product of $A_1, \dots, A_n \in \mathcal{A}$.*
(1) If every $\mathcal{G}(A_i)$ is strongly r-connected, then $\mathcal{G}(R)$ is strongly r-connected.
(2) If every $\mathcal{G}(A_i)$ is strongly rb-connected, then $\mathcal{G}(R)$ is strongly rb-connected.
(3) If every $\mathcal{G}(A_i)$ is strongly ry-connected and such that every r-connected component of $\mathcal{G}(A_i)$ is strongly r-connected, then $\mathcal{G}(R)$ is strongly ry-connected.

PROOF. We prove the proposition by induction in the sizes of $A_1, \dots, A_n$. If all of them are simple, we use the Almost Trivial Proposition 4.7. Otherwise we choose A_i which is not simple, factorize by its maximal congruence, and apply the induction hypothesis. □

LEMMA 5.2 (GENERALIZED CONNECTEDNESS LEMMA). *Let R be a subdirect product of $A_1, \dots, A_n \in \mathcal{A}$ and $A_i^{\max}$ denote the set of r-maximal elements of A_i.*
(1) Let C_i be a strongly r-connected component of $\mathcal{G}(A_i^{\max})$, $i \in \underline{n}$. Then if $R' = R \cap (C_1 \times \dots \times C_n)$ is non-empty then it is a subdirect product of $C_1, \dots, C_n$ and $\mathcal{G}(R')$ is strongly r-connected.
(2) Let C_i be a strongly ry-connected component of $\mathcal{G}(A_i^{\max})$, $i \in \underline{n}$. Then if $R' = R \cap (C_1 \times \dots \times C_n)$ is non-empty then it is a subdirect product of $C_1, \dots, C_n$ and $\mathcal{G}(R')$ is strongly ry-connected.
(3) Let C_i be an rb-maximal component of $\mathcal{G}(A_i)$ (not $A_i^{\max}$!), $i \in \underline{n}$. Then if $R' = R \cap (C_1 \times \dots \times C_n)$ is non-empty then it is a subdirect product of $C_1, \dots, C_n$ and $\mathcal{G}(R')$ is strongly rb-connected.

Remark. In Lemma 5.2(3) the sets C_i are in general larger than strongly rb-connected components of $A_i^{\max}$. Indeed, rb-maximal components can contain elements that are connected to the r-maximal elements with blue edges, but that are not r-maximal themselves and therefore do not belong to $A_i^{\max}$.

PROOF. (2) Since $A_i^{\max}$ is the set of r-maximal elements from A_i, every its r-connected component is strongly r-connected. Take $i \in \underline{n}$; without loss of generality we may assume $i = 1$. Suppose that $C = \mathrm{pr}_1 R' \neq C_1$. Since $\mathcal{G}(C_1)$ is strongly ry-connected, there are $a \in C$ and $b \in C_1 - C$ such that $a \leq b$ or $\langle a, b\rangle \in \beta$. Let $a = \mathbf{a}[1]$, $b = \mathbf{b}[1]$ for some $\mathbf{a} \in R'$ and $\mathbf{b} \in R$. By the Maximal Expansion Lemma 3.8(1), we may assume $\mathbf{b}$ to be r-maximal. This means that if $\langle \mathbf{a}[i], \mathbf{b}[i]\rangle \in \beta$ or $\mathbf{a}[i] \leq \mathbf{b}[i]$ then $\mathbf{b}[i] \in C_i$. For $\mathbf{c} = p(\mathbf{a}, \mathbf{b})$ we have

$$\mathbf{c}[i] = \begin{cases} \mathbf{a}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i]\rangle \in \gamma \text{ or } \mathbf{b}[i] \leq \mathbf{a}[i], \\ \mathbf{b}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i]\rangle \in \beta \text{ or } \mathbf{a}[i] \leq \mathbf{b}[i]; \end{cases}$$

in particular, $\mathbf{c}[1] = \mathbf{b}[1] = b$. As is easily seen, $\mathbf{c} \in R'$, a contradiction with $b \notin C$. Thus, R' is a subdirect product of $C_1, \dots, C_n$.

Since every $\mathcal{G}(C_i)$ is strongly ry-connected and, by the definition of $A_i^{\max}$, every its r-connected component is strongly r-connected, R' satisfies the conditions of the Connectedness Proposition 5.1(3). Thus it is strongly ry-connected.

For (1) the proof is quite similar; one just needs to assign $\mathbf{c} = f(\mathbf{a}, \mathbf{b})$ rather than $\mathbf{c} = p(\mathbf{a}, \mathbf{b})$.

(3) Take $i \in \underline{n}$; without loss of generality we may assume $i = 1$. Suppose that $B = \mathrm{pr}_1 R' \neq C_1$. Since C_1 is strongly rb-connected, there are $a \in B$ and $b \in C_1 - B$ such that $a \leq b$ or $\langle a, b\rangle \in \gamma$. Let $a = \mathbf{a}[1]$, $b = \mathbf{b}[1]$ for some $\mathbf{a} \in R'$ and $\mathbf{b} \in R$. For $\mathbf{c} = p(\mathbf{b}, \mathbf{a})$ we have

$$\mathbf{c}[i] = \begin{cases} \mathbf{a}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i]\rangle \in \beta \text{ or } \mathbf{b}[i] \leq \mathbf{a}[i], \\ \mathbf{b}[i], & \text{if } \langle \mathbf{a}[i], \mathbf{b}[i]\rangle \in \gamma \text{ or } \mathbf{a}[i] \leq \mathbf{b}[i]; \end{cases}$$

in particular, $\mathbf{c}[1] = \mathbf{b}[1] = b$. As is easily seen, $\mathbf{c} \in R'$. Thus, R' is a subdirect product of $C_1, \dots, C_n$.

Since every C_i is strongly rb-connected, R' satisfies the conditions of the Connectedness Proposition 5.1(2). Thus $\mathcal{G}(R')$ is strongly rb-connected. □

5.2 Rectangularity

In the rest of Section 5, $R \in \Gamma$ is a subdirect product of $A_1, \dots, A_n \in \mathcal{A}$, unless otherwise is explicitly stated. Let x be one of the symbols ry, rb, $j \in \underline{n}$, and let $B \subseteq A_j$ be a strongly x-connected component of $\mathcal{G}(A_j)$. We define the set $J_R(\mathrm{x}, j, B) \subseteq \underline{n}$ (usually we will omit R) as follows:

$$\begin{aligned} J_R(\mathrm{x}, j, B) \;=\; \{ i \in \underline{n} \mid & \text{ for any } \mathbf{a}, \mathbf{b} \in R \text{ such that } \mathbf{a}[j] \in B, \text{ the element } \mathbf{b}[j] \\ & \text{belongs to } B \text{ if and only if } \mathbf{a}[i], \mathbf{b}[i] \text{ are strongly x-connected in } \mathcal{G}(A_i)\} \end{aligned}$$

Let $j \in \underline{n}$, and B be a strongly x-connected component of $\mathcal{G}(A_j)$. The definition of $J_R(\mathrm{x}, j, B)$ implies that for each $i \in J_R(\mathrm{x}, j, B)$ there is a strongly x-connected component B_i of $\mathcal{G}(A_i)$ such that, for any $\mathbf{a} \in R$, $\mathbf{a}[i] \in B_i$ whenever $\mathbf{a}[j] \in B$; in particular, $B_j = B$ (see Fig. 5.1(1)). Note that, for another strongly x-connected

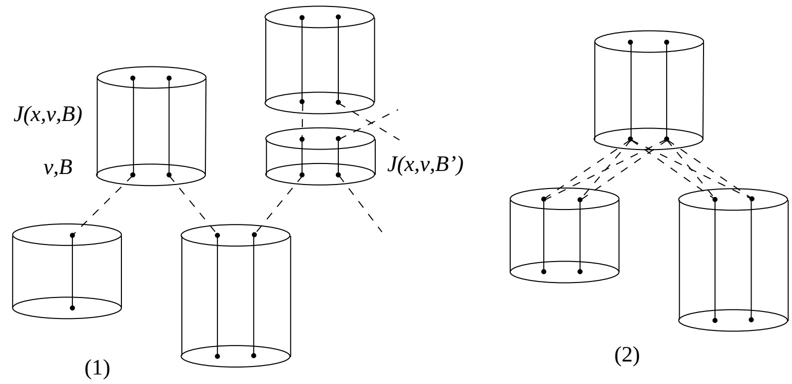


Fig. 5.1. Sets $J(\mathrm{x}, v, B)$. Solid vertical lines represent tuples or their chunks; dotted lines indicate that two solid lines show pieces of the same tuple

component B' of the graph $\mathcal{G}(A_j)$ of the same domain, the set $J_R(\mathrm{x}, j, B')$ can be quite different from $J_R(\mathrm{x}, j, B)$.

Lemma 5.3. *If B is an x-maximal component of $\mathcal{G}(A_j)$ then, for any $i \in J_R(\mathrm{x}, j, B)$, B_i is an x-maximal component of $\mathcal{G}(A_i)$.*

Proof. If, say B_i, is not x-maximal, then take $\mathbf{a} \in R$ such that $\mathbf{a}[j] \in B_j$. By the Maximal Expansion Lemma 3.8(2), there is $\mathbf{b} \in R$ such that $\mathbf{b}[j] = \mathbf{a}[j]$, but $\mathbf{b}[i]$ belongs to an x-maximal component, i.e. $\mathbf{b}[i] \notin B_i$. $\square$

Now we define a restricted version of a direct product decomposition of a relation. This type of decomposition proposes that if we choose $i_1, \dots, i_k \in \underline{n}$ and $B_1, \dots, B_k$, where B_j is an x-maximal component of $\mathcal{G}(A_{i_j})$, in such a way that the sets $J(\mathrm{x}, i_j, B_j)$ constitute a partition of $\underline{n}$, then the restriction of R to the tuples with entries in the B_j is a direct product with projections onto the $J(\mathrm{x}, i_j, B_j)$ as factors (see Fig. 5.1(2)). More formally, the relation R is said to be x-*rectangular* if for any x-maximal $\mathbf{a} \in R$, any $j \in \underline{n}$, and any $\mathbf{b} \in \mathrm{pr}_{J_R(\mathrm{x},j,B)} R$, where B is the x-maximal component containing $\mathbf{a}[j]$, and $\mathbf{b}[j] \in B$, the tuple $\mathbf{c}$ belongs to R:

$$\mathbf{c}[i] = \begin{cases} \mathbf{b}[i], & \text{if } i \in J_R(\mathrm{x}, j, B), \\ \mathbf{a}[i], & \text{otherwise.} \end{cases}$$

It may seem that all these conditions are very hard to satisfy. However, if $\mathbf{a} \in R$ is an x-maximal tuple of $\mathcal{G}(R)$ then the collection of different sets of the form $J(\mathrm{x}, i, B_i)$, B_i is the strongly x-connected component of $\mathcal{G}(A_i)$ containing $\mathbf{a}[i]$, satisfies the conditions. So, such a restricted direct product decomposition can be built around every x-maximal tuple from R.

In this section we prove two rectangularity results which allow us to isolate ry- and rb-connected components.

Proposition 5.4 (Rectangularity Proposition). *(1) If for any $i \in \underline{n}$, every r-connected component of $\mathcal{G}(A_i)$ is strongly r-connected then R is ry-rectangular. (2) R is rb-rectangular.*

Proof. (1) First, observe that, as every r-connected component from $\mathcal{G}(A_i)$, $i \in \underline{n}$, is strongly r-connected, every element of $\mathcal{G}(A_i)$ is r-maximal. Take an ry-maximal tuple $\mathbf{a} \in R$, $q \in \underline{n}$, set $I = J(\mathrm{ry}, q, B)$, where B is the ry-connected component of $\mathcal{G}(A_q)$ containing $\mathbf{a}[q]$, and take $\mathbf{b} \in \mathrm{pr}_I R$ such that $\mathbf{b}[i], \mathbf{a}[i]$ are strongly ry-connected for $i \in I$. Since all elements of R are r-maximal, by the Generalized Connectedness Lemma 5.2(2), $\mathrm{pr}_I \mathbf{a}$ and $\mathbf{b}$ are strongly ry-connected. We prove the proposition by induction on n. If $I = \underline{n}$, in particular, if $n = 1$, then the result follows trivially. So, suppose that $I \neq \underline{n}$.

As $I = J(\mathrm{ry}, q, B) \neq \underline{n}$, there are $\mathbf{c} \in R$ and $t \in \underline{n}$ such that either $\mathbf{c}[q], \mathbf{a}[q]$ are strongly ry-connected while $\mathbf{c}[t], \mathbf{a}[t]$ are not, or $\mathbf{c}[q], \mathbf{a}[q]$ are not strongly ry-connected while $\mathbf{c}[t], \mathbf{a}[t]$ are. Without loss of generality, suppose that the former case holds. Let J_1, J_2 be the partition of $\underline{n}$ such that $\mathbf{c}[i], \mathbf{a}[i]$ are strongly ry-connected for $i \in J_1$, $\mathbf{c}[i], \mathbf{a}[i]$ are not strongly ry-connected (and therefore $\langle \mathbf{c}[i], \mathbf{a}[i] \rangle \in \gamma$ as every r-connected component of $\mathcal{G}(A_i)$ is strongly r-connected) for $i \in J_2$. Since $I \subseteq J_1$, $t \in J_2$, these sets are nonempty. By the Generalized Connectedness Lemma 5.2(2), $\mathrm{pr}_{J_1} \mathbf{a}$ and $\mathrm{pr}_{J_1} \mathbf{c}$ are strongly ry-connected, moreover, we can assume that $\mathrm{pr}_{J_1} \mathbf{a} \le \mathrm{pr}_{J_1} \mathbf{c}$ or $\langle \mathrm{pr}_{J_1} \mathbf{a}, \mathrm{pr}_{J_1} \mathbf{c} \rangle \in \beta$. Indeed, note that the only property of $\mathbf{a}$ we use is $\mathbf{a}[q] \in B$. This means that we can safely replace $\mathbf{a}$ with any tuple $\mathbf{a}' \in R$ such that $\mathbf{a}'[q] \in B$. Now suppose that $\mathbf{a}, \mathbf{c}$ are such that (i) $\mathbf{a}[q], \mathbf{c}[q] \in B$, (ii) $\mathbf{a}[i], \mathbf{c}[i]$ are strongly ry-connected for $i \in J_1$ and $\mathbf{a}[i], \mathbf{c}[i]$ are not strongly ry-connected for $i \in J_2$, and there is an ry-path $\mathrm{pr}_{J_1} \mathbf{a} = \mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^k = \mathrm{pr}_{J_1} \mathbf{c}$ that is shortest possible for pairs satisfying conditions (i) and (ii). Let also $\mathbf{b}^1 \in \mathrm{pr}_{J_2} R$ be such that $(\mathbf{a}^1, \mathbf{b}^1) \in R$. Then the tuples $p\left(\mathbf{a}, \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{b}^1 \end{pmatrix}\right)$ and $\mathbf{c}$ also satisfy conditions (i) and (ii) above, but are connected with a shorter ry-path. The obtained contradiction implies that $k = 2$ and therefore $\mathrm{pr}_{J_1} \mathbf{a} \le \mathrm{pr}_{J_1} \mathbf{c}$ or $\langle \mathrm{pr}_{J_1} \mathbf{a}, \mathrm{pr}_{J_1} \mathbf{c} \rangle \in \beta$.

Set $\mathrm{pr}_{J_1} \mathbf{a} = \mathbf{a}_1$, $\mathrm{pr}_{J_1} \mathbf{c} = \mathbf{c}_1$, $\mathrm{pr}_{J_2} \mathbf{a} = \mathbf{a}_2$, $\mathrm{pr}_{J_2} \mathbf{c} = \mathbf{c}_2$. Let B_i denote the ry-connected component of $\mathcal{G}(A_i)$ containing $\mathbf{a}[i]$, and C_i the ry-connected component containing $\mathbf{c}[i]$. By the choice of J_1, J_2, we have $B_i = C_i$ for $i \in J_1$, and $B_i \neq C_i$ for $i \in J_2$. Let us denote

$$R' = \mathrm{pr}_{J_1} R \cap \prod_{i \in J_1} B_i, \quad R'' = \mathrm{pr}_{J_2} R \cap \prod_{i \in J_2} B_i, \quad R''' = \mathrm{pr}_{J_2} R \cap \prod_{i \in J_2} C_i.$$

By Lemma 5.2(2), $\mathcal{G}(R'), \mathcal{G}(R''), \mathcal{G}(R''')$ are strongly ry-connected.

Claim 1. If $(\mathbf{a}_1', \mathbf{a}_2'), (\mathbf{c}_1', \mathbf{c}_2') \in R$, where $\mathbf{a}_1', \mathbf{c}_1' \in R'$ and $\mathbf{a}_2' \in R''$, $\mathbf{c}_2' \in R'''$ (thus, $\mathbf{a}_1', \mathbf{c}_1'$ are strongly ry-connected while $\mathbf{a}_2', \mathbf{c}_2'$ are not) are such that either $\mathbf{a}_1' \le \mathbf{c}_1'$, or $\mathbf{c}_1' \le \mathbf{a}_1'$, or $\langle \mathbf{a}_1', \mathbf{c}_1' \rangle \in \beta$, then, for every $\mathbf{b}_1 \in R'$ we have $(\mathbf{b}_1, \mathbf{a}_2'), (\mathbf{b}_1, \mathbf{c}_2') \in R$.

Prove first that $(\mathbf{a}_1', \mathbf{c}_2') \in R$ or $(\mathbf{c}_1', \mathbf{a}_2') \in R$. If $\mathbf{a}_1' \le \mathbf{c}_1'$ then we have $f\left(\begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{pmatrix}, \begin{pmatrix} \mathbf{c}_1' \\ \mathbf{c}_2' \end{pmatrix}\right) = \begin{pmatrix} \mathbf{c}_1' \\ \mathbf{a}_2' \end{pmatrix} \in R$. If $\mathbf{c}_1' \le \mathbf{a}_1'$ we get $(\mathbf{a}_1', \mathbf{c}_2') \in R$. If $\langle \mathbf{a}_1', \mathbf{c}_1' \rangle \in \beta$ then

$$\begin{pmatrix} \mathbf{a}_1' \\ \mathbf{c}_2' \end{pmatrix} = p\left(\begin{pmatrix} \mathbf{c}_1' \\ \mathbf{c}_2' \end{pmatrix}, \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{pmatrix}\right), \quad \begin{pmatrix} \mathbf{c}_1' \\ \mathbf{a}_2' \end{pmatrix} = p\left(\begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{pmatrix}, \begin{pmatrix} \mathbf{c}_1' \\ \mathbf{c}_2' \end{pmatrix}\right) \in R.$$

Now let B be the set of tuples $\mathbf{b}'$ from R' for which $(\mathbf{b}', \mathbf{a}_2'), (\mathbf{b}', \mathbf{c}_2') \in R$. If $B \neq R'$ then there is $\mathbf{b}_1 \in B$ and $\mathbf{d}_1 \in R' - B$ such that $\mathbf{b}_1 \le \mathbf{d}_1$ or $\langle \mathbf{b}_1, \mathbf{d}_1 \rangle \in \beta$.

There is $\mathbf{d}_2 \in \mathrm{pr}_{J_2} R$ with $(\mathbf{d}_1, \mathbf{d}_2) \in R$. By the P-Lemma 3.4(3), in the tuple

$$\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}'_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{c}'_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix}\right)$$

$\mathbf{c}'_2[i] \le \mathbf{d}'_2[i]$ or $\langle \mathbf{c}'_2[i], \mathbf{d}'_2[i]\rangle \in \beta$ for any $i \in J_2$, and therefore $\langle \mathbf{d}'_2[i], \mathbf{a}[i]\rangle \in \gamma$ (as $\mathbf{a}'_2[i]$ and $\mathbf{c}'_2[i]$ belong to different strongly ry-connected components). Then

$$\begin{pmatrix}\mathbf{d}_1\\ \mathbf{a}'_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{a}'_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}'_2\end{pmatrix}\right) \in R \quad \text{and} \quad \begin{pmatrix}\mathbf{d}_1\\ \mathbf{c}'_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{b}_1\\ \mathbf{c}'_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{a}'_2\end{pmatrix}\right) \in R,$$

a contradiction. The claim is proved.

Claim 2. For any $\mathbf{e}_1 \in R'$ and any $\mathbf{a}'_2 \in R''$ the tuple $(\mathbf{e}_1, \mathbf{a}'_2)$ belongs to R.

Let $C \subseteq R''$ be the set of those $\mathbf{a}'_2$ for which $(\mathbf{e}_1, \mathbf{a}'_2) \in R$. As the tuples $\mathbf{a}, \mathbf{c}$ fulfill the conditions of Claim 1, $(\mathbf{e}_1, \mathbf{a}_2) \in R$, and this set is non-empty. Since $\mathcal{G}(R'')$ is strongly ry-connected, there are $\mathbf{a}''_2 \in C$ and $\mathbf{a}'_2 \in R'' - C$ such that $\mathbf{a}''_2 \le \mathbf{a}'_2$ or $\langle \mathbf{a}''_2, \mathbf{a}'_2\rangle \in \beta$. There is $\mathbf{e}'_1 \in \mathrm{pr}_{J_1} R$ such that $(\mathbf{e}'_1, \mathbf{a}'_2) \in R$. Set

$$\begin{pmatrix}\mathbf{e}''_1\\ \mathbf{a}'_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{e}_1\\ \mathbf{a}''_2\end{pmatrix}, \begin{pmatrix}\mathbf{e}'_1\\ \mathbf{a}'_2\end{pmatrix}\right)$$

where $\mathbf{e}_1[i] \le \mathbf{e}''_1[i]$ or $\langle \mathbf{e}_1[i], \mathbf{e}''_1[i]\rangle \in \beta$ for any $i \in J_1$. By Claim 1 $(\mathbf{e}'_1, \mathbf{c}_2) \in R$. Since $\langle \mathbf{a}'_2, \mathbf{c}_2\rangle \in \gamma$ and

$$\begin{pmatrix}\mathbf{e}''_1\\ \mathbf{c}_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{e}_1\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{e}''_1\\ \mathbf{a}'_2\end{pmatrix}\right) \in R,$$

applying Claim 1 to the pair $(\mathbf{e}''_1, \mathbf{a}'_2), (\mathbf{e}''_1, \mathbf{c}_2)$ and tuple $\mathbf{e}_1$, we get what is required.

By the induction hypothesis, there is the tuple $\mathbf{d} \in R'$ such that

$$\mathbf{d}[i] = \begin{cases} \mathbf{b}[i], & \text{if } i \in I,\\ \mathbf{a}[i], & \text{if } i \in J_1 - I.\end{cases}$$

By Claims 1,2 $(\mathbf{d}, \mathbf{a}_2) \in R$, and the result is proved.

(2) In this case the first part of the proof is close to that for (1). We take an rb-maximal $\mathbf{a} \in R$ and choose $\mathbf{c}$ in the same way. We shall use the same notation in this case, that is I, J_1, J_2, B_i, C_i, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{c}_1, \mathbf{c}_2$, and R', R'', R'''. Since $\mathbf{a}$ is rb-maximal, for every $i \in J_1$, the component B_i is rb-maximal. We again can assume that either $\mathbf{a}_1 \le \mathbf{c}_1$, or $\mathbf{c}_1 \le \mathbf{a}_1$, or $\langle \mathbf{a}_1, \mathbf{c}_1\rangle \in \gamma$. Note that we can exclude the second option. Indeed, suppose that $\mathbf{c}_1 \le \mathbf{a}_1$ and $\mathbf{c}_1 \ne \mathbf{a}_1$. By the Generalized Connectedness Lemma 5.2(3), $\mathcal{G}(R \cap (R' \times R''))$ is strongly rb-connected. Therefore, there is $\mathbf{a}' \in R \cap (R' \times R'')$ such that $\mathrm{pr}_{J_1} \mathbf{a}' \le \mathbf{c}_1$ or $\langle \mathrm{pr}_{J_1} \mathbf{a}, \mathbf{c}_1\rangle \in \gamma$. Replacing $\mathbf{a}$ with $\mathbf{a}'$ we get what is needed. Observe that, as $\mathbf{c}$ is not necessarily rb-maximal, we cannot claim that $\mathcal{G}(R''')$ is strongly rb-connected, and that $\langle \mathbf{a}_2, \mathbf{c}_2\rangle \in \beta$.

Since $B_i \ne C_i$ for $i \in J_2$, in this case $\mathbf{c}[i] \le \mathbf{a}[i]$ or $\langle \mathbf{a}[i], \mathbf{c}[i]\rangle \in \beta$. If neither $\mathbf{c}_2 \le \mathbf{a}_2$ nor $\langle \mathbf{a}_2, \mathbf{c}_2\rangle \in \beta$, we replace $\mathbf{c}$ with $p(\mathbf{c}, \mathbf{a})$ (for this tuple we would have $\langle \mathbf{a}_2, \mathbf{c}_2\rangle \in \beta$). Note that this transformation results in different sets J_1, J_2, B_i, and C_i. Thus we can assume that $\mathbf{c}_2 \le \mathbf{a}_2$ or $\langle \mathbf{a}_2, \mathbf{c}_2\rangle \in \beta$. Finally, note that $\mathbf{c}' = (\mathbf{c}_1, \mathbf{a}_2) = p(\mathbf{c}, \mathbf{a}) \in R$.

We prove by induction on $|R''|$ that $R' \times R'' \subseteq R$. First, observe that by Lemma 3.10 $\overline{R} = R \cap (R' \times R'')$ is a subdirect product of R' and R''. The base case for induction, when R'' is a singleton, is obvious.

For induction step we assume that the result is true for all relations smaller than R''. Take a maximal projection congruence θ of R''. Then $R''/_\theta$ is simple, hence, $R''/_\theta$ can be treated as a unary relation. We show first that $\overline{R} = R \cap (R' \times R''/_\theta)$ is not the graph of any mapping $\pi\colon R' \to R''/_\theta$. Suppose that $\overline{R}$ is the graph of such a π, and denote $D = \pi^{-1}(\mathbf{a}_2^\theta)$ the set of all preimages of $\mathbf{a}_2^\theta[n]$. Clearly, $\mathbf{c}_1 \in D$. Since $\mathcal{G}(R')$ is strongly rb-connected, there is $\mathbf{d}_1 \in R' - D$ and $\mathbf{d}_1' \in D$ such that $\mathbf{d}_1 \le \mathbf{d}_1'$ or $\langle \mathbf{d}_1, \mathbf{d}_1' \rangle \in \gamma$. As $\overline{R}$ is the graph of a mapping, $\pi(\mathbf{d}_1) \le \pi(\mathbf{d}_1')$ or $\langle \pi(\mathbf{d}_1), \pi(\mathbf{d}_1')\rangle \in \gamma$, respectively. There are also $\mathbf{d}_2, \mathbf{d}_2' \in R''$ such that $\mathbf{d}_2^\theta = \pi(\mathbf{d}_1)$, ${\mathbf{d}_2'}^\theta = \pi(\mathbf{d}_1')$ and $(\mathbf{d}_1, \mathbf{d}_2), (\mathbf{d}_1', \mathbf{d}_2') \in R$. For $\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{c}_1\\ \mathbf{a}_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix}\right)$ we have $\mathbf{e}_2^\theta = (\mathbf{d}_2')^\theta = \mathbf{a}_2^\theta$, hence, $\mathbf{e}_1 \in \pi^{-1}({\mathbf{d}_2'}^\theta)$. Thus, in $\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2'\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{pmatrix}, \begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix}\right)$, element ${\mathbf{e}_2'}^\theta$ equals ${\mathbf{d}_2'}^\theta$, while, as is easily seen, ${\mathbf{e}_2'}^\theta = \mathbf{d}_1^\theta$, a contradiction. Finally, by the Semi-Simple Double Connected Corollary 4.4 and Lemma 3.5 we have $\overline{R} = R' \times R''/_\theta$.

Take a class D of congruence θ. The equality $\overline{R} = R' \times R''/_\theta$ implies that, for any $\mathbf{d} \in R'$, there is $\mathbf{d}' \in D$ such that $(\mathbf{d}, \mathbf{d}') \in R$. Moreover, if D' denotes the set of rb-maximal elements from $\mathcal{G}(D)$, then by Lemma 3.10(2), $R \cap (R' \times D')$ is a subdirect product of R' and D'. Applying induction hypothesis we get $R' \times D' \subseteq R$. By Lemma 3.5, we conclude the result. $\Box$

5.3 Strongly rb-connected components

In the following two sections we show how the Rectangularity Proposition 5.4 can be used to solve problems over domains whose graphs are not rb-connected (this section) or the graphs of the sets of r-maximal elements of which are not strongly ry-connected (next section).

First we extend the notation $J(\mathrm{x}, i, B)$ from relations to problem instances. Let $\mathcal{P} = (V, \mathcal{A}, \delta, \mathcal{C}) \in \mathrm{MCSP}(\Gamma)$ be a 3-minimal problem instance. Let also x be one of rb, ry. Take a variable $v \in V$, an rb- [ry-] strongly connected component B of $\mathcal{G}(\mathcal{S}_v)$, and set $J(\mathrm{x}, v, B)$ to be the union of $J_R(\mathrm{x}, v, B)$ where R runs over all constraint relations whose scope contains v. Note that, since $\mathcal{P}$ is 3-minimal, if for a certain $w \in J(\mathrm{x}, v, B)$, w is contained in the scopes of a constraint $\langle s, R\rangle$ then, for any $\mathbf{a}, \mathbf{b} \in R$ such that $\mathbf{a}[v], \mathbf{b}[v] \in B$, the elements $\mathbf{a}[w], \mathbf{b}[w]$ are also strongly rb-connected [strongly ry-connected]. This allows us to define B_w for every $w \in J(\mathrm{x}, v, B)$ in the same way as for relations. If $w \in J(\mathrm{x}, v, B)$ then, since the instance is 3-minimal, $J_R(\mathrm{x}, w, B_w) \subseteq J(\mathrm{x}, v, B)$ for any constraint $\langle s, R\rangle$ such that $w \in s$.

The following property follows straightforwardly from the construction of $J(\mathrm{x}, v, B)$.

Lemma 5.5. *Using the notation above, if φ is a solution to $\mathcal{P}$ such that $\varphi(v) \in B$ then $\varphi(w) \in B_w$ for any $w \in J(\mathrm{x}, v, B)$.*

We start with the case when there is $v \in V$ such that $\mathcal{G}(\mathcal{S}_v)$ is not strongly rb-connected.

We define the subproblem corresponding to $J(\mathrm{rb}, v, B)$. The problem $\mathcal{P}_{\mathrm{rb},v,B}$ is defined to be the problem instance $(U; \mathcal{A}; \delta'; \mathcal{C}')$ where $U = J(\mathrm{rb}, v, B)$, $A_{\delta'(w)} = B_w$

for $w \in U$ and, for every $C = \langle s, R\rangle \in \mathcal{C}$, there is $C' = \langle s \cap U; R'\rangle \in \mathcal{C}$ such that

$$R' = \{\mathrm{pr}_{s\cap U}\mathbf{a} \mid \mathbf{a} \in R,\ \mathbf{a}[w] \in B_w \text{ for any } w \in s \cap U\}.$$

Since the graphs of the domains of problems of the form $\mathcal{P}_{\mathrm{rb},v,B}$ are strongly rb-connected, such problems belong to another type of problems and are to be solved by methods introduced in other sections. Here we are going to prove that if all problems of this form have solutions, then the existence of a solution to $\mathcal{P}$ follows from 3-minimality. The proof is very similar to that for constraints invariant with respect to a majority operation.

In [Baker and Pixley 1975] (see also [Jeavons et al. 1998]), it was proved that relations invariant with respect to a so-called *near-unanimity operation* (a majority operation is a ternary near-unanimity operation) are *decomposable*, that is they are completely defined by their projections of bounded arity. For example if R is an n-ary relation invariant under a majority operation, then $\mathbf{a} \in R$ if and only if, for any $i, j \in \underline{n}$, $\mathrm{pr}_{i,j}\mathbf{a} \in \mathrm{pr}_{i,j}R$. We show that a similar property holds also for some types of relations in Γ.

LEMMA 5.6 (2-DECOMPOSITION LEMMA). *Let $R \in \Gamma$ be a subdirect product of $A_1, \ldots, A_n \in \mathcal{A}$, and let $\mathbf{a} \in A_1 \times \ldots \times A_n$. If, for any $i, j \in \underline{n}$, $(\mathbf{a}[i], \mathbf{a}[j]) \in \mathrm{pr}_{i,j}R$ and, for any $i \in \underline{n}$, $\mathbf{a}[i]$ belongs to an rb-maximal component B_i of $\mathcal{G}(A_i)$ and $\mathrm{pr}_{J(\mathrm{rb},i,B_i)}\mathbf{a} \in \mathrm{pr}_{J(\mathrm{rb},i,B_i)}R$, then $\mathbf{a} \in R$.*

PROOF. Let $I_1, \ldots, I_k$ be the sets of the form $J(\mathrm{rb}, i, B_i)$. We use $\mathbf{a}_j$ to denote the tuple $\mathrm{pr}_{I_j}\mathbf{a}$. We prove by induction on $l \le k$ that, for any choice of $I_{j_1}, \ldots, I_{j_l}$, the tuple $\mathrm{pr}_{I_{j_1}\cup\ldots\cup I_{j_l}}\mathbf{a}$ belongs to $\mathrm{pr}_{I_{j_1}\cup\ldots\cup I_{j_l}}R$. The base case of induction is provided by the conditions of the lemma. Indeed, if $l = 1$ then the required property follows from the condition $\mathrm{pr}_{J(\mathrm{rb},i,B_i)}\mathbf{a} \in \mathrm{pr}_{J(\mathrm{rb},i,B_i)}R$. If $l = 2$ then take $i_1 \in I_{j_1}$, $i_2 \in I_{j_2}$. By the conditions, $(\mathbf{a}[i_1], \mathbf{a}[i_2]) \in \mathrm{pr}_{\{i_1,i_2\}}R$ and, therefore, there is $\mathbf{b} \in \mathrm{pr}_{I_{j_1}\cup I_{j_2}}R$ such that $\mathrm{pr}_{I_{j_1}}\mathbf{a}$, $\mathrm{pr}_{I_{j_1}}\mathbf{b}$ and $\mathrm{pr}_{I_{j_2}}\mathbf{a}$, $\mathrm{pr}_{I_{j_2}}\mathbf{b}$ are strongly rb-connected. By the Rectangularity Proposition 5.4(2), $\mathrm{pr}_{I_{j_1}\cup I_{j_2}}\mathbf{a} \in \mathrm{pr}_{I_{j_1}\cup I_{j_2}}R$.

Suppose that for any subset $J \subseteq \{1, \ldots, k\}$ containing less than $l > 2$ element the tuple $\mathrm{pr}_{I_J}\mathbf{a}$, $I_J = \bigcup_{j\in J} I_j$, belongs to $\mathrm{pr}_J R$. Take $K \subseteq \underline{k}$ with $|K| = l$, and $s, t, q \in K$. Without loss of generality we may assume $K = \underline{l}$, $s = 1, t = 2, q = 3$. We have to show that $\mathrm{pr}_{I_K}\mathbf{a} \in \mathrm{pr}_{I_K}R$.

Since $\mathrm{pr}_{I_{K-\{s\}}}\mathbf{a}, \mathrm{pr}_{I_{K-\{t\}}}\mathbf{a}, \mathrm{pr}_{I_{K-\{q\}}}\mathbf{a}$ belong to $\mathrm{pr}_{I_{K-\{s\}}}R$, $\mathrm{pr}_{I_{K-\{t\}}}R$, $\mathrm{pr}_{I_{K-\{q\}}}R$, respectively, the tuples $(\mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_l), (\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_4, \ldots, \mathbf{a}_l), (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4, \ldots, \mathbf{a}_l)$ belong to $\mathrm{pr}_{I_J - I_1}R, \mathrm{pr}_{I_J - I_2}R, \mathrm{pr}_{I_J - I_3}R$ respectively. Therefore, there are $\mathbf{b}_1 \in \mathrm{pr}_{I_1}R$, $\mathbf{b}_2 \in \mathrm{pr}_{I_2}R$, $\mathbf{b}_3 \in \mathrm{pr}_{I_3}R$ such that $(\mathbf{b}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \ldots, \mathbf{a}_l), (\mathbf{a}_1, \mathbf{b}_2, \mathbf{a}_3, \mathbf{a}_4, \ldots, \mathbf{a}_l)$, $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_3, \mathbf{a}_4, \ldots, \mathbf{a}_l) \in \mathrm{pr}_{I_K}R$. By the Maximal Expansion Lemma 3.8(2), $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ can be chosen from rb-maximal components of $\mathcal{G}(\mathrm{pr}_{I_1}R), \mathcal{G}(\mathrm{pr}_{I_2}R), \mathcal{G}(\mathrm{pr}_{I_3}R)$, respectively. If, say, $\mathbf{b}_1$ is in the same strongly rb-connected component as $\mathbf{a}_1$, then by the Rectangularity Proposition 5.4(2), $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_l) \in \mathrm{pr}_{I_K}R$, and we are done. So, we may assume that the pairs $\mathbf{a}_1, \mathbf{b}_1$, $\mathbf{a}_2, \mathbf{b}_2$, $\mathbf{a}_3, \mathbf{b}_3$ lie in different strongly rb-connected components. Therefore, for every $j \in \{1, 2, 3\}$ and every $v \in I_j$, the pair

$(\mathbf{a}_j[v], \mathbf{b}_j[v])$ is yellow or $\mathbf{b}_j[v] \le \mathbf{a}_j[v]$. Hence,

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \\ \vdots \\ \mathbf{a}_l \end{pmatrix} = g\left(\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \\ \vdots \\ \mathbf{a}_l \end{pmatrix}, \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{b}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \\ \vdots \\ \mathbf{a}_l \end{pmatrix}, \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{b}_3 \\ \mathbf{a}_4 \\ \vdots \\ \mathbf{a}_l \end{pmatrix}\right) \in \mathrm{pr}_K R.$$

□

Now we are in a position to prove the main result of this subsection.

PROPOSITION 5.7 (RED-BLUE DECOMPOSITION PROPOSITION). *If, for any $v \in V$, and any rb-maximal component B of $\mathcal{G}(\mathcal{S}_v)$, the restricted problem $\mathcal{P}_{\mathrm{rb},v,B}$ has a solution, then $\mathcal{P}$ also has a solution.*

PROOF. The proof we give here is a modification of the proof of Theorem 3.5 from [Jeavons et al. 1998].

For any $W \subseteq V$ and any $v \in V - W$, we prove that if φ_W is a partial solution to $\mathcal{P}_W$ such that, for any $w \in W$, $\varphi_W(w)$ belongs to an rb-maximal component B_w, and there is a solution ψ_I to $\mathcal{P}_{\mathrm{rb},v,B_v}$ such that $\psi_I|_{W\cap I} = \varphi_W|_{W\cap I}$ where $I = J(\mathrm{rb}, v, B_v)$, then φ_W can be extended to a solution $\varphi_{W\cup\{v\}}$ of $\mathcal{P}_{W\cup\{v\}}$ satisfying the same conditions.

In the base case for induction we show that, for any set W of the form $J(\mathrm{rb}, w, B_w)$, where B_w is an rb-maximal component of $\mathcal{G}(\mathcal{S}_w)$, any solution φ_W of $\mathcal{P}_{\mathrm{rb},w,B_w}$ can be extended to a solution on the set $W \cup J(\mathrm{rb}, v, B_v)$ for any $v \in V - W$ and some rb-maximal component B_v of $\mathcal{G}(A_{\delta(v)})$. Take $w \in W$; by the Maximal Expansion Lemma 3.8(2), there is a from an rb-maximal component of $\mathcal{G}(\mathcal{S}_v)$ such that $(\varphi_W(w), a) \in \mathcal{S}_{w,v}$. There is a solution ψ to $\mathcal{P}_{\mathrm{rb},v,B_v}$, where B_v is the rb-maximal component containing a. Let $I = J(\mathrm{rb}, v, B_v)$. Again, by the Maximal Expansion Lemma 3.8(2) we may assume that $\psi(v) = a$. We shall prove that a tuple $\mathbf{c} \in \prod_{u\in W\cup I} \mathcal{S}_u$ such that $\mathbf{c}[u] = \varphi_W(u)$ if $u \in W$ and $\mathbf{c}[u] = \psi(u)$ if $u \in I$ is a solution to $\mathcal{P}_{W\cup J(\mathrm{rb},v,B_v)}$ extending φ_W. Let also $C = \langle s, R\rangle \in \mathcal{C}$ be a constraint such that $v' \in s$ for some $v' \in I$, $a' = \psi(v')$, and $w' \in s \cap W$. Since $\mathcal{P}$ is 3-minimal, the tuple $(\varphi_W(w), a') \in \mathcal{S}_{w,v'}$ can be extended to a tuple $(\varphi_W(w), a', b) \in \mathcal{S}_{w,v',w'}$. Moreover, by the Rectangularity Proposition 5.4(2) applied to $\mathcal{S}_{w,v',w'}$, we may choose $b = \varphi_W(w')$. Therefore, $(\varphi_W(w'), a') \in \mathrm{pr}_{\{w',v'\}} R$ can be expanded to a tuple $\mathbf{b} \in R$. For any $u \in s \cap I$, the element $\mathbf{b}[u]$ is a member of the same rb-maximal component as $\psi(u)$, and, for any $u \in s \cap W$, the element $\mathbf{b}[u]$ is a member of the same rb-maximal component as $\varphi_W(u)$. Thus, $\mathrm{pr}_{s\cap I}\mathbf{b}$ is strongly rb-connected with $\psi|_{s\cap I}$ and $\mathrm{pr}_{s\cap W}\mathbf{b}$ is strongly rb-connected with $\varphi_W|_{s\cap W}$. By the Rectangularity Proposition 5.4(2), $\mathrm{pr}_{s\cap(W\cup I)}\mathbf{c} \in \mathrm{pr}_{s\cap(W\cup I)} R$. This means that $\mathbf{c}$ is a solution to $\mathcal{P}_{W\cup J(\mathrm{rb},v,B_v)}$ extending φ_W.

Now suppose that the result is proved for every $U \subseteq V$ with $|U| < k$. Take $W \subseteq V$ with $W = \{w_1, \ldots, w_k\}$ and $v \in V - W$, and let φ_W be a partial solution satisfying the conditions above. Assume for contradiction that φ cannot be extended to a solution to $\mathcal{P}_{W\cup\{v\}}$.

Let the constraints of $\mathcal{P}_{W\cup\{v\}}$ be $\langle s_1, R_1\rangle, \dots, \langle s_q, R_q\rangle$. To obtain the desired contradiction we shall construct a problem $\mathcal{P}'$ which also has q constraints, with the same constraint relations, but with different constraint scopes.

We define the set of variables of $\mathcal{P}'$ to be the union of $\{v'\}$ and q disjoint copies $W_1, \dots, W_q$ of W, where $W_i = \{w_1^i, \dots, w_k^i\}$. The domain function is defined by the equalities $\delta'(w_j^i) = \delta(w_j)$, $\delta'(v') = \delta(v)$. Now, for each $i \in \underline{q}$, we define a mapping $f_i\colon W \to W_i$ by setting $f_i(w_j) = w_j^i$, and extend each f_i to v by setting $f_i(v) = v'$. The set of constraints of $\mathcal{P}'$ is then defined as $\{\langle f_1(s_1), R_1\rangle, \dots, \langle f_q(s_q), R_q\rangle\}$.

Then let the $q \cdot k$-ary relation R be defined as follows

$$\begin{aligned} R = \{&(\sigma(f_1(w_1)), \dots, \sigma(f_1(w_k)), \dots, \sigma(f_q(w_1)), \dots, \sigma(f_q(w_k))) \mid \\ &\sigma \text{ is a solution to } \mathcal{P}'\}. \end{aligned}$$

Note that R is a subdirect product of $\mathcal{S}_{w_1}, \dots, \mathcal{S}_{w_k}, \dots, \mathcal{S}_{w_1}, \dots, \mathcal{S}_{w_k}$. Indeed, since $\mathcal{P}$ is 3-minimal, for any i and any $a \in \mathcal{S}_{w_i}$, there is $b \in \mathcal{S}_v$ such (a, b) is a partial solution of $\mathcal{P}$ on $\{w, v\}$. Furthermore, for any constraint $\langle s_j, R_j\rangle$, this partial solution can be extended to a tuple $\mathbf{a}$ from R_j. Then we assign values to $f_j(w_1), \dots, f_j(w_k)$ accordingly to $\mathbf{a}$ (the variables that are not in the constraint scope $f_j(s_j)$ can be assigned values arbitrarily).

The tuple $\mathbf{a} = (\varphi_W(w_1), \dots, \varphi_W(w_k), \dots, \varphi_W(w_1), \dots, \varphi_W(w_k))$ does not belong to R, since φ_W cannot be extended to a solution to $\mathcal{P}_{W\cup\{v\}}$. However, we shall show that $\mathbf{a}$ satisfies the conditions of the 2-Decomposition Lemma 5.6 for R, and thus derive a contradiction.

Let B_j denote the rb-maximal component containing $\varphi_W(w_j)$. The condition $\mathrm{pr}_{J_R(\mathrm{rb}, w_j^i, B_j)}\mathbf{a} \in \mathrm{pr}_{J_R(\mathrm{rb}, w_j^i, B_j)} R$ is satisfied automatically, because $\mathbf{a}[w_j^i]$ belongs to B_j, an rb-maximal component. For any pair of indices $w_{j_1}^{i_1}, w_{j_2}^{i_2}$, we claim that $\mathrm{pr}_{\{w_{j_1}^{i_1}, w_{j_2}^{i_2}\}}\mathbf{a} \in \mathrm{pr}_{\{w_{j_1}^{i_1}, w_{j_2}^{i_2}\}} R$. Since $\mathcal{P}$ is 3-minimal the tuple $(\varphi_W(w_{j_1}), \varphi_W(w_{j_2})) \in \mathcal{S}_{w_{j_1}, w_{j_2}}$ can be extended to a solution $(\varphi_W(w_{j_1}), \varphi_W(w_{j_2}), a) \in \mathcal{S}_{w_{j_1}, w_{j_2}, v}$ of $\mathcal{P}_{w_{j_1}, w_{j_2}, v}$. Furthermore, for this solution, we can construct a corresponding solution, σ, to $\mathcal{P}'$, such that $\sigma(f_{i_1}(w_{j_1})) = \varphi_W(w_{j_1})$, $\sigma(f_{i_2}(w_{j_2})) = \varphi_W(w_{j_2})$, by the construction of $\mathcal{P}'$. Hence $\mathbf{a}$ agrees with some element of R in $w_{j_1}^{i_1}, w_{j_2}^{i_2}$, which establishes the claim.

Finally, let φ' be an extension of φ to a solution of $\mathcal{P}_{W\cup\{v\}}$. We need to show that it can be chosen such that $\varphi'(v)$ belongs to an rb-maximal component. However, this follows straightforwardly from the Maximal Expansion Lemma 3.8(2) applied to the relation $\mathcal{S}_{W\cup\{v\}}$ and sets $I_1 = W$, $I_2 = \{v\}$. $\square$

Moreover, if the conditions of the Red-Blue Decomposition Proposition 5.7 hold, then a solution to $\mathcal{P}$ can be found in polynomial time by the algorithm Block-3-Width (see Fig. 5.2).

Thus if all the subproblems $\mathcal{P}_{\mathrm{rb},v,B}$ have solutions then the Red-Blue Decomposition Proposition 5.7 allows one to solve the problem $\mathcal{P}$. Otherwise we have the following

Lemma 5.8. *If for one of the problems $\mathcal{P}_{\mathrm{rb},v,B}$ there is a variable w and r-maximal element a from the domain of w such that no solution φ of $\mathcal{P}_{\mathrm{rb},v,B}$ satisfies $\varphi(v) = a$, then $\mathcal{P}$ can be tightened by removing a from the domain of w. More precisely $\varphi(w) = a$ for no solution φ of $\mathcal{P}$.*

INPUT: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.
OUTPUT: A solution to $\mathcal{P}$ if it has one, $\varnothing$ otherwise.

Step 1. **invoke** 3-MINIMALITY($\mathcal{P}$)
Step 2. take $v \in V$, an rb-maximal component B of $\mathcal{G}(\mathcal{S}_v)$, and a solution ψ to $\mathcal{P}_{\mathrm{rb},v,B}$;
set $W = J(\mathrm{rb}, v, B)$
Step 3. **set** $\mathcal{P}_{W,\psi}$ to be the problem instance $(V'; \mathcal{A}; \delta'; \mathcal{C}')$ where
- $V' = V - W$,
- $\delta' = \delta|_{V'}$,
- $\mathcal{C}' = \{C' = \langle s \cap V', R'\rangle \mid \langle s, R\rangle \in \mathcal{C}\}$ where $R' = R$ if $W \cap s = \varnothing$, and $R' = \{\mathbf{a} \in R \mid \mathbf{a}[w] = \psi(w)$ for all $w \in W \cap s\}$ otherwise.

Step 7. **return** BLOCK-3-WIDTH($\mathcal{P}_{W,\psi}$)$\cup\{\psi(w) \mid w \in W\}$.

Fig. 5.2. Algorithm BLOCK-3-WIDTH

PROOF. If $\mathcal{G}(\mathcal{S}_v)$ is not strongly ry-connected for some $v \in V$ and $\mathcal{P}_{\mathrm{ry},w,B}$ has no solution φ such that $\varphi(w) = a$ for some w,B, and an r-maximal element a, then by the definition of $I = J(\mathrm{ry}, w, B)$, for any $u \in I$ and any $\mathbf{a} \in \mathcal{S}_{w,u}$, if $\mathbf{a}[w] \in B$ then $\mathbf{a}[u] \in B_u$, where B_u is the corresponding rb-maximal component. Therefore, for any solution φ of $\mathcal{P}$, if $\varphi(w) = a$ then $\varphi|_I$ is a solution of $\mathcal{P}_{\mathrm{ry},w,B}$. The elements a can be removed from $A_{\delta(w)}$. $\square$

5.4 Strongly ry-connected components

In this section we show how the Rectangularity Proposition 5.4 allows one to solve problems such that, for any $v \in V$, $\mathcal{G}(\mathcal{S}_v)$ is strongly rb-connected and every its r-connected component is strongly r-connected. More precisely, we show how problems of this type can be reduced to problems over smaller strongly ry-connected domains and a problem whose constraints are invariant with respect to a Mal'tsev operation.

The Rectangularity Proposition 5.4(1) reduces the problem $\mathcal{P}$ to a collection of subproblems $\mathcal{P}_{\mathrm{ry},v,B}$, defined analogously to those for strongly rb-connected components, and a *skeleton problem.* The latter problem is defined as follows. We may assume that each problem of the form $\mathcal{P}_{\mathrm{ry},v,B}$ has a solution $\varphi_{\mathrm{ry},v,B}$ found by the algorithm. Moreover, if $w \in J(\mathrm{ry}, v, B)$, then $J(\mathrm{ry}, v, B) = J(\mathrm{ry}, w, B_w)$ where B_w is the ry-connected component corresponding to B, $\mathcal{P}_{\mathrm{ry},v,B} = \mathcal{P}_{\mathrm{ry},w,B_w}$, and $\varphi_{\mathrm{ry},v,B} = \varphi_{\mathrm{ry},w,B_w}$.

Then the skeleton problem is the problem $\mathcal{P}^s = (V; \mathcal{A}; \delta; \mathcal{C}^s)$ obtained by removing all tuples from the constraint relations but those consisting of the solutions $\varphi_{\mathrm{ry},v,B}$: for every $C = \langle s, R\rangle \in \mathcal{C}$ there is $C^s = \langle s, R^s\rangle \in \mathcal{C}^s$ where $\mathbf{a} \in R^s$ if and only if $\mathbf{a} \in R$ and, for any $v \in s$, there is an ry-connected component B of $\mathcal{G}(\mathcal{S}_v)$ such that $\mathbf{a}[v] = \varphi_{\mathrm{ry},v,B}(v)$.

LEMMA 5.9. *Let $\mathcal{P}^s = (V; \mathcal{A}; \delta; \mathcal{C}^s)$ be the skeleton problem of instance $\mathcal{P}$. Then every constraint relation R^s is invariant under all polymorphisms of constraint relations of $\mathcal{P}$.*

PROOF. Let C be the set of polymorphisms of constraint relations of $\mathcal{P}$ and let R' denote the relation generated by R^s by applying operations from C. Let also $\langle s, R\rangle$ be the constraint of $\mathcal{P}$ relation R^s is derived from. Clearly $R' \subseteq R$. Therefore if $R' \neq R^s$ then for a certain variable v, an ry-connected component B of $\mathcal{G}(\mathcal{S}_v)$, and a tuple $\mathbf{a} \in R'$ with $\mathbf{a}[v] \in B$ we have $\mathrm{pr}_{J(\mathrm{ry},v,B)\cap s}\mathbf{a} \notin \mathrm{pr}_{J(\mathrm{ry},v,B)\cap s}R^s$, or more precisely, since every unary projection of R^s is invariant under C, $\mathrm{pr}_{J(\mathrm{ry},v,B)\cap s}\mathbf{a} \neq \varphi_{\mathrm{ry},v,B}|_{J(\mathrm{ry},v,B)\cap s}$. However, this means that there is $w \in J(\mathrm{ry},v,B) \cap s$ such that $\mathbf{a}[w] \notin B_w$ that contradicts the definition of the set $J(\mathrm{ry},v,B)$. $\square$

LEMMA 5.10 (SKELETON DECOMPOSITION LEMMA). *If, for any $v \in V$ and any ry-connected component B of $\mathcal{G}(\mathcal{S}_v)$, the problem $\mathcal{P}_{\mathrm{ry},v,B}$ has a solution, then the skeleton problem has a solution if and only if $\mathcal{P}$ has a solution. Moreover, a solution to $\mathcal{P}$ can be chosen to be a solution of the skeleton problem.*

PROOF. If ψ is a solution of $\mathcal{P}$ then, for any $v \in V$, $\psi|_{J(\mathrm{ry},v,B_v)}$, where $\psi(v) \in B_v$, is a solution of $\mathcal{P}_{\mathrm{ry},v,B_v}$. It follows straightforwardly from the Rectangularity Proposition 5.4(1) that, for any mapping φ with $\varphi|_{\mathrm{ry},v,B_v} = \varphi_{\mathrm{ry},v,B_v}$ and $\varphi|_{V-J(\mathrm{ry},v,B_v)} = \psi|_{V-J(\mathrm{ry},v,B_v)}$, and any constraint $\langle s, R\rangle$, the tuple $\varphi|_s$ belongs to R. Therefore φ is a solution of $\mathcal{P}$. Processing in this way for every $v \in V$ we get a solution of $\mathcal{P}$ composed of solutions of the form $\varphi_{\mathrm{ry},v,B_v}$. $\square$

Note that if $\mathcal{S}^s_v$ denotes the partial solution to the skeleton problem on $v \in V$, then every edge of $\mathcal{S}^s_v$ is blue. Therefore the skeleton problem can be solved by the algorithm MALTSEV described in [Bulatov 2002b; Bulatov and Dalmau 2006].

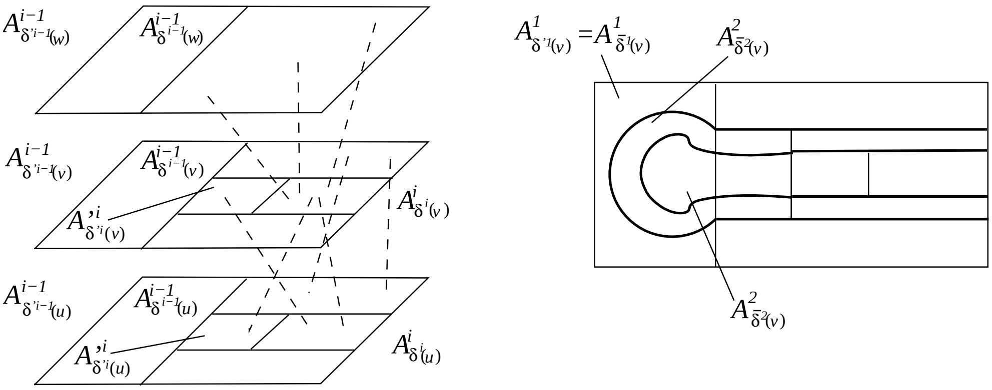
LEMMA 5.11. *If for one of the problems $\mathcal{P}_{\mathrm{ry},v,B}$ there is a variable w and r-maximal element a from the domain of w such that no solution φ of $\mathcal{P}_{\mathrm{ry},v,B}$ satisfies $\varphi(v) = a$, then $\mathcal{P}$ can be tightened by removing a from the domain of w. More precisely $\varphi(w) = a$ for no solution ψ of $\mathcal{P}$.*

PROOF. If $\mathcal{P}_{\mathrm{ry},w,B}$ has no solution φ such that $\varphi(w) = a$ for some w, B, and an r-maximal element a, then by the definition of $I = J(\mathrm{ry},w,B)$, for any $u \in I$ and any $\mathbf{a} \in \mathcal{S}_{w,u}$, if $\mathbf{a}[w] = a$ then $\mathbf{a}[u] \in B_u$, where B_u is the corresponding ry-maximal component. Therefore, for any solution φ of $\mathcal{P}$, if $\varphi(w) = a$ then $\varphi|_I$ is a solution of $\mathcal{P}_{\mathrm{ry},w,B}$. The element a can be removed from $A_{\delta(w)}$. $\square$

5.5 Problems over strongly ry/rb-connected domains

In this subsection we return to problems from Section 4 and consider 3-minimal CSPs, in which, for any $w \in V$, the graph $\mathcal{G}(\mathcal{S}^{\max}_w)$, where $\mathcal{S}^{\max}_w$ is the set of all r-maximal elements from $\mathcal{G}(\mathcal{S}_w)$ is strongly ry/rb-connected, but for some $v \in V$ the set $\mathcal{S}^{\max}_v$ is not hereditarily strongly ry/rb-connected. The requirement of the set of r-maximal elements to be ry/rb-connected rather than the domains themselves comes from the fact that this is how problems of this type occur in the algorithm given in Section 7.2. Although such problems cannot be solved using only results from Sections 4 and 5, we show how these results allow one to reduce solving a problem of this type to solving smaller problems.

We use notation introduced in Section 4.3. Since $\mathcal{S}^{\max}_v$ is not hereditarily strongly ry/rb-connected, there are sequences $\mathcal{S}^{\max}_v \times \mathcal{S}^{\max}_v \supseteq \theta_0 \supseteq \theta_1 \supseteq \ldots \supseteq \theta_{k-1}$, $B'_0, B'_1, \ldots, B'_k$, and $B_0, B_1, \ldots, B_k$ such that $B'_0 = \mathcal{S}_v$, $B_0 = \mathcal{S}^{\max}_v$, θ_i is a maximal congruence of B_i, the set B'_{i+1} is a class of θ_i, the set B_{i+1} is the set of r-maximal elements of $\mathcal{G}(B'_{i+1})$, and the graphs $\mathcal{G}(B_0), \ldots, \mathcal{G}(B_{k-1})$ are strongly ry/rb-connected while $\mathcal{G}(B_k)$ is not. For the sake of brevity, we denote these sequences by $\overline{\theta}$, $\overline{B}'$ and $\overline{B}$, respectively.



(1) (2)

Figure 5.3

We also define a sequence of problems $\mathcal{P}^0_{v,\overline{B}',\overline{\theta}}, \mathcal{P}^1_{v,\overline{B}',\overline{\theta}}, \dots, \mathcal{P}^k_{v,\overline{B}',\overline{\theta}}$ as follows: $\mathcal{P}^0_{v,\overline{B}',\overline{\theta}} = \mathcal{P}^{\max} = (V^0; \mathcal{A}; \delta^0; \mathcal{C}^0)$ with $V^0 = V$, $A_{\delta^0(w)} = \mathcal{S}^{\max}_w$ and, for any $\langle s, R\rangle \in \mathcal{C}$, there is $\langle s, R^0\rangle \in \mathcal{C}^0$, $R^0 = R \cap \prod_{w\in V} A_{\delta^0(w)}$. Then, we intend to define $\mathcal{P}^{i+1}_{v,\overline{B}',\overline{\theta}}$ to be $(\mathcal{P}^i_{v,\overline{B}',\overline{\theta}})_{v,\theta_i,B_{i+1}}$ (recall that such problems were defined in the beginning of Section 4.3), however, it may happen that the graphs of some domains of $\mathcal{P}^i_{v,\overline{B}',\overline{\theta}}$ are not strongly ry/rb-connected and these domains should be excluded. Thus let $\mathcal{P}^i_{v,\overline{B}',\overline{\theta}} = (V^i; \mathcal{A}; \delta^i; \mathcal{C}^i)$, then let $\mathcal{P}'^i = (\mathcal{P}^i_{v,\overline{B}',\overline{\theta}})_{V^i - W^i}$, where W^i is the set of those variables from V^i, for which the graph of the domain in $\mathcal{P}^i$ is not strongly ry/rb-connected, and $\mathcal{P}^{i+1}_{v,\overline{B}',\overline{\theta}} = \mathcal{P}'^i_{v,\theta_i,B_{i+1}}$. We also denote by U_i variables $w \in V^i$ connected to v, θ_{i-1}. Then for each such variable the set $\mathcal{S}^i_{w,v}$ of partial solutions of $\mathcal{P}^i_{v,\overline{B}',\overline{\theta}}$ on w, v is the graph of a mapping $\pi^i_w\colon A_{\delta^i(w)} \to A_{\delta^i(v)}/_{\theta_{i-1}}$. It is not hard to see that $\ker \pi^{i-1}_w$ is a maximal congruence of $A_{\delta^{i-1}(w)}$ and $(\pi^{i-1}_w)^{-1}(B'_i)$ is a class of this congruence. We denote this class by $A_{\delta'^i(w)}$ (see Fig. 5.3(1)). Finally, let $W_{v,\overline{B}',\overline{\theta}} = W^k$.

Lemma 5.12. *If the problems $\mathcal{P}^i_{v,\overline{B}',\overline{\theta}}$ are as defined above then $W_{v,\overline{B}',\overline{\theta}} \subseteq U_k \subseteq \ldots \subseteq U_1$.*

Proof. The inclusion $W_{v,\overline{B}',\overline{\theta}} \subseteq U_k$ follows from an easy observation that if $w \in U_i$ is such that $\mathcal{G}(A_{\delta^i(w)})$ is not strongly ry/rb-connected then by construction $w \notin V^{i+1}$. In order to derive a contradiction we assume that $U_{i+1} \not\subseteq U_i$, $w \in U_{i+1} - U_i$ and $a \in A_{\delta^{i-1}(w)} = A_{\delta^i(w)}$ is such that $a \notin C = A_{\delta'^{i+1}(w)} = (\pi^i_w)^{-1}(B'_{i+1})$ but, for any $b \in C$, we have $b \leq a$ or $\langle a, b\rangle \in \beta$. This is possible, because $\mathcal{G}(A_{\delta^i(w)}/_{\ker \pi^i_w})$ is strongly ry/rb-connected and C is a class of a congruence. Let D be a class of θ_{i-1} such that, for any $c \in B'_i$ and $d \in D$, we have $d \leq c$ or $\langle c, d\rangle \in \gamma$. Such a class exists by the same reason. Since $w \notin U_i$, there is $d \in D$ with $(a, d) \in \mathrm{pr}_{w,v} R$, and there is $b \in C$ and $c \in B'_i$ with $(b,c) \in \mathrm{pr}_{w,v}R$. Then

$$\begin{pmatrix} a \\ c \end{pmatrix} = p\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}\right) \in \mathrm{pr}_{w,v}R.$$

As $a \notin C$ while $c \in B'_i$, we get a contradiction. $\square$

We shall be interested in solutions of problems of the form $(\mathcal{P}^k_{v,\overline{B}',\overline{\theta}})_{W_{v,\overline{B}',\overline{\theta}}}$, that is the problem $\mathcal{P}^k_{v,\overline{B}',\overline{\theta}}$ restricted onto set $W_{v,\overline{B}',\overline{\theta}}$. Let us denote this problem by $\mathcal{P}^*_{v,\overline{B}',\overline{\theta}}$. First, we study how problems of this form change, or rather why they do not change, if we build sequences defined above for two variables simultaneously or sequentially. In Lemma 5.13 we look at what happens if we make one step in each sequence; then in Lemma 5.14 one of sequences moves on one step and the other is built to the end; finally, in Lemma 5.15 both sequence completely unfold.

LEMMA 5.13. *Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a 3-minimal problem such that the graph of every its domain is strongly ry/rb-connected. Let $v, w \in V$, and let θ, η be maximal congruences of $A_{\delta(v)}, A_{\delta(w)}$ respectively, and B, C be classes of θ, η respectively. Let also U and W denote the sets of variables connected to v, θ and w, η in $\mathcal{P}$, and let U' and W' denote the sets of variables connected to v, θ and w, η in the problems $\mathcal{P}_{w,C,\eta}$ and $\mathcal{P}_{v,B,\theta}$ respectively. If $v \notin W$ then*
- $v \notin W'$,
- for any $u \notin W$, we have $u \in U$ if and only if $u \in U'$,
- if $u \in U \cap W$ and θ', η' denote the kernels of mappings $\pi : A_{\delta(u)} \to A_{\delta(v)}/_\theta$ and $\varrho : A_{\delta(u)} \to A_{\delta(w)}/_\eta$ such that $\mathcal{S}_{u,v}/_\theta$ and $\mathcal{S}_{u,w}/_\eta$ are the graphs of π and ϱ, then $\theta' \neq \eta'$ and the graph of every class of both congruences is strongly ry/rb-connected.

Remark. Since $v \notin W$, the variable v has the same domain in the problems $\mathcal{P}_{w,C,\eta}$ and $\mathcal{P}$. However, w can belong to U, and thus the proper definition of W' should be: the set of all variables connected to w, η'', where $\eta'' = \eta \cap (\pi_w(B) \times \pi_w(B))$. Although, by what the lemma claims, this does not change the result.

PROOF. Let $\mathcal{P}' = \mathcal{P}_{v,B,\theta}$, let δ' be the domain function of this problem, and let $\mathcal{S}_{v,w}, \mathcal{S}'_{v,w}$ be the sets of partial solutions on $\{v, w\}$ of $\mathcal{P}, \mathcal{P}'$ respectively. By Lemma 4.10, $\mathcal{S}'_{v,w} = \mathcal{S}_{v,w} \cap (A_{\delta'(v)} \times A_{\delta'(w)})$. Now as $v \notin W$, we have $\mathcal{S}_{v,w}/_\eta = A_{\delta(v)} \times A_{\delta(w)}/_\eta$. Therefore $\mathcal{S}'_{v,w}/_\eta = A_{\delta'(v)} \times A_{\delta(w)}/_\eta$, and $v \notin W'$.

Take $u \notin W$ and consider $\mathcal{S}_{v,u,w}$. Again by the Double Connected Rectangularity Lemma 4.8, $\mathcal{S}_{v,u,w}/_\eta = \mathcal{S}_{v,u} \times A_{\delta(w)}/_\eta$. This immediately implies that $u \in U$ if and only if $u \in U'$.

Now take $u \in U \cap W$ and define $\pi, \varrho, \theta', \eta'$ as in the lemma. Let us consider, for $a \in A_{\delta(v)}$, any tuple $(a, b, c) \in \mathcal{S}_{v,u,w}$. Since $v \notin W$, element c can be chosen from any η-class, hence b can belong to any η'-class. However, $b \in \pi(a)$, and therefore every θ'-class intersects with any η'-class.

Let D be a θ'-class and $d_1, d_2 \in D$. We show that d_1, d_2 are both ry- and rb-connected. In order to do this we consider the congruence $\eta'' = \eta' \cap D^2$ of D. Recall that as $\mathcal{G}(A_{\delta(u)})$ is strongly ry/rb-connected, $\mathcal{G}(A_{\delta(u)}/_{\eta'})$ is also strongly ry/rb-connected, and, as D intersects with every η'-class, $\mathcal{G}(D/_{\eta''})$ is also strongly ry/rb-connected. Moreover, every ry- [rb-] path in $\mathcal{G}(D/_{\eta'})$ gives rise to an ry- [rb-]

path for any representatives of the corresponding η''-classes. Therefore there are an ry-path and an rb-path from d_1 to d_2. Thus $\mathcal{G}(D)$ is strongly ry/rb-connected. $\square$

LEMMA 5.14. *Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a 3-minimal problem such that the graph of every its domain is strongly ry/rb-connected, and $v, \overline{B}', \overline{\theta}$ defined as in the beginning of Section 5.5. Let $w \in V$, η be a congruence of $\mathcal{S}_w$ and C' a class of η. Let also $\mathcal{P}'$ denote the problem $\mathcal{P}_{w,C',\eta}$ and let U' denote the set of variables connected to w, η. Then if $v \notin U'$ then $\mathcal{P}'^{*}_{v,\overline{B}',\overline{\theta}} = \mathcal{P}^{*}_{v,\overline{B}',\overline{\theta}}$.*

PROOF. Notice first that problem $\mathcal{P}'^{*}_{v,\overline{B}',\overline{\theta}}$ is defined correctly. To see this we just need to observe that since $v \notin U'$ the domain of v in $\mathcal{P}'$ equals that in $\mathcal{P}$. Therefore the congruences from $\overline{\theta}$ and their classes from $\overline{B}'$ are the same for both problems.

We prove the lemma by induction on k. Suppose first that $k = 1$. Let W and W' denote the sets of the form $W_{v,\overline{B}',\overline{\theta}}$ defined for $\mathcal{P}$ and $\mathcal{P}'$ respectively. Then, by Lemma 5.13, $W \cap U' = W' \cap U' = \varnothing$, and by Lemma 4.10, for any $u \notin U'$ the domains of u in $\mathcal{P}^{1}_{v,\overline{B}',\overline{\theta}}$ and $\mathcal{P}'^{1}_{v,\overline{B}',\overline{\theta}}$ are the same. This implies $\mathcal{P}'^{*}_{v,\overline{B}',\overline{\theta}} = \mathcal{P}^{*}_{v,\overline{B}',\overline{\theta}}$.

Now suppose that the lemma is true for any problem $\mathcal{P}''$, for which $\mathcal{P}''^{*}_{v,\overline{B}',\overline{\theta}}$ is a result of a sequence of problems shorter than k. Take $\mathcal{P}''$ to be $\mathcal{P}^{1}_{v,\overline{B}',\overline{\theta}}$. Obviously, $\mathcal{P}''^{*}_{v,\overline{B}',\overline{\theta}} = \mathcal{P}^{*}_{v,\overline{B}',\overline{\theta}}$, so the only thing remaining to prove is that $\mathcal{P}''$ satisfies the conditions of the lemma. However, this straightforwardly follows from Lemma 5.13. $\square$

The following two lemmas show that we either can reduce $\mathcal{P}$ to smaller problems, or we are able to tighten it.

LEMMA 5.15. *If, for any $v \in V$ such that $A_{\delta(v)}$ is not hereditarily ry/rb-connected, and any $B'_1, \dots, B'_k$ and $\theta_1, \dots, \theta_{k-1}$ witnessing this, the problem $\mathcal{P}^{*}_{v,\overline{B}',\overline{\theta}}$ has a solution φ, then $\mathcal{P}$ has a solution.*

PROOF. We choose v, $B'_1, \dots, B'_k$, and $\theta_1, \dots, \theta_{k-1}$ such that k is minimal possible. In this case $V^1 = \ldots = V^k = V$. First, we show that, for any solution ψ of $(\mathcal{P}^{k}_{v,\overline{B}',\overline{\theta}})_U$, where $U = V - W_{v,\overline{B}',\overline{\theta}}$, then the mapping χ on V defined by

$$\chi(w) = \begin{cases} \psi(w), & \text{if } w \in U, \\ \varphi(w), & \text{if } w \in W_{v,\overline{B}',\overline{\theta}} \end{cases}$$

is a solution of $\mathcal{P}^{k}_{v,\overline{B}',\overline{\theta}}$ and therefore of $\mathcal{P}$.

By Lemma 4.10, $\mathcal{P}^{k}_{v,\overline{B}',\overline{\theta}}$ is 3-minimal.

Take a constraint $\langle s, R \rangle \in \mathcal{C}^k$; we show that $\mathrm{pr}_{s \cap W_{v,\overline{B}',\overline{\theta}}} R \times \mathrm{pr}_{s \cap U} R \subseteq R$. It is not hard to see that this suffices to obtain the required result. Let W^{ry} be the set of variables $w \subseteq W_{v,\overline{B}',\overline{\theta}}$ for which $\mathcal{G}(A_{\delta^k(w)})$ is strongly ry-connected (but not rb-connected) and W^{rb} the set of $w \in W_{v,\overline{B}',\overline{\theta}}$ such that $\mathcal{G}(A_{\delta^k(w)})$ is strongly rb-connected (but not ry-connected). By the Rectangularity Proposition 5.4(1),(2), R is ry- and rb-rectangular. In particular, for any tuple $\mathbf{a} \in R$ and tuple $\mathbf{b} \in \mathrm{pr}_{s \cap W} R$ such that $\mathbf{b}[w]$ is strongly ry-connected with $\mathbf{a}[w]$ for $w \in W^{\mathrm{rb}}$ and $\mathbf{b}[w]$ is strongly rb-connected with $\mathbf{a}[w]$ for $w \in W^{\mathrm{ry}}$, the tuple $(\mathbf{b}, \mathrm{pr}_{s \cap U} \mathbf{a})$ belongs to R.

Then take $\mathbf{d} \in \mathrm{pr}_{s \cap U} R$, we show that, for any $U' \subseteq s \cap W_{v,\overline{B}',\overline{\theta}}$ there is $\mathbf{b} \in \mathrm{pr}_{U'} R$ such that $\mathbf{b}[w]$ is strongly rb-connected with $\mathbf{a}[w]$ if $w \in W^{\mathrm{ry}}$ and $\mathbf{b}[w]$ is strongly ry-connected with $\mathbf{a}[w]$ if $w \in W^{\mathrm{rb}}$, and $(\mathbf{b}, \mathbf{d}) \in \mathrm{pr}_{U' \cup (s \cap U)} R$. Suppose for contradiction that, for $U' \subseteq s \cap W_{v,\overline{B}',\overline{\theta}}$ and for any $\mathbf{d} \in \mathrm{pr}_{s\cap U} R$, there is a tuple $\mathbf{b}$ satisfying the required conditions, but this is not true for $U'' = U' \cup \{w\} \subseteq s \cap W_{v,\overline{B}',\overline{\theta}}$. We assume first that $w \in W^{\mathrm{rb}}$, that is, for any $\mathbf{b} \in \mathrm{pr}_{U''} R$ such that $\mathrm{pr}_{U'} \mathbf{b}$ is strongly ry/rb- connected with $\mathrm{pr}_{U'} \mathbf{a}$, we have $\langle \mathbf{b}[w], \mathbf{a}[w] \rangle \in \gamma$. Let B be the strongly ry/rb-connected component of $\mathrm{pr}_{U''} R$ containing $\mathrm{pr}_{U''} \mathbf{a}$ and $C \subseteq R' = \mathrm{pr}_{s\cap U} R$ the set all tuples from R' extendible with an element from B. Since by the Double Connectedness Proposition 4.9 R' is strongly ry-connected, there are $\mathbf{c}_1 \in C$, $\mathbf{c}_2 \in R' - C$ such that $\mathbf{c}_1 \le \mathbf{c}_2$ or $\langle \mathbf{c}_1, \mathbf{c}_2 \rangle \in \beta$. By the assumption made there are $\mathbf{b}_1, \mathbf{b}_2 \in \mathrm{pr}_{U''} R$ such that $\mathbf{b}_1 \in B$, $\mathrm{pr}_{U'} \mathbf{b}_2 \in \mathrm{pr}_{U'} B$, and $(\mathbf{b}_1, \mathbf{c}_1), (\mathbf{b}_2, \mathbf{c}_2) \in \mathrm{pr}_{U'' \cup (s\cap U)} R$. As $\mathbf{b}_2 \notin B$, we get $\langle \mathbf{b}_1[w], \mathbf{b}_2[w] \rangle \in \gamma$ and $\begin{pmatrix} \mathbf{b} \\ \mathbf{c}_2 \end{pmatrix} = p\left(\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{c}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{b}_2 \\ \mathbf{c}_2 \end{pmatrix} \right) \in \mathrm{pr}_{U'' \cup (s\cap U)} R$. For the obtained tuple we have $\mathbf{b}[u] \in \{\mathbf{b}_1[u], \mathbf{b}_2[u]\}$ for every $u \in U'$, and therefore is strongly ry/rb-connected with $\mathbf{a}[u]$; then, $\mathbf{b}[w] = \mathbf{b}_1[w]$ and is strongly ry/rb-connected with $\mathbf{a}[w]$ as well. By the Double Connectedness Proposition 4.9, $\mathbf{b}$ is strongly ry/rb-connected with $\mathrm{pr}_{U''} \mathbf{a}$, a contradiction.

For the variables from W^{ry}, the proof is similar.

Then we prove by induction on the number of variables in $\mathcal{P}$ that $(\mathcal{P}^k_{v,\overline{B}',\overline{\theta}})_U$ has a solution of the required form. If $|V| = 1$ then $U = \varnothing$ and there is nothing to prove. So, suppose that the lemma holds for any problem of the specified type with fewer variables than $\mathcal{P}$. Hence we just need to prove that $\mathcal{P}' = (\overline{\mathcal{P}}^k_{v,\overline{B}',\overline{\theta}})_U$ satisfies the conditions of Lemma 5.15.

Take $w \in U$ such that $A_{\delta^k(w)}$ is not hereditarily ry/rb-connected; this implies that so is $A_{\delta(w)}$. Suppose first that $w \notin U_1$. Then applying Lemma 5.14 inductively to problems $\mathcal{P}^0_{v,\overline{B}',\overline{\theta}}, \mathcal{P}^1_{v,\overline{B}',\overline{\theta}}, \dots, \mathcal{P}^k_{v,\overline{B}',\overline{\theta}}$ we get $\mathcal{P}^{k*}_{w,\overline{B}',\overline{\xi}} = \mathcal{P}^*_{w,\overline{B}',\overline{\xi}}$, which has a solution by the assumption made. If $w \in U_i - U_{i+1}$ then, as is easily seen, $\mathcal{P}^i_{v,\overline{B}',\overline{\theta}} = \mathcal{P}^i_{w,\overline{D}',\overline{\eta}}$, where $D'_j = (\pi^j_w)^{-1}(B'_j)$ and $\eta_j = \ker \pi^j_w$ for $j \in \{0, \dots, i\}$. As before, $\mathcal{P}^{i*}_{w,\overline{D}',\overline{\eta}} = \mathcal{P}^*_{w,\overline{D}',\overline{\eta}}$. Thus applying Lemma 5.14 to the problems $\mathcal{P}^i_{v,\overline{B}',\overline{\theta}}, \dots, \mathcal{P}^k_{v,\overline{B}',\overline{\theta}}$ we obtain the result. $\square$

To prove that if the conditions of Lemma 5.15 are not satisfied then the problem can be tightened, we need another sequence of problems, $\overline{\mathcal{P}}^0_{v,\overline{B},\overline{\theta}}, \overline{\mathcal{P}}^1_{v,\overline{B},\overline{\theta}}, \dots, \overline{\mathcal{P}}^k_{v,\overline{B},\overline{\theta}}$, which are defined as follows: $\overline{\mathcal{P}}^i_{v,\overline{B},\overline{\theta}} = (V^i; \mathcal{A}; \overline{\delta}^i; \overline{\mathcal{C}}^i)$, where $A_{\overline{\delta}^0(w)} = A_{\delta(w)}$ for all $w \in V$, and, for $i > 0$, $A_{\overline{\delta}^i(w)} = A_{\overline{\delta}^{i-1}(w)}$ if $w \in V^i - U_i$ and $A_{\overline{\delta}^i(w)} = A_{\delta'^i(w)} \cup \{a \in A_{\overline{\delta}^{i-1}(w)} - A_{\delta^{i-1}(w)} \mid$ there is $b \in A_{\delta'^i(w)}$ such that $a \prec b$ in $A_{\overline{\delta}^{i-1}(w)}\}$ if $w \in U_i$ (see Fig. 5.3(2)).

Lemma 5.16 (Double Connected Tightening Lemma). *If a problem of the form $\mathcal{P}^*_{v,\overline{B},\overline{\theta}}$ has no solution φ such that $\varphi(w) = a$ for some variable w and an r-maximal element a, then $\mathcal{P}$ can be tightened by removing elements from some of the domains. More precisely $\varphi(w) = a$ for no solution of $\mathcal{P}$.*

Proof. First, notice that φ is a solution of $\mathcal{P}^*_{v,\overline{B},\overline{\theta}}$ if and only if it is a solution of $(\overline{\mathcal{P}}^k_{v,\overline{B},\overline{\theta}})_{W_{v,\overline{B},\overline{\theta}}}$ such that the value of every variable w belongs to the set of r-

Input: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ such that, for all $v \in V$, $\mathcal{G}(\mathcal{S}_v)$ is strongly ry/rb-connected.

Output: A solution to $\mathcal{P}$.

Step 1. **if**, for any $v \in V$, $\mathcal{S}_v$ is hereditarily ry/rb-connected, **then return** 3-Width($\mathcal{P}$)
Step 2. take $v \in V$ of minimal depth and sequences $\overline{B}_v$ of $\overline{\theta}_v$, and invoke Conserv($\mathcal{P}^*_{v,\overline{B}_v,\overline{\theta}_v}$); let ψ be its solution
Step 3. **set** $\mathcal{P}'$ to be the problem instance $(\mathcal{P}^k_{v,\overline{B}_v,\overline{\theta}_v})_{V-W_{v,\overline{B}_v,\overline{\theta}_v}}$
Step 4. **return(**Ry/rb-Conn($\mathcal{P}'$)$\cup\{w = \psi(w) \mid w \in W_{v,\overline{B}_v,\overline{\theta}_v}\}$).

Fig. 5.4. Algorithm Ry/rb-Conn

maximal elements of $\mathcal{G}(A_{\overline{\delta}^k(w)})$. If there are $w \in W_{v,\overline{B},\overline{\theta}}$ and $a \in A_{\delta^k(w)}$ such that $\varphi(w) = a$ for no solution of $\mathcal{P}^*_{v,\overline{B},\overline{\theta}}$, then this also holds for $\overline{\mathcal{P}}^k_{v,\overline{B},\overline{\theta}}$. It is not hard to see that without loss of generality we may assume $w = v$. The following claim implies induction step.

Claim. If the problem $\overline{\mathcal{P}}^i$ has a solution φ such that $\varphi(v) \in B_{i+1}$, then φ is a solution of $\overline{\mathcal{P}}^{i+1}$.

We show that, for any constraint $C = \langle s, R\rangle \in \overline{\mathcal{C}}^i$ and any tuple $\mathbf{a} \in R$, if $\mathbf{a}[v] \in B_{i+1}$, then, for any $u \in s \cap U_{i+1}$, $\mathbf{a}[u] \in A_{\overline{\delta}^{i+1}(u)}$.

Suppose first that $\mathbf{a}[u] \in A_{\delta^i(u)}$. Then by the construction of $\overline{\mathcal{P}}^{i+1}$, we have $\mathbf{a}[u] \in (\pi^i_u)^{-1}(B'_i) = A_{\delta'^{i+1}(u)} \subseteq A_{\overline{\delta}^{i+1}(u)}$. If $\mathbf{a}[u] \notin A_{\delta^i(u)}$, then, by the Maximality Lemma 3.7(2), there is $\mathbf{b} \in R$ such that $\mathbf{b}[v] = a$, $\mathbf{b}[u]$ is an r-maximal element in $\mathcal{G}(A_{\overline{\delta}^i(u)})$ and $\mathbf{a}[u] \prec \mathbf{b}[u]$. If $\mathbf{b}[u] \notin A_{\delta^{i+1}(u)}$ then we get a contradiction with the construction of $\overline{\mathcal{P}}^{i+1}$. Otherwise, $\mathbf{a}[u] \in A_{\overline{\delta}^i(u)}$ as required.

As $\mathcal{P} = \overline{\mathcal{P}}^0$, it has no solution φ with $\varphi(v) = a$. $\square$

We will use $\overline{\theta}_v$ and $\overline{B}_v$ to denote the shortest sequences of congruences and their classes witnessing that $\mathcal{S}_v$ is not hereditarily strongly ry/rb-connected. The length k of these sequences is called the *depth* of v. If $\mathcal{S}_v$ is hereditarily strongly ry/rb-connected, then the depth of v is defined to be ∞.

Thus if $\mathcal{P}^*_{v,\overline{B},\overline{\theta}}$ has no solution for some $v, \overline{B}$ and $\overline{\theta}$, then the Double Connected Tightening Lemma) 5.16 claims that $\mathcal{P}$ can be tightened. Otherwise $\mathcal{P}$ can be solved using the algorithm Ry/rb-Conn (see Fig. 5.4).

6. RED/BLUE-CONNECTED RELATIONS

In this section we consider the most difficult case of conservative constraint problems. We saw in the previous sections that such a problem can be reduced to smaller problems if either all the domains in the problem are both strongly ry- and strongly rb-connected, or there is a domain which is not strongly rb-connected, or all elements of all the domains are r-maximal. Thus in the remaining case we assume that every domain is strongly rb-connected, and there is a domain, for which the quasi-order $\prec$ is non-trivial in the sense that there are a, b such that $a \prec b$, but not $b \prec a$. It is not difficult to come up with an example that shows that ry-rectangularity does not hold in this case, even for the sets of r-maximal elements of the domains. Our main goals in this section are: to introduce another type of rectangularity, max-rectangularity, that uses the non-triviality of $\prec$; to prove that in the case considered this type of rectangularity holds; and finally to use max-rectangularity to solve conservative constraint problems.

In order to achieve these goals we first study certain properties of r-paths in relations (Section 6.1). Then in Section 6.2 we define the filter of an element a with respect to quasi-order $\prec$ as the set of elements b with $a \prec b$, and the set $\max(a)$ of r-maximal elements associated with a as the set of elements strongly ry-connected to the r-maximal elements from the filter of a. We also introduce another quasi-order $\sqsubseteq$ induced by the reverse inclusion of sets $\max(a)$. We use these notions to prove several auxiliary results that basically amount to say that under certain conditions the set of r-maximal elements associated with some tuple can be decomposed into the direct product of sets of r-maximal elements associated with fragments of the tuple. The culmination of this section is the Maximal Rectangularity Proposition 6.11, which is very similar to the Rectangularity Proposition 5.4, but uses sets $\max(a)$ for a's that are *minimal* with respect to $\sqsubseteq$ instead of rb- (ry-) maximal components, and sets $J(\mathrm{m}, v, B)$ instead of $J(\mathrm{rb}, v, B)$, $J(\mathrm{ry}, v, B)$. Finally, in Section 6.4, we use the Maximal Rectangularity Proposition 6.11 to reduce problems over domains of the described type to problems over strongly ry-connected domains.

We will call a strongly ry-connected component simply an *ry-component*.

6.1 Paths

In this section we prove an auxiliary property roughly stating that if $(a_1, b_1), \dots, (a_k, b_k)$ is an r-path in a subdirect product of two domains then either, for any tuple $\mathbf{c}$ from the product such that $\mathbf{c}[1] = a_i$ and $\mathbf{c}[2] \in \{b_1, \dots, b_k\}$, we have $\mathbf{c}[2] = b_i$, or for every i, j, there is a tuple $\mathbf{c}$ in the product such that $\mathbf{c}[1] = a_i$ and $\mathbf{c}[2] = b_j$.

Let $a_1, \dots, a_k$ be an r-path in $\mathcal{G}(A)$, $A \in \mathcal{A}$. It is said to be *irreducible* if, for any $i \in \underline{k-2}$, if $a_i \le a_{i+2}$ then $a_{i+1} = \dots = a_k$. In other words, an r-path is irreducible if none of the intermediate elements can be omitted without destroying the r-path.

LEMMA 6.1. *Let R be an n-ary relation and $\mathbf{a}_1, \dots, \mathbf{a}_k \in R$ an r-path. Then, for any $i, j \in \underline{n}$, there is an r-path $\mathbf{b}_1, \dots, \mathbf{b}_m$ such that $\mathbf{b}_1 = \mathbf{a}_1$, $\mathbf{b}_1[l], \dots, \mathbf{b}_m[l] \in \{\mathbf{a}_1[l], \dots, \mathbf{a}_k[l]\}$, for $l \in \underline{n}$, $\mathbf{b}_m[i] = \mathbf{a}_k[i], \mathbf{b}_m[j] = \mathbf{a}_k[j]$, and both $\mathbf{b}_1[i], \dots, \mathbf{b}_m[i]$ and $\mathbf{b}_1[j], \dots, \mathbf{b}_m[j]$ are irreducible.*

PROOF. By induction on the the number of different elements in the sequences $\mathbf{a}_1[1], \dots, \mathbf{a}_k[1]$ and $\mathbf{a}_1[2], \dots, \mathbf{a}_k[2]$. $\square$

Now we prove the main result of Section 6.1.

LEMMA 6.2 (PATH ALIGNMENT LEMMA). *Let $\mathbf{a}_1, \dots, \mathbf{a}_k \in R$ be an r-path such that $\mathbf{a}_1[1], \dots, \mathbf{a}_k[1]$ and $\mathbf{a}_1[2], \dots, \mathbf{a}_k[2]$ are irreducible, and $\mathbf{a}_{k-1}[1] \neq \mathbf{a}_k[1]$. If there is $\mathbf{b} \in R$ such that $\mathbf{b}[1] = \mathbf{a}_k[1]$ and $\mathbf{b}[2] = \mathbf{a}_{k-1}[2]$, then*
(1) there is $\mathbf{c} \in R$ such that $\mathbf{c}[1] = \mathbf{a}_1[1]$ and $\mathbf{c}[2] = \mathbf{a}_{k-1}[2]$;
(2) if $k > 2$ then, for any $1 \le i, j \le k$ there is $\mathbf{c}_i^j \in R$ such that $\mathbf{c}_i^j[1] = \mathbf{a}_i[1]$ and $\mathbf{c}_i^j[2] = \mathbf{a}_j[2]$.
If, in addition, for any $l \in \{3, \dots, n\}$, $\mathbf{b}[l] \in \{\mathbf{a}_1[l], \dots, \mathbf{a}_k[l]\}$, then the tuples $\mathbf{c}$, $\mathbf{c}_i^j$ can be chosen such that $\mathbf{c}[l], \mathbf{c}_i^j[l] \in \{\mathbf{a}_1[l], \dots, \mathbf{a}_k[l]\}$ for any $l \in \{3, \dots, n\}$ (see Fig. 6.1).

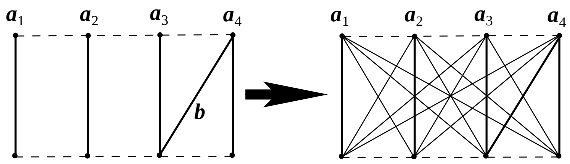


Fig. 6.1. Skew tuples in a crossed r-path

Note that if, say, $\mathbf{a}_{k-1}[2] = \mathbf{a}_k[2]$ then $\mathbf{b}$ can chosen to be $\mathbf{a}_k$.

Proof. (1) We begin with constructing a subsequence $j_1, \ldots, j_l$ from $1, \ldots, k$ such that $j_1 = k-1$; let p be such that $\mathbf{a}_{k-p-1}[2] \neq \mathbf{a}_{k-p}[2] = \mathbf{a}_{k-p+1}[2] = \ldots = \mathbf{a}_{k-1}[2]$. Then set $j_2 = k-p-2$, and if j_i is defined then $j_{i+1} = j_i - 2$. Thus, $\langle \mathbf{a}_{j_{i+1}}[2], \mathbf{a}_{j_i}[2]\rangle \in \beta \cup \gamma$ or $\mathbf{a}_{j_i}[2] \le \mathbf{a}_{j_{i+1}}[2]$ for any $i < l$.

Claim 1. For any $r \in \underline{l-1}$, there are $\mathbf{b}^r_{j_{r-1}+1}, \ldots, \mathbf{b}^r_k \in R$ such that $\mathbf{b}^r_s[1] = \mathbf{a}_s[1]$ and $\mathbf{b}^r_s[2] = \mathbf{a}_{j_r}[2]$ for any $i \in \{3, \ldots, n\}$ and any $s \in \{j_{r-1}+1, \ldots, k\}$. There are $\mathbf{b}^l_1, \ldots, \mathbf{b}^l_k \in R$ such that $\mathbf{b}^l_s[1] = \mathbf{a}_s[1]$ and $\mathbf{b}^l_s[2] = \mathbf{a}_{j_l}[2]$. Moreover, if for any $i \in \{3, \ldots, n\}$, $\mathbf{b}[i] \in \{\mathbf{a}_1[i], \ldots, \mathbf{a}_k[i]\}$, then $\mathbf{b}^r_s[i] \in \{\mathbf{a}_1[i], \ldots, \mathbf{a}_k[i]\}$ for any $i \in \{3, \ldots, n\}$, $r \in \underline{l}$ and $s \in \{j_{r-1}+1, \ldots, k\}$.

We prove the claim by induction on r. To avoid repetitions we start with proving induction step, and then observe that the base case for induction also follows from our arguments. First we show the existence of $\mathbf{b}^r_s$ for $s \in \{j_r, \ldots, k\}$. Clearly, we do not need to do this for $r = 1$, since in this case $\mathbf{b}^r_{j_r+1} = \mathbf{b}^1_k$ can be chosen to be $\mathbf{b}$, and $\mathbf{b}^r_{j_r}$ can be chosen to be $\mathbf{a}_{k-1}$.

The tuple $\mathbf{b}^r_{j_r}$ can be chosen to be $\mathbf{a}_{j_r}$. Furthermore, if $\mathbf{b}^r_s$ is already obtained, then set $\mathbf{b}^r_{s+1} = f(\mathbf{b}^r_s, \mathbf{b}^{r-1}_{s+1})$. Since $\mathbf{b}^r_s[1] = \mathbf{a}_s[1] \le \mathbf{a}_{s+1}[1] = \mathbf{b}^{r-1}_{s+1}[1]$ and $\langle \mathbf{b}^r_s[2], \mathbf{b}^{r-1}_{s+1}[2]\rangle \in \beta \cup \gamma$ or $\mathbf{b}^{r-1}_{s+1}[2] \le \mathbf{b}^r_s[2]$ where $\mathbf{b}^r_s[2] = \mathbf{a}_{j_r}[2]$, $\mathbf{b}^{r-1}_{s+1}[2] = \mathbf{a}_{j_{r-1}}[2]$, the obtained tuple satisfies the required conditions.

Notice that $\mathbf{b}^r_{j_r-1}$ can be chosen to be $f(\mathbf{a}_{j_r-1}, \mathbf{b}^r_{j_r+1})$. If $r \neq 1$ then we are done. In the base case of induction, if $r = 1$, we need to obtain $\mathbf{b}^1_s$ for $s \in \{j_1 - 2, \ldots, j_2 + 1\} = \{k-p-1, \ldots, k-3\}$. This can be done by setting, $\mathbf{b}^1_s = f(\mathbf{a}_s, \mathbf{b}^1_{s+2})$, for each $s = k-3, \ldots, k-p-1$.

The last statement of the claim also follows from the argument above.

Claim 2. For any t, $j_l \le t < k$, there are $\mathbf{c}^t_1, \ldots, \mathbf{c}^t_k \in R$ such that $\mathbf{c}^t_1[2] = \ldots = \mathbf{c}^t_k[2] = \mathbf{a}_t[2]$ and $\mathbf{c}^t_s[1] = \mathbf{a}_s[1]$ for $s \in \underline{k}$.

We prove the claim by induction on t. By Claim 1, there exist $\mathbf{b}^l_1, \ldots, \mathbf{b}^l_k$ such that $\mathbf{b}^l_1[2] = \ldots = \mathbf{b}^l_k[2] = \mathbf{a}_{j_l}[2]$ and $\mathbf{b}^l_s[1] = \mathbf{a}_s[1]$, which proves the base case of induction, that is $t = j_l$. So, suppose that $\mathbf{c}^t_1, \ldots, \mathbf{c}^t_k$ are found. Then, $\mathbf{c}^{t+1}_{t+1} = \mathbf{a}_{t+1}$ and, for $j \in \{t+2, \ldots, k\}$, we get $\mathbf{c}^{t+1}_j = f(\mathbf{c}^t_j, \mathbf{c}^{t+1}_{j-1})$. Finally, for $j \in \{t, t-1, \ldots, 1\}$, we set $\mathbf{c}^{t+1}_j = f(\mathbf{c}^t_j, \mathbf{c}^{t+1}_{j+2})$.

Claim 2, implies, in particular, that there is $\mathbf{c} = \mathbf{c}^{k-1}_1$ satisfying the conditions specified in part (1) of the lemma.

(2) If $k > 2$ then using Claim 2 it suffices to construct tuples $\mathbf{c}^1_1, \ldots, \mathbf{c}^1_k$ and, possibly, $\mathbf{c}^k_1, \ldots, \mathbf{c}^k_k$. First, we set $\mathbf{c}^k_k = \mathbf{a}_k$, $\mathbf{c}^k_{k-2} = f(\mathbf{c}^{k-1}_{k-2}, \mathbf{c}^k_k)$, and $\mathbf{c}^k_{k-1} = f(\mathbf{c}^k_{k-2}, \mathbf{a}_{k-1})$. Then, for any $j \in \{k-3, \ldots, 1\}$, $\mathbf{c}^k_j = f(\mathbf{c}^{k-1}_j, \mathbf{c}^k_{j+2})$. The tuples

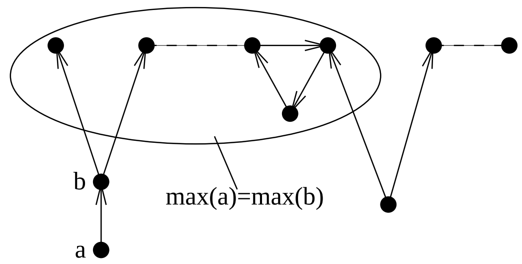


Figure 6.2

satisfy the conditions: $\mathbf{c}_j^k[1] = \mathbf{a}_j[1]$, $j \in \underline{k}$, and $\mathbf{c}_1^k[2] = \ldots = \mathbf{c}_k^k[2] = \mathbf{a}_k[2]$. As is easily seen, $\mathbf{c}_1^1$ can be chosen to be $\mathbf{a}_1$, and, for any $1 < i \leq k$, $\mathbf{c}_i^1 = f(\mathbf{c}_{i-1}^1, \mathbf{c}_i^3)$. $\Box$

6.2 Maximal elements

We use $R^{\max}$ to denote the union of all r-maximal components from $\mathcal{G}(R)$, $R \in \Gamma$. Recall that $\mathbf{a} \prec \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in R$ indicates the fact that there is an r-path connecting $\mathbf{a}$ with $\mathbf{b}$. For an element $\mathbf{a} \in R$, the *filter* $\mathcal{F}(\mathbf{a})$ generated by $\mathbf{a}$ is defined to be the set of all $\mathbf{b} \in R$ such that $\mathbf{a} \prec \mathbf{b}$. For any $a \in A$, $A \in \mathcal{A}$, we define the set of corresponding r-maximal elements as follows (see Fig. 6.2):

$$\begin{aligned}\max(a) \; = \; \{ & b \in A^{\max} \mid \text{there is } c \in A^{\max} \cap \mathcal{F}(a) \text{ such that} \\ & c, b \text{ are strongly ry-connected in } \mathcal{G}(A^{\max}).\end{aligned}$$

If $R \in \Gamma$ is a subdirect product of $A_1, \ldots, A_n$, then, for any $\mathbf{a} \in R$, we set $\max(\mathbf{a}) = \{\mathbf{b} \in R \mid \mathbf{b}[i] \in \max(\mathbf{a}[i]) \text{ for any } i \in \underline{n}\}$.

For $\mathbf{a}, \mathbf{b} \in R$, we write $\mathbf{a} \sqsubseteq \mathbf{b}$ if and only if $\max(\mathbf{b}) \subseteq \max(\mathbf{a})$. Clearly, $\prec$ refines $\sqsubseteq$, as $\mathbf{a} \prec \mathbf{b}$ implies $\max(\mathbf{b}) \subseteq \max(\mathbf{a})$. If $\mathbf{a} \sqsubseteq \mathbf{b}$ and $\mathbf{b} \sqsubseteq \mathbf{a}$, that is if $\max(\mathbf{a}) = \max(\mathbf{b})$, then $\mathbf{a}, \mathbf{b}$ are said to be *indistinguishable*. Obviously, if $\mathbf{a}, \mathbf{b}$ lie in the same strongly r-connected component, then they are indistinguishable. The classes of the equivalence relation $\sqsubseteq \cap \sqsubseteq^{-1}$ are called the *i-components*.

We first make an easy observation.

LEMMA 6.3. *If a, b are not indistinguishable, say $a \not\sqsubseteq b$, then there is $c \in \mathcal{F}(b)$ such that $c \notin \mathcal{F}(a)$.*

The next lemma shows that $\max(\mathbf{a})$ for a relation can be defined in the same way as for a single set.

LEMMA 6.4. *If $\mathbf{a}, \mathbf{b} \in R$ are such that $\mathbf{a} \in \max(\mathbf{b})$, then there is $\mathbf{c} \in \mathcal{F}(\mathbf{b}) \cap \max(\mathbf{b})$ such that $\mathbf{c}$ is strongly ry-connected to $\mathbf{a}$ in $\mathcal{G}(R^{\max})$.*

PROOF. By induction on the arity of R. $\Box$

For a set $B \subseteq R$ we denote by $\max(B)$ the set $\bigcup_{\mathbf{a} \in B} \max(\mathbf{a})$. The next lemma follows straightforwardly from the definition of $\max(\mathbf{a})$ and Lemma 3.7.

LEMMA 6.5. *Let $B \subseteq R$, and $I \subseteq \underline{n}$. Then $\max(\mathrm{pr}_I B) = \mathrm{pr}_I \max(B)$.*

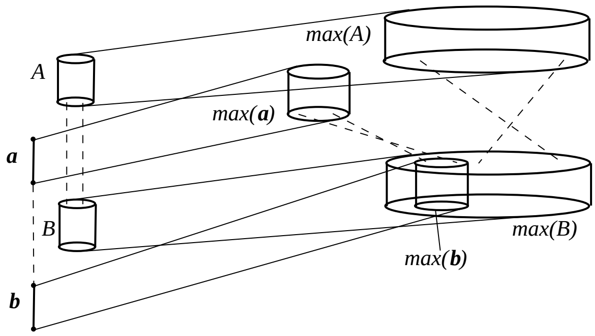


Fig. 6.3. Lemma 6.6. Cylinders represent sets of tuples or their fragments. Crossed dotted lines indicate direct products of two sets.

As we declared in the beginning of this section, our goal is to prove some kind of rectangularity for sets of r-maximal elements. The following two lemmas provide some weak and preliminary version of this property.

Lemma 6.6. *Let R be a subdirect product of $R^1, R^2 \in \Gamma$, and $A \subseteq R^1$, $B \subseteq R^2$ such that* $\max(A) \times \max(B) \subseteq R$. *If for* $\mathbf{a} \in R^1$ *there is* $\mathbf{b} \in R^2$ *with* $(\mathbf{a}, \mathbf{b}) \in R$ *and* $\max(\mathbf{b}) \subseteq \max(B)$, *then* $\max(\mathbf{a}) \times \max(B) \subseteq R$ *(see Fig. 6.3).*

Proof. Suppose that R^1, R^2 are subdirect products of $A_1, \dots, A_m$ and $A_{m+1}, \dots, A_{m+n}$ respectively. Take $\mathbf{a}' \in \max(\mathbf{a})$. We have to show that, for any $\mathbf{c} \in \max(B)$, the tuple $(\mathbf{a}', \mathbf{c}) \in R$. By Lemma 6.5, $\max(\mathbf{a}) = \mathrm{pr}_{\underline{m}} \max(\mathbf{a}, \mathbf{b})$, i.e. there is $\mathbf{b}' \in \max(B)$ with $(\mathbf{a}', \mathbf{b}') \in R$.

Take $\mathbf{d} \in \max(A)$ such that the set $I = \{i \in \underline{m} \mid \mathbf{a}'[i], \mathbf{d}[i]$ are in different ry-component$\}$ is minimal. If $\mathbf{a}'[i], \mathbf{d}[i] \in C_i$ for $i \in \underline{m} - I$, where C_i is a strongly ry-connected component of $\mathcal{G}(\max(A_i))$ then by Lemma 6.5,

$$\mathrm{pr}_{\underline{m}-I} \max(\mathbf{a}) \cap \prod_{i \in \underline{m}-I} C_i = \mathrm{pr}_{\underline{m}-I} \max(A) \cap \prod_{i \in \underline{m}-I} C_i.$$

Therefore, $\mathbf{d}$ can be chosen such that $\mathrm{pr}_{\underline{m}-I} \mathbf{d} = \mathrm{pr}_{\underline{m}-I} \mathbf{a}'$. Since $\max(A) \times \max(B) \subseteq R$, we have $(\mathbf{d}, \mathbf{b}'), (\mathbf{d}, \mathbf{c}) \in R$.

Consider the tuple $\begin{pmatrix} \mathbf{d}' \\ \mathbf{e} \end{pmatrix} = h\left(\begin{pmatrix} \mathbf{d} \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} \mathbf{d} \\ \mathbf{b}' \end{pmatrix}, \begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix}\right)$. If $i \in \underline{m}$ is such that $\langle \mathbf{a}'[i], \mathbf{d}[i] \rangle \in \alpha \cup \beta$ then $\mathbf{a}'[i] = \mathbf{d}[i]$ and $\mathbf{d}'[i] = \mathbf{a}'[i]$. If $\langle \mathbf{a}'[i], \mathbf{d}[i] \rangle \in \gamma$ then $\mathbf{d}'[i] = \mathbf{a}'[i]$, because h is a Mal'tsev operation on $\{\mathbf{a}'[i], \mathbf{d}[i]\}$. Thus $\mathbf{d}' = \mathbf{a}'$. Furthermore, $\mathbf{e}[i] = \mathbf{c}[i]$ whenever $\langle \mathbf{b}'[i], \mathbf{c}[i] \rangle \in \beta \cup \gamma$ or $\mathbf{b}'[i] \le \mathbf{c}[i]$, $i \in \{m+1, \dots, m+n\}$. Therefore $\mathbf{c}[i] \le \mathbf{e}[i]$; and, since both $\mathbf{e}, \mathbf{c}$ are r-maximal, the Generalized Connectedness Lemma 5.2(1) implies that $\mathbf{e}, \mathbf{c}$ are in the same strongly r-connected component.

There is an r-path $\mathbf{e} = \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_q = \mathbf{c}$. We have $(\mathbf{a}', \mathbf{e}_1) \in R$; and since $(\mathbf{d}, \mathbf{e}_i) \in R$ for all $i \in \underline{q}$, if $(\mathbf{a}', \mathbf{e}_i) \in R$ then $\begin{pmatrix} \mathbf{a}' \\ \mathbf{e}_{i+1} \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}' \\ \mathbf{e}_i \end{pmatrix}, \begin{pmatrix} \mathbf{d} \\ \mathbf{e}_{i+1} \end{pmatrix}\right) \in R$. Thus $(\mathbf{a}', \mathbf{c}) \in R$, and the lemma is proved. □

Corollary 6.7. *Let R be a subdirect product of $R^1, R^2 \in \Gamma$, $A \subseteq R^1$, $B \subseteq R^2$, and $(\mathbf{a}_1, \mathbf{a}_2) \in R$ is such that* $\max(\mathbf{a}_1) \subseteq \max(A)$, $\max(\mathbf{a}_2) \subseteq \max(B)$. *If*

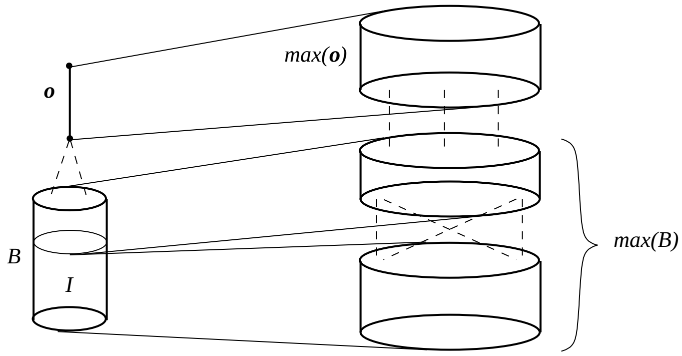


Fig. 6.4. The Fork Lemma 6.8.

$\max(\mathbf{a}_1) \times \max(B) \subseteq R$ *and, for any* $\mathbf{b}_1 \in \max(A)$*, there is* $\mathbf{b}_2 \in \max(\mathbf{a}_2)$ *such that* $(\mathbf{b}_1, \mathbf{b}_2) \in R$*, then* $\max(A) \times \max(B) \subseteq R$.

PROOF. For any $\mathbf{b}_1 \in \max(A)$, there is $\mathbf{b}_2 \in \max(\mathbf{a}_2) \subseteq \max(B)$ such that $(\mathbf{b}_1, \mathbf{b}_2) \in R$; moreover, as $\max(\mathbf{a}_1) \times \max(B) \subseteq R$, we are in the conditions of Lemma 6.6, and therefore $\max(\mathbf{b}_1) \times \max(B) \subseteq R$. Since this holds for every tuple $\mathbf{b}_1 \in \max(A)$, the result follows. □

Now we are in a position to state and prove (see Appendix) the main technical lemma.

LEMMA 6.8 (FORK LEMMA). *Let* R *be a subdirect product of* $R^1, R^2 \in \Gamma$*, which are subdirect products of* $A_1, \dots, A_m \in \mathcal{A}$ *and* $A_{m+1}, \dots, A_{m+n} \in \mathcal{A}$*, respectively, and* $\mathbf{o} \in R^1$*,* $B \subseteq R^2$ *such that* $\{\mathbf{o}\} \times B \subseteq R$*. There is* $I \subseteq K = \{m+1, \dots, m+n\}$ *such that* $\max(\{\mathbf{o}\} \times \mathrm{pr}_{K-I} B) \times \max(\mathrm{pr}_I B) \subseteq R$*, and all members of* $\mathrm{pr}_{K-I} B$ *are indistinguishable (see Fig. C.2).*

6.3 Max-rectangularity

Let R be a subdirect product R of $A_1, \dots, A_n$. An i-component of $\mathcal{G}(R)$ minimal with respect to $\sqsubseteq$ will be called *minimal*, a member of a minimal i-component will be called a *minimal element.* For a coordinate position $j \in \underline{n}$ and a minimal i-component B of $\mathcal{G}(A_j)$, we define the set $J_R(\mathrm{m}, j, B)$ (usually we will omit R) as follows

$$\begin{aligned} J_R(\mathrm{m}, j, B) \;=\; \{i \in \underline{n} \mid & \text{ for any } \mathbf{a}, \mathbf{b} \in R, \text{ such that } \mathbf{a}[j] \in B, \text{ the element } \mathbf{b}[j] \\ & \text{belongs to } B \text{ if and only if } \mathbf{a}[i], \mathbf{b}[i] \text{ are minimal and indistinguishable}\} \end{aligned}$$

The relation R is said to be *max-rectangular* if, for any r-maximal $\mathbf{a} \in R$, any minimal i-components $B_1, \dots, B_n$ of $\mathcal{G}(A_1), \dots, \mathcal{G}(A_n)$, respectively, such that $\mathbf{a}[i] \in \max(B_i)$, $i \in \underline{n}$, any $j \in \underline{n}$ and any r-maximal $\mathbf{b} \in \mathrm{pr}_{J(\mathrm{m},j,B_j)} R \cap \left(\prod_{i \in J(\mathrm{m},j,B_j)} \max(B_j)\right)$, the tuple $\mathbf{c}$:

$$\mathbf{c}[i] = \begin{cases} \mathbf{b}[i], & \text{if } i \in J(\mathrm{m}, j, B), \\ \mathbf{a}[i], & \text{otherwise,} \end{cases}$$

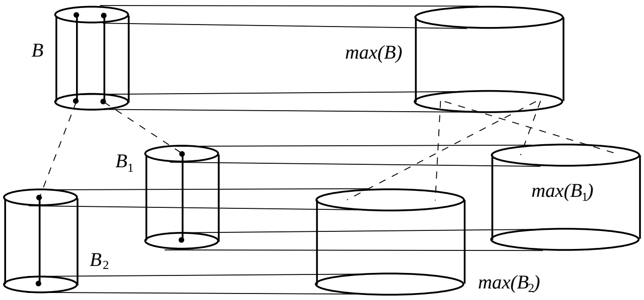


Figure 6.5

belongs to R (see Fig. 6.5). We will need two auxiliary lemmas.

LEMMA 6.9. *Let $j \in \underline{n}$, $l \in J(\mathrm{m}, j, B)$, and B, C be minimal i-components of $\mathcal{G}(A_j), \mathcal{G}(A_l)$, respectively, such that, for any $\mathbf{o} \in R$ with $\mathbf{o}[j] \in B$, we have $\mathbf{o}[l] \in C$. For any $\mathbf{a} \in R$ such that $\mathbf{a}[j] \in \mathcal{F}(B)$, the element $\mathbf{a}[l]$ belongs to $\mathcal{F}(C)$.*

PROOF. Without loss of generality we may assume that $j = 1$, $l = 2$. Let us suppose that $\mathbf{a} \in R$ is such that $\mathbf{a}[1] \in \mathcal{F}(B)$ while $\mathbf{a}[2] \notin \mathcal{F}(C)$. Notice that if $\mathbf{a}[1] \in B$ then $\mathbf{a}[2] \in C$; therefore, we may assume that $\mathbf{a}[1] \in \mathcal{F}(B) - B$. There is $\mathbf{b} \in R$ such that $\mathbf{b}[1] \in B$ and $\mathbf{a}[1] \in \mathcal{F}(\mathbf{b}[1])$. This means that there is also an r-path $\mathbf{b}[1] = b_1, b_2, \dots, b_k = \mathbf{a}[1]$. This r-path can be extended to an r-path $\mathbf{b} = \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k$ with $\mathbf{b}_i[1] = b_i$, $i \in \underline{k}$. By Lemma 6.1, $\mathbf{b}_1[1], \dots, \mathbf{b}_k[1]$ and $\mathbf{b}_1[2], \dots, \mathbf{b}_k[2]$ can be assumed to be irreducible. Since $\mathbf{b}_1[2], \dots, \mathbf{b}_k[2] \in \mathcal{F}(\mathbf{b}[2])$, $\mathbf{b}_i[2] \le \mathbf{a}[2]$ for no $i \in \underline{k}$. Therefore, this r-path satisfies the conditions of the Path Alignment Lemma 6.2 for the tuple $\mathbf{c} = f(\mathbf{b}_{k-1}, \mathbf{a})$ with $\mathbf{c}[1] = \mathbf{a}[1] = \mathbf{b}_k[1]$ and $\mathbf{c}[2] = \mathbf{b}_{k-1}[2]$. Hence, there is $\mathbf{d} \in R$ with $\mathbf{d}[1] = \mathbf{b}[1]$, $\mathbf{d}[2] = \mathbf{b}_{k-1}[2]$. If $\mathbf{b}_{k-1}[2] \notin C$ then this contradicts the assumption $2 \in J(\mathrm{m}, 1, B)$. If $\mathbf{b}_{k-1}[2] \in C$ then as $\mathbf{c}[1] = \mathbf{a}[1] \notin B$, we again get a contradiction with the assumption $1 \in J(\mathrm{m}, 2, C)$. $\Box$

LEMMA 6.10. *Let $j \in \underline{n}$, $l \in J(\mathrm{m}, j, B)$, and B, C be minimal i-components of $\mathcal{G}(A_j), \mathcal{G}(A_l)$, respectively, such that, for any $\mathbf{o} \in R$ with $\mathbf{o}[j] \in B$, we have $\mathbf{o}[l] \in C$. For any $\mathbf{a} \in R$ such that $\mathbf{a}[j] \in \max(B)$, the element $\mathbf{a}[l]$ is such that $C \sqsubseteq \mathbf{a}[l]$.*

PROOF. We may assume that $j = 1$, $l = 2$. Let us suppose that $\mathbf{a} \in R$ is such that $\mathbf{a}[1] \in \max(B)$ while $C \not\sqsubseteq \mathbf{a}[2]$. By the Maximality Lemma 3.7(1), we may assume that $\mathbf{a}[2]$ is r-maximal. Take $a \in \max(B) \cap \mathcal{F}(B)$ such that $\mathbf{a}[1]$ is strongly ry-connected with a in $\max(B)$, and let $\mathbf{a}[1] = a_0, a_1 \dots a_k = a$ be an ry-path such that $a_1, \dots, a_k \in \max(B)$. Let also $\mathbf{a} = \mathbf{a}_0, \mathbf{a}_1 \dots \mathbf{a}_k$ be an expansion of this r-path. If all the elements $\mathbf{a}_1[2] \dots \mathbf{a}_k[2]$ are r-maximal then, by Lemma 6.9, $\mathbf{a}_k[2] \in \max(C) \cap \mathcal{F}(C)$, a contradiction with the assumption $C \not\sqsubseteq \mathbf{a}[2]$. Thus, we have to show that the r-path can be chosen such that $\mathbf{a}_1[2] \dots \mathbf{a}_k[2]$ are r-maximal.

Let i be minimal such that $\mathbf{a}_i[2]$ is not r-maximal. Then $i \ge 1$, and $\langle \mathbf{a}_{i-1}, \mathbf{a}_i \rangle \in \beta$. Take a r-maximal b such that $\mathbf{a}_i[2] \prec b$. By the Maximality Lemma 3.7(1), there is $\mathbf{b} \in R$ such that $\mathbf{b}[1] = \mathbf{a}_i[1]$ and $\mathbf{b}[2] = b$. If $\mathbf{a}_{i-1} \le b$ then replace $\mathbf{a}_i$ with $\mathbf{c}_1 = f(\mathbf{a}_{i-1}, \mathbf{b})$ and $\mathbf{c}_2 = p(\mathbf{c}_1, \mathbf{b})$. By the P-Lemma $\mathbf{a}_{i-1}, \mathbf{c}_1, \mathbf{c}_2$ is an ry-path, and $\mathbf{c}_1[1] = \mathbf{a}_{i-1}[1], \mathbf{c}_2[1] = \mathbf{a}_i[1]$ and $\mathbf{c}_1[2], \mathbf{c}_2[2] = b$. If $b \le \mathbf{a}_{i-1}[2]$ or $\langle \mathbf{a}_{i-1}[2], b \rangle \in \gamma$

then replace $\mathbf{a}_i$ with $\mathbf{c} = p(\mathbf{a}_{i-1}, \mathbf{b})$. We have that $\mathbf{a}_{i-1}$ is strongly ry-connected with $\mathbf{c}$, and $\mathbf{c}[1] = \mathbf{a}_i[1]$, $\mathbf{c}[2] = \mathbf{a}_{i-1}[2]$. Finally, if $\langle \mathbf{a}_{i-1}[2], b\rangle \in \beta$, then $\mathbf{a}_{i-1}$ and $\mathbf{c} = p(\mathbf{a}_{i-1}, \mathbf{b})$ are strongly ry-connected, and $\mathbf{c}[1] = \mathbf{a}_i[1]$ and $\mathbf{c}[2] = b$. Then the ry-path $a_i \dots a_k = a$ can be expanded with $\mathbf{c}$ (or $\mathbf{c}_2$)$= \mathbf{a}'_i \dots \mathbf{a}'_k$. □

Now we are in a position to prove the main result of Section 6.

Proposition 6.11 (Maximal Rectangularity Proposition). *Let the $\mathcal{G}(A_i)$ be strongly rb-connected. Then R is max-rectangular.*

Proof. Let $\mathbf{a} \in \max(R)$ and $B_1, \dots, B_n$ be minimal i-components of $\mathcal{G}(A_1), \dots, \mathcal{G}(A_n)$, respectively, such that $\mathbf{a}[i] \in \max(B_i)$. Let also $I_1, \dots, I_k \subseteq \underline{n}$ be the sets of the form $J(\mathrm{m}, i, B_i)$, and $I_1 = J(\mathrm{m}, 1, B_1)$. Without loss of generality we have $I_1 = \{1, \dots, m\}$. We prove that, for any $\mathbf{a}' \in \mathrm{pr}_{\underline{m}} R \cap (\max(B_1) \times \dots \times \max(B_m))$, the tuple $(\mathbf{a}', \mathrm{pr}_{\underline{n}-I_1} \mathbf{a})$ belongs to R.

Claim 1. If $\mathbf{b} \in R$ is such that, for certain $j \in \underline{k}$ and $i \in I_j$, $\mathbf{b}[i] \in B_i$, then, for any $l \in I_j$, $\mathbf{b}[l] \in B_l$.

Note that the definition of sets $J(\mathrm{m}, i, B_i)$ only implies that $\mathbf{b}[l]$ belongs to some minimal i-component, which is the same for all $\mathbf{b}$ with $\mathbf{b}[i] \in B_i$.

By Lemma 6.9, $\mathbf{b}[l] \in \mathcal{F}(B_l)$. On the other hand, $\mathbf{b}[l]$ belongs to a minimal i-component B of $\mathcal{G}(R_l)$. Since $B_l \sqsubseteq B$ and both are minimal, we get $B = B_l$.

Claim 1 implies that, for any $j \in \underline{k}$, $\mathrm{pr}_{I_j} R \cap \prod_{i \in I_j} \max(B_i) = \max(\mathrm{pr}_{I_j} \mathbf{b})$, where $\mathbf{b} \in R$ is an arbitrary tuple such that $\mathbf{b}[i] \in B_i$ for $i \in I_j$.

If B_i contains an r-maximal element, then $\max(B_i) \subseteq B_i$ and $\mathcal{G}(\max(B_i))$ is strongly ry-connected. Indeed, r-maximal elements from different ry-components of $\mathcal{G}(\max(B_i))$ are not indistinguishable.

Claim 2. Let B_j, for a certain $j \in \underline{n}$, contain an r-maximal element. Then, for every $i \in J(\mathrm{m}, j, B_j)$, B_i contains an r-maximal element.

Let $d \in \max(B_j)$. By the Maximal Expansion Lemma 3.8, it can be expanded to an r-maximal $\mathbf{d} \in R$; in particular, $\mathbf{d}[i]$ is r-maximal. Since $i \in J(\mathrm{m}, j, B_j)$, we have $\mathbf{d}[i] \in B_i$, a contradiction.

Claim 3. Let B_j, for a certain $j \in \underline{n}$, contain an r-maximal element. Then $J(\mathrm{ry}, j, \max(B_j)) \subseteq J(\mathrm{m}, j, B_j)$.

We have to show that, for any $i \in \underline{n} - J(\mathrm{m}, j, B_j)$, we have $i \notin J(\mathrm{ry}, j, C)$, where $C = \max(B_j)$. Without loss of generality, let us assume that $j = 1$, $i = n$. Since $n \notin J(\mathrm{m}, 1, B_1)$, either
(1) there is $\mathbf{c} \in R$ such that $\mathbf{c}[1] \in B_1$ and $\mathbf{c}[n]$ is not in a minimal i-component; or
(2) there are $\mathbf{c}, \mathbf{d}$ such that $\mathbf{c}[1], \mathbf{d}[1] \in B_1$, $\mathbf{c}[n], \mathbf{d}[n]$ belong to different minimal i-components; or
(3) there is a minimal i-component B of $\mathcal{G}(R_n)$ such that, for any $\mathbf{c} \in R$ with $\mathbf{c}[1] \in B_1$, $\mathbf{c}[n] \in B$, but there is $\mathbf{d} \in R$ with $\mathbf{d}[n] \in B$, $\mathbf{d}[1] \notin B_1$.

(1) Take $\mathbf{d} \in R$ such that $\mathbf{d}[n]$ is minimal and $\mathbf{d}[n] \sqsubseteq \mathbf{c}[n]$. Since $\max(\mathbf{d}[n]) \not\subseteq \max(\mathbf{c}[n])$, there is $d \in \mathcal{F}(\mathbf{d}[n]) - \max(\mathbf{c}[n])$. This element can be chosen to be r-maximal and can be expanded to an r-maximal tuple $\mathbf{e} \in \max(\mathbf{d}) \cap \mathcal{F}(\mathbf{d})$, in particular, there exists an r-path $\mathbf{d} = \mathbf{d}_1 \le \dots \le \mathbf{d}_r = \mathbf{e}$.

Suppose first that $\mathbf{d}[1] \in B_1$, that is $\max(\mathbf{c}[1]) = \max(\mathbf{d}[1])$ and $\mathbf{e}[1] \in \max(\mathbf{c}[1])$. Therefore, $\mathcal{F}(\mathbf{c}) \cap \max(\mathbf{c})$ contains a tuple $\mathbf{e}'$ such that $\mathbf{e}[1], \mathbf{e}'[1]$ are strongly ry-connected. Clearly, $\mathbf{e}[n], \mathbf{e}'[n]$ are in different strongly ry-connected components of $\mathcal{G}(\max(R_n))$. Thus, $\mathbf{e}, \mathbf{e}'$ witness that $n \notin J(\mathrm{ry}, 1, \max(B_1))$.

Now, let $\mathbf{d}[1] \notin B_1$. Since B_1 is a minimal i-component, by Lemma 6.3, there is $e \in \mathcal{F}(\mathbf{c}[1]) - \max(\mathbf{d}[1])$. The element e can be assumed to be r-maximal, and there is an r-maximal tuple $\mathbf{e} \in \mathcal{F}(\mathbf{c})$ with $\mathbf{e}[1] = e$. Since $\max(\mathbf{c}[n]) \subseteq \max(\mathbf{d}[n])$, there is an r-maximal tuple $\mathbf{e}' \in \mathcal{F}(\mathbf{d})$ such that $\mathbf{e}[n], \mathbf{e}'[n]$ are strongly ry-connected in $\max(\mathbf{d}[n])$. Furthermore, as $\mathbf{e}[1] \notin \max(\mathbf{d}[1])$, $\mathbf{e}[1], \mathbf{e}'[1]$ are in different strongly ry-connected components of $\mathcal{G}(\max(A_1))$; this implies $n \notin J(\mathrm{ry}, 1, \max(B_1))$.

(2) Since $\mathbf{c}[n], \mathbf{d}[n]$ belong to different minimal i-components, there is $d \in \max(\mathbf{c}[n]) - \max(\mathbf{d}[n])$. This element can be expanded to an r-maximal tuple $\mathbf{e}$ from $\mathcal{F}(\mathbf{c})$. Furthermore, as $\max(\mathbf{c}[1]) = \max(\mathbf{d}[1])$, there is an r-maximal $\mathbf{e}' \in \mathcal{F}(\mathbf{d})$ such that $\mathbf{e}'[1]$ is in the same strongly ry-connected component with $\mathbf{e}[1]$. The elements $\mathbf{e}[n], \mathbf{e}'[n]$ are not strongly ry-connected, that implies $n \notin J(\mathrm{ry}, 1, \max(B_1))$.

(3) There is $d \in \mathcal{F}(\mathbf{c}[1]) - \max(\mathbf{d}[1])$ which can be expanded to an r-maximal tuple $\mathbf{e} \in \mathcal{F}(\mathbf{c})$. Since $\mathbf{e}[n] \in \max(\mathbf{d}[n])$, there is an r-maximal $\mathbf{e}' \in \mathcal{F}(\mathbf{d})$ such that $\mathbf{e}[n], \mathbf{e}'[n]$ are strongly ry-connected. As $\mathbf{e}[1], \mathbf{e}'[1]$ are in different strongly ry-connected components, we get what is required. The claim is proved.

Claim 3 implies that $J(\mathrm{m}, 1, B_1)$ is a union of sets of the form $J(\mathrm{ry}, i, \max(B_i))$.

We proceed by induction on n. If $n = 1$ then the result is obvious. So, let us assume that it is proved for any number less than n.

If B_1 contains an r-maximal element then, by Claim 3, I_1 is the union of $J(\mathrm{ry}, i, \max(B_i))$, $i \in I_1$. Moreover, by Claim 2, every B_i, $i \in \underline{m}$, contains an r-maximal element. Let $\mathbf{a}'$ be a tuple strongly ry-connected to $\mathrm{pr}_{I_1}\mathbf{a}$, that is, by the Generalized Connectedness Lemma 5.2(2), for any $\mathbf{a}' \in \mathrm{pr}_{\underline{m}} R \cap (\max(B_1) \times \ldots \times \max(B_m))$. By the Rectangularity Proposition 5.4(1), the tuple $\mathbf{a}''$ corresponding to $\mathbf{a}'$ belongs to R. So, suppose that B_1 contains no r-maximal elements.

First we show that there are $\mathbf{b} \in R$ with $\mathbf{b}[1] \in B_1$ and $I_1 \subseteq J \subseteq \underline{n}$, $n \in K = \underline{n} - J$, such that $\max(\mathrm{pr}_J \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) \subseteq R$. There are three cases to consider similar to cases (1)–(3) in the proof of Claim 3.

Case 1. There are $\mathbf{b} \in R$ and $j \in \underline{n} - I_1$, such that $\mathbf{b}[1] \in B_1$ and $\mathbf{b}[j]$ does not belong to a minimal i-component.

Without loss of generality we may assume that $j = n$. Take $\mathbf{c} \in R$ such that $\mathbf{c}[n]$ is minimal and $\mathbf{c}[n] \sqsubseteq \mathbf{b}[n]$.

Subcase 1.1. $\mathbf{c}[1] \in B_1$.

Since $\mathbf{b}[n] \not\sqsubseteq \mathbf{c}[n]$, there is $d \in \mathcal{F}(\mathbf{c}[n]) - \max(\mathbf{b}[n])$. This element can be chosen to be r-maximal and can be expanded to an r-maximal tuple $\mathbf{d} \in \max(\mathbf{c}) \cap \mathcal{F}(\mathbf{c})$. As $\max(\mathbf{b}[1]) = \max(\mathbf{c}[1])$, $\mathbf{d}[1] \in \max(\mathbf{b}[1])$. Therefore, $\mathcal{F}(\mathbf{b}) \cap \max(\mathbf{b})$ contains a tuple $\mathbf{e}$ such that $\mathbf{e}[1], \mathbf{d}[1]$ are strongly ry-connected and, by Lemma 6.1, there exists an r-path $\mathbf{b} = \mathbf{b}_1 \le \ldots \le \mathbf{b}_r = \mathbf{e}$ such that $\mathbf{b}_1[1], \ldots, \mathbf{b}_r[1]$ and $\mathbf{b}_1[n], \ldots, \mathbf{b}_r[n]$ are irreducible.

Clearly, $\mathbf{e}[n], \mathbf{d}[n]$ are in different strongly ry-connected components of $\mathcal{G}(\max(A_n))$. Therefore, $n \notin J(\mathrm{ry}, 1, C)$ where C is the ry-maximal component containing $\mathbf{e}[1], \mathbf{d}[1]$, and, by the Rectangularity Proposition 5.4(1), there exists $\mathbf{e}' \in R$ such that $\mathbf{e}'[1] = \mathbf{e}[1]$, $\mathbf{e}'[n] = \mathbf{d}[n]$. The r-path $\mathbf{b}_1, \ldots, \mathbf{b}_r$ and the tuple $f(\mathbf{b}_{r-1}, \mathbf{e}')$ satisfy the conditions of the Path Alignment Lemma 6.2, hence, there is $\mathbf{b}' \in R$ with $\mathbf{b}'[1] = \mathbf{e}[1]$,

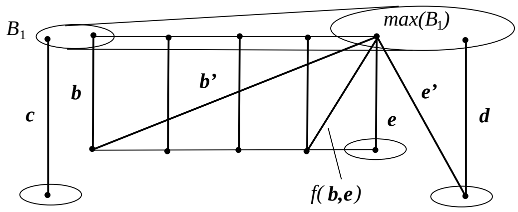


Figure 6.6

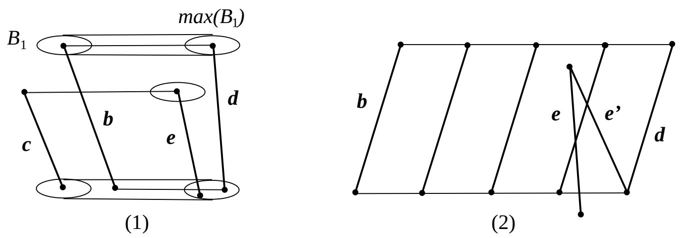


Figure 6.7

$\mathbf{b}'[n] = \mathbf{b}[n]$ (see Fig. 6.6). Set $J = \{i \in \underline{n} \mid \mathbf{b}[i], \mathbf{b}'[i] \text{ are indistinguishable}\}$, $K = \underline{n} - J$. Clearly, both sets are non-empty, as $n \in J$ and $1 \in K$. By the Fork Lemma 6.8, where in the notation of Lemma 6.8 $\mathbf{o} = (\mathbf{b}[n])$, $K = \underline{n-1}$, $I = J - \{n\}$, and $B = \{\mathrm{pr}_{\underline{n-1}}\mathbf{b}, \mathrm{pr}_{\underline{n-1}}\mathbf{b}'\}$, we have $\max(\mathrm{pr}_J\mathbf{b}) \times \max(\mathrm{pr}_K\mathbf{b}) \subseteq R$. By induction hypothesis, $\max(\mathrm{pr}_K\mathbf{b}) = \max(\mathrm{pr}_{I_1}\mathbf{b}) \times \max(\mathrm{pr}_{K-I_1}\mathbf{b})$; consequently, $\max(\mathrm{pr}_{I_1}\mathbf{b}) \times \max(\mathrm{pr}_{\underline{n}-I_1}\mathbf{b}) \subseteq R$.

SUBCASE 1.2. $\mathbf{c}[1] \notin B_1$.

Since B_1 is a minimal i-component, by Lemma 6.3, there is $d \in \mathcal{F}(\mathbf{b}[1]) - \max(\mathbf{c}[1])$. The element d can be assumed to be r-maximal and there is an r-maximal tuple $\mathbf{d} \in \mathcal{F}(\mathbf{b})$ with $\mathbf{d}[1] = d$. Since $\max(\mathbf{b}[n]) \subseteq \max(\mathbf{c}[n])$, there is an r-maximal tuple $\mathbf{e} \in \mathcal{F}(\mathbf{c})$ such that $\mathbf{d}[n], \mathbf{e}[n]$ are strongly ry-connected in $\max(\mathbf{c}[n])$. Furthermore, as $\mathbf{d}[1] \notin \max(\mathbf{c}[1])$, $\mathbf{d}[1], \mathbf{e}[1]$ are in different strongly ry-connected components of $\mathcal{G}(\max(R_1))$.

By the Rectangularity Proposition 5.4(1) and the Generalized Connectedness Lemma 5.2(2), there is an r-maximal $\mathbf{e}' \in R$ such that $\mathbf{e}'[1] = \mathbf{e}[1]$, $\mathbf{e}'[n] = \mathbf{d}[n]$ (see Fig. 6.7(1)). Consider an r-path $\mathbf{b} = \mathbf{d}_1 \leq \mathbf{d}_2 \leq \ldots \leq \mathbf{d}_r = \mathbf{d}$ such that $\mathbf{d}_1[1], \ldots, \mathbf{d}_r[1]$ and $\mathbf{d}_1[n], \ldots, \mathbf{d}_r[n]$ are irreducible. Considering the subcases below we show that there exist $\mathbf{b}' \in R$ or $\mathbf{b}'' \in R$ such that $\mathbf{b}'[n] = \mathbf{b}[n]$ and $\mathbf{b}'[1]$ equals either $\mathbf{d}[1]$ or $\mathbf{e}[1]$, or $\mathbf{b}''[1] = \mathbf{b}[1]$ and $\mathbf{b}''[n] = \mathbf{d}[n]$.

SUBCASE 1.2.1. $\mathbf{d}_{r-1}[1] \not\leq \mathbf{e}[1]$.

The r-path $\mathbf{d}_1, \ldots, \mathbf{d}_r$ and the tuple $\mathbf{d}' = f(\mathbf{d}_{r-1}, \mathbf{e}')$ satisfy the conditions of the Path Alignment Lemma 6.2. Therefore, if $r > 2$ then there is $\mathbf{b}' \in R$ such that $\mathbf{b}'[n] = \mathbf{b}[n]$ and $\mathbf{b}'[1] = \mathbf{d}[1]$ (see Fig. 6.7(2)). If $r = 2$ then the tuple $\mathbf{b}'' = \mathbf{d}'$ satisfies the required conditions.

Let l be the least number such that $\mathbf{d}_l[1] \leq \mathbf{e}[1]$.

SUBCASE 1.2.2. $\mathbf{d}_l[n] \not\leq \mathbf{e}[n]$.

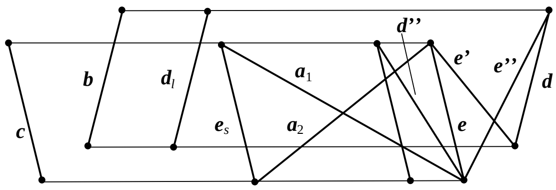


Figure 6.8

Consider the r-path $\mathbf{d}_1, \dots, \mathbf{d}_l, \mathbf{d}'$, where $\mathbf{d}' = f(\mathbf{d}_l, \mathbf{e})$. Since $\mathbf{d}'[1] = \mathbf{e}[1]$, $\mathbf{d}'[n] = \mathbf{d}_l[n]$, this r-path and the tuple $\mathbf{d}'$ satisfy the conditions of the Path Alignment Lemma 6.2 and there is $\mathbf{b}'$ with the required properties.

Let m be the least number such that $\mathbf{d}_m[n] \leq \mathbf{e}[n]$. By Subcase 1.2.2, we may assume $m \leq l$.

Subcase 1.2.3. $m < l$.

For $\mathbf{d}' = f(\mathbf{d}_m, \mathbf{e})$ we have $\mathbf{d}'[n] = \mathbf{e}[n]$, $\mathbf{d}'[1] = \mathbf{d}_m[1]$. If $m = 1$ then we may choose $\mathbf{b}'' = \mathbf{d}'$. Otherwise, the r-path $\mathbf{d}_1, \dots, \mathbf{d}_m, \mathbf{d}'$ and $\mathbf{d}'$ satisfy the conditions of the Path Alignment Lemma 6.2. Therefore, there exists a tuple $\mathbf{d}'' \in R$ such that $\mathbf{d}''[1] = \mathbf{d}_{m-1}[1]$, $\mathbf{d}''[n] = \mathbf{d}_m[n]$.

By induction on $j \in \{m, \dots, r\}$ we show that there is a tuple $\mathbf{d}''_j$ such that $\mathbf{d}''_j[1] = \mathbf{d}_{j-1}[1]$, $\mathbf{d}''_j[n] = \mathbf{d}_j[n]$. For the base case of induction we may set $\mathbf{d}''_m = \mathbf{d}''$. If the existence of $\mathbf{d}''_j$ is proved, then set $\mathbf{d}''_{j+1} = f(\mathbf{d}_j, f(\mathbf{d}''_j, \mathbf{d}_{j+1}))$. It is not hard to verify that it satisfies the required conditions.

The r-path $\mathbf{d}_1, \dots, \mathbf{d}_r$ and the tuple $\mathbf{d}''_r$ satisfies the conditions of the Path Alignment Lemma 6.2, which implies that there exists $\mathbf{b}' \in R$ with $\mathbf{b}'[1] = \mathbf{d}_r[1]$, $\mathbf{b}'[n] = \mathbf{b}[n]$.

Subcase 1.2.4. $m = l$.

Let $\mathbf{c} = \mathbf{e}_1 \leq \mathbf{e}_2 \leq \dots \leq \mathbf{e}_q = \mathbf{e}$ be an r-path connecting $\mathbf{c}$ and $\mathbf{e}$, such that $\mathbf{e}_1[1], \dots, \mathbf{e}_q[1]$ and $\mathbf{e}_1[n], \dots, \mathbf{e}_q[n]$ are irreducible, and let s be the maximal number such that either $\mathbf{d}_l[1] \not\leq \mathbf{e}_s[1]$ or $\mathbf{d}_l[n] \not\leq \mathbf{e}_s[n]$ (see Fig. 6.9). Since $\mathbf{b}[n] \not\sqsubseteq \mathbf{c}[n]$, we have $\mathbf{b}[n]$ is not r-connected to $\mathbf{c}[n]$, therefore $\mathbf{d}_l[n] \not\leq \mathbf{c}[n]$ and such an s exists. Analogously to $\mathbf{e}'$ there is a r-maximal $\mathbf{e}'' \in R$ such that $\mathbf{e}''[1] = \mathbf{d}[1]$, $\mathbf{e}''[n] = \mathbf{e}[n]$. As $\mathbf{d}[1] \notin \mathcal{F}(\mathbf{c}[1])$, we have $\mathbf{e}_{q-1}[1] \not\leq \mathbf{d}_j[1]$, $j \in \underline{r}$. Therefore, the tuple $\mathbf{d}'' = f(\mathbf{e}_{q-1}, \mathbf{e}'')$ satisfies the conditions: $\mathbf{d}''[1] = \mathbf{e}_{q-1}[1]$, $\mathbf{d}''[n] = \mathbf{e}[n] = \mathbf{e}_q[n]$. If $q > 2$ then the Path Alignment Lemma 6.2 can be applied to $\mathbf{e}_1, \dots, \mathbf{e}_q$ and $\mathbf{d}''$. There are $\mathbf{a}_1, \mathbf{a}_2 \in R$ such that $\mathbf{a}_1[1] = \mathbf{e}_s[1]$, $\mathbf{a}_1[n] = \mathbf{e}_q[n]$ and $\mathbf{a}_2[1] = \mathbf{e}_q[1]$, $\mathbf{a}_2[n] = \mathbf{e}_s[n]$.

If $\mathbf{d}_l[1] \not\leq \mathbf{e}_s[1]$ then consider the r-path $\mathbf{d}_1, \dots, \mathbf{d}_l, f(\mathbf{d}_l, \mathbf{e})$ and the tuple $\mathbf{d}' = f(\mathbf{d}_l, \mathbf{a}_1)$. Since $\mathbf{d}'[1] = \mathbf{d}_l[1]$ and $\mathbf{d}'[n] = \mathbf{e}[n]$, by the Path Alignment Lemma 6.2, there is $\mathbf{b}'$ with $\mathbf{b}'[1] = \mathbf{e}[1]$, $\mathbf{b}'[n] = \mathbf{b}[n]$. If $\mathbf{d}_l[n] \not\leq \mathbf{e}_s[n]$ then consider the r-path $\mathbf{d}_1, \dots, \mathbf{d}_l, f(\mathbf{d}_l, \mathbf{e})$ and the tuple $\mathbf{d}' = f(\mathbf{d}_l, \mathbf{a}_2)$. Since $\mathbf{d}'[1] = \mathbf{e}[1]$ and $\mathbf{d}'[n] = \mathbf{d}_l[n]$, again by the Path Alignment Lemma 6.2, there is $\mathbf{b}'$ with $\mathbf{b}'[1] = \mathbf{e}[1]$, $\mathbf{b}'[n] = \mathbf{b}[n]$.

Finally, let $q = 2$; then, since $\mathbf{d}[1] \leq \mathbf{e}[1]$ and $\mathbf{d}_l[n] \leq \mathbf{e}[n]$, we have $s = 1$. As before, we get $\mathbf{a}_1 = f(\mathbf{c}, \mathbf{e}'')$ with $\mathbf{a}_1[1] = \mathbf{e}_s[1]$, $\mathbf{a}_1[n] = \mathbf{e}_q[n]$. If $\mathbf{c}[n] \not\leq \mathbf{d}[n]$ then we also get $\mathbf{a}_2 = f(\mathbf{c}, \mathbf{e}')$ with $\mathbf{a}_2[1] = \mathbf{e}_q[1]$, $\mathbf{a}_2[n] = \mathbf{e}_s[n]$. Then we proceed as in the previous paragraph. If $\mathbf{c}[n] \le \mathbf{d}[n]$ then set $\mathbf{d}' = f(\mathbf{d}_{r-1}, f(\mathbf{c}, \mathbf{d}))$. Since $\mathbf{d}'[1] = \mathbf{d}_{r-1}[1]$, $\mathbf{d}'[n] = \mathbf{d}_r[n]$, the r-path $\mathbf{d}_1, \dots, \mathbf{d}_r$ and the tuple $\mathbf{d}'$ satisfy the conditions of the Path Alignment Lemma 6.2 and there is $\mathbf{b}''$ with $\mathbf{b}''[1] = \mathbf{b}[1]$ and $\mathbf{b}''[n] = \mathbf{d}[n]$.

Now, setting $J = \{i \in \underline{n} \mid \mathbf{b}[i], \mathbf{b}'[i]$ are indistinguishable$\}$ or $J = \{i \in \underline{n} \mid \mathbf{b}''[i], \mathbf{d}[i]$ are indistinguishable$\}$, $K = \underline{n} - J$ we can argue as in Subcase 1.1.

Case 2. For any $\mathbf{b} \in R$ such that $\mathbf{b}[1] \in B_1$ and any $i \in \underline{n}$, $\mathbf{b}[i]$ belongs to a minimal i-component, but there are $\mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{b}[1], \mathbf{c}[1] \in B_1$ and $\mathbf{b}[n], \mathbf{c}[n]$ are not indistinguishable.

We shall not use the fact that $\mathbf{c}[n]$ is in a minimal i-component. However, $\mathbf{b}[n]$ is assumed to be a member of a minimal i-component. There is $d \in \mathcal{F}(\mathbf{b}[n]) - \max(\mathbf{c}[n])$. The element d can be expanded to an r-maximal tuple $\mathbf{d} \in \mathcal{F}(\mathbf{b})$ so that $\mathbf{d}[n] = d$. There is an r-path $\mathbf{c} = \mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_l = \mathbf{e}$ such that $\mathbf{e}$ is r-maximal, $\mathbf{e}[1], \mathbf{d}[1]$ are strongly ry-connected and $\mathbf{c}_1[1], \dots, \mathbf{c}_l[1]$, $\mathbf{c}_1[n], \dots, \mathbf{c}_l[n]$ are irreducible.

As in Case 1, it can be shown that there is a tuple $\mathbf{e}'$ such that $\mathbf{e}'[1] = \mathbf{d}[1]$, $\mathbf{e}'[n] = \mathbf{e}[n]$. Again we shall show that there exist non-empty $J, K = \underline{n} - J \subseteq \underline{n}$ such that $\max(\mathrm{pr}_J \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) \subseteq R$.

Since $\mathbf{d}[n] \notin \max(\mathbf{c}[n])$, we have $\mathbf{c}_{l-1}[n] \not\le \mathbf{d}[n]$. The r-path $\mathbf{c}_1, \dots, \mathbf{c}_l$ and the tuple $\mathbf{d}' = f(\mathbf{c}_{l-1}, \mathbf{e}')$ satisfy the conditions of the Path Alignment Lemma 6.2; hence, there is $\mathbf{c}' \in R$ such that $\mathbf{c}'[1] = \mathbf{e}[1]$ and $\mathbf{c}'[n] = \mathbf{c}[n]$.

We set $J = \{i \in \underline{n} \mid \mathbf{c}[i], \mathbf{c}'[i]$ are indistinguishable$\}$, $K = \underline{n} - J$. Clearly, both sets are non-empty, as $n \in J$ and $1 \in K$. By the Fork Lemma 6.8, where $\mathbf{o} = \mathbf{c}[n]$ and $I = J - \{n\}$, we get $\max(\mathrm{pr}_J \mathbf{c}) \times \max(\mathrm{pr}_K \mathbf{c}) \subseteq R$. Since $\mathbf{c}[1] \in B_1$, by induction hypothesis, $\max(\mathrm{pr}_{K-I_1} \mathbf{c}) \times \max(\mathrm{pr}_{I_1} \mathbf{c}) \subseteq R$. Therefore, $\max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{c}) \times \max(\mathrm{pr}_{I_1} \mathbf{c}) = \max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{c}) \times \max(\mathrm{pr}_{I_1} \mathbf{b}) \subseteq R$. Finally, making use of Lemma 6.6, we get $\max(\mathrm{pr}_J \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) \subseteq R$ where $J = \underline{n} - I_1$ and $K = I_1$.

Case 3. For any $\mathbf{b} \in R$ such that $\mathbf{b}[1] \in B_1$ and any $i \in \underline{n}$, $\mathbf{b}[i]$ belongs to a minimal i-component, but there are $\mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{b}[1] \in B_1$, $\mathbf{b}[n], \mathbf{c}[n]$ belong to the same minimal i-component B and $\mathbf{b}[1], \mathbf{c}[1]$ are not indistinguishable.

Subcase 3.1. B contains an r-maximal element.

Let $\mathbf{d} \in \mathcal{F}(\mathbf{b})$ be an r-maximal tuple. Let us denote the set $J(\mathrm{m}, n, B)$ by I. By Claim 3, I is the union of the sets $J(\mathrm{ry}, i, C_i)$ where $i \in I$ and C_i is the strongly ry-connected component of $\mathcal{G}(\max(A_i))$ containing $\mathbf{d}[i]$. Since $\mathcal{G}(\max(\mathrm{pr}_I \mathbf{d}))$ is strongly ry-connected, by the Rectangularity Proposition 5.4(1), $\mathrm{pr}_{\underline{n}-I} \mathbf{d} \times \max(\mathrm{pr}_I \mathbf{d}) = \mathrm{pr}_{\underline{n}-I} \mathbf{d} \times \max(\mathrm{pr}_I \mathbf{b}) \subseteq R$. Furthermore, by Lemma 6.6, $\max(\mathrm{pr}_{\underline{n}-I} \mathbf{b}) \times \max(\mathrm{pr}_I \mathbf{b}) \subseteq R$ and we may set $J = \underline{n} - I$, $K = I$.

Subcase 3.2. B contains no r-maximal element.

In this case the proof is similar to that in Case 2.

Now, by induction hypothesis $\max(\mathrm{pr}_{J-I_1} \mathbf{b}) \times \max(\mathrm{pr}_{I_1} \mathbf{b}) \subseteq \mathrm{pr}_J R$. Therefore, $\max(\mathrm{pr}_{J-I_1} \mathbf{b}) \times \max(\mathrm{pr}_{I_1} \mathbf{b}) \times \max(\mathrm{pr}_K \mathbf{b}) = \max(\mathrm{pr}_{\underline{n}-I_1} \mathbf{b}) \times \max(\mathrm{pr}_{I_1} \mathbf{b}) \in R$.

Since $\max(\mathrm{pr}_{I_1}\mathbf{b}) = \left(R \cap \prod_{i\in I_1} \max(B_i)\right)$, we get

$$\max(\mathrm{pr}_{\underline{n}-I_1}\mathbf{b}) \times \left(R \cap \prod_{i\in I_1} \max(B_i)\right) \subseteq R.$$

Finally, as $\mathrm{pr}_{\underline{n}-I_1}\mathbf{a} \in \max(\mathrm{pr}_{\underline{n}-I_1}\mathbf{b})$ and $\mathbf{a}' \in \left(R \cap \prod_{i\in I_1} \max(B_i)\right)$, making use of Lemma 6.6, we get the required result. □

6.4 Maximal strongly r-connected components

We complete Section 6 by showing how the results of this section can be used to solve constraint satisfaction problems, in which the graph of every domain $\mathcal{S}_v$ is strongly rb-connected, but, for a certain $v \in V$, there is an r-connected component of $\mathcal{G}(\mathcal{S}_v)$ which is not strongly r-connected.

Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be such a problem, and let $\mathcal{S}_v^0$ denote the set of all minimal elements from $\mathcal{S}_v$, $v \in V$. First, similar to Section 5.3, we extend the notation $J(\mathrm{m}, i, B)$ from relations to problem instances. The following property follows straightforwardly by the construction of $J(\mathrm{m}, v, B)$. By B_w we denote the minimal i-component corresponding to $w \in J(\mathrm{m}, v, B)$.

Lemma 6.12. *If φ is a solution to $\mathcal{P}$ such that $\varphi(v) \in B$ then $\varphi(w) \in B_w$ for any $w \in J(\mathrm{m}, v, B)$.*

We consider 4 cases: first, $J(\mathrm{m}, v, B) \neq V$ for some $v \in V$ and minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$; second, $J(\mathrm{m}, v, B) = V$ for all v and B, but $\mathcal{S}_v^0 \neq \mathcal{S}_v$ for some $v \in V$; third, $J(\mathrm{m}, v, B) = V$ and $\mathcal{S}_v^0 = \mathcal{S}_v$ for all v and B, but $\mathcal{G}(\mathcal{S}_v)$ contains more than one i-component for some $v \in V$; and forth, $\mathcal{S}_v^0 = \mathcal{S}_v$ and $\mathcal{G}(\mathcal{S}_v)$ has only one i-component for all v, this implies $J(\mathrm{m}, v, B) = V$ for all v and (a unique) B. As before, we construct subproblems of $\mathcal{P}$, and prove that if some of those subproblems has no solutions assigning an r-maximal element a from some domain, then $\mathcal{P}$ can be tightened by removing a from the corresponding domain.

Case 1. In this case the subproblems of $\mathcal{P}$ correspond to the sets of the form $J(\mathrm{m}, v, B)$ and are defined as follows. For $v \in V$ and a minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$, by $\mathcal{P}^+_{v,B}$ we denote the constraint satisfaction problem $(W; \mathcal{A}; \delta'; \mathcal{C}')$ where $W = J(\mathrm{m}, v, B)$, $A_{\delta'(w)} = \{a \in \mathcal{S}_w \mid B_w \sqsubseteq a\}$ and, for each constraint $\langle s, R\rangle \in \mathcal{C}$, we include $C' = \langle s \cap W, R'\rangle$ in $\mathcal{C}'$, with $R' = \{\mathrm{pr}_{s\cap W}\mathbf{a} \mid \mathbf{a} \in R,\ \mathbf{a}[w] \in A_{\delta'(w)}$ for $w \in s \cap W\}$. As is easily seen, for any v, w such that $w \in J(\mathrm{m}, v, B_v)$, we have $\mathcal{P}^+_{v,B_v} = \mathcal{P}^+_{w,B_w}$.

Lemma 6.13 (Maximal Solution Lemma). *If $\mathcal{P}$ has a solution and every problem of the form $\mathcal{P}^+_{v,B}$ has a solution $\varphi_{v,B}$ such that, for any $w \in J(\mathrm{m}, v, B)$, $\varphi_{v,B}(w) \in \max(B_w)$, then there is a solution ψ to $\mathcal{P}$ such that $\psi(w)$ is r-maximal for any $w \in V$.*

Proof. Let φ be a solution to $\mathcal{P}$ and B_v, $v \in V$, a minimal i-component such that $B_v \sqsubseteq \varphi(v)$. By Lemma 6.9, the B_v can be chosen such that for any $v \in V$ any $w \in J(\mathrm{m}, v, B_v)$, the minimal i-component B_w corresponds to B_v, that is, for any $\mathbf{a} \in \mathcal{S}_{v,w}$, if $\mathbf{a}[v] \in B_v$ then $\mathbf{a}[w] \in B_w$. Indeed, if it is not the case, let B'_w be the minimal i-component corresponding to B_v. By Lemma 3.7 there is $(a,b) \in \mathcal{F}((\varphi(v),\varphi(w))$ in $\mathcal{S}_{v,w}$ such that $a \in \mathcal{F}(B_v)$ and $b \in \mathcal{F}(B'_w) - \mathcal{F}(B_w)$. However, by Lemma 6.9 this contradicts the assumption $w \in J(\mathrm{m}, v, B_v)$. By the assumption, for any $v \in V$, the problem $\mathcal{P}^+_{v,B_v}$ has a solution φ_{v,B_v} such that $\varphi_{v,B_v}(w) \in \max(B_w)$ for any $w \in J(\mathrm{m}, v, B_v)$. Since $\mathcal{P}^+_{v,B_v} = \mathcal{P}^+_{w,B_w}$ for any v, w such that $w \in J(\mathrm{m}, v, B_v)$, we may assume that $\varphi_{v,B_v} = \varphi_{w,B_w}$.

Let $W_1, \dots, W_k$ be the sets of the form $J(\mathrm{m}, v, B_v)$, let $\psi_1, \dots, \psi_k$ be the solutions of corresponding problems $\mathcal{P}^+_{v,B}$, and let the mapping ψ be defined by $\psi(v) = \psi_i(v)$ whenever $v \in W_i$. Since, for any $\langle s, R\rangle \in \mathcal{C}$, $(\varphi(v))_{v\in s} \in R$ and $(\psi(v))_{v\in s\cap W_j} \in \max((\varphi(v))_{v\in s\cap W_j})$, by the Maximal Rectangularity Proposition 6.11, we have $(\psi(v))_{v\in s} \in R$, and therefore ψ is a solution to $\mathcal{P}$ with the required properties. $\square$

LEMMA 6.14. *If for one of the problems $\mathcal{P}^+_{v,B}$ there is a variable w an r-maximal element a from the domain of w such that no solution φ of $\mathcal{P}^+_{v,B}$ satisfies $\varphi(v) = a$, then $\mathcal{P}$ can be tightened by removing a from the domain of w. More precisely $\varphi(w) = a$ for no solution ψ of $\mathcal{P}$.*

PROOF. Assume there is $v \in V$ and a minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$ such that $J(\mathrm{m}, v, B) \neq V$, and for some $w \in V$ and a minimal i-component D of $\mathcal{G}(\mathcal{S}_w)$, the problem $\mathcal{P}^+_{w,D}$ has no solution such that the value of some variable w' equals certain r-maximal element a. In order to get a contradiction let us assume that there exists a solution φ of $\mathcal{P}$ such that $\varphi(w') = a$. By Lemma 6.10, for any $u \in I$ and any $\mathbf{a} \in \mathcal{S}_{w,u}$, if $\mathbf{a}[w] \in \max(D)$ then $B_u \sqsubseteq \mathbf{a}[u]$, where B_u is the minimal i-component of $\mathcal{G}(\mathcal{S}_u)$ corresponding to D, that is $\mathbf{a}[u] \in A_{\delta'(u)}$. Therefore, $\varphi|_I$ is a solution of $\mathcal{P}^+_{w,D}$. $\square$

By Lemma 6.14, if a problem of the form $\mathcal{P}^+_{v,B}$ has no r-maximal solution, then $\mathcal{P}$ can be tightened. Otherwise, the Maximal Solution Lemma 6.13 allows us to reduce $\mathcal{P}$ to the smaller problems $\mathcal{P}^+_{v,B}$, and the problem $\mathcal{P}^{\max} = (V; \mathcal{A}; \delta; \mathcal{C}')$ in which every constraint relation $\langle s, R\rangle$ is replaced with $\langle s, R'\rangle$, $R' = \{\mathbf{a} \in R \mid$ for any $v \in s$, $\mathbf{a}[v]$ is r-maximal$\}$.

However, Lemma 6.13 cannot help us in the case when, for any $v \in V$ and any minimal i-component B, $J(\mathrm{m}, v, B) = V$. In this case we need another method.

CASE 2. Let $\mathcal{P}^0$ denote the problem $(V; \mathcal{A}; \delta^0; \mathcal{C}^0)$, where $A_{\delta^0(v)} = \mathcal{S}^0_v$ and for each $\langle s, R\rangle \in \mathcal{C}$ the set $\mathcal{C}^0$ contains $\langle s, R^0\rangle$ with $R^0 = \{\mathbf{a} \in R \mid \mathbf{a}[v] \in \mathcal{S}^0_v$ for every $v \in s\}$.

LEMMA 6.15 (SINGLE MINIMAL LEMMA). *If $J(\mathrm{m}, v, B) = \mathcal{S}_v$ for every $v \in V$ and every minimal i-component of $\mathcal{G}(\mathcal{S}_v)$, then every solution φ of $\mathcal{P}$ such that $\varphi(v) \in \mathcal{S}^0_v$ for some $v \in V$ is also a solution of $\mathcal{P}^0$.*

PROOF. Let φ be a solution of $\mathcal{P}$ such that $\varphi(v) \in \mathcal{S}^0_v$. Since $J(\mathrm{m}, v, B) = V$, where B is the minimal i-component containing $\varphi(v)$, for any $w \in V - \{v\}$ and any $\mathbf{a} \in \mathcal{S}_{v,w}$ such that $\mathbf{a}[v] = \varphi(v)$ we have $\mathbf{a}[w] \in \mathcal{S}^0_w$. This implies the result. $\square$

Let $\mathcal{P}^+$ denote the problem $(V; \mathcal{A}; \delta^+; \mathcal{C}^+)$, where $A_{\delta^+(v)} = \mathcal{S}_v - \mathcal{S}^0_v$ and for any $\langle s, R\rangle \in \mathcal{C}$ the set $\mathcal{C}^+$ contains $\langle s, R^+\rangle$ with $R^+ = \{\mathbf{a} \in R \mid \mathbf{a}[v] \in \mathcal{S}_v - \mathcal{S}^0_v$ for every $v \in s\}$.

Corollary 6.16. *Every solution of $\mathcal{P}$ is a solution of $\mathcal{P}_0$ or of $\mathcal{P}^+$. Therefore $\mathcal{P}$ is reducible to $\mathcal{P}_0$ and $\mathcal{P}^+$.*

Lemma 6.17. *If for the problem $\mathcal{P}^+$ there is a variable w an r-maximal element a from the domain of w such that no solution φ of $\mathcal{P}^+$ satisfies $\varphi(v) = a$, then $\mathcal{P}$ can be tightened by removing a from the domain of w. More precisely $\varphi(w) = a$ for no solution ψ of $\mathcal{P}$.*

Proof. Every solution of $\mathcal{P}$ is either a solution of $\mathcal{P}^+$ or a solution of $\mathcal{P}^0$. The latter problem cannot have a solution ψ with $\psi(w) = a$, therefore, a can be removed from $A_{\delta(w)}$. $\square$

Case 3. By the assumption $\mathcal{S}_v^0 = \mathcal{S}_v$ and $J(\mathrm{m}, v, B) = V$ for all $v \in V$, and for some $w \in V$ the graph $\mathcal{G}(\mathcal{S}_w)$ contains more than one i-component (all of them are minimal).

Lemma 6.18 (Many Minimal Lemma). *Let $\mathcal{S}_v^0 = \mathcal{S}_v$ and $J(\mathrm{m}, v, B) = V$ for all $v \in V$. (1) For any $v, w \in V$ there is a one-to-one correspondence $\xi_{v,w}$ between the i-components of $\mathcal{G}(\mathcal{S}_v)$ and the i-components of $\mathcal{G}(\mathcal{S}_w)$ such that, for any i-component B of $\mathcal{G}(\mathcal{S}_v)$, we have $\xi_{v,w} = B_w$.*
(2) The graph $\mathcal{G}(\mathcal{S}_v)$ contains the same number of i-components for each $v \in V$; denote this number by ℓ.
(3) There are ℓ different problems of the form $\mathcal{P}^+_{v,B}$, which can be represented as follows: fix $v \in V$ and let $B^1, \dots, B^\ell$ be the i-components of $\mathcal{G}(\mathcal{S}_v)$; then the different problems are $\mathcal{P}^+_{v,B^1}, \dots, \mathcal{P}^+_{v,B^\ell}$.
(4) for any solution φ of $\mathcal{P}$, the mapping φ is solution of $\mathcal{P}^+_{v,B}$ for some $v \in V$ and an i-component B of $\mathcal{G}(\mathcal{S}_v)$.

Proof. (1) follows straightforwardly from the condition $J(\mathrm{m}, v, B) = V$ for all v and B, and (2) is a direct consequence of (1). Statement (3) is again a direct implication of the definition of a problem $\mathcal{P}^+_{v,B}$. If φ is a solution to $\mathcal{P}$ and, for some $v \in V$, $\varphi(v) \in B$, where B is an i-component of $\mathcal{G}(\mathcal{S}_v)$, then $\varphi(w) \in \xi_{v,w}(B)$ for every $w \in V$. This proves (4). $\square$

Lemma 6.19. *If for one of the problems $\mathcal{P}^+_{v,B}$ there is a variable w an r-maximal element a from the domain of w such that no solution φ of $\mathcal{P}^+_{v,B}$ satisfies $\varphi(v) = a$, then $\mathcal{P}$ can be tightened by removing a from the domain of w. More precisely $\varphi(w) = a$ for no solution ψ of $\mathcal{P}$.*

Proof. Follows straightforwardly from the Many Minimal Lemma 6.18(4). $\square$

Thus, if any of the problems $\mathcal{P}^+_{v,B}$ has an r-maximal solution then it is also a solution to $\mathcal{P}$. Otherwise the problem can be tightened.
Case 4. Recall that the set of r-maximal elements of $\mathcal{S}_v$ is denoted by $\mathcal{S}_v^{\max}$. By the assumption $\mathcal{S}_v^0 = \mathcal{S}_v$, therefore $\mathcal{G}(\mathcal{S}_v^{\max})$ is strongly ry-connected. Note that, although $\mathcal{S}_v$ is strongly rb-connected, $\mathcal{G}(\mathcal{S}_v^{\max})$ may be not strongly rb-connected. First suppose that, for any $v \in V$, it is also strongly rb-connected. In this case, $\mathcal{P}$ satisfies the conditions studied in Sections 4.3 and 5.5.

Then, suppose that there is $v \in V$ such that $\mathcal{G}(\mathcal{S}_v^{\max})$ is not strongly rb-connected. We consider the sets of the form $J_{\max}(\mathrm{rb}, v, B)$ defined as $J(\mathrm{rb}, v, B)$ for the problem $\mathcal{P}^{\max}$, where B is a strongly rb-connected component of $\mathcal{G}(\mathcal{S}_v^{\max})$. As usual, having fixed v and B we get a strongly rb-connected component B_w for each $w \in J_{\max}(\mathrm{rb}, v, B)$.

We define subproblems of $\mathcal{P}$ corresponding to the sets of the form $J_{\max}(\mathrm{rb}, v, B)$ as follows. By $\mathcal{P}^{\max}_{v,B}$ we denote the constraint satisfaction problem $(W; \mathcal{A}; \delta'; \mathcal{C}')$ where $W = J_{\max}(\mathrm{rb}, v, B)$, $A_{\delta'(w)}$ is the set of all $a \in \mathcal{S}_w$ such that $\max(a) \cap \mathcal{F}(a) \subseteq B_w$ and, for each constraint $\langle s, R\rangle \in \mathcal{C}$, we include $C' = \langle s \cap W, R'\rangle$ in $\mathcal{C}'$, with $R' = \{\mathrm{pr}_{s\cap W}\mathbf{a} \mid \mathbf{a} \in R,\ \mathbf{a}[w] \in A_{\delta'(w)} \text{ for } w \in s \cap W\}$. Note that $\max(A_{\delta'(w)}) = B_w$.

LEMMA 6.20 (ALL MINIMAL LEMMA). *Suppose that*
(1) $\mathcal{P}$ is 3-minimal;
(2) $\mathcal{S}_v = \mathcal{S}^0_v$ for any $v \in V$;
(3) there is $v \in V$ such that $\mathcal{G}(A_{\delta'(v)})$ is not strongly rb-connected;
(4) for any $v \in V$ and rb-maximal component B of $\mathcal{G}(A_{\delta'(v)})$, the problem $\mathcal{P}^{\max}_{v,B}$ has a solution $\psi_{v,B}$ such that $\psi_{v,B}(w) \in B_w$ for any $w \in J_{\max}(\mathrm{rb}, v, B)$.
Then $\mathcal{P}$ has a solution φ such that $\varphi(v)$ is r-maximal for any $v \in V$.

PROOF. If (1)–(4) hold, the problem $\mathcal{P}^{\max}$ defined in Case 1 satisfies the conditions of Red-Blue Decomposition Proposition 5.7, and therefore has a solution. □

LEMMA 6.21. *If for one of the problems $\mathcal{P}^{\max}_{v,B}$ there is a variable w and r-maximal element a from the domain of w such that no solution φ of $\mathcal{P}^{\max}_{v,B}$ satisfies $\varphi(v) = a$, then $\mathcal{P}$ can be tightened by removing a from the domain of w. More precisely $\varphi(w) = a$ for no solution ψ of $\mathcal{P}$.*

PROOF. We have to show that if, for a certain $v \in V$, a strongly rb-connected component B of $\mathcal{G}(\mathcal{S}^{\max}_v)$, and $a \in B$, the problem $\mathcal{P}^{\max}_{v,B}$ has no solution ψ such that $\psi(v) = a$, then, for any solution φ to $\mathcal{P}$, we have $\varphi(v) \neq a$.

Supposing the contrary, let φ be a solution to $\mathcal{P}$ such that $\varphi(v) = a$. Then there is $w \in J_{\max}(\mathrm{rb}, v, B)$ such that $\mathcal{F}(\varphi(w)) \cap \max(\varphi(w)) \not\subseteq B_w$, where B_w is the strongly rb-connected component of $\mathcal{G}(\mathcal{S}^{\max}_w)$ corresponding to B. Let $b \in \mathcal{F}(\varphi(w)) \cap \max(\varphi(w))$ be such that $b \notin B_w$. Take an r-path in $\mathcal{S}_{v,w}$,

$$\begin{pmatrix}\varphi(v)\\ \varphi(w)\end{pmatrix} = \begin{pmatrix}a_1\\ b_1\end{pmatrix} \leq \begin{pmatrix}a_2\\ b_2\end{pmatrix} \leq \cdots \leq \begin{pmatrix}a_k\\ b_k\end{pmatrix} = \begin{pmatrix}a_k\\ b\end{pmatrix}.$$

Clearly, $a_k \in B$ while $b \notin B_w$ that contradicts the assumption $w \in J_{\max}(\mathrm{rb}, v, B)$. Thus the elements of B can be removed from $A_{\delta(v)}$. □

7. ALGORITHM

We are now ready to pull all the results obtained so far together and present an algorithm that solves conservative constraint satisfaction problems. In Section 7.1, we recall constructions used in previous sections, and prove the Tightening Proposition 7.1 that allows us to remove some elements from the domains of the original problems if the derived smaller problems have no solution. Then, in Section 7.2, we give a formal description of the algorithm. Finally, in Section 7.3, we prove that the algorithm is correct and estimate its time complexity.

As usual, we assume all problem instances to be 1-minimal, and all constraint relations to be subdirect products of their domains.

7.1 Nine types of subproblems

Before giving a formal description of the algorithm we summarize all the cases it distinguishes and all the constructions it uses. In some sense this review is similar to that in Section 2.7. Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a problem instance from CSP(Γ). We distinguish 8 main cases.

A non-rb-connected domain. If the graph of one of the domains is not strongly rb-connected then we construct problems of the form $\mathcal{P}_{\mathrm{rb},v,B}$, for $v \in V$ and an rb-maximal component B of $\mathcal{G}(\mathcal{S}_v)$, that is defined to be $(U; \mathcal{A}; \delta'; \mathcal{C}')$ where $U = J(\mathrm{rb}, v, B)$, $A_{\delta'(w)} = B_w$ for $w \in U$ and, for every $C = \langle s, R\rangle \in \mathcal{C}$, there is $C' = \langle s \cap U; R'\rangle \in \mathcal{C}$ such that $R' = \{\mathrm{pr}_{s\cap U}\mathbf{a} \mid \mathbf{a} \in R,\ \mathbf{a}[w] \in B_w \text{ for any } w \in s \cap U\}$. The graph of every domain of $\mathcal{P}_{\mathrm{rb},v,B}$ is strongly rb-connected. By the Red-Blue Decomposition Proposition 5.7, if every problem of this form has a solution then $\mathcal{P}$ has a solution. The Tightening Proposition 7.1 claims that otherwise $\mathcal{P}$ can be tightened by removing some elements from the domains. We use algorithm Block-3-Width in this case.

Note that this approach works even when $J(\mathrm{rb}, v, B) = V$ for some v and B. In this case, the domains of the problem $\mathcal{P}_{\mathrm{rb},v,B}$ are strictly smaller than those of $\mathcal{P}$, as one of the domains of $\mathcal{P}$ is not strongly rb-connected.

In the next 3 cases every domain of $\mathcal{P}$ is strongly rb-connected, but there is a domain $\mathcal{S}_v$ such that the set of r-maximal elements of $\mathcal{S}_v$ is not strongly ry/rb-connected.

Every r-connected component of every domain is strongly r-connected. We use two types of restricted problems in this case. Firstly, for every $v \in V$ and every strongly ry-connected component B of $\mathcal{G}(\mathcal{S}_v)$, we define the problem $\mathcal{P}_{\mathrm{ry},v,B}$ to be $(U; \mathcal{A}; \delta'; \mathcal{C}')$ where $U = J(\mathrm{ry}, v, B)$, $A_{\delta'(w)} = B_w$ for $w \in U$ and, for every $C = \langle s, R\rangle \in \mathcal{C}$, there is $C' = \langle s \cap U; R'\rangle \in \mathcal{C}$ such that $R' = \{\mathrm{pr}_{s\cap U}\mathbf{a} \mid \mathbf{a} \in R,\ \mathbf{a}[w] \in B_w \text{ for any } w \in s \cap U\}$. Every domain of $\mathcal{P}_{\mathrm{ry},v,B}$ is strongly ry-connected (although it may be not strongly rb-connected) and every element is r-maximal. If one of such problems $\mathcal{P}_{\mathrm{ry},v,B}$ has no solution then, as is easily seen, no solution φ of $\mathcal{P}$ has $\varphi(v) \in B$. Therefore $\mathcal{P}$ can be tightened. If every such problem has a solution then, by the Skeleton Decomposition Lemma 5.10, $\mathcal{P}$ has a solution if and only if the skeleton problem $\mathcal{P}^s$ has a solution. The skeleton problem is defined as follows (see also Section 5.4): Let $\varphi_{\mathrm{ry},v,B}$ be a solution of $\mathcal{P}_{\mathrm{ry},v,B}$. Then $\mathcal{P}^s = (V; \mathcal{A}; \delta; \mathcal{C}^s)$, where for every $C = \langle s, R\rangle \in \mathcal{C}$ there is $C^s = \langle s, R^s\rangle \in \mathcal{C}^s$ such that $\mathbf{a} \in R^s$ if and only if $\mathbf{a} \in R$ and, for any $v \in s$, there is an ry-connected component B of $\mathcal{G}(\mathcal{S}_v)$ such that $\mathbf{a}[v] = \varphi_{\mathrm{ry},v,B}(v)$. Note that every edge in every domain of this problem is blue.

Non-trivial max-decomposition. In this case there is $v \in V$ and a minimal i-component of $\mathcal{G}(\mathcal{S}_v)$ such that $J(\mathrm{m}, v, B) \neq V$. Then, for every $v \in V$ and every minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$ we define the problem $\mathcal{P}^+_{v,B} = (W; \mathcal{A}; \delta'; \mathcal{C}')$ where $W = J(\mathrm{m}, v, B)$, $A_{\delta'(w)} = \{a \in \mathcal{S}_w \mid B_w \sqsubseteq a\}$ and, for each constraint $\langle s, R\rangle \in \mathcal{C}$, we include $C' = \langle s \cap W, R'\rangle$ in $\mathcal{C}'$, with $R' = \{\mathrm{pr}_{s\cap W}\mathbf{a} \mid \mathbf{a} \in R,\ \mathbf{a}[w] \in A_{\delta'(w)} \text{ for } w \in s \cap W\}$. For every problem $\mathcal{P}^+_{v,B}$ we have $J(\mathrm{m}, v, B) = W$ for any v and B, although the graphs of some domains can be not strongly rb-connected. If one of such problems has no solution then, by the Tightening Proposition 7.1, $\mathcal{P}$ can be tightened. Otherwise the Maximal Solution Lemma 6.13 claims that $\mathcal{P}$ has a solution if and only if $\mathcal{P}^{\max}$ has a solution, where $\mathcal{P}^{\max} = (V; \mathcal{A}; \delta; \mathcal{C}')$ in which every constraint relation $\langle s, R\rangle \in \mathcal{C}$ is replaced with $\langle s, R'\rangle$, $R' = \{\mathbf{a} \in R \mid$ for any $v \in s$, $\mathbf{a}[v]$ is r-maximal $\}$.

Trivial max-decomposition. Recall that $\mathcal{S}_v^0$ denotes the set of all elements from minimal i-components of $\mathcal{G}(\mathcal{S}_v)$. If $J(\mathrm{m}, v, B) = V$ for all v and B, and $\mathcal{S}_v \neq \mathcal{S}_v^0$ for some v, then the Single Minimal Lemma 6.15 shows that $\mathcal{P}$ has a solution if and only if one of $\mathcal{P}^0$ or $\mathcal{P}^+$ has a solution. Problem $\mathcal{P}^0$ is defined to be $(V; \mathcal{A}; \delta^0; \mathcal{C}^0)$, where $A_{\delta^0(v)} = \mathcal{S}_v^0$ and for each $\langle s, R\rangle \in \mathcal{C}$ the set $\mathcal{C}^0$ contains $\langle s, R^0\rangle$ with $R^0 = \{\mathbf{a} \in R \mid \mathbf{a}[v] \in \mathcal{S}_v^0$ for every $v \in s\}$. Then $\mathcal{P}^+ = (V; \mathcal{A}; \delta^+; \mathcal{C}^+)$, where $A_{\delta^+(v)} = \mathcal{S}_v - \mathcal{S}_v^0$ and for any $\langle s, R\rangle \in \mathcal{C}$ the set $\mathcal{C}^+$ contains $\langle s, R^+\rangle$ with $R^+ = \{\mathbf{a} \in R \mid \mathbf{a}[v] \in \mathcal{S}_v - \mathcal{S}_v^0$ for every $v \in s\}$.

All elements are minimal, there are multiple i-minimal components. If $J(\mathrm{m}, v, B) = V$ and $\mathcal{S}_v = \mathcal{S}_v^0$ for all v and B, and the graph $\mathcal{G}(\mathcal{S}_v)$ contains more than one minimal i-component for some v, then the Many Minimal Lemma 6.18 states that $\mathcal{P}$ has an r-maximal solution if and only if one of the problems $\mathcal{P}_{v,B}^+$ has, where v is a fixed variable from V and B goes over the set of minimal i-components of $\mathcal{G}(\mathcal{S}_v)$.

All elements are minimal, there is only one minimal i-component. In this case $\mathcal{S}_v^0 = \mathcal{S}_v$ for all $v \in V$, but for some v the graph $\mathcal{G}(\mathcal{S}_v^{\max})$ is not strongly rb-connected. We use $J_{\max}(\mathrm{rb}, v, B)$ to denote the set $J(\mathrm{rb}, v, B)$ for the problem $\mathcal{P}^{\max}$, where B is a strongly rb-connected component of $\mathcal{G}(\mathcal{S}_v^{\max})$, and by $\mathcal{P}_{v,B}^{\max}$ we denote the problem $(W; \mathcal{A}; \delta'; \mathcal{C}')$ where $W = J_{\max}(\mathrm{rb}, v, B)$, $A_{\delta'(w)}$ is the set of all $a \in \mathcal{S}_w$ such that $\max(a) \cap \mathcal{F}(a) \subseteq B_w$ and, for each constraint $\langle s, R\rangle \in \mathcal{C}$, we include $C' = \langle s \cap W, R'\rangle$ in $\mathcal{C}'$, with $R' = \{\mathrm{pr}_{s\cap W}\mathbf{a} \mid \mathbf{a} \in R,\ \mathbf{a}[w] \in A_{\delta'(w)}$ for $w \in s \cap W\}$. The All Minimal Lemma 6.20 implies that if all the problems $\mathcal{P}_{v,B}^{\max}$ have a solution then $\mathcal{P}$ has a solution such that the value of each variable is r-maximal in its domain. Otherwise, by the Tightening Proposition 7.1, some elements can be removed from the domains of $\mathcal{P}$.

In the remaining cases the set of r-maximal elements of every domain is strongly ry/rb-connected.

A non-hereditarily double-connected domain. In this case, we check if for every variable $v \in V$ of finite depth and any sequences of congruences $\overline{\theta}$ and their classes $\overline{B}$, the problem $\mathcal{P}^*_{v,\overline{B},\overline{\theta}}$ has a solution (see Section 5.5 for definitions). If not then the Double-Connected Tightening Lemma 5.16 claims that $\mathcal{P}$ can be tightened. Otherwise a solution for $\mathcal{P}$ is assembled from solutions of problems of that form by algorithm Ry/rb-Conn.

Hereditarily double-connected domains. Proposition 4.11 claims that in this case every 3-minimal problem without empty constraint relations has a solution. We use algorithm 3-Width.

Same colour domains. In this case one of the previously known algorithms can be used: 3-Width if all edges of all domains are yellow, or all edges are red, and Maltsev if all edges of all domains are blue.

We complete this section by proving that in all the cases considered problem $\mathcal{P}$ can be tightened.

PROPOSITION 7.1 (TIGHTENING PROPOSITION). *If one of the problems* $\mathcal{P}_{\mathrm{rb},v,B}$ *(*$\mathcal{P}_{\mathrm{ry},v,B}$*,* $\mathcal{P}^+_{v,B}$*,* $\mathcal{P}^+$*,* $\mathcal{P}^{\max}_{v,B}$*,* $\mathcal{P}^*_{v,\overline{B},\overline{\theta}}$*, depending on the type of the problem) has no (r-maximal) solution, then* $\mathcal{P}$ *can be tightened by removing r-maximal elements from some of the domains. More precisely there are* $w \in V$ *and* $a \in A^{\max}_{\delta(w)}$ *such that* $\varphi(w) = a$ *for no solution* φ *of* $\mathcal{P}$*.*

PROOF. We proceed by induction on the size of $\mathcal{P}$. In the base case of induction either every edge of the graph of every domain is yellow, or every edge is blue, or the set of r-maximal elements of every domain is hereditarily ry/rb-connected. In the first two cases we assume all the listed derived problems to coincide with $\mathcal{P}$ (although from the formal point of view it is not so, the derived problems that do not equal $\mathcal{P}$ are on 1-element domains and therefore trivial). Such problem can be solved by algorithms 3-WIDTH or MALTSEV and, if they do not have a solution then $\mathcal{P}$ does not have a solution; therefore $\mathcal{P}$ can be tightened by removing *all* elements from *all* domains. In the third case, again 3-WIDTH solves $\mathcal{P}$, and the problem can be tightened in the same fashion.

The induction step follows from Lemmas 5.8, 5.11, 5.16, 6.14, 6.17, 6.19, 6.21. The proposition is proved. □

We complete this section by proving that if the domains from $\mathcal{A}$ contain no blue edge then the problem MCSP(Γ) is of bounded relational width.

LEMMA 7.2. *If, for any* $A \in \mathcal{A}$*, every edge of the graph* $\mathcal{G}(A)$ *is red or yellow then* MCSP(Γ) *has relational width 3.*

PROOF. Take a problem $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C}) \in$ MCSP(Γ). Since $\mathcal{G}(\mathcal{S}_v)$ is not strongly rb-connected, we should construct the problems $\mathcal{P}_{\mathrm{rb},v,B}$. However, as all the strongly rb-connected components are 1-element, the 3-minimality of $\mathcal{P}$ implies that all of them have solutions. Finally, by the Red-Blue Decomposition Proposition 5.7 and the 3-minimality of $\mathcal{P}$ have a solution. □

7.2 Algorithm

Now we in are a position to give a formal description of the algorithm.

INPUT: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.

OUTPUT: Whether $\mathcal{P}$ has a solution.

Step 1. **invoke** CONSERV($\mathcal{P}$) (see Fig. 7.2)
Step 2. **if** $\mathcal{P} \neq \varnothing$ **output** "YES", **otherwise output** "NO".
Step 3. **stop.**

Fig. 7.1. Algorithm solving conservative constraint satisfaction problems

INPUT: Problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.
OUTPUT: A solution to $\mathcal{P}$ and a tightened problem instance if $\mathcal{P}$ has a solution, $(\varnothing, \varnothing)$ otherwise.
Step 1. **invoke** 3-MINIMALITY($\mathcal{P}$)
Step 2. **find** ry-, rb-, r- connected and i-minimal components of $\mathcal{G}(\mathcal{S}_v)$, $v \in V$

Step 3. **ifcase:**
there exists $v \in V$ such that $\mathcal{G}(\mathcal{S}_v)$ is not strongly rb-connected:
Step 3.1. **do**
Step 3.1.1. **for each** $v \in V$, **each** rb-maximal component $B \subseteq \mathcal{G}(\mathcal{S}_v)$, **do**
Step 3.1.1.1. $(\varphi, \mathcal{P}'_{\mathrm{rb},v,B})$:=CONSERV($\mathcal{P}_{\mathrm{rb},v,B}$)
Step 3.1.1.2. **for every** constraint $C = \langle s, R\rangle$ of $\mathcal{P}$, **remove** from R every tuple $\mathbf{a}$ such that $\mathrm{pr}_{s\cap W}\mathbf{a} \notin R'$, where $C' = \langle s \cap W, R'\rangle$ is the corresponding constraint of $\mathcal{P}'_{\mathrm{rb},v,B}$ and $W = J(\mathrm{rb}, v, B)$
endfor
Step 3.1.2. **if** $\mathcal{P}$ changed **then return**(CONSERV($\mathcal{P}$))
Step 3.1.3. **else return**(BLOCK-3-WIDTH($\mathcal{P}$),$\mathcal{P}$)
enddo
there exists $v \in V$ and an r-connected component C of $\mathcal{G}(\mathcal{S}_v)$ such that $\mathcal{G}(C)$ is not strongly r-connected, there is $u \in V$ such that $\mathcal{G}(\mathcal{S}_u^{\max})$ is not strongly ry/rb-connected, and, for any $w \in V$, $\mathcal{S}_w = \mathcal{S}_w^0$ and $\mathcal{G}(\mathcal{S}_w)$ has a unique minimal i-component:
Step 3.2. **do**
Step 3.2.1. **for each** $v \in V$, **each** strongly rb-connected component B of $\mathcal{G}(\mathcal{S}_v^{\max})$, **do**
Step 3.2.1.1. $(\varphi, \mathcal{P}'_{v,B})$:=CONSERV($\mathcal{P}^{\max}_{v,B}$)
Step 3.2.1.2. **for every** constraint $C = \langle s, R\rangle$ of $\mathcal{P}$, **remove** from R every tuple $\mathbf{a}$ such that $\mathrm{pr}_{s\cap W}\mathbf{a} \notin R'$ and $\mathbf{a}[v] \in B$, where $C' = \langle s \cap W, R'\rangle$ is the corresponding constraint of $\mathcal{P}'_{v,B}$ and $W = J_{\max}(\mathrm{rb}, v, B)$
endfor
Step 3.2.2. **if** $\mathcal{P}$ changed **then return**(CONSERV($\mathcal{P}$))
Step 3.2.3. **else return**(BLOCK-3-WIDTH($\mathcal{P}^{\max}$),$\mathcal{P}$)
enddo
there exists $v \in V$ and an r-connected component C of $\mathcal{G}(\mathcal{S}_v)$ such that $\mathcal{G}(C)$ is not strongly r-connected, there is $u \in V$ such that $\mathcal{G}(\mathcal{S}_u^{\max})$ is not strongly ry/rb-connected, for any $w \in V$, $\mathcal{S}_w = \mathcal{S}_w^0$, for some $w \in V$ such that $\mathcal{G}(\mathcal{S}_w)$ contains more than one minimal i-component, and $J(\mathrm{m}, w, B) = V$ for all $w \in V$ and all minimal i-components B of $\mathcal{G}(\mathcal{S}_w)$:
Step 3.3. **do**
Step 3.3.1. **fix** $v \in V$
Step 3.3.1. **for each** minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$, **do**
Step 3.3.1.1. $(\varphi, \mathcal{P}'_{v,B})$:=CONSERV($\mathcal{P}^{+}_{v,B}$)
Step 3.3.1.2. **for every** constraint $C = \langle s, R\rangle$ of $\mathcal{P}$, **remove** from R every tuple $\mathbf{a}$ such that $\mathbf{a} \notin R'$ and $\mathbf{a}[v] \in B$, where $C' = \langle s, R'\rangle$ is the corresponding constraint of $\mathcal{P}'_{v,B}$
endfor
Step 3.3.2. **if** $\mathcal{P}$ changed **then return**(CONSERV($\mathcal{P}$))
Step 3.3.3. **else return**(φ,$\mathcal{P}$), where φ is a solution of $\mathcal{P}^{+}_{v,B}$ for an arbitrary minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$
enddo
there exists $v \in V$ and an r-connected component C of $\mathcal{G}(\mathcal{S}_v)$ such that $\mathcal{G}(C)$ is not strongly r-connected, $J(\mathrm{m}, v, B) = V$ for all $v \in B$ and minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$, and there exists $w \in V$ such that $\mathcal{S}_w \neq \mathcal{S}_w^0$:
Step 3.4. **do**
Step 3.4.1. **set** $(\varphi, \mathcal{P}')$:=CONSERV($\mathcal{P}^{+}$)
Step 3.4.2. **if** $\mathcal{P}' \neq \varnothing$ **then**
Step 3.4.2.1. **for every** constraint $C = \langle s, R\rangle$ of $\mathcal{P}$, **remove** from R every tuple $\mathbf{a}$ such that $\mathbf{a} \notin R'$, where $C' = \langle s, R'\rangle$ is the corresponding constraint of $\mathcal{P}'_{v,B}$
Step 3.4.2.2. **return**($\varphi, \mathcal{P}$)
endif
Step 3.4.3. **else for every** constraint $C = \langle s, R\rangle$ of $\mathcal{P}$, **remove** from R every tuple $\mathbf{a}$ such that $\mathbf{a}[v] \notin \mathcal{S}_v^0$ for some $v \in s$

Step 3.4.4. **return**(CONSERV($\mathcal{P}$))
enddo
there exists $v \in V$ and an r-connected component C of $\mathcal{G}(\mathcal{S}_v)$ such that $\mathcal{G}(C)$ is not strongly r-connected and $J(\mathrm{m}, v, B) \neq V$ for some $v \in B$ and minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$:
Step 3.5. **do**
Step 3.5.1. **for each** $v \in V$, **each** minimal i-component B of $\mathcal{G}(\mathcal{S}_v)$, **do**
Step 3.5.1.1. $(\varphi, \mathcal{P}'_{v,B})$:=CONSERV($\mathcal{P}^+_{v,B}$)
Step 3.5.1.2. **for every** constraint $C = \langle s, R\rangle$ of $\mathcal{P}$, **remove** from R every tuple $\mathbf{a}$ such that $\mathrm{pr}_{s\cap W}\mathbf{a} \notin R'$, where $C' = \langle s \cap W, R'\rangle$ is the corresponding constraint of $\mathcal{P}'_{v,B}$ and $W = J(\mathrm{m}, v, B)$
endfor
Step 3.5.2. **if** $\mathcal{P}$ changed **then return**(CONSERV($\mathcal{P}$))
Step 3.5.3. **else**
Step 3.5.3.1. $(\varphi, \mathcal{P}')$:=CONSERV($\mathcal{P}^{\max}$))
Step 3.5.3.2. **return**(φ,$\mathcal{P}$)
endif
enddo
there exists $v \in V$ such that $\mathcal{G}(\mathcal{S}_v)$ is not ry-connected:
Step 3.6. **do**
Step 3.6.1. **for each** $v \in V$, **each** ry-connected component B **do**
Step 3.6.1.1. $(\varphi, \mathcal{P}'_{\mathrm{ry},v,B})$:=CONSERV($\mathcal{P}_{\mathrm{ry},v,B}$)
Step 3.6.1.2. **for every** constraint $C = \langle s, R\rangle$ of $\mathcal{P}$, **remove** from R all tuples $\mathbf{a}$ such that $\mathrm{pr}_{s\cap W}\mathbf{a} \notin R'$, where $C' = \langle s \cap W, R'\rangle$ is the corresponding constraint of $\mathcal{P}'_{\mathrm{ry},v,B}$ and $W = J(\mathrm{ry}, v, B)$
endfor
Step 3.6.2. **if** $\mathcal{P}$ changed **then return**(CONSERV($\mathcal{P}$))
Step 3.6.3. **else return**(MALTSEV($\mathcal{P}^s$),$\mathcal{P}$)
enddo
$\mathcal{G}(\mathcal{S}_v)$ is strongly ry/rb-connected for any $v \in V$ and there exists $u \in V$ such that $\mathcal{S}_u$ is not hereditarily strongly ry/rb-connected:
Step 3.7. **do**
Step 3.7.1. **for each** $v \in V$ such that $\mathcal{S}_v$ is not hereditarily strongly ry/rb connected, **each** sequence $\overline{B} = B_0, B_1, \ldots, B_k$ and $\overline{\theta} = \theta_0, \ldots, \theta_{k-1}$ witnessing that $\mathcal{S}_v$ is not hereditarily strongly ry/rb connected, **do**
Step 3.7.1.1. $(\varphi, \mathcal{P}'')$:=CONSERV($\mathcal{P}^*_{v,\overline{B},\overline{\theta}}$)
Step 3.7.1.2. **for every** constraint $C = \langle s, R\rangle$ of $\mathcal{P}$, **remove** from R every tuple $\mathbf{a}$ such that there is $w \in s \cap W_{v,\overline{B},\overline{\theta}}$ such that $\mathbf{a}[w]$ is r-maximal and $\mathbf{a}[w] \notin \mathcal{S}'_w$
endfor
Step 3.7.2. **if** $\mathcal{P}$ changed **then return**(CONSERV($\mathcal{P}$))
Step 3.7.3. **else return**(RY/RB-CONN($\mathcal{P}$),$\mathcal{P}$)
enddo
$\mathcal{S}_v$ is hereditarily strongly ry/rb-connected for any $v \in V$:
Step 3.8. **return**(3-WIDTH($\mathcal{P}$),$\mathcal{P}$)
endifcase
Step 4. **invoke** 3-MINIMALITY($\mathcal{P}$)

Fig. 7.2. Algorithm CONSERV

7.3 Soundness and complexity

To prove soundness of the algorithms from Section 7.1 we have to show that the algorithm CONSERV works correctly. However, this is actually done in Section 7.1 and in the Tightening Proposition 7.1.

Proposition 7.3. *The algorithm* Conserv *returns a solution of the given problem instance if and only if a solution exists.*

Then we estimate the complexity of the algorithm.

Proposition 7.4. *If the size of the domains in $\mathcal{A}$ is bounded, then the algorithm* Conserv *is polynomial time.*

Proof. We use $|\mathcal{P}|$ to denote the size of a problem instance $\mathcal{P}$. Suppose that Conserv is applied to a problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$. We assume first, that the algorithm never changes $\mathcal{P}$, that is Steps 3.1.2, 3.2.2, 3.3.2, 3.4.4, 3.5.2, 3.6.2 and 3.7.2 are never performed. Let s denote the maximal size of domains in $\mathcal{A}$.

Claim 1. The depth of recursion when performing Conserv($\mathcal{P}$) does not exceed $4s$.

Notice that the size of $\mathcal{S}_v$ decreases as follows.
– If $\mathcal{G}(\mathcal{S}_v)$ is not strongly rb-connected then its size decreases on Step 3.1.1.1, as this domain is split between the domains of two or more problems of the form $\mathcal{P}_{\mathrm{rb},v,B}$
– If there is an r-connected component of $\mathcal{G}(\mathcal{S}_v)$ which is not strongly r-connected and $J(\mathrm{m}, v, B) \neq V$ for some v and B, then the size of $\mathcal{S}_v$ may stay unchanged when Step 3.1.1.1 is performed if $\mathcal{G}(\mathcal{S}_v)$ is strongly rb-connected, as all variables like this go to the same problem $\mathcal{P}_{\mathrm{rb},v,B}$ while there are more problems of this form for some other variables, but it decreases during the next round of recursion, since v in one of the sets $J(\mathrm{m}, w, B)$.
– If there is an r-connected component of $\mathcal{G}(\mathcal{S}_v)$ which is not strongly r-connected, $J(\mathrm{m}, v, B) = V$ for all v and B, and $\mathcal{S}_v^0 \neq \mathcal{S}_v$ for some v, then the size of $\mathcal{S}_v$ may stay unchanged when Step 3.1.1.1 or 3.5.1.1 are performed. However, for every $w \in J(\mathrm{rb}, v, \mathcal{S}_v)$, $\mathcal{G}(\mathcal{S}_w)$ is strongly rb-connected and, for every $w \in J(\mathrm{rb}, v, \mathcal{S}_v)$, $J(\mathrm{m}, w, \mathcal{S}_w^0) = J(\mathrm{rb}, v, \mathcal{S}_v)$. This means that the size of $\mathcal{S}_v$ decreases during the next or during the third round of recursion.
– If there is an r-connected component of $\mathcal{G}(\mathcal{S}_v)$ which is not strongly r-connected and $\mathcal{S}_v^0 = \mathcal{S}_v$ for all v, then the size of $\mathcal{S}_v$ may stay unchanged when Step 3.1.1.1, 3.4.1 or 3.5.1.1 are performed. However, for every $w \in J(\mathrm{rb}, v, \mathcal{S}_v)$, $\mathcal{G}(\mathcal{S}_w)$ is strongly rb-connected and, for every $w \in J(\mathrm{m}, v, \mathcal{S}_v)$, $\mathcal{S}_w = \mathcal{S}_w^0$. This means that the size of $\mathcal{S}_v$ decreases during the next or during the third round of recursion.
– If $\mathcal{G}(\mathcal{S}_v)$ is strongly rb-connected, is not ry-connected and every r-connected component is strongly r-connected, then $\mathcal{G}(\mathcal{S}_v)$ contains more than one minimal i-components (every ry-connected component is a minimal i-component). Therefore, the size of $\mathcal{S}_v$ may stay unchanged when Step 3.1.1.1 is performed, but v is not included into any set of the form $J(\mathrm{m}, w, B)$ or $J_{\max}(\mathrm{rb}, w, B)$ on Step 3.2.1.1 or Step 3.4.1. Therefore, for any variable $v \in V$, $|\mathcal{S}_v|$ decreases during at least each third iteration.
– If $\mathcal{G}(\mathcal{S}_v^{\max})$ is strongly ry/rb-connected, it may stay unchanged on Step 3.1.1.1, 3.2.1.1, 3.3.1, 3.4.1, 3.5.1.1 and 3.6.1.1 (at most three of them can be performed), but then it will change on Step 3.7.1.1.
– Finally, algorithm Conserv on Step 3.5.3 is applied to a problem in which, for any domain, every r-connected component is strongly r-connected, which means that for any variable $v \in V$, $|\mathcal{S}_v|$ decreases during at least each second iteration.

Let $p(n)$ be a polynomial that bounds the time complexity of 3-Minimality, 3-Width, Block-3-Width, and Maltsev, r its degree; and ζ the maximal number of sets of the form $\{a \in \mathcal{S}_v \mid B \sqsubseteq a\}$, where B is a minimal i-connected component of $\mathcal{G}(\mathcal{S}_v)$, sharing an element.

Claim 2. The number of operations performed by Conserv($\mathcal{P}$) does not exceed $\xi^{lr}p(|\mathcal{P}|)$ where l is the depth of recursion and $\xi = \zeta + 3$.

We prove Claim 2 by induction on the depth l of recursion. The base case for induction, Steps 3.8, 3.1.3, 3.2.3, and 3.6.3 is obvious, in this case the time complexity is bounded with $p(|\mathcal{P}|)$.

Step 2 can be performed in linear time, so we neglect its contribution. By the assumption made, Steps 3.1.1.2, 3.1.2, 3.2.1.2, 3.2.2, 3.3.2, 3.4.3, 3.4.4, 3.5.1.2, 3.5.2, 3.6.1.2, 3.6.2, 3.7.1.2 and 3.7.2 are never performed. Therefore, it is enough to estimate the time complexity of Steps 3.1.1.1, 3.2.1.1, 3.3.1.1, 3.4.1, 3.5.1.1, 3.6.1.1 and 3.7.1.1.

Suppose first, that the conditions for Step 3.6 are valid. Then $|\mathcal{P}_{\mathrm{ry},v_1,B_1}| + \ldots + |\mathcal{P}_{\mathrm{ry},v_k,B_k}| \le |\mathcal{P}|$ where $B_1, \ldots, B_k$ are the strongly ry-connected components of $\mathcal{G}(\mathcal{S}_{v_1}), \ldots, \mathcal{G}(\mathcal{S}_{v_k})$. Therefore, the number of operations required on Step 3.6.1.1 equals $\xi^{r(l-1)}p(|\mathcal{P}_{v_1,B_1}|) + \ldots + \xi^{r(l-1)}p(|\mathcal{P}_{v_k,B_k}|)$. As is easily seen

$$\xi^{r(l-1)}p(|\mathcal{P}_{v_1,B_1}|) + \ldots + \xi^{r(l-1)}p(|\mathcal{P}_{v_k,B_k}|) \le \xi^{r(l-1)}p(|\mathcal{P}|).$$

Since $|\mathcal{P}_{\mathrm{rb},v_1,B_1}| + \ldots + |\mathcal{P}_{\mathrm{rb},v_k,B_k}| \le |\mathcal{P}|$ where $B_1, \ldots, B_k$ are the maximal rb-connected components of $\mathcal{G}(\mathcal{S}_{v_1}), \ldots, \mathcal{G}(\mathcal{S}_{v_k})$, the same bound is valid for Step 3.1.1.1 and the same arguments work for Steps 3.2.1.1, 3.3.1.1, 3.4.1 and 3.7.1.1.

In Step 3.5.1.1, we have $\zeta|\mathcal{P}| \ge |\mathcal{P}^+_{v_1,B_1}| + \ldots + |\mathcal{P}^+_{v_k,B_k}|$ where $B_1, \ldots, B_k$ are the minimal i-components of $\mathcal{G}(\mathcal{S}_1), \ldots, \mathcal{G}(\mathcal{S}_k)$. Therefore, the number of operations required on Step 3.5.1.1 is bounded with

$$\xi^{r(l-1)}p(|\mathcal{P}^+_{v_1,B_1}|) + \ldots + \xi^{r(l-1)}p(|\mathcal{P}^+_{v_k,B_k}|) \le \xi^{r(l-1)}p(\zeta|\mathcal{P}|) \le \xi^{r(l-1)}\zeta^r p(|\mathcal{P}|).$$

Thus, the overall time complexity is bounded with

$$\xi^{r(l-1)}\zeta^r p(|\mathcal{P}|) + 3p(|\mathcal{P}|) \le \xi^{rl}p(|\mathcal{P}|),$$

as required.

In the case of Step 3.5.3, $|\mathcal{P}^{\max}| < |\mathcal{P}|$.

To complete the proof, notice that in the general case, including performing Steps 3.1.1.2, 3.1.2, 3.2.1.2, 3.2.2, 3.3.1.2, 3.3.2, 3.4.3, 3.4.4, 3.5.1.2, 3.5.2, 3.6.1.2, 3.6.2, 3.7.1.2 and 3.7.2, the problem $\mathcal{P}$ can be changed at most $s|V|$ times. Therefore, the time complexity of Conserv($\mathcal{P}$) is bounded with $|V| \cdot s\xi^{rl}p(|\mathcal{P}|)$. $\square$

8. CONCLUSION

We have continued the research project aiming to distinguish those constraint languages that give rise to tractable constraint satisfaction problems from those which do not. We achieve this goal in the case of conservative constraint languages which, besides its own importance, including applications, provides a possible prototype for the most general case of arbitrary constraint languages. The main result of the paper has become possible due to the algebraic approach to constraints. This approach being originally developed for unifying and expanding known tractable classes has grown up to one the most prolific methods of studying the constraint satisfaction problem. It allows one to tackle problems which can hardly be solved by other existing methods, and we strongly believe that eventually it will make it possible to completely solve the indicated research problems.

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Complexity of conservative Constraint Satisfaction Problems

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A. PROOFS FROM SECTION 3

PROPOSITION A.1 (THREE OPERATIONS PROPOSITION, PROPOSITION 3.1, P. 17). *There are polymorphisms $f(x,y), g(x,y,z), h(x,y,z) \in \mathsf{MPol}(\Gamma)$ such that, for every $A \in \mathcal{A}$ and every two-element subset $B \subseteq A$,*

—$f^A|_B$ *is a semilattice operation whenever B is red, and $f^A|_B(x,y) = x$ otherwise;*
—$g^A|_B$ *is a majority operation if B is yellow, $g^A|_B(x,y,z) = x$ if B is blue, and $g^A|_B(x,y,z) = f^A|_B(f^A|_B(x,y),z)$ if B is red;*
—$h^A|_B$ *is the affine operation if B is blue, $h^A|_B(x,y,z) = x$ if B is yellow, and $h^A|_B(x,y,z) = f^A|_B(f^A|_B(x,y),z)$ if B is red.*

There is also a polymorphism $p(x,y)$ such that $p^A|_B = f^A|_B$ if B is red, $p^A|_B(x,y) = y$ if B is yellow, and $p^A|_B(x,y) = x$ if B is blue.

PROOF. Show first that there is an operation f that is semilattice on each red edge. Let $B_1,\dots,B_n$ be a list of all red edges of graphs $\mathcal{G}(A)$ for all $A \in \mathcal{A}$. To avoid clumsy notation we shall denote the operation $f^A|_{B_j}$, $B_j \subseteq A$, simply by $f|_{B_j}$. Let also $f_1,\dots,f_n$ be the list of polymorphisms of Γ such that $f_i|_{B_i}$ is a semilattice operation. Notice that every binary idempotent operation on a 2-element set is either a projection or a semilattice operation. Since each f_i is idempotent, for any i,j, $f_i|_{B_j}$ is either a projection or a semilattice operation. We prove by induction that the operation f^i constructed via the following rules is a semilattice operation on $B_1,\dots,B_i$: $f^1 = f_1$, $f^i(x,y) = f_i(f^{i-1}(x,y), f^{i-1}(y,x))$.

The base case for induction, $i = 1$, holds by the choice of f_1. Suppose that f^{i-1} satisfies the required conditions. If $f^{i-1}|_{B_i}$ is a projection, say, $f^{i-1}|_{B_i}(x,y) = x$, then $f^i(x,y) = f_i(f^{i-1}(x,y), f^{i-1}(y,x)) = f_i(x,y)$, i.e. it is a semilattice operation on B_i. Let $B_i = \{a,b\}$, and f^{i-1} a semilattice operation such that $f^{i-1}(a,b) = f^{i-1}(b,a) = a$. Then

$$\begin{aligned} f^i(a,b) &= f_i(f^{i-1}(a,b), f^{i-1}(b,a)) = f_i(a,a) = a,\\ f^i(b,a) &= f_i(f^{i-1}(b,a), f^{i-1}(a,b)) = f_i(a,a) = a, \end{aligned}$$

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hence, f^i is again a semilattice operation. As is easily seen, $f^i|_{B_j} = f^{i-1}|_{B_j}$, and so $f^i|_{B_j}$ is a semilattice operation for $j < i$.

Thus, for each edge B, $f^n|_B$ is a semilattice operation if B is red and a projection otherwise. Finally, it is easy to check that $f(x,y) = f^n(f^n(x,y),x)$ satisfies the required conditions, as it is first projection on every yellow or blue edge.

Now let $B_1,\dots,B_k$, $C_1,\dots,C_l$ be the lists of all yellow and all blue edges respectively of graphs $\mathcal{G}(A)$ for all $A \in \mathcal{A}$, and $g_1,\dots,g_k$, $h_1,\dots,h_l$ the lists of polymorphisms of Γ such that $g_i|_{B_i}$ is a majority operation, and $h_i|_{C_i}$ is an affine operation. Notice first that every binary polymorphism is a projection on the B_i and C_i. Therefore, for any i,j, $g_i|_{B_j}(x,y,y), g_i|_{B_j}(y,x,y), g_i|_{B_j}(y,y,x)$, $h_i|_{B_j}(x,y,y), h_i|_{B_j}(y,x,y), h_i|_{B_j}(y,y,x)$, $g_i|_{C_j}(x,y,y), g_i|_{C_j}(y,x,y), g_i|_{C_j}(y,y,x)$, $h_i|_{C_j}(x,y,y), h_i|_{C_j}(y,x,y), h_i|_{C_j}(y,y,x) \in \{x,y\}$. This means that the operations $g_i|_{B_j}, h_i|_{B_j}, g_i|_{C_j}, h_i|_{C_j}$ are of one of the following types: a projection, a minority operation (that is the affine operation), a majority operation, or a 2/3-*minority operation*, that is an operation satisfying the equalities $m(x,y,y) = y$, $m(y,x,y) = m(y,y,x) = x$ or those obtained by permuting the arguments. Moreover, as is shown in [Post 1941], a majority operation on a 2-element set can be derived by means of superposition from any 2/3-minority operation. Therefore if $g_i|_{C_j}$ or $h_i|_{C_j}$ is a 2/3-minority operation, then there is g such that $g|_{C_j}$ is a majority operation, that contradicts the assumption that C_j is blue. Hence, $g_i|_{C_j}, h_i|_{C_j}$ are either minority operations or projections.

First we prove by induction that for every $j \in \underline{k}$ there is an operation $g^j(x,y,z)$ which is a majority operation on B_i for $i \le j$. The operation $g^1 = g_1$ gives the base case for induction. Let us assume that g^{j-1} is already found. If $g^{j-1}|_{B_j}$ is a majority operation, set $g^j = g^{j-1}$. Otherwise, it is either a projection, or a 2/3-minority operation, or a minority operation. In all these case its variables can be permuted such that $g^{j-1}|_{B_j}(x,y,y) = x$. Then the operation $r(x,y) = g^{j-1}(x,y,y)$ satisfies the conditions $r|_{B_j}(x,y) = x$, and $r|_{B_i}(x,y) = y$ for all $i < j$. It is not hard to see that the operation $g^j(x,y,z) = r(g_j(x,y,z), g^{j-1}(x,y,z))$ satisfies the required conditions.

Consider the operation g^k. Its restriction $g^k|_{C_j}$, $1 \le j \le l$, is either a projection, or the minority operation. If $g^k|_{C_j}$ is the minority operation, then as above we can derive an operation $r(x,y)$ such that $r|_{B_i}(x,y) = y$ for all $1 \le i \le k$, and $r|_{C_j}(x,y) = x$. The operation $g'(x,y,z) = r(x, g^k(x,y,z))$ is a majority operation on B_i, $1 \le i \le k$, the first projection on C_j, and a projection on each C_i such that $g^k|_{C_i}$ is a projection. Therefore, $g^k|_{C_i}$ can be assumed to be a projection for all $i \in \underline{l}$. Then for the operation $\overline{g}(x,y,z) = g^k(x, g^k(y,x,y), g^k(z,z,x))$ we have

$$
\begin{aligned}
\overline{g}|_{B_i}(x,y,z) &= g^k|_{B_i}(x, g^k|_{B_i}(y,x,y), g^k|_{B_i}(z,z,x)) = g^k|_{B_i}(x,y,z), \text{ for any } i \in \underline{k};\\
\overline{g}|_{C_i}(x,y,z) &= g^k|_{C_i}(x, g^k|_{C_i}(y,x,y), g^k|_{C_i}(z,z,x)) = x, \quad \text{for any } i \in \underline{l}\\
&\qquad \text{such that } g^k|_{C_i}(x,y,z) = x;\\
\overline{g}|_{C_i}(x,y,z) &= g^k|_{C_i}(x, g^k|_{C_i}(y,x,y), g^k|_{C_i}(z,z,x)) = g^k|_{C_i}(y,x,y) = x,\\
&\qquad \text{for any } i \in \underline{l} \text{ such that } g^k|_{C_i}(x,y,z) = y;
\end{aligned}
$$

$$\overline{g}|_{C_i}(x,y,z) \;=\; g^k|_{C_i}(x, g^k|_{C_i}(y,x,y), g^k|_{C_i}(z,z,x)) = g^k|_{C_i}(z,z,x) = x,$$
$$\text{for any } i \in \underline{l} \text{ such that } g^k|_{C_i}(x,y,z) = z.$$

Finally, to make $\overline{g}$ acting correctly on red edges we set $g(x,y,z) = \overline{g}(f(x,f(y,z)), f(y,f(z,x)), f(z,f(x,y)))$. The operation g is as required.

Next we show that, for any $j \in \underline{l}$, there is h^j such that $h^j|_{C_i}$ is the minority operation for $i \le j$. As usual, $h^1 = h_1$ gives the base case for induction. If h^{j-1} is obtained, then if $h^{j-1}|_{C_j}$ is the minority operation then set $h^j = h^{j-1}$. Otherwise, we may assume $h^{j-1}|_{C_j}(x,x,y) = x$, and h^j can be chosen to be $r(h_j(x,y,z), h^{j-1}(x,y,z))$ where $r(x,y) = h^{j-1}(x,x,y)$. Finally, set $p(x,y) = g(x,y,y)$, $h''(x,y,z) = p(h^l(x,y,z),x)$ and $h(x,y,z) = h''(f(x,f(y,z)), f(y,f(z,x)), f(z,f(x,y)))$. As is easily seen h satisfies the conditions required.

Finally, note that operation p defined above also satisfies the conditions required. $\square$

B. PROOFS FROM SECTION 5

Proposition B.1 (Connectedness Proposition, Proposition 5.1, p. 34). *Let R be a subdirect product of $A_1,\dots,A_n \in \mathcal{A}$.*
(1) If every $\mathcal{G}(A_i)$ is strongly r-connected, then $\mathcal{G}(R)$ is strongly r-connected.
(2) If every $\mathcal{G}(A_i)$ is strongly rb-connected, then $\mathcal{G}(R)$ is strongly rb-connected.
(3) If every $\mathcal{G}(A_i)$ is strongly ry-connected and such that every r-connected component of $\mathcal{G}(A_i)$ is strongly r-connected, then $\mathcal{G}(R)$ is strongly ry-connected.

Proof. We start with (2). Let us suppose first that $A_1,\dots,A_n$ are simple. Without loss of generality, let us assume that $\mathcal{G}(A_1),\dots,\mathcal{G}(A_m)$ contain a red or yellow edge while $\mathcal{G}(A_{m+1}),\dots,\mathcal{G}(A_n)$ do not. We use R_1 to denote $\mathrm{pr}_{\{1,\dots,m\}}R$ and R_2 to denote $\mathrm{pr}_{\{m+1,\dots,n\}}R$. It is not hard to see that the Almost Trivial Proposition 4.7 implies that $\mathcal{G}(R_1)$ is strongly rb-connected.

Take $(\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d}) \in R \subseteq R_1 \times R_2$. There is an rb-path $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k = \mathbf{c}$ and tuples $\mathbf{b} = \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k \in R_2$ such that $(\mathbf{a}_1,\mathbf{b}_1),(\mathbf{a}_2,\mathbf{b}_2),\dots,(\mathbf{a}_k,\mathbf{b}_k) \in R$. Clearly, any two elements of R_2 constitute a blue edge. We build an rb-path from $(\mathbf{a},\mathbf{b})$ to $(\mathbf{c},\mathbf{d})$ as follows. Set $(\mathbf{a}_1,\mathbf{d}_1) = (\mathbf{a}_1,\mathbf{b}_1) = (\mathbf{a},\mathbf{b})$. Then, if $(\mathbf{a}_i,\mathbf{d}_{2i-1})$ is already constructed and $\langle \mathbf{a}_i, \mathbf{a}_{i+1}\rangle \in \gamma$ then set $(\mathbf{a}_{i+1},\mathbf{d}_{2i}) = (\mathbf{a}_{i+1},\mathbf{b}_{i+1})$ and $(\mathbf{a}_{i+1},\mathbf{d}_{2i+1}) = (\mathbf{a}_{i+1},\mathbf{d}_{2i})$. If $\mathbf{a}_i \le \mathbf{a}_{i+1}$, then set $\begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{d}_{2i}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_i\\ \mathbf{d}_{2i-1}\end{pmatrix}, \begin{pmatrix}\mathbf{a}_{i+1}\\ \mathbf{b}_{i+1}\end{pmatrix}\right)$ (observe that $(\mathbf{a}_{i+1},\mathbf{d}_{2i}) = (\mathbf{a}_{i+1},\mathbf{d}_{2i-1})$ and $(\mathbf{a}_{i+1},\mathbf{d}_{2i+1}) = (\mathbf{a}_{i+1},\mathbf{b}_{i+1})$. As is easily seen, the sequence $(\mathbf{a}_1,\mathbf{d}_1),(\mathbf{a}_2,\mathbf{d}_2),(\mathbf{a}_2,\mathbf{d}_3),\dots,(\mathbf{a}_k,\mathbf{b}_{2k-1}) = (\mathbf{c},\mathbf{d}_{2k-1})$ is an rb-path. Finally, as $(\mathbf{c},\mathbf{d}_{2k-1})$ and $(\mathbf{c},\mathbf{d})$ constitute a blue edge, $(\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d})$ are strongly rb-connected.

We prove the proposition by induction on the number of non-simple factors.

Let R be a subdirect product of domains $A_1,\dots,A_n \in \mathcal{A}$ whose graphs are strongly rb-connected, and A_n is not simple. Take a maximal congruence θ of R_n, and consider the relation $R' = \{(\mathbf{a}[1],\dots,\mathbf{a}[n-1],(\mathbf{a}[n])^\theta) \mid \mathbf{a} \in R\}$. The relation R' is a subdirect product of $A_1,\dots,A_{n-1},A_n/_\theta$; all of these domains have strongly rb-connected graphs, and $A_n/_\theta$ is simple. Hence, they have one non-simple domain less than $A_1,\dots,A_n$. Therefore, by inductive hypothesis, $\mathcal{G}(R')$ is strongly rb-connected.

For any $(\mathbf{a},a),(\mathbf{b},b)\in R$ (here $\mathbf{a},\mathbf{b}\in \mathrm{pr}_{1,\dots,n-1}R$), the corresponding tuples $(\mathbf{a},a^\theta),(\mathbf{b},b^\theta)$ are rb-connected. Suppose first $a^\theta\neq b^\theta$. First we prove two claims.

Claim 1. There is $(\mathbf{c},c)\in R$ such that $c^\theta\neq b^\theta$, and $\mathbf{c}\le\mathbf{b}$, $c\le b$ or $\langle\mathbf{c},\mathbf{b}\rangle\in\gamma,\langle c,b\rangle\in\gamma$.

By the Semi-Simple Double Connected Corollary 4.4 and Lemma 3.5 the relation R' is either the direct product of $\mathrm{pr}_{1,\dots,n-1}R$ and $A_n/_\theta$, or the graph of a mapping $\varphi\colon \mathrm{pr}_{1,\dots,n-1}R\to A_n/_\theta$. If $R'=\mathrm{pr}_{1,\dots,n-1}R\times A_n/_\theta$ then the claim is obvious, so suppose that R' is the graph of a mapping φ.

Let $I\subseteq\{1,\dots,n-1\}$ be a maximal set such that $\mathrm{pr}_{I\cup\{n\}}R'$ is the direct product of pr_IR and $A_n/_\theta$. If $I=\varnothing$ then for any $(\mathbf{c},c)\in R$ such that $c^\theta\le b^\theta$ [or $\langle c,b\rangle\in\gamma$] we have $\mathbf{c}\le\mathbf{b}$ [$\langle\mathbf{c},\mathbf{b}\rangle\in\gamma$, correspondingly]. Since $A_n/_\theta$ is strongly rb-connected, there exists c with $c^\theta\neq b^\theta$ and $c^\theta\le b^\theta$ or $\langle c,b\rangle\in\gamma$, and the required tuple is found.

Now suppose that $I\neq\varnothing$. Suppose first that there is $c\in A_n$ with $c^\theta\neq b^\theta$ and $c^\theta\le b^\theta$. By the choice of I there is $\mathbf{c}\in\mathrm{pr}_{1,\dots,n-1}R$ such that $(\mathbf{c},c^\theta)\in R'$ and $\mathrm{pr}_I\mathbf{c}=\mathrm{pr}_I\mathbf{b}$. Take $i\in\underline{n-1}-I$ and suppose that $\mathbf{c}[i]\not\le\mathbf{b}[i]$. Let $\begin{pmatrix}\mathbf{d}\\ b^\theta\end{pmatrix}=f\left(\begin{pmatrix}\mathbf{c}\\ c^\theta\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ b^\theta\end{pmatrix}\right)\in R'$. We have $\mathrm{pr}_I\mathbf{d}=\mathrm{pr}_I\mathbf{b}$ and $\mathbf{d}[i]=\mathbf{c}[i]$. Therefore $(\mathrm{pr}_{I\cup\{i\}}\mathbf{d},b),(\mathrm{pr}_{I\cup\{i\}}\mathbf{d},c)\in\mathrm{pr}_{I\cup\{i,n\}}R'$ implying $\mathrm{pr}_{I\cup\{i,n\}}R'$ is not the graph of a mapping; a contradiction with the choice of I.

Next, suppose that that there is $c\in A_n$ with $c^\theta\neq b^\theta$ and $\langle c^\theta,b^\theta\rangle\in\gamma$. By the choice of I there is $\mathbf{c}\in\mathrm{pr}_{1,\dots,n-1}R$ such that $(\mathbf{c},c^\theta)\in R'$ and $\mathrm{pr}_I\mathbf{c}=\mathrm{pr}_I\mathbf{b}$. Take $i\in\underline{n-1}-I$ and suppose that $\langle\mathbf{c}[i],\mathbf{b}[i]\rangle\not\in\gamma$. If $\mathbf{c}[i]\le\mathbf{b}[i]$ then setting $\begin{pmatrix}\mathbf{d}\\ c^\theta\end{pmatrix}=f\left(\begin{pmatrix}\mathbf{c}\\ c^\theta\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ b^\theta\end{pmatrix}\right)\in R'$ we get $(\mathrm{pr}_{I\cup\{i\}}\mathbf{d},b),(\mathrm{pr}_{I\cup\{i\}}\mathbf{d},c)\in\mathrm{pr}_{I\cup\{i,n\}}R'$, a contradiction. Otherwise set Let $\begin{pmatrix}\mathbf{d}\\ b^\theta\end{pmatrix}=p\left(\begin{pmatrix}\mathbf{c}\\ c^\theta\end{pmatrix},\begin{pmatrix}\mathbf{b}\\ b^\theta\end{pmatrix}\right)\in R'$. We have $\mathrm{pr}_I\mathbf{d}=\mathrm{pr}_I\mathbf{b}$ and $\mathbf{d}[i]=\mathbf{c}[i]$. Therefore $(\mathrm{pr}_{I\cup\{i\}}\mathbf{d},b),(\mathrm{pr}_{I\cup\{i\}}\mathbf{d},c)\in\mathrm{pr}_{I\cup\{i,n\}}R'$, a contradiction again. Claim 1 is proved.

Claim 2. Let $(\mathbf{a},a^\theta)=(\mathbf{a}_0,a_0^\theta),(\mathbf{a}_1,a_1^\theta),\dots,(\mathbf{a}_k,a_k^\theta)$ be an rb-path in R' such that $a_0=a$ and $(\mathbf{a}_i,a_i)\in R$ for all i. Then there are $a=c_0,c_1,\dots,c_\ell\in\{a_0,\dots,a_k\}$ and $\mathbf{a}=\mathbf{c}_0,\mathbf{c}_1,\dots,\mathbf{c}_\ell\in\{\mathbf{a}_0,\dots,\mathbf{a}_k\}$ such that $(\mathbf{a},a)=(\mathbf{c}_0,c_0),(\mathbf{c}_1,c_1),\dots,(\mathbf{c}_\ell,c_\ell)$ is an rb-path in R, $\mathbf{c}_\ell=\mathbf{a}_k$, and $c_\ell^\theta=a_k^\theta$.

We proceed by induction on k. If $k=0$ there is nothing to prove. Suppose that the claim is proved for $k-1$; that is there are $a=c_0,c_1,\dots,c_{\ell'}\in\{a_0,\dots,a_{k-1}\}$ and $\mathbf{a}=\mathbf{c}_0,\mathbf{c}_1,\dots,\mathbf{c}_{\ell'}\in\{\mathbf{a}_0,\dots,\mathbf{a}_{k-1}\}$ such that $(\mathbf{a},a)=(\mathbf{c}_0,c_0),(\mathbf{c}_1,c_1),\dots,(\mathbf{c}_{\ell'},c_{\ell'}^\theta)$ is an rb-path in R, $\mathbf{c}_{\ell'}=\mathbf{a}_{k-1}$, and $c_\ell^\theta=a_{k-1}^\theta$. If $a_k^\theta\neq a_{k-1}^\theta$ then by Lemma 4.1 $c_{\ell'}\le a_k$ [$\langle c_{\ell'},a_k\rangle\in\gamma$] whenever $a_{k-1}^\theta\le a_k^\theta$ [$\langle a_{k-1}^\theta,a_k^\theta\rangle\in\gamma$], and we can choose $\ell=\ell'+1$, $\mathbf{c}_\ell=\mathbf{a}_k$, and $c_\ell=a_k$. If $a_k^\theta=a_{k-1}^\theta$ and $\mathbf{a}_{k-1}\le\mathbf{a}_k$, $c_{\ell'}\le a_k$ or $\langle\mathbf{a}_{k-1},\mathbf{a}_k\rangle\in\gamma$, $\langle c_{\ell'},a_k\rangle\in\gamma$ then we can do the same.

If $a_k^\theta=a_{k-1}^\theta$, and $a_k\le c_{\ell'}$ or $\langle a_k,c_{\ell'}\rangle\in\beta$, then we choose again $\ell=\ell'$, $\mathbf{c}_\ell=\mathbf{a}_k$ and $c_\ell=c_{\ell'}$. This is possible because $\begin{pmatrix}\mathbf{a}_k\\ c_{\ell'}\end{pmatrix}=p\left(\begin{pmatrix}\mathbf{a}_k\\ a_k\end{pmatrix},\begin{pmatrix}\mathbf{a}_{k-1}\\ c_{\ell'}\end{pmatrix}\right)\in R$. If $\mathbf{a}_{k-1}\le\mathbf{a}_k$ while $\langle c_{\ell'},a_k\rangle\in\gamma$, then set $\ell=\ell'+2$, $\mathbf{c}_{\ell'+1}=\mathbf{c}_{\ell'+2}=\mathbf{a}_k$, and $c_{\ell'+1}=c_{\ell'}$, $c_{\ell'+2}=a_k$. If $\langle\mathbf{a}_{k-1},\mathbf{a}_k\rangle\in\gamma$ while $c_{\ell'}\le a_k$, then set $\ell=\ell'+2$,

$\mathbf{c}_{\ell'+1} = \mathbf{c}_{\ell'}$, $\mathbf{c}_{\ell'+2} = \mathbf{a}_k$, and $c_{\ell'+1} = c_{\ell'+2} = a_k$. In both cases it is straightforward that the tuples $(\mathbf{c}_{\ell'+1}, c_{\ell'+1}), (\mathbf{c}_{\ell'+2}, c_{\ell'+2})$ belong to R and satisfy the conditions required. Claim 2 is proved.

Now we are ready to prove that there is an rb-path from $(\mathbf{a}, a)$ to $(\mathbf{b}, b)$ in R. Choose $(\mathbf{c}, c)$ satisfying the conditions of Claim 1. By Claim 2 there is an rb-path in $(\mathbf{a}, a) = (\mathbf{a}_0, a_0), (\mathbf{a}_1, a_a), \dots, (\mathbf{a}_k, a_k)$ such that $\mathbf{a}_k = \mathbf{c}$ and $a_k^\theta = c^\theta$. Thus we may assume $a_k = c$. Clearly, $(\mathbf{a}, a) = (\mathbf{a}_0, a_0), (\mathbf{a}_1, a_a), \dots, (\mathbf{a}_k, a_k), (\mathbf{b}, b)$ is an rb-path.

(1) can be proved in a very similar way, but the base case of induction follows straightforwardly from the Almost Trivial Proposition 4.7.

(3) We prove the lemma by induction on n. The base case of induction, $n = 1$, is obvious. So, suppose that the result is proved for any subdirect product of any $B_1, \dots, B_{n-1} \in \mathcal{A}$ such that the $\mathcal{G}(B_i)$ are strongly ry-connected.

Claim 3. Let $B_1, \dots, B_n$ be strongly r-connected components of $\mathcal{G}(A_1), \dots, \mathcal{G}(A_n)$ respectively. Then either $R' = R \cap (B_1 \times \dots \times B_n) = \varnothing$ or $\mathcal{G}(R')$ is strongly r-connected.

Let us suppose that $R' \neq \varnothing$, say, $(\mathbf{a}, a) \in R'$ where $\mathbf{a} \in R'' = \mathrm{pr}_{\underline{n-1}} R \cap (B_1 \times \dots \times B_{n-1})$, and $a \in B'_n = \mathrm{pr}_n R'$. We show that $B'_n = B_n$. Since $\mathcal{G}(B_n)$ is strongly r-connected it is enough to prove that if $b, c \in B_n$ are such that $b \le c$ and $(\mathbf{b}, b) \in R$ for some $\mathbf{b} \in R''$ then there exists $\mathbf{c} \in R''$ with $(\mathbf{c}, c) \in R$. Take $\mathbf{d} \in \mathrm{pr}_{\underline{n-1}} R$ with $(\mathbf{d}, c) \in R$. For $\begin{pmatrix} \mathbf{c} \\ c \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{b} \\ b \end{pmatrix}, \begin{pmatrix} \mathbf{d} \\ c \end{pmatrix}\right) \in R$ we have $\mathbf{b} \le \mathbf{c}$, and therefore $\mathbf{c} \in R''$.

We have proved that R' is a subdirect product of $B_1, \dots, B_n$. By the Connectedness Proposition 5.1(1), $\mathcal{G}(R')$ is strongly r-connected.

Claim 4. Let R' denote $\mathrm{pr}_{\underline{n-1}} R$. Any tuples of the form $(\mathbf{a}, a), (\mathbf{a}, b) \in R$ where $\mathbf{a} \in R'$ are strongly ry-connected.

Suppose the contrary, there are $(\mathbf{a}, a), (\mathbf{a}, b) \in R$ such that $(\mathbf{a}, a)$ is not strongly ry-connected to $(\mathbf{a}, b)$. Notice first that if $a \le b$ or $\langle a, b \rangle \in \beta$ the tuples are strongly ry-connected by definition, and if $b \le a$ then they are strongly ry-connected by Claim 1. So, we may assume $\langle a, b \rangle \in \gamma$. Since $\mathcal{G}(R_n)$ is strongly ry-connected, there are $a = a_1, a_2, \dots, a_k = b$ such that $a_i \le a_{i+1}$ or $\langle a_i, a_{i+1} \rangle \in \beta$. Without loss of generality, we may assume that a, b and k are such that k is the least possible. Let $\mathbf{a}_1, \dots, \mathbf{a}_k \in R'$ be such that $\mathbf{a}_1 = \mathbf{a}$ and $(\mathbf{a}_i, a_i) \in R$.

Case 1. $k > 3$.

Then $a_2 \not\le b$ and $\langle a_2, b \rangle \notin \beta$. We have two subcases to consider.

Subcase 1.1. $b \le a_2$.

In this case a_2, b are in the same strongly r-connected component. If also $a \le a_2$, then a, b in the same strongly r-connected component, and we get the result by Claim 3. Therefore, $\langle a, a_2 \rangle \in \beta$, and $(\mathbf{a}', a), (\mathbf{a}'', a), (\mathbf{a}', b)$ belong to R, where

$$\begin{pmatrix} \mathbf{a}' \\ a \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a} \\ a \end{pmatrix}, \begin{pmatrix} \mathbf{a}_2 \\ a_2 \end{pmatrix}\right), \quad \begin{pmatrix} \mathbf{a}'' \\ a_2 \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}_2 \\ a_2 \end{pmatrix}, \begin{pmatrix} \mathbf{a} \\ a \end{pmatrix}\right), \quad \begin{pmatrix} \mathbf{a}' \\ b \end{pmatrix} = p\left(\begin{pmatrix} \mathbf{a} \\ b \end{pmatrix}, \begin{pmatrix} \mathbf{a}' \\ a \end{pmatrix}\right).$$

Since $\mathbf{a} \le \mathbf{a}'$, if we prove that $(\mathbf{a}', a), (\mathbf{a}', b)$ are strongly ry-connected, then by Claim 1, $(\mathbf{a}, a)$ and $(\mathbf{a}', a)$ are strongly ry-connected, and we get the result. As

$\langle \mathbf{a}'[i], \mathbf{a}''[i]\rangle \in \beta \cup \gamma$ for any $i \in \underline{n-1}$, we get

$$\begin{pmatrix}\mathbf{a}'\\ a_2\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}'\\ b\end{pmatrix}, \begin{pmatrix}\mathbf{a}''\\ a_2\end{pmatrix}\right) \in R.$$

By Claim 3, $(\mathbf{a}', a_2)$ and $(\mathbf{a}', b)$ are strongly ry-connected. Hence, $(\mathbf{a}', a), (\mathbf{a}', b)$ and therefore $(\mathbf{a}, a), (\mathbf{a}, b)$ are strongly ry-connected, a contradiction.

Subcase 1.2. $\langle a_2, b\rangle \in \gamma$.

Observe that $p\left(\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}\right) = \begin{pmatrix}p(\mathbf{a}_2, \mathbf{a})\\ a_2\end{pmatrix}$. Therefore, by replacing $\mathbf{a}_2$ with $p(\mathbf{a}_2, \mathbf{a})$ we may assume that $\langle \mathbf{a}[i], \mathbf{a}_2[i]\rangle \in \beta$ or $\mathbf{a}[i] \le \mathbf{a}_2[i]$ for all $i \in \underline{n-1}$. For the tuple $\begin{pmatrix}\mathbf{a}'\\ a'\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right)$ we have $(\mathbf{a}, a) \le (\mathbf{a}', a')$ and $\langle(\mathbf{a}', a'), (\mathbf{a}_2, a_2)\rangle \in \beta$. This means that $(\mathbf{a}, a)$ and $(\mathbf{a}_2, a_2)$ are strongly ry-connected. Furthermore, as is easily seen

$$\begin{pmatrix}\mathbf{a}_2\\ b\end{pmatrix} = p\left(f\left(\begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right), \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right) = p\left(\begin{pmatrix}\mathbf{a}'\\ b\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right) \in R.$$

Since a_2 and b are connected with an rb-path shorter than k, by the assumption made, $(\mathbf{a}_2, a_2)$ and $(\mathbf{a}_2, b)$ are strongly ry-connected, and therefore $(\mathbf{a}, a), (\mathbf{a}, b)$ are strongly ry-connected, a contradiction.

Case 2. $k = 3$.

If $a \le a_2 \le b$ then the result follows from Claim 3. Therefore, either $\langle a, a_2\rangle \in \beta$ or $\langle a_2, b\rangle \in \beta$.

Subcase 2.1. $\langle a, a_2\rangle \in \beta$.

As before, replacing $(\mathbf{a}_2, a_2)$ with $\begin{pmatrix}\mathbf{a}_2'\\ a_2\end{pmatrix} = p\left(\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right)$, we may assume that $\langle \mathbf{a}[i], \mathbf{a}_2[i]\rangle \in \gamma$ and $\mathbf{a}_2[i] \le \mathbf{a}[i]$ for no $i \in \underline{n-1}$. Then $\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix} \le \begin{pmatrix}\mathbf{a}'\\ a\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right)$ and $\langle(\mathbf{a}', a), (\mathbf{a}_2, a_2)\rangle \in \beta$.

If $a_2 \le b$ then $\begin{pmatrix}\mathbf{a}_2\\ b\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}\right) \in R$ and $\begin{pmatrix}\mathbf{a}_2\\ b\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}'\\ b\end{pmatrix}\right) \in R$. We have $(\mathbf{a}_2, a_2) \le (\mathbf{a}_2, b)$ and $\langle(\mathbf{a}_2, b), (\mathbf{a}', b)\rangle \in \beta$, and hence $(\mathbf{a}, a)$ is strongly ry-connected to $(\mathbf{a}', b)$. Since $\mathbf{a}'$ and $\mathbf{a}$ are strongly r-connected, by Claim 1, the result follows.

If $\langle a_2, b\rangle \in \beta$ then set $\begin{pmatrix}\mathbf{a}''\\ a_2\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}\right)$. We have $(\mathbf{a}_2, a_2) \le (\mathbf{a}'', a_2)$, $\langle \mathbf{a}'', a_2), (\mathbf{a}b)\rangle \in \beta$. Thus, $(\mathbf{a}_2, a_2), (\mathbf{a}, b)$ are strongly ry-connected, and so are $(\mathbf{a}, a), (\mathbf{a}, b)$, a contradiction.

Subcase 2.2. $\langle a_2, b\rangle \in \beta$.

If $\langle a, a_2\rangle \in \beta$ then we are in the conditions of Subcase 2.1. If $a \le a_2$ then

$$\begin{pmatrix}\mathbf{a}'\\ a_2\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}\\ a\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right) \in R \quad \text{and} \quad \begin{pmatrix}\mathbf{a}'\\ b\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}\\ b\end{pmatrix}, \begin{pmatrix}\mathbf{a}_2\\ a_2\end{pmatrix}\right) \in R.$$

As is easily seen, $(\mathbf{a}, a) \le (\mathbf{a}', a_2)$, $\langle(\mathbf{a}', a_2), (\mathbf{a}', b)\rangle \in \beta$ and, by Claim 1, $(\mathbf{a}', b)$ and

$(\mathbf{a}, b)$ are strongly r-connected. The claim is proved.

Finally, take $(\mathbf{a}, a), (\mathbf{b}, b) \in R$ where $\mathbf{a}, \mathbf{b} \in R'$. Since $\mathcal{G}(R')$ is strongly ry-connected, there are $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k = \mathbf{b}$ such that $\mathbf{a}_i \leq \mathbf{a}_{i+1}$ or $\langle \mathbf{a}_i, \mathbf{a}_{i+1} \rangle \in \beta$. These tuples can be expanded to tuples from R: there are $a_1, \dots, a_k \in A_n$ with $(\mathbf{a}_i, a_i) \in R$, $i \in \underline{k}$. We prove that, for every i, $(\mathbf{a}_i, a_i)$ and $(\mathbf{a}_{i+1}, a_{i+1})$ are strongly ry-connected.

First, suppose $\mathbf{a}_i \leq \mathbf{a}_{i+1}$. If $a_i \leq a_{i+1}$, then $(\mathbf{a}_i, a_i) \leq (\mathbf{a}_{i+1}, a_{i+1})$. If $a_{i+1} \leq a_i$ then the result follows by Claim 1. If $\langle a_i, a_{i+1} \rangle \in \beta$, then $\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix} \leq \begin{pmatrix} \mathbf{a}_{i+1} \\ a_i \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}\right)$ and $\langle (\mathbf{a}_{i+1}, a_i), (\mathbf{a}_{i+1}, a_{i+1}) \rangle \in \beta$. Finally, if $\langle a_i, a_{i+1} \rangle \in \gamma$ then $\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix} \leq \begin{pmatrix} \mathbf{a}_{i+1} \\ a_i \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}\right)$ and $(\mathbf{a}_{i+1}, a_i), (\mathbf{a}_{i+1}, a_{i+1})$ are strongly ry-connected by Claim 4.

Then, let $\langle \mathbf{a}_i, \mathbf{a}_{i+1} \rangle \in \beta$. If $\langle a_i, a_{i+1} \rangle \in \beta$ then $\langle (\mathbf{a}_i, a_i), (\mathbf{a}_{i+1}, a_{i+1}) \rangle \in \beta$. If $\langle a_i, a_{i+1} \rangle \in \gamma$ or $a_i \leq a_{i+1}$ then $\langle (\mathbf{a}_i, a_{i+1}), (\mathbf{a}_{i+1}, a_{i+1}) \rangle \in \beta$, where $(\mathbf{a}_i, a_i), (\mathbf{a}_i, a_{i+1})$ are strongly ry-connected by Claim 2 and $\begin{pmatrix} \mathbf{a}_i \\ a_{i+1} \end{pmatrix} = p\left(\begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}, \begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}\right)$. If $a_{i+1} \leq a_i$ then $\langle (\mathbf{a}_i, a_i), (\mathbf{a}_{i+1}, a_i) \rangle \in \beta$, where $\begin{pmatrix} \mathbf{a}_{i+1} \\ a_i \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}_{i+1} \\ a_{i+1} \end{pmatrix}, \begin{pmatrix} \mathbf{a}_i \\ a_i \end{pmatrix}\right)$, and $(\mathbf{a}_{i+1}, a_i)$, $(\mathbf{a}_{i+1}, a_{i+1})$ are strongly ry-connected by Claim 3. Finally, $(\mathbf{a}_k, a_k) = (\mathbf{b}, \mathbf{a}_k)$ and $(\mathbf{b}, b)$ are strongly ry-connected by Claim 4. $\square$

C. PROOFS FROM SECTION 6

Lemma C.1 (Lemma 6.1, p. 50). *Let R be an n-ary relation and $\mathbf{a}_1, \dots, \mathbf{a}_k \in R$ an r-path. Then, for any $i, j \in \underline{n}$, there is an r-path $\mathbf{b}_1, \dots, \mathbf{b}_m$ such that $\mathbf{b}_1 = \mathbf{a}_1$, $\mathbf{b}_1[l], \dots, \mathbf{b}_m[l] \in \{\mathbf{a}_1[l], \dots, \mathbf{a}_k[l]\}$, for $l \in \underline{n}$, $\mathbf{b}_m[i] = \mathbf{a}_k[i], \mathbf{b}_m[j] = \mathbf{a}_k[j]$, and both $\mathbf{b}_1[i], \dots, \mathbf{b}_m[i]$ and $\mathbf{b}_1[j], \dots, \mathbf{b}_m[j]$ are irreducible.*

Proof. Without loss of generality we may assume that $i = 1$, $j = 2$. We prove the lemma by induction on the sum of the numbers of alternations in the sequences $\mathbf{a}_1[1], \dots, \mathbf{a}_k[1]$ and $\mathbf{a}_1[2], \dots, \mathbf{a}_k[2]$ (that is positions i such that $\mathbf{a}_i[1] \neq \mathbf{a}_{i+1}[1]$ and $\mathbf{a}_i[2] \neq \mathbf{a}_{i+1}[2]$, respectively). In the base case for induction when this number equals 2, we have $k = 1$, and the result holds trivially. We also may assume that either $\mathbf{a}_{k-1}[1] \neq \mathbf{a}_k[1]$ or $\mathbf{a}_{k-1}[2] \neq \mathbf{a}_k[2]$, and by induction hypothesis, that $\mathbf{a}_1[1], \dots, \mathbf{a}_{k-1}[1]$, $\mathbf{a}_1[2], \dots, \mathbf{a}_{k-1}[2]$ are irreducible, since the set $\{\mathbf{a}_1[1], \dots, \mathbf{a}_{k-1}[1], \mathbf{a}_1[2], \dots, \mathbf{a}_{k-1}[2]\}$ contains fewer elements than $\{\mathbf{a}_1[1], \dots, \mathbf{a}_k[1], \mathbf{a}_1[2], \dots, \mathbf{a}_k[2]\}$. Note that it suffices to consider only the first two components of the tuples. Indeed, if we obtain a sequence of tuples $\mathbf{b}_1, \dots, \mathbf{b}_m$ such that $\mathrm{pr}_{1,2}\mathbf{b}_1, \dots, \mathrm{pr}_{1,2}\mathbf{b}_m$ is an irreducible r-path, then the sequence $\mathbf{b}'_1, \dots, \mathbf{b}'_m$ with $\mathbf{b}'_1 = \mathbf{b}_1$ and $\mathbf{b}'_{i+1} = f(\mathbf{b}'_i, \mathbf{b}_{i+1})$, for $i > 1$, is an r-path and $\mathrm{pr}_{1,2}\mathbf{b}'_i = \mathrm{pr}_{1,2}\mathbf{b}_i$. There are two cases to consider.

Case 1. $\mathbf{a}_{k-2}[1] \neq \mathbf{a}_{k-1}[1]$ and $\mathbf{a}_{k-2}[2] \neq \mathbf{a}_{k-1}[2]$.

Subcase 1a. $\mathbf{a}_{k-2}[1] \not\leq \mathbf{a}_k[1]$, $\mathbf{a}_{k-2}[2] \not\leq \mathbf{a}_k[2]$.

In this case $\mathbf{a}_1[1], \dots, \mathbf{a}_{k-1}[1], \mathbf{a}_k[1]$ and $\mathbf{a}_1[2], \dots, \mathbf{a}_{k-1}[2], \mathbf{a}_k[2]$ are irreducible and we are done.

Subcase 1b. $\mathbf{a}_{k-2}[1] \not\leq \mathbf{a}_k[1]$, $\mathbf{a}_{k-2}[2] \leq \mathbf{a}_k[2]$ or $\mathbf{a}_{k-2}[1] \leq \mathbf{a}_k[1]$, $\mathbf{a}_{k-2}[2] \not\leq \mathbf{a}_k[2]$.

Without loss of generality we assume the former case. If $\mathbf{a}_{k-1}[2] = \mathbf{a}_k[2]$ then $\mathbf{a}_1[1], \ldots, \mathbf{a}_{k-1}[1], \mathbf{a}_k[1]$ and $\mathbf{a}_1[2], \ldots, \mathbf{a}_{k-1}[2], \mathbf{a}_k[2]$ are irreducible and we are done. Otherwise, by setting $\mathbf{a}'_{k-2} = f(\mathbf{a}_{k-2}, \mathbf{a}_k)$ and $\mathbf{a}'_{k-1} = f(\mathbf{a}_{k-1}, \mathbf{a}'_{k-2})$, we get $\mathbf{a}'_{k-2}[1] = \mathbf{a}_{k-2}[1]$, $\mathbf{a}'_{k-2}[2] = \mathbf{a}_k[2]$, $\mathbf{a}'_{k-1}[1] = \mathbf{a}_{k-1}[1]$, $\mathbf{a}'_{k-1}[2] = \mathbf{a}_k[2]$. In the r-path $\mathbf{a}_1, \ldots, \mathbf{a}_{k-2}, \mathbf{a}'_{k-1}, \mathbf{a}_k$ the number of alternations in $\mathbf{a}_1[1], \ldots, \mathbf{a}_{k-2}[1], \mathbf{a}'_{k-1}[1], \mathbf{a}_k[1]$ and $\mathbf{a}_1[2], \ldots, \mathbf{a}_{k-2}[2], \mathbf{a}'_{k-1}[2], \mathbf{a}_k[2]$ is less than that for the original r-path, as $\mathbf{a}_{k-1}[2]$ is excluded. Therefore we can apply induction hypothesis.

Subcase 1c. $\mathbf{a}_{k-2}[1] \leq \mathbf{a}_k[1]$ and $\mathbf{a}_{k-2}[2] \leq \mathbf{a}_k[2]$.

In this case it suffices to remove $\mathbf{a}_{k-1}$ and apply induction hypothesis.

Case 2. $\mathbf{a}_{k-2}[1] \neq \mathbf{a}_{k-1}[1]$ and $\mathbf{a}_{k-l}[2] = \ldots = \mathbf{a}_{k-2}[2] = \mathbf{a}_{k-1}[2]$, or $\mathbf{a}_{k-l}[1] = \ldots = \mathbf{a}_{k-2}[1] = \mathbf{a}_{k-1}[1]$ and $\mathbf{a}_{k-2}[2] \neq \mathbf{a}_{k-1}[2]$.

We assume that $\mathbf{a}_{k-2}[2] \neq \mathbf{a}_{k-1}[2]$ and $\mathbf{a}_{k-l}[1] = \ldots = \mathbf{a}_{k-2}[1] = \mathbf{a}_{k-1}[1]$. Clearly, $\mathbf{a}_{k-l}[1] = \ldots = \mathbf{a}_{k-2}[1] = \mathbf{a}_{k-1}[1] \leq \mathbf{a}_k[1]$.

Subcase 2a. $\mathbf{a}_{k-l-1}[1] \not\leq \mathbf{a}_k[1]$, $\mathbf{a}_{k-2}[2] \not\leq \mathbf{a}_k[2]$.

By setting $\mathbf{a}'_{k-2} = f(\mathbf{a}_{k-2}, \mathbf{a}_k)$, $\mathbf{a}'_{k-1} = f(\mathbf{a}'_{k-2}, \mathbf{a}_{k-1})$, and then $\mathbf{a}'_j = f(\mathbf{a}_j, \mathbf{a}'_{j+2})$ for all $k-l+1 \leq j \leq k-3$, we get $\mathbf{a}'_{k-l+1}[1] = \ldots = \mathbf{a}'_{k-1}[1] = \mathbf{a}_k[1]$ and $\mathbf{a}'_{k-l+1}[2] = \mathbf{a}_{k-l+1}[2], \ldots, \mathbf{a}'_{k-1}[2] = \mathbf{a}_{k-1}[2]$. Then, for the r-path $\mathbf{a}_1, \ldots, \mathbf{a}_{k-l}, \mathbf{a}'_{k-l+1}, \ldots, \mathbf{a}'_{k-1}$, $\mathbf{a}_k$, the r-paths $\mathbf{a}_1[1], \ldots, \mathbf{a}_{n-l}[1], \mathbf{a}'_{n-l+1}[1], \ldots, \mathbf{a}'_{k-1}[1], \mathbf{a}_k[1]$ and $\mathbf{a}_1[2], \ldots, \mathbf{a}_{n-l}[2]$, $\mathbf{a}'_{n-l+1}[2], \ldots, \mathbf{a}'_{k-1}[2], \mathbf{a}_k[2]$ are irreducible.

Subcase 2b. $\mathbf{a}_{k-l-1}[1] \leq \mathbf{a}_k[1]$, $\mathbf{a}_{k-2}[2] \not\leq \mathbf{a}_k[2]$.

As in Subcase 2a we may get $\mathbf{a}'_{k-l}, \ldots, \mathbf{a}'_{k-1}$ such that $\mathbf{a}'_{k-l}[1] = \ldots = \mathbf{a}'_{k-1} = \mathbf{a}_k[1]$ and $\mathbf{a}'_{k-l}[2] = \mathbf{a}_{k-l}[2], \ldots, \mathbf{a}'_{k-1}[2] = \mathbf{a}_{k-1}[2]$. Then in the r-path $\mathbf{a}_1, \ldots, \mathbf{a}_{k-l-1}$, $\mathbf{a}'_{k-l}, \ldots, \mathbf{a}'_{k-1}, \mathbf{a}_k$ the number of alternations in $\mathbf{a}_1[1], \ldots, \mathbf{a}_{k-l-1}[1], \mathbf{a}'_{k-l}[1], \ldots$, $\mathbf{a}'_{k-1}[1], \mathbf{a}_k[1]$ and $\mathbf{a}_1[2], \ldots, \mathbf{a}_{k-l-1}[2], \mathbf{a}'_{k-l}[2], \ldots, \mathbf{a}'_{k-1}[2], \mathbf{a}_k[2]$ is less than that for the original r-path, as $\mathbf{a}_{k-l}[1] = \ldots = \mathbf{a}_{k-1}[1]$ is excluded. Therefore we may apply induction hypothesis.

Subcase 2c. $\mathbf{a}_{k-2}[2] \leq \mathbf{a}_k[2]$.

Replace the original r-path with $\mathbf{a}_1, \ldots, \mathbf{a}_{k-2}, f(\mathbf{a}_{k-2}, \mathbf{a}_k)$ and apply induction hypothesis. □

Lemma C.2 (Lemma 6.4, p. 52). *If $\mathbf{a}, \mathbf{b} \in R$ are such that $\mathbf{a} \in \max(\mathbf{b})$, then there is $\mathbf{c} \in \mathcal{F}(\mathbf{b}) \cap \max(\mathbf{b})$ such that $\mathbf{c}$ is strongly ry-connected to $\mathbf{a}$ in $\mathcal{G}(R^{\max})$.*

Proof. We should prove that every strongly ry-connected component of $\mathcal{G}(\max(\mathbf{b}))$ contains a tuple $\mathbf{a}$ such that $\mathbf{a} \in \mathcal{F}(\mathbf{b})$. By the Generalized Connectedness Lemma 5.2(2), the strongly ry-connected components of $\mathcal{G}(\max(\mathbf{b}))$ are of the form $R' = R \cap (C_1 \times \ldots \times C_n)$ where $C_1, \ldots, C_n$ are strongly ry-connected components of $\mathcal{G}(\max(\mathbf{b}[1])), \ldots, \mathcal{G}(\max(\mathbf{b}[n]))$ respectively.

We proceed by induction on n. The base case for induction, $n = 1$, is obvious by definition of r-maximal elements. So, suppose the lemma is proved for all $m < n$.

By induction hypothesis, there is $\mathbf{c} \in \max(\mathbf{b})$ such that $\mathbf{c}[i] \in C_i$ for $i \in \underline{n-1}$ and $\mathrm{pr}_{\underline{n-1}}\mathbf{b} \prec \mathrm{pr}_{\underline{n-1}}\mathbf{c}$. Moreover, by the Maximal Expansion Lemma 3.8(1), $\mathbf{c}$ can be chosen to be r-maximal. We also may assume that $\mathbf{b} \prec \mathbf{c}$. Indeed, let $\mathrm{pr}_{\underline{n-1}}\mathbf{b} = \mathbf{b}'_1 \leq \mathbf{b}'_2 \leq \ldots \leq \mathbf{b}'_k = \mathrm{pr}_{\underline{n-1}}\mathbf{c}$ be an r-path, and $\mathbf{b} = \mathbf{b}_1 \leq \mathbf{b}_2 \leq \ldots \leq$

$\mathbf{b}_k \in R$ its extension that exists by the Path Extension Lemma 3.6(1). Then, by the Maximality Lemma 3.7(1), there is $\mathbf{c}'$ such that $\mathbf{b}_k \prec \mathbf{c}'$ and $\mathbf{c}'[n]$ is r-maximal. Since $\mathrm{pr}_{\underline{n-1}}\mathbf{b}_k = \mathrm{pr}_{\underline{n-1}}\mathbf{c}$ and $\mathrm{pr}_{\underline{n-1}}\mathbf{c}'$ belong to the same strongly r-connected component, there is $\mathbf{c}''$ such that $\mathrm{pr}_{\underline{n-1}}\mathbf{c}'' = \mathrm{pr}_{\underline{n-1}}\mathbf{c}$ and $\mathbf{c}' \prec \mathbf{c}''$. Thus $\mathbf{b} \prec \mathbf{c}''$, and $\mathbf{c}''$ satisfies the same condition as $\mathbf{c}$. Let us replace $\mathbf{c}$ with $\mathbf{c}''$ if needed.

If $\mathbf{c}[n] \in C_n$ then we are done. Otherwise, let $\mathbf{a} \in R'$. Since $J(\mathrm{ry}, n, C_n) = \{n\}$, by the Rectangularity Proposition 5.4(1), $\max(\mathbf{b})$ is ry-rectangular, which implies that the tuple $\mathbf{d} = (\mathrm{pr}_{\underline{n-1}}\mathbf{c}, \mathbf{a}[n])$ belongs to R'. Thus we may assume that there is $\mathbf{a} \in R' \subseteq \max(\mathbf{b})$ such that $\mathrm{pr}_{\underline{n-1}}\mathbf{a} = \mathrm{pr}_{\underline{n-1}}\mathbf{c}$, $\mathbf{a}[n] \in C_n$ and $\langle \mathbf{c}[n], \mathbf{a}[n]\rangle \in \gamma$. We shall show that $\mathbf{b} \prec \mathbf{a}$. Let $\mathbf{b} = \mathbf{b}_1 \le \mathbf{b}_2 \le \ldots \le \mathbf{b}_k = \mathbf{c}$ be an r-path connecting $\mathbf{b}$ and $\mathbf{c}$.

Notice that if $\langle \mathbf{a}[n], \mathbf{b}_j[n]\rangle \notin \beta \cup \gamma$ for a certain $j \in \underline{k}$ then $\mathbf{b} \prec \mathbf{a}$. Indeed, suppose first $\mathbf{a}[n] \le \mathbf{b}_l[n]$ for a certain l. Consider the tuples $\mathbf{b}'_l = f(\mathbf{a}, \mathbf{b}_l)$, and if $\mathbf{b}'_j$ is obtained, $\mathbf{b}'_{j+1} = f(\mathbf{b}'_j, \mathbf{b}_{j+1})$. It is not hard to see that $\mathbf{b}'_j[i] \in \{\mathbf{b}_j[i], \mathbf{b}_k[i]\}$. Therefore, $\mathbf{b}'_k = \mathbf{b}_k$. Since $\mathbf{a} \le \mathbf{b}'_j$, we have $\mathbf{a} \prec \mathbf{b}_k$. Since $\mathbf{a}$ is r-maximal and by the Generalized Connectedness Lemma 5.2(1), we have $\mathbf{b}_k \prec \mathbf{a}$, a contradiction with $\mathbf{c}[n] \notin C_n$. So, suppose that, for any $j \le k$, $\mathbf{b}_j[n] \le \mathbf{a}[n]$ or $\langle \mathbf{b}_j[n], \mathbf{a}[n]\rangle \in \beta \cup \gamma$. If $\mathbf{b}_l[n] \le \mathbf{a}[n]$ for a certain $l \le k$, then the r-path $\mathbf{a}_l = f(\mathbf{b}_l, \mathbf{a})$, $\mathbf{a}_{j+1} = f(\mathbf{a}_j, \mathbf{b}_{j+1})$, for $j > l$, ends up with $\mathbf{a}$, which proves that $\mathbf{b} \prec \mathbf{a}$.

Then we prove by induction on $m < n$ that there exists $\mathbf{d} \in \mathcal{F}(\mathbf{b})$ such that $\mathrm{pr}_{\underline{m}}\mathbf{d}, \mathrm{pr}_{\underline{m}}\mathbf{c}$ are strongly r-connected, $\mathbf{d}[n] = \mathbf{b}[n]$, and $\mathbf{d}[i] \in \{\mathbf{b}_1[i], \ldots, \mathbf{b}_k[i]\}$ for $i \in \{m+1, \ldots, n-1\}$. In the base case for induction, $m = 0$, we may choose $\mathbf{d} = \mathbf{b}$.

By Lemma 6.1, the r-paths $\mathbf{b}_1[m], \ldots, \mathbf{b}_k[m]$, $\mathbf{b}_1[n], \ldots, \mathbf{b}_k[n]$ can be assumed to be irreducible. Clearly, by truncating the sequence $\mathbf{b}_1, \ldots, \mathbf{b}_k$, either all $\mathbf{b}_1[m], \ldots, \mathbf{b}_k[m]$ or all $\mathbf{b}_1[n], \ldots, \mathbf{b}_k[n]$ can be assumed to be different.

There are two cases.

CASE 1. All $\mathbf{b}_1[m], \ldots, \mathbf{b}_k[m]$ are different.

If $k \le 2$ then for the tuple $\mathbf{a}' = f(\mathbf{b}_1, \mathbf{a})$, we have $\mathbf{a}'[m] = \mathbf{c}[m]$, $\mathbf{a}'[n] = \mathbf{b}_1[n]$ and $\mathbf{b} = \mathbf{b}_1 \le \mathbf{a}'$. If $k > 2$ then set $\mathbf{a}' = f(\mathbf{b}_{k-1}, \mathbf{a})$. By the Path Alignment Lemma 6.2(2), we obtain tuples $\mathbf{c}^1_1, \ldots, \mathbf{c}^1_k \in \mathcal{F}(\mathbf{b})$ such that $\mathbf{c}^1_1[n] = \ldots = \mathbf{c}^1_k[n] = \mathbf{b}_1[n]$, $\mathbf{c}^1_j[m] = \mathbf{b}_j[m]$ for $j \in \underline{k}$, and $\mathbf{c}^1_1[i], \ldots, \mathbf{c}^1_l[i] \in \{\mathbf{b}_1[i], \ldots, \mathbf{b}_k[i]\}$ for $i \in \underline{n} - \{m, n\}$.

CASE 2. All $\mathbf{b}_1[n], \ldots, \mathbf{b}_k[n]$ are different.

Let l be such that $\mathbf{b}_{k-l}[m] = \ldots = \mathbf{b}_k[m]$, but $\mathbf{b}_{k-l-1}[m] \ne \mathbf{b}_{k-l}[m]$. Consider the r-path $\mathbf{b}_1, \ldots, \mathbf{b}_{k-l}$. As in the previous case, there are $\mathbf{c}^1_1 \le \ldots \le \mathbf{c}^1_{k-l}$ with $\mathbf{c}^1_1[n] = \ldots = \mathbf{c}^1_{k-l} = \mathbf{b}_1[n]$ and $\mathbf{c}^1_j[m] = \mathbf{b}_j[m]$ for $j \in \underline{k-l}$.

In both cases, there are $\mathbf{b} = \mathbf{c}^1_1 \le \ldots \le \mathbf{c}^1_k$ where in Case 2 $\mathbf{c}^1_{n-l+1}, \ldots, \mathbf{c}^1_k$ can be taken to be equal $\mathbf{c}^1_{k-l}$. Let also $\mathbf{d}'$ be the tuple existing by induction hypothesis, that is such that $\mathrm{pr}_{\underline{m-1}}\mathbf{d}'$ is strongly r-connected with $\mathrm{pr}_{\underline{m-1}}\mathbf{c}$ and $\mathbf{d}'[n] = \mathbf{b}[n]$. There is $j \in \underline{k}$ such that $\mathbf{d}'[m] = \mathbf{b}_j[m]$. Let us define tuples $\mathbf{d}_j, \ldots, \mathbf{d}_k$ as follows. Set $\mathbf{d}_j = \mathbf{d}'$, and if $\mathbf{d}_i$ is already defined then set $\mathbf{d}_{i+1} = f(\mathbf{d}_i, \mathbf{c}^1_{i+1})$. Clearly, $\mathbf{d}_k[m] = \mathbf{c}[m]$ and $\mathrm{pr}_{\underline{m-1}}\mathbf{d}_k$ is strongly r-connected with $\mathrm{pr}_{\underline{m-1}}\mathbf{c}$. By the Generalized Connectedness Lemma 5.2(1), $\mathrm{pr}_{\underline{m}}\mathbf{d}_k$ is strongly r-connected with $\mathrm{pr}_{\underline{m}}\mathbf{c}$.

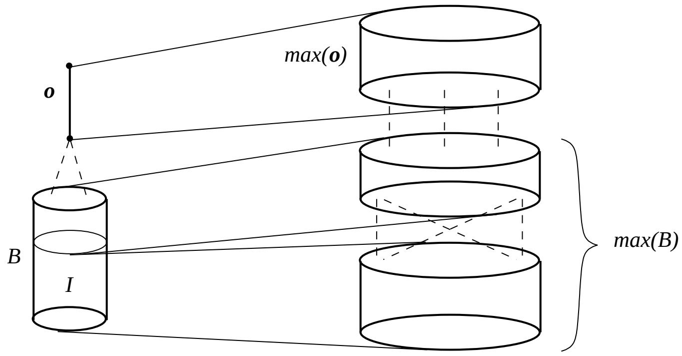

Fig. C.2. The Fork Lemma 6.8.

Thus, $\mathcal{F}(\mathbf{b})$ contains a tuple $\mathbf{d}$ such that $\mathbf{d}[n] = \mathbf{b}[n]$ and $\mathrm{pr}_{\underline{n-1}}\mathbf{d}$ is strongly r-connected with $\mathrm{pr}_{\underline{n-1}}\mathbf{c}$. Take an r-path $\mathbf{b}[n] = e_1 \le e_2 \le \ldots \le e_r = \mathbf{a}[n]$, and its extension to an r-path $\mathbf{b} = \mathbf{e}_1 \le \mathbf{e}_2 \le \ldots \le \mathbf{e}_r$. Then set $\mathbf{e}'_1 = f(\mathbf{d}, \mathbf{e}_1)$, and if $\mathbf{e}'_j$ is already obtained, then $\mathbf{e}'_{j+1} = f(\mathbf{e}'_j, \mathbf{e}_{j+1})$. It is not hard to see, that $\mathbf{e}'_1, \ldots, \mathbf{e}'_r$ form an r-path. Since $\mathrm{pr}_{\underline{n-1}}\mathbf{a} \prec \mathrm{pr}_{\underline{n-1}}\mathbf{d}_k \prec \mathrm{pr}_{\underline{n-1}}\mathbf{e}'_r$, $\mathbf{a}[i]$ is in a r-maximal component and $\mathbf{e}'_r[n] = \mathbf{a}[n]$, by the Generalized Connectedness Lemma 5.2(1), $\mathbf{a}$ and $\mathbf{e}'_r$ are strongly r-connected. Finally, as $\mathbf{b} \prec \mathbf{d}_k \prec \mathbf{e}'_r \prec \mathbf{a}$, we get $\mathbf{a} \in \mathcal{F}(\mathbf{b})$, as required. □

LEMMA C.3 (FORK LEMMA, LEMMA 6.8, P.54). *Let R be a subdirect product of $R^1, R^2 \in \Gamma$, which are subdirect products of $A_1, \ldots, A_m \in \mathcal{A}$ and $A_{m+1}, \ldots, A_{m+n} \in \mathcal{A}$, respectively, and $\mathbf{o} \in R^1$, $B \subseteq R^2$ such that $\{\mathbf{o}\} \times B \subseteq R$. There is $I \subseteq K = \{m+1, \ldots, m+n\}$ such that $\max(\{\mathbf{o}\} \times \mathrm{pr}_{K-I}B) \times \max(\mathrm{pr}_I B) \subseteq R$, and all members of $\mathrm{pr}_{K-I}B$ are indistinguishable (see Fig. C.2).*

PROOF. We proceed by induction on 3 parameters: n, $|\mathcal{F}(\mathbf{o})|$, $\sum_{\mathbf{b}\in B} |\mathcal{F}(\mathbf{b})|$. The base case for induction, $n = 1$ and the other parameters any, is trivial.

Since $\max(B) = \bigcup_{\mathbf{b}\in B} \max(\mathbf{b})$, it is enough to prove that there is $I \subseteq K$ such that, for any $\mathbf{b} \in B$, we have $\max(\mathbf{o}, \mathrm{pr}_{K-I}\mathbf{b}) \times \max(\mathrm{pr}_I \mathbf{b}) \subseteq R$ and the members of $\mathrm{pr}_{K-I}B$ are indistinguishable. In fact, this condition can be weakened even further.

CLAIM 1. If there is $\mathbf{b}^0 \in B$ and $\varnothing \ne J \subseteq K$ such that $\max(\mathbf{o}, \mathrm{pr}_{K-J}\mathbf{b}^0) \times \max(\mathrm{pr}_J \mathbf{b}^0) \subseteq R$, then the result follows.

Consider $\mathrm{pr}_{\underline{m}\cup(K-J)}R$, $\mathbf{o} \in R^1$, and $\mathrm{pr}_{K-J}B \subseteq \mathrm{pr}_{K-J}R^2$. Since $J \ne \varnothing$, by induction hypothesis, there is $I \subseteq K - J$ such that $\max(\{\mathbf{o}\} \times \mathrm{pr}_{K-J-I}B) \times \max(\mathrm{pr}_I B) \subseteq R$, and the members of $\mathrm{pr}_{K-J-I}B$ are indistinguishable. In particular, $\max(\mathbf{o}, \mathrm{pr}_{K-J-I}\mathbf{b}^0) \times \max(\mathrm{pr}_{J\cup I}\mathbf{b}^0) \subseteq R$.

Since, for any $\mathbf{b} \in B$, $(\mathbf{o}, \mathrm{pr}_{K-J-I}\mathbf{b})$ and $(\mathbf{o}, \mathrm{pr}_{K-J-I}\mathbf{b}^0)$ are indistinguishable, we get $\max(\mathbf{o}, \mathrm{pr}_{K-J-I}\mathbf{b}) \times \max(\mathrm{pr}_{J\cup I}\mathbf{b}^0) \subseteq R$. Finally, by Corollary 6.7, where $A = \{\mathrm{pr}_{J\cup I}\mathbf{b}^0, \mathrm{pr}_{J\cup I}\mathbf{b}\}$, $B = \{(\mathbf{o}, \mathrm{pr}_{K-J-I}\mathbf{b}^0), (\mathbf{o}, \mathrm{pr}_{K-J-I}\mathbf{b})\}$, we get $\max(\mathbf{o}, \mathrm{pr}_{K-J-I}\mathbf{b}) \times \max(\mathrm{pr}_{J\cup I}\mathbf{b}) \subseteq R$. The claim is proved.

As is easily seen, by Claim 1, we may restrict ourselves with the case $|B| = 2$. So, let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathbf{b}_1, \mathbf{b}_2$ are not indistinguishable. The base case for induction, where n is any, $\mathbf{o}$ and $\mathbf{b}_1, \mathbf{b}_2$ are r-maximal, follows from the next claim.

Claim 2. Let $\mathbf{b}_1, \mathbf{b}_2$ be r-maximal elements of $\mathcal{G}(R^2)$, let $\mathbf{o}$ be an r-maximal element of $\mathcal{G}(R^1)$, let B_i^1, B_i^2 be the strongly ry-connected components of $\mathcal{G}(\max(A_i))$, $i \in K$, containing $\mathbf{b}_1[i], \mathbf{b}_2[i]$ respectively, and let $I = \{i \in K \mid B_i^1 \neq B_i^2\}$. If $I = \varnothing$ then $\mathbf{b}_1, \mathbf{b}_2$ are indistinguishable, otherwise, $\max(\mathbf{o}, \mathrm{pr}_{K-I}\mathbf{b}_1) \times \max(\mathrm{pr}_I \mathbf{b}_1) \subseteq R$.

By the Generalized Connectedness Lemma 5.2(2), the relation $E^j = R^2 \cap (B^j_{m+1} \times \ldots \times B^j_{m+n})$, $j = 1, 2$, is a subdirect product of $B^j_{m+1}, \ldots, B^j_{m+n}$, $E^j = \max(\mathbf{b}_j)$; and $\mathcal{G}(E^j)$ is strongly ry-connected.

Notice that $\max(\mathbf{o})$, $\max(\mathbf{b}_1)$, $\max(\mathbf{b}_2)$ equal the strongly ry-connected components of $\mathcal{G}(R^1), \mathcal{G}(R^2)$ containing $\mathbf{o}, \mathbf{b}_1, \mathbf{b}_2$ respectively. Therefore, if $I = \varnothing$ then $\max(\mathbf{b}_1) = \max(\mathbf{b}_2)$, and $\mathbf{b}_1, \mathbf{b}_2$ are indistinguishable. Otherwise, we apply the Rectangularity Proposition 5.4(1) to the relation

$$R \cap \left(\max(\mathbf{o}) \times \left(R^2 \cap ((B^1_{m+1} \cup B^2_{m+1}) \times \ldots \times (B^1_{m+n} \cup B^2_{m+n}))\right)\right).$$

This relation is ry-rectangular, which means that for any $\mathbf{a} \in \max(\mathbf{o}, \mathrm{pr}_{K-I}\mathbf{b}_1)$ and any $\mathbf{b} \in \mathrm{pr}_I \mathbf{b}_1$, the tuple $(\mathbf{a}, \mathbf{b})$ belongs to R. The claim is proved.

Suppose that $\mathbf{b}_1$ is not r-maximal. If $\mathbf{b}_2$ is r-maximal, then $\mathbf{b}_2 \not\sqsubseteq \mathbf{b}_1$. Indeed, $\max(\mathbf{b}_2)$ is the strongly ry-connected component of $\mathcal{G}(R^2)$ containing $\mathbf{b}_2$, therefore, if $\max(\mathbf{b}_1) \subseteq \max(\mathbf{b}_2)$ then $\max(\mathbf{b}_1) = \max(\mathbf{b}_2)$, that is $\mathbf{b}_1, \mathbf{b}_2$ are indistinguishable. Now suppose that there is $\mathbf{c} \in \mathcal{F}(\mathbf{b}_1) - \mathcal{F}(\mathbf{b}_2)$. We may assume that $\mathbf{c}$ is r-maximal and $\mathbf{c}, \mathbf{b}_1$ are in different strongly r-connected components. In other words, $|\mathcal{F}(\mathbf{c})| \le |\mathcal{F}(\mathbf{b}_1)|$.

There is an r-path $\mathbf{b}_1 = \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k = \mathbf{c}$ and, by Lemma 6.3, $\mathbf{b}_2 \le \mathbf{d}$ for no $\mathbf{d} \in R^2$ indistinguishable with one of $\mathbf{c}_1, \ldots, \mathbf{c}_k$. We proceed by induction on k.

Claim 3. Either there is $\mathbf{o}' \in \mathcal{F}(\mathbf{o})$ such that $(\mathbf{o}', \mathbf{c}_2), (\mathbf{o}', \mathbf{b}_2) \in R$, or there is $J \subseteq K$ with $\max(\mathbf{o}, \mathrm{pr}_{K-J}\mathbf{b}_2) \times \max(\mathrm{pr}_J \mathbf{b}_2) \subseteq R$.

There is $\mathbf{o}' \in R^1$ with $(\mathbf{o}', \mathbf{c}_2) \in R$. Replacing $(\mathbf{o}', \mathbf{c}_2)$ with $f\left(\begin{pmatrix}\mathbf{o}\\ \mathbf{b}_1\end{pmatrix}, \begin{pmatrix}\mathbf{o}'\\ \mathbf{c}_2\end{pmatrix}\right)$, we may assume that $\mathbf{o} \le \mathbf{o}'$.

Consider the tuple $\begin{pmatrix}\mathbf{o}'\\ \mathbf{d}\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{o}\\ \mathbf{b}_2\end{pmatrix}, \begin{pmatrix}\mathbf{o}'\\ \mathbf{c}_2\end{pmatrix}\right)$. Clearly, $\mathbf{b}_2 \le \mathbf{d}$. Let us denote by L the set $\{i \in K \mid \mathbf{d}[i] = \mathbf{b}_2[i]\}$. If $L = K$ then $(\mathbf{o}', \mathbf{c}_2), (\mathbf{o}', \mathbf{b}_2) \in R$. If $L = \varnothing$ then $\mathbf{d} = \mathbf{c}_2$ which means $\mathbf{b}_2 \le \mathbf{c}_2$, a contradiction with the condition $\mathbf{c} \in \mathcal{F}(\mathbf{b}_1) - \mathcal{F}(\mathbf{b}_2)$. Otherwise, $\mathbf{d}[i] = \mathbf{c}_2[i]$ for any $i \in K - L$, hence, we may use induction hypothesis for the pair $(\mathbf{a}, \mathrm{pr}_L \mathbf{b}_2), (\mathbf{a}, \mathrm{pr}_L \mathbf{c}_2)$ where $\mathbf{a} = (\mathbf{o}', \mathrm{pr}_{K-L}\mathbf{c}_2)$. Therefore there is $\varnothing \neq J \subseteq L$ such that $\max(\mathbf{a}, \mathrm{pr}_{L-J}\mathbf{b}_2) \times \max(\mathrm{pr}_J \mathbf{b}_2) \subseteq R$. Since $(\mathbf{o}, \mathrm{pr}_{K-L-J}\mathbf{b}_2) \le \mathbf{a} = (\mathbf{o}', \mathrm{pr}_{K-L-J}\mathbf{c}_2)$, by Corollary 6.7, $\max(\mathbf{o}, \mathrm{pr}_{K-L-J}\mathbf{b}_2) \times \max(\mathrm{pr}_J \mathbf{b}_2) \subseteq R$ (see Fig. C.3).

If the second possibility (a direct decomposition) from Claim 3 is the case then, by Claim 1, the result follows. Otherwise, if $k = 2$ then, since $|\mathcal{F}(\mathbf{c}_2)| + |\mathcal{F}(\mathbf{b}_2)| < |\mathcal{F}(\mathbf{b}_1)| + |\mathcal{F}(\mathbf{b}_2)|$ and $|\mathcal{F}(\mathbf{o}')| \le |\mathcal{F}(\mathbf{o})|$, by induction hypothesis, there is $J \subseteq K$ such that $\max(\mathbf{o}', \mathrm{pr}_{K-J}\mathbf{b}_2) \times \max(\mathrm{pr}_J \mathbf{b}_2) \subseteq R$. Moreover, as $(\mathbf{a}, \mathrm{pr}_J \mathbf{b}_2) \in R$, where $\mathbf{a} = (\mathbf{o}, \mathrm{pr}_{K-J}\mathbf{b}_2)$, we have, by Lemma 6.6, $\max(\mathbf{o}, \mathrm{pr}_{K-J}\mathbf{b}_2) \times \max(\mathrm{pr}_J \mathbf{b}_2) \subseteq R$. Therefore, we get what is required.

In the case $k > 2$, the argument is the same but we use induction hypothesis of

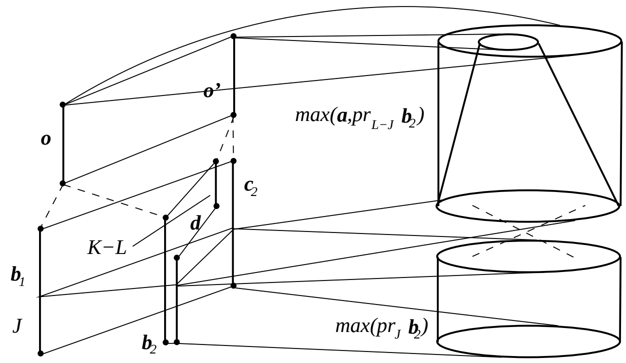

Figure C.3

the second induction process, on k. This completes the proof. □