*The Annals of Probability*
2013, Vol. 41, No. 6, 4214–4247
DOI: 10.1214/13-AOP862


# COMPLEXITY OF RANDOM SMOOTH FUNCTIONS ON THE HIGH-DIMENSIONAL SPHERE

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We analyze the landscape of general smooth Gaussian functions on the sphere in dimension $N$, when $N$ is large. We give an explicit formula for the asymptotic complexity of the mean number of critical points of finite and diverging index at any level of energy and for the mean Euler characteristic of level sets. We then find two possible scenarios for the bottom landscape, one that has a layered structure of critical values and a strong correlation between indexes and critical values and another where even at levels below the limiting ground state energy the mean number of local minima is exponentially large. We end the paper by discussing how these results can be interpreted in the language of spin glasses models.

**1. Introduction.** This work deals with the number of critical points of Gaussian smooth functions on the $N$ dimensional sphere. The questions addressed in this paper can be phrased as: What does a random Morse function look like on a high-dimensional sphere? How many critical values of given index, or below a given level? What can be said about the topology of its level sets? We investigate the number of critical points of given index in level sets below a given value, as well as the topology of the level sets through their mean Euler characteristic. Our main result is that these functions have an exponentially large number of critical points of given index, and that the Euler characteristic of the level sets have a very interesting oscillatory behavior. Moreover we find an invariant to distinguish between two very different classes of these functions that we describe below.

Let us know describe the functions that we will analyze. For $N \geq 1$, let $S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$ be the Euclidean sphere of radius $\sqrt{N}$,

$$S^{N-1}(\sqrt{N}) := \left\{ \boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 = 1 \right\}.$$

Consider the Gaussian function defined on $S^{N-1}(\sqrt{N})$ by

$$H_{N,p}(\boldsymbol{\sigma}) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p} \sigma_{i_1} \cdots \sigma_{i_p}, \tag{1.1}$$

Received October 2011; revised March 2013.

[1]Supported in part by NSF Grants DMS-08-06180 and DMS-05-00923.

[2]Supported in part by NSF Grants DMS-08-06180 and OISE-0730136.

*MSC2010 subject classifications.* 15A52, 82D30, 60G60.

*Key words and phrases.* Sample, spin glasses, critical points, random matrices, Parisi formula.

where $J_{i_1,\dots,i_p}$ are independent centered standard Gaussian random variables.

Equivalently, $H_{N,p}$ is the centered Gaussian process on the sphere $S^{N-1}(\sqrt{N})$ whose covariance is given by

$$\mathbb{E}[H_{N,p}(\boldsymbol{\sigma})H_{N,p}(\boldsymbol{\sigma}')] = N^{1-p}\left(\sum_{i=1}^{N}\sigma_i\sigma_i'\right)^p = NR(\boldsymbol{\sigma},\boldsymbol{\sigma}')^p, \tag{1.2}$$

where $R$ is the normalized inner product $R(\boldsymbol{\sigma},\boldsymbol{\sigma}') := \frac{1}{N}\langle\boldsymbol{\sigma},\boldsymbol{\sigma}'\rangle = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'$.

Given a sequence $\boldsymbol{\beta} = (\beta_p)_{p\in\mathbb{N},p\geq 2}$ of positive real numbers such that

$$\sum_{p=2}^{\infty} 2^p\beta_p < \infty, \tag{1.3}$$

let

$$H_N(\boldsymbol{\sigma}) = \sum_{p=2}^{\infty}\beta_p H_{N,p}(\boldsymbol{\sigma}), \tag{1.4}$$

where for any pair of values $p \neq p'$, the Hamiltonians $H_{N,p}$, $H_{N,p'}$ are independent. Condition (1.3) is more than enough to guarantee that the above sum is a.s. finite, and the Hamiltonian $H_N$ is a.s. smooth and Morse; see Theorem 11.3.1 of [1].

In this case, we have that

$$\mathbb{E}[H_N(\boldsymbol{\sigma})H_N(\boldsymbol{\sigma}')] = N\sum_{p=2}^{\infty}\beta_p^2(R(\boldsymbol{\sigma},\boldsymbol{\sigma}'))^p = N\nu(R(\boldsymbol{\sigma},\boldsymbol{\sigma}')), \tag{1.5}$$

where

$$\nu(t) := \sum_{p=2}^{\infty}\beta_p^2 t^p. \tag{1.6}$$

We will fix the variance of $H_N$ by assuming

$$\nu(1) = \sum_{p=2}^{\infty}\beta_p^2 = 1.$$

A word of comment is needed here. By Schoenberg's theorem [12], if $\nu(R(\boldsymbol{\sigma},\boldsymbol{\sigma}'))$ is a positive-definite function for all $N$ and all $\boldsymbol{\sigma},\boldsymbol{\sigma}' \in S^{N-1}(\sqrt{N})$, then $\nu$ can be written as a linear sum as in (1.6). This remark implies that we are exhausting all possible covariances given as (1.5) that satisfy (1.3). The importance of (1.3) is to ensure smoothness of the process $H_N$.

From now on, we call the function $\nu$ a mixture. If $\nu = \beta_p^2 t^p$, for some $p \geq 2$, we call $\nu$ a pure mixture. Note that $\nu$ is smooth with

$$\nu'(1) := \nu' \neq 0, \qquad \nu''(1) := \nu'' > 0. \tag{1.7}$$

If we consider the random variable $X$ that assigns probability $\beta_p^2$ to the integer $p$, then its probability measure is given by $\mu_X = \sum \beta_p^2 \delta_p$ and

$$\mathbb{E}X = \nu' \quad \text{and} \quad \alpha^2 := \operatorname{Var} X = \nu'' + \nu' - \nu'^2. \tag{1.8}$$

A mixture is pure if and only if $\alpha = 0$. Furthermore, note that $\nu'' \geq \nu'$ with equality only in the pure case with $p = 2$. The parameters $\nu'$, $\nu''$ and $\alpha^2$ will be fundamental in our analysis.

We now introduce the main object of our study. For any open set $B \subset \mathbb{R}$ and any integer $0 \leq k < N$, we consider the (random) number $\mathrm{Crt}_{N,k}(B)$ of critical values of the function $H_N$ in the set $NB = \{Nx : x \in B\}$ with index equal to $k$,

$$\mathrm{Crt}_{N,k}(B) = \sum_{\boldsymbol{\sigma} : \nabla H_N(\boldsymbol{\sigma}) = 0} \mathbf{1}\{H_N(\boldsymbol{\sigma}) \in NB\} \mathbf{1}\{i(\nabla^2 H_N(\boldsymbol{\sigma})) = k\}. \tag{1.9}$$

Here $\nabla$, $\nabla^2$ are the gradient and the Hessian restricted to $S^{N-1}(\sqrt{N})$, and $i(\nabla^2 H_N(\boldsymbol{\sigma}))$ is the number of negative eigenvalues of the Hessian $\nabla^2 H_N$, called the index of the Hessian at $\boldsymbol{\sigma}$. We will also consider the total number $\mathrm{Crt}_N(B)$ of critical values of the function $H_N$ in the set $NB$ (whatever their index)

$$\mathrm{Crt}_N(B) = \sum_{\boldsymbol{\sigma} : \nabla H_N(\boldsymbol{\sigma}) = 0} \mathbf{1}\{H_N(\boldsymbol{\sigma}) \in NB\}. \tag{1.10}$$

Our first results will give exact and asymptotic formulas for the mean values $\mathbb{E}\mathrm{Crt}_{N,k}(B)$ and $\mathbb{E}\mathrm{Crt}_N(B)$, when $N \to \infty$ and $k$, $B$ and $\nu$ are fixed. This initial computation uses the method developed in [2], where this study was initiated for pure mixtures.

THEOREM 1.1. *For any fixed integer $k \geq 0$, there exists a continuous function $\theta_{k,\nu}(u)$, called the $k$-complexity function, explicitly given in* (2.10), *such that, for any open set $B \subseteq \mathbb{R}$,*

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k}(B) = \sup_{u \in B} \theta_{k,\nu}(u). \tag{1.11}$$

We decide to postpone to Section 2.2 the explicit expression of the $k$-complexity functions $\theta_{k,\nu}(u)$. However, we describe some important properties of these functions (see Figure 1) in the proposition below. We first fix four important thresholds that depend on $\nu$. Let

$$E'_\infty := \frac{2\nu'\sqrt{\nu''}}{\nu' + \nu''}, \qquad E_\infty := \frac{2\nu'' - \alpha^2}{\nu'\sqrt{\nu''}} \tag{1.12}$$

and

$$E_\infty^\pm := \frac{2\nu'\sqrt{\nu''} \pm \sqrt{4\nu''\nu'^2 - (\nu'' + \nu')(2(\nu'' - \nu' + \nu'^2) - \alpha^2 \log \nu''/\nu')}}{\nu' + \nu''}. \tag{1.13}$$

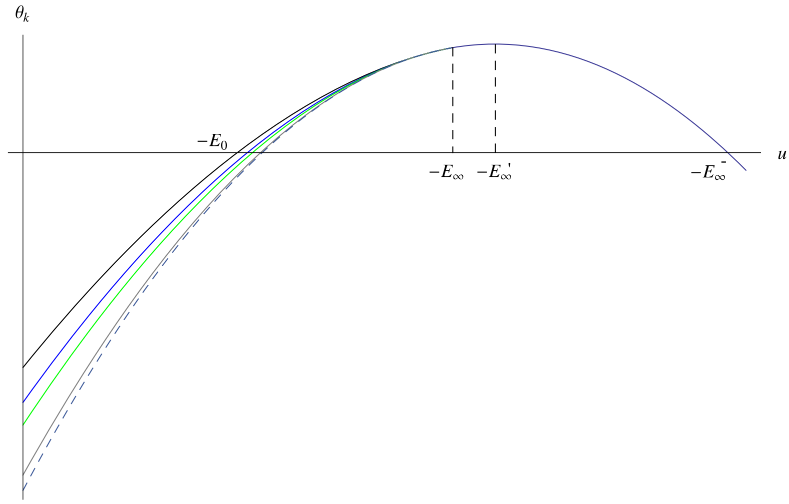


FIG. 1. *$k$-complexity functions $\theta_{k,\nu}(u)$ for $-6 \le u \le -1$, $k = 1, 2, 3, 5$ in the case where $\nu$ is pure-like, that is, $\theta_{k,\nu}(-E_\infty) > 0$. The dashed line is the continuation of the parabola that describes $\theta_{k,\nu}(u)$ in the interval $[-E_\infty, \infty)$ where they all agree.*

Note that

$$E_\infty^- \le E_\infty' \le E_\infty. \tag{1.14}$$

Furthermore, $E_\infty' = E_\infty$ if and only if $E_\infty = E_\infty^-$ if and only if $\alpha^2 = 0$; that is, any equality in (1.14) implies a triple equality. It occurs if and only if the mixture is pure; see (1.8).

PROPOSITION 1. *For any mixture $\nu$ and any $k \ge 0$, the $k$-complexity functions $\theta_{k,\nu}(u)$ satisfy the following*:

(1) *$\theta_{k,\nu}(u)$ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{-E_\infty\}$.*

(2) *$\theta_{k,\nu}(u)$ is strictly increasing on $(-\infty, -E_\infty')$ and strictly decreasing on $(-E_\infty', \infty)$. Its unique maximum is independent of $k$ and equal to*

$$\Sigma_\nu := \theta_{k,\nu}(-E_\infty') = \frac{1}{2}\log\frac{\nu''}{\nu'} - \frac{\nu'' - \nu'}{\nu'' + \nu'} > 0. \tag{1.15}$$

(3) *$\theta_{k,\nu}(u)$ has exactly two distinct zeros. The largest zero is given by $-E_\infty^-$ and therefore is independent of $k$.*

(4) *For any $k, k' \ge 0$ with $k < k'$, $\theta_{k,\nu}(u) > \theta_{k',\nu}(u)$ for all $u \in (-\infty, -E_\infty)$.*

(5) *For any $k, k' \ge 0$ with $k < k'$, $\theta_{k,\nu}(u) = \theta_{k',\nu}(u)$ for all $u \in [-E_\infty, \infty)$.*

From Theorem 1.1 and Proposition 1 we obtain:

COROLLARY 1.1. *The mean total number of critical points of index k satisfies*

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\mathrm{Crt}_{N,k}(\mathbb{R})=\Sigma_\nu. \tag{1.16}$$

*Furthermore, if* $B=(-\infty,u)$ *with* $u\le -E'_\infty$, *then*

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\mathrm{Crt}_{N,k}(-\infty,u)=\theta_{k,\nu}(u). \tag{1.17}$$

REMARK 1. By symmetry, Theorem 1.1 also holds as stated for the random variables $\mathrm{Crt}_{N,N-l}(B)$, with $l\ge 1$ fixed if one replaces $\theta_{k,\nu}(u)$ by $\theta_{k,\nu}(-u)$.

We now use Theorem 1.1 and Proposition 1 to describe the bottom landscape of $H_N$. For any integer $k\ge 0$, we introduce $E_k=E_k(\nu)>0$ as the unique solution in $(E_\infty,\infty)$ to (see Figure 1 again)

$$\theta_{k,\nu}\bigl(-E_k(\nu)\bigr)=0. \tag{1.18}$$

That is, $-E_k(\nu)$ is the smallest zero of the $k$-complexity function. It is important to note that, by items (4) and (5) of Proposition 1, the sequence $(E_k(\nu))_{k\in\mathbb{N}}$ is nonincreasing. Its structure is of extreme importance and will be also explored in Section 4. We have the following consequence of Theorem 1.1:

THEOREM 1.2. *For* $k\ge 0$ *and* $\varepsilon>0$, *let* $A_{N,k}(\varepsilon)$ *be the event "there is a critical value of* $H_N$ *below the level* $-N(E_k(\nu)+\varepsilon)$ *and with index larger or equal to k," that is,*

$$A_{N,k}(\varepsilon)=\left\{\sum_{i=k}^{\infty}\mathrm{Crt}_{N,i}\bigl((-\infty,-E_k(\nu)-\varepsilon)\bigr)>0\right\}$$

*and* $B_{N,k}(\varepsilon)$ *be the event "there is a critical value of index k of* $H_N$ *above the level* $-N(E_\infty^- -\varepsilon)$," *that is,*

$$B_{N,k}(\varepsilon)=\bigl\{\mathrm{Crt}_{N,k}\bigl((-E_\infty^-+\varepsilon,\infty)\bigr)>0\bigr\}.$$

*Then for all* $k\ge 0$ *and* $\varepsilon>0$,

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\bigl(A_{N,k}(\varepsilon)\bigr)<0\quad\textit{and}\quad\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}\bigl(B_{N,k}(\varepsilon)\bigr)<0. \tag{1.19}$$

Theorem 1.2 says that with overwhelming probability all critical values of $H_N$ of index $k$ are inside the interval $[-NE_k,-NE_\infty^-]$. A similar result was derived for the pure case in [2]. However, in the pure case it was shown (Theorem 2.2 of [2]) that the probability of finding a critical point of finite index above the level $-NE_\infty$ is asymptotically of order $\exp(-N^2C)$.

We now study the number of critical points with diverging index and the total number of critical points (regardless of index). Let $k = k(N)$ be a sequence of integers such that as $N$ goes to infinity,

$$\frac{k(N)}{N} \to \gamma \in (0, 1). \tag{1.20}$$

Let $s_\gamma \in (-\sqrt{2}, \sqrt{2})$ be defined as solution of

$$\frac{1}{\pi} \int_{-\sqrt{2}}^{-s_\gamma} \sqrt{2 - x^2}\, \mathrm{d}x = \gamma. \tag{1.21}$$

Our next result is the analogue of Theorem 1.1 for critical points of diverging index.

THEOREM 1.3. *For any sequence $k(N)$ satisfying* (1.20), *as $N$ goes to infinity,*

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k(N)}(B) \\
&\quad = \sup_{y\in B} \left\{ \frac{1}{2} \log \frac{\nu''}{\nu'} + \frac{1}{2} \left( s_\gamma^2 - \frac{2\nu''}{\alpha^2} \left( s_\gamma - \frac{\nu' y}{(2\nu'')^{1/2}} \right)^2 - y^2 \right) \right\} \\
&\quad := \sup_{y\in B} \theta_{\gamma,\nu}(u).
\end{aligned}$$

REMARK 2. From Theorem 1.3 one can easily get analogues of Theorem 1.2 and Corollary 1.1 for the case of critical points with diverging index. Its statements are adapted rewrites of the respective results. We leave this to the reader.

We also provide the complexity for the expected total number of critical values at a level of energy. Precisely, define

$$\theta_\nu(u) = \begin{cases} \theta_{0,\nu}(u) & \text{if } u \le -E'_\infty, \\ \theta_{0,\nu}(-u) & \text{if } u \ge E'_\infty, \\ \frac{1}{2}\left( \log \frac{\nu''}{\nu'} - \frac{\nu'' - \nu'}{\nu'^2 - \nu' + \nu''} u^2 \right) & \\ \qquad = \sup_{\gamma\in(0,1)} \theta_{\gamma,\nu}(u), & \text{otherwise}. \end{cases} \tag{1.22}$$

THEOREM 1.4. *The total number of critical points satisfies*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_N(B) = \sup_{u\in B} \theta_\nu(u) := \Theta_\nu(u). \tag{1.23}$$

REMARK 3. The last result can be interpreted as follows: the mean number of critical points at levels of the form $Nu + o(N)$ is asymptotically given by the mean number of local minima, local maxima or critical points of index $k(N) \sim \gamma(u)N$ if $u \le -E'_\infty$, $u \ge E'_\infty$, $-E'_\infty \le u \le E'_\infty$, respectively. Here, $\gamma(u) \in (0, 1)$ is such that $s_{\gamma(u)} = \sqrt{2}\frac{u}{E'_\infty}$; see (1.21).

We also investigate the landscape of the Hamiltonian $H_N$ by analyzing the mean Euler characteristic of level sets as $N$ goes to infinity. In order to state our results we need further notation. The Hermite functions $\phi_j$, $j \in \mathbb{N}$, are defined by

$$\phi_j(x) = \big(2^j j!\sqrt{\pi}\big)^{-1/2} h_j(x) e^{-x^2/2}, \tag{1.24}$$

where $h_j$, $j \in \mathbb{N}$ are Hermite polynomials,

$$h_j(x) = e^{x^2}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^j e^{-x^2}. \tag{1.25}$$

In particular, $h_0(x) = 1$, $h_1(x) = 2x$, $h_2(x) = 4x^2 - 2x$. The Hermite functions are orthonormal functions in $\mathbb{R}$ with respect to Lebesgue measure.

We denote by $\chi(A_u)$ the Euler characteristic of a level set

$$A_u := \big\{\boldsymbol{\sigma} \in S^{N-1}(\sqrt{N}) : H_N(\boldsymbol{\sigma}) \leq Nu\big\}.$$

$\chi(\cdot)$ is a topological invariant, integer valued function that is defined for any CW-complex as the alternating sum of Betti's numbers [16]. It is a functional that is invariant under homotopies and satisfies

$$\begin{aligned} &\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B), \qquad \chi(\mathbb{B}) = 1 \quad \text{and} \\ &\chi(S_N) = 1 + (-1)^{N-1}, \end{aligned} \tag{1.26}$$

where $\mathbb{B}$ denotes a $N$-dimensional unit ball, $S_N$ the $N$-dimensional unit sphere and $A$, $B$ are CW-complexes. $\chi(\cdot)$ roughly measures the number of connected components and its number of attached cylindrical holes and handles. Since we are only interested in Euler characteristics of level sets of functions that are almost surely Morse, we use the equivalent definition that follows from Morse's theorem (see [1], Theorem 9.3.2),

$$\chi(A_u) := \sum_{k=0}^{N-1} (-1)^k \mathrm{Crt}_k(A_u).$$

The strategy of using Rice's formula to compute Euler characteristics of level sets was developed in [1, 14, 15] and also explored in [3]. In fact, in a similar fashion, we prove the following proposition:

PROPOSITION 2.

$$\begin{aligned} &\mathbb{E}\chi(A_u) \\ &\quad = (-1)^{N-1}\left(\frac{\nu''}{\nu'}\right)^{(N-1)/2} \frac{2^{-(N-1)}N}{\sqrt{\pi}\,\Gamma(N/2)} \\ &\qquad \times \int_{-\infty}^{\infty}\int_{-\infty}^{u} h_{N-1}\left(\frac{\sqrt{N}(\nu' x - \alpha y)}{\sqrt{2\nu''}}\right) e^{-N/2(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{1.27}$$

This allows us to derive the asymptotic formula for $\mathbb{E}\chi(A_u)$ and its relation to the asymptotic complexity of the total number of critical points; see (1.23).

THEOREM 1.5. *The mean Euler–Poincaré characteristic* $\mathbb{E}\chi(A_u)$ *satisfies the following*:

(1) *If* $u \le -E'_\infty$,

$$\mathbb{E}\chi(A_u) = C(N, \nu, u) N^{-1/2} e^{N\Theta_\nu(u)} \big(1 + O\big(N^{-1}\big)\big), \tag{1.28}$$

*where* $C(N, \nu, u)$ *is a positive constant given in* (3.23).

(2) *If* $-E'_\infty < u \le 0$, *with* $u = -E'_\infty \cos\omega$, $\omega \in (0, \pi)$

$$\begin{aligned} \mathbb{E}\chi(A_u) &= (-1)^{N-1} \frac{c(N,\nu)}{2^{1/4}\pi^{1/2}N^{5/4}} \frac{e^{N\Theta_\nu(u)}}{f(\omega)(\sin\omega)^{1/2}} \sin\big[N\tau(\omega) + \rho(\omega)\big] \\ &\quad \times \big(1 + O\big(N^{-1}\big)\big), \end{aligned} \tag{1.29}$$

*where*

$$\tau(\omega) = \frac{1}{2}(\sin 2\omega - 2\omega), \qquad \rho(\omega) = -\frac{1}{2}\tau(\omega) + \frac{3\pi}{4} + \alpha(\omega),$$

$c(N, \nu)$ *is given in* (3.21) *and* $f(\omega)$, $\alpha(\omega)$ *are given in* (3.22).

(3) *If* $u > 0$, *we have* $\mathbb{E}\chi(A_u) = \mathbb{E}\chi(A_{-u})$ *for* $N$ *even and* $\mathbb{E}\chi(A_u) = 2 - \mathbb{E}\chi(A_{-u})$ *for* $N$ *odd.*

Let us now describe in words the landscape picture emerging from Theorem 1.5. Roughly speaking, Theorem 1.5 says that the mean Euler characteristic of $A_u$ is in absolute value asymptotically equal to the total number of critical points *at level* $Nu$ if $u < E_0$. This picture is fairly intuitive and easy to explain in the bottom of the landscape. As we increase the energy level $u$ from negative infinity to $-E'_\infty$, the level set $A_u$ is "essentially" a union of disjoint simply connected neighborhoods of local minima. Since these are exponentially large and dominate the total number of critical points, the mean Euler characteristic is positive and of the same size. As we cross the level $-E'_\infty$, local minima cease to dominate. The total number of critical points and the Euler characteristic (in absolute value) is given by the critical values of dominant divergent index. The landscape is then hard to visualize. By increasing a tiny amount of energy it oscillates from a large positive to a large negative Euler characteristic (and vice versa). This oscillation continues up to level $E'_\infty$. It would be of interest to find a simple and intuitive geometric reason for this large oscillation. By symmetry above $E'_\infty$ we have "essentially" covered the whole sphere minus an exponentially large number of disjoint simply connected sets.

The rest of the paper is organized as follows. In Section 2 we prove all Theorems about the complexity function. Their proofs follow the same strategy of [2]. Namely, they will follow from an exact formula for the mean number of critical points of index $k$ that translates the problem to a Random Matrix Theory question. This formula is more involved than the pure case since in a mixture the Hessian matrix gains an independent Gaussian component on the diagonal. This leads to a different variational principle that we analyze. In Section 3 we prove the results related to the Euler's characteristic. In Sections 4 and 5 we explain our interest in such functions, and we relate $H_N$ to Hamiltonians of classical models in statistical physics.

## 2. Complexity of critical points.

2.1. *Main identity.* In this section, we introduce the main identity that relates the mean number of critical points of index $k$ with the $k$th smallest eigenvalue of the Gaussian orthogonal ensemble. This identity, given in Proposition 3, is the analogous of Theorem 2.1 of [2] and it is the first step of the proofs of Theorems 1.1, 1.2, 1.4 and Proposition 5.

We fix our notation for the Gaussian orthogonal ensemble (GOE). The GOE is a probability measure on the space of real symmetric matrices. Namely, it is the probability distribution of the $N \times N$ real symmetric random matrix $M^N$, whose entries $(M_{ij}, i \le j)$ are independent centered Gaussian random variables with variance

$$\mathbb{E} M_{ij}^2 = \frac{1 + \delta_{ij}}{2N}. \tag{2.1}$$

We will denote by $\mathbb{E}_{\mathrm{GOE}}^N$ the expectation under the GOE ensemble of size $N \times N$.

Let $\lambda_0^N \le \lambda_1^N \le \cdots \le \lambda_{N-1}^N$ be the ordered eigenvalues of $M^N$.

PROPOSITION 3. *The following identity holds for all $N, \nu, k \in \{0, \ldots, N-1\}$, and for all open sets $B \subset \mathbb{R}$:*

$$\begin{aligned} &\mathbb{E}[\mathrm{Crt}_{N,k}(B)] \\ &\quad = C(N, \nu', \nu'') \\ &\qquad \times \int_B \mathbb{E}_{\mathrm{GOE}}^N \left[ \exp\left\{ \frac{N}{2} \left( (\lambda_k^N)^2 - y^2 - \frac{2\nu''}{\alpha^2} \left( \lambda_k^N - \frac{\nu' y}{(2\nu'')^{1/2}} \right)^2 \right) \right\} \right] \mathrm{d}y, \end{aligned} \tag{2.2}$$

*where* $C(N, \nu', \nu'') = 2\sqrt{\frac{2\nu'' N}{\nu' \pi \alpha^2}} (\frac{\nu''}{\nu'})^{N/2} \frac{\nu'}{\sqrt{2\nu''}}$.

PROOF. Proof of Proposition 3 is a rewrite of the proof of Theorem 2.1 of [2] with one subtle difference: the law of the Hessian in the mixed case gains an independent Gaussian component on its diagonal. In this proof, we use $H$ to denote $H_N$.

The hypothesis on $\nu$ allows us to apply Rice's formula, in the form of Lemma 3.1 of [2]. It says that using $\mathrm{d}\boldsymbol{\sigma}$ to denote the usual surface measure on $S^{N-1}(\sqrt{N})$,

$$\begin{aligned}\mathbb{E}\mathrm{Crt}_{N,k}(B) \\ = \int_{S^{N-1}(\sqrt{N})} \mathbb{E}\bigl[\bigl|\det \nabla^2 H(\boldsymbol{\sigma})\bigr| \mathbf{1}\bigl\{H(\boldsymbol{\sigma}) \in NB, i\bigl(\nabla^2 H(\boldsymbol{\sigma})\bigr) = k\bigr\} \bigm| \\ \nabla H(\boldsymbol{\sigma}) = 0\bigr] \phi_{\boldsymbol{\sigma}}(0)\, \mathrm{d}\boldsymbol{\sigma},\end{aligned} \tag{2.3}$$

where $\phi_{\boldsymbol{\sigma}}$ is the density of the gradient vector of $H$.

Now, since $H$ is invariant under rotations, to compute the above expectation it is enough to study the joint distribution of $(H, \nabla H, \nabla^2 H)$ at the north pole $\mathbf{n}$. We fix a orthogonal base for the tangent plane at the north pole, and we consider $\nabla H(\mathbf{n})$, $\nabla^2 H(\mathbf{n})$ with respect to that base. Denoting subscript by a derivative according to a orthonormal basis in $T_\sigma S^{N-1}(\sqrt{N})$ we have that

LEMMA 1. *For all* $1 \le i \le j \le N-1$,

$$\mathbb{E}\bigl[H(\mathbf{n})^2\bigr] = N, \qquad \mathbb{E}\bigl[H(\mathbf{n})H_i(\mathbf{n})\bigr] = \mathbb{E}\bigl[H_i(\mathbf{n})H_{jk}(\mathbf{n})\bigr] = 0,$$

$$\mathbb{E}\bigl[H(\mathbf{n})H_{ij}(\mathbf{n})\bigr] = -\nu'\delta_{ij}, \qquad \mathbb{E}\bigl[H_i(\mathbf{n})H_j(\mathbf{n})\bigr] = \nu'\delta_{ij}$$

*and*

$$\mathbb{E}\bigl[H_{ij}(\mathbf{n})H_{kl}(\mathbf{n})\bigr] = \frac{1}{N}\bigl[\nu''(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \bigl(\nu'' + \nu'\bigr)\delta_{ij}\delta_{kl}\bigr].$$

*Furthermore, under the conditional distribution* $\mathbb{P}[\cdot|H(\mathbf{n}) = x]$ *the random variables* $H_{ij}(\mathbf{n})$ *are Gaussian variables with*

$$\mathbb{E}\bigl[H_{ij}(\mathbf{n})\bigr] = -\frac{x}{N}\nu'\delta_{ij}$$

*and*

$$\mathbb{E}\bigl[H_{ij}(\mathbf{n})H_{kl}(\mathbf{n})\bigr] = \frac{1}{N}\bigl[\nu''(1 + \delta_{ij})\delta_{ik}\delta_{jl} + \alpha^2\delta_{ij}\delta_{kl}\bigr],$$

*that is, if* $M^{N-1}$ *is distributed as a* $(N-1) \times (N-1)$ *GOE matrix*

$$\begin{aligned}\mathbb{E}\bigl[\nabla^2 H \,|\, H(\mathbf{n})\bigr] &\stackrel{d}{=} \left(\frac{N-1}{N} 2\nu''\right)^{1/2} M^{N-1} \\ &\quad + \frac{1}{\sqrt{N}}\left(\alpha Z - \frac{1}{\sqrt{N}}\nu' H(\mathbf{n})\right) I,\end{aligned}$$

*where* $Z$ *is an independent standard Gaussian.*

The above lemma implies that (2.3) can be rewritten as

$$
\begin{aligned}
\mathbb{E}\mathrm{Crt}_{N,k}(B) \\
&= \omega_N \mathbb{E}\biggl[\mathbb{E}\biggl[\biggl|\det\biggl(\biggl(\frac{N-1}{N}2\nu''\biggr)^{1/2} M^{N-1} + \frac{1}{N}(\sqrt{N}\alpha Z - \nu' H(\mathbf{n}))I\biggr)\biggr| \\
&\qquad \times \mathbf{1}\biggl\{i\biggl[\biggl(\frac{N-1}{N}2\nu''\biggr)^{1/2} M^{N-1} + \biggl(\alpha\frac{Z}{\sqrt{N}} - \nu'\frac{H(\mathbf{n})}{N}\biggr)I\biggr] = k\biggr\} \\
&\qquad \times \mathbf{1}\{H(\mathbf{n}) \in NB\} | H(\mathbf{n})\biggr]\biggr] \\
&\quad \times \phi_{\mathbf{n}}(\mathbf{n}),
\end{aligned} \tag{2.4}
$$

where $\omega_N$, the volume of the sphere $S^{N-1}(\sqrt{N})$ and $\phi_{\mathbf{n}}(\mathbf{n})$ are given by

$$
\omega_N = (\sqrt{N})^{N-1}\frac{2\pi^{N/2}}{\Gamma(N/2)}, \qquad \phi_{\mathbf{n}}(\mathbf{n}) = (2\pi\nu')^{-(N-1)/2}. \tag{2.5}
$$

Since we can assume $\alpha \neq 0$ (the case $\alpha = 0$, that is, the pure p-spin was treated in [2]), we can rewrite the conditional expectation in (2.4) as

$$
\begin{aligned}
&\frac{\sqrt{N}}{\sqrt{2\pi}}\biggl(2\nu''\frac{N-1}{N}\biggr)^{(N-1)/2} \\
&\quad \times \int_B e^{-Ny^2/2}\mathbb{E}|\det(M^{N-1} - X(y))I|\mathbf{1}\{i[M^{N-1} - X(y)I] = k\}\,\mathrm{d}y,
\end{aligned} \tag{2.6}
$$

where $X(y)$ is a Gaussian random variable with mean $m = \frac{\sqrt{N}\nu' y}{(2\nu''(N-1))^{1/2}}$ and variance $t^2 = \frac{\alpha^2}{2\nu''(N-1)}$. Hence, we can apply Lemma 3.3 of [2] with $G = \mathbb{R}$ to get that (2.6) is equal to

$$
\begin{aligned}
&\frac{\Gamma(N/2)((N-1)/N)^{-N/2}}{\sqrt{\pi t^2}} \\
&\quad \times \int_B \mathbb{E}^N_{\mathrm{GOE}}\biggl[\exp\biggl\{\frac{N}{2}\biggl((\lambda^N_k)^2 - y^2 - \frac{2\nu''}{\alpha^2}\biggl(\lambda^N_k - \frac{\nu' y}{(2\nu'')^{1/2}}\biggr)^2\biggr)\biggr\}\,\mathrm{d}y.
\end{aligned} \tag{2.7}
$$

Putting (2.4), (2.5) and (2.7) together, we end the proof of Proposition 3. □

2.2. *Proof of Theorems* 1.1, 1.2, 1.3 *and* 1.4.

2.2.1. *Proving Theorem* 1.1 *and Proposition* 1. In this subsection, we will compute the logarithm asymptotics of the left-hand side of (2.2).

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$
F(\lambda, y) = \frac{1}{2}\biggl(-\frac{\nu'' + \nu'}{\nu'' + \nu' - \nu'^2}y^2 + \frac{2\sqrt{2}\sqrt{\nu''}\nu'}{\nu'' + \nu' - \nu'^2}\lambda y - \frac{\nu'' - \nu' + \nu'^2}{\nu'' + \nu' - \nu'^2}\lambda^2\biggr). \tag{2.8}
$$

Note that $F(\lambda, y) = -ay^2 + by\lambda - c\lambda^2$ for some constants $a, b, c > 0$. Let

$$I_1(x) = \int_{\sqrt{2}}^{x} \sqrt{z^2 - 2}\,\mathrm{d}z = \frac{1}{2}\big(x\sqrt{x^2-2} + \log[2] - 2\log\big[\big(x + \sqrt{x^2-2}\big)\big]\big). \tag{2.9}$$

For any $k \in \mathbb{N}$ fixed, let

$$\theta_{k,\nu}(u) = \begin{cases} \frac{1}{2}\log\frac{\nu''}{\nu'} + F(-\sqrt{2}, u), & \text{if } -E_\infty \le u, \\ \frac{1}{2}\log\frac{\nu''}{\nu'} + F\big(\lambda_k^*[u], u\big) - (k+1)I_1\big(\big|\lambda_k^*[u]\big|\big), & \\ & \text{if } u \le -E_\infty, \end{cases} \tag{2.10}$$

where $\frac{\nu'\sqrt{2\nu''}u}{\nu''-\nu'+\nu'^2} < \lambda_k^*[u] \le -\sqrt{2}$ is given by

$$\Psi'\big(\lambda_k^*[u]\big) = 0, \qquad \Psi(x) = \frac{2\nu'\sqrt{2\nu''}}{\alpha^2}ux - \frac{\nu''-\nu'+\nu'^2}{\alpha^2}x^2 - 2(k+1)I_1\big(|x|\big),$$

that is, $\lambda_k^*[u]$ is a solution on $(-\infty, -\sqrt{2}]$ of

$$\frac{\nu'\sqrt{2\nu''}}{\alpha^2}u - \frac{\nu''-\nu'+\nu'^2}{\alpha^2}\lambda_k^*[u] + (k+1)\sqrt{\big(\lambda_k^*[u]\big)^2 - 2} = 0. \tag{2.11}$$

Our goal in this section is to prove that $\theta_{k,\nu}$ is the $k$-complexity function. When $k = 0$ the formula for $\theta_{0,\nu}$ simplifies as follows.

PROPOSITION 4. *For all $u \in \mathbb{R}$,*

$$\theta_{0,\nu}(u) = \begin{cases} \frac{1}{2}\Big(\log\Big[\frac{\nu''}{\nu'}\Big] - \frac{u^2(\nu'+\nu'')}{\nu'-\nu'^2+\nu''} + \frac{4u\nu'\sqrt{\nu''}}{\nu'-\nu'^2+\nu''} \\ \qquad\qquad - \frac{2(-\nu'+\nu'^2+\nu'')}{\nu'-\nu'^2+\nu''}\Big), \\ \quad \text{if } -E_\infty \le u, \\ \frac{1}{2}\log[\nu'-1] - \frac{u^2(\nu'-2)}{4(\nu'-1)} - I_1\Big(-\frac{u\nu'}{\sqrt{2}\sqrt{\nu'(\nu'-1)}}\Big), \\ \quad \text{if } u \le -E_\infty. \end{cases} \tag{2.12}$$

REMARK 4. It is possible to recover all complexity functions of the pure case by taking $\alpha$ to zero (i.e., recover the first results of [2]). In particular, if $\alpha = 0$, $E'_\infty = E_\infty$, and we do not have the intermediate regions where the $k$-complexity functions are equal for different $k$ and nonconstant.

We postpone the proof of Proposition 4 to the end of this subsection since we will need another characterization of $\theta_{k,\nu}$.

PROOF OF THEOREM 1.1. To prove Theorem 1.1 it suffices to show that $\theta_{k,\nu}(u)$ is the logarithm asymptotic limit of the left-hand side of (2.2).

First, note that we can rewrite (2.2) as

$$C_N \mathbb{E} e^{-N\Lambda(\lambda_k^N, Y_N)} \mathbf{1}\{Y_N \in B\}, \tag{2.13}$$

where $Y_N$ is a Gaussian random variable of mean zero and variance $N$ independent of $\lambda_k^N$, $\mathbb{E}$ is the expectation with respect to GOE and $Y_N$ and

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \log C_N = \frac{1}{2} \log \frac{\nu''}{\nu'}, \\ &\Lambda(\lambda, y) = F(\lambda, y) + \frac{y^2}{2} = \frac{1}{2}\left(\lambda^2 - \frac{2\nu''}{\alpha^2}\left(\lambda - \frac{\nu' y}{(2\nu'')^{1/2}}\right)^2\right). \end{aligned} \tag{2.14}$$

By the independence of $Y_N$ and $\lambda_k^N$ and Theorem A.1 of [2], the sequence of random variables $(\lambda_k^N, Y_N)$ satisfies a large deviation principle of speed $N$ and rate function

$$I_k(\lambda, x) = \begin{cases} \dfrac{x^2}{2} + (k+1) I_1(|\lambda|), & \text{if } \lambda \le -\sqrt{2}, \\ \infty, & \text{otherwise.} \end{cases}$$

Therefore, in view of (2.13) and (2.14), we can apply Laplace–Varadhan lemma (see, e.g., [8], Theorem 4.3.1 and Exercise 4.3.11) and get that

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k}(B) \\ &\quad = \frac{1}{2}\left[\log \frac{\nu''}{\nu'} + \max_{x\in B, \lambda \le -\sqrt{2}} \left\{\lambda^2 - \frac{1}{\alpha^2}\left(\nu' x - \sqrt{2\nu''}\lambda\right)^2 - 2 I_k(\lambda, x)\right\}\right]. \end{aligned} \tag{2.15}$$

We will now analyze the above variational principle. We start with the case of $B = (-\infty, u)$. We want to find

$$\max_{x \le u, \lambda \le -\sqrt{2}} \left\{-x^2 + \lambda^2 - \frac{1}{\alpha^2}\left(\nu' x - \sqrt{2\nu''}\lambda\right)^2 - 2(k+1) I_1(|\lambda|)\right\}. \tag{2.16}$$

*Case* $u \ge -E'_\infty$: If $u \ge -E'_\infty$, then we maximize (2.16) in $x$ first. The maximum is obtained at $x = x_\lambda := \frac{\nu'\sqrt{2\nu''}}{\nu''+\nu'}\lambda \le u$. Plugging $x_\lambda$ back in (2.16), we get an increasing function in $\lambda$, since $I_1(|\lambda|)$ is itself decreasing. Thus the maximum is realized at

$$x = x_\lambda, \qquad \lambda = -\sqrt{2}.$$

This together with (2.15) proves Theorem 1.1 in the case $B = (-\infty, u)$ with $-E'_\infty \le u$.

*Case* $u \le -E'_\infty$: In the case $u \le -E'_\infty$, $x_\lambda \le u$ if and only if $\lambda \le \frac{\sqrt{2}u}{E'_\infty}$. Therefore if $x^*$ maximizes (2.16), then

$$x^* = x_\lambda \Leftrightarrow \lambda \le \frac{\sqrt{2}u}{E'_\infty} \quad \text{and} \quad x^* = u \Leftrightarrow \frac{\sqrt{2}u}{E'_\infty} \le \lambda \le -\sqrt{2}. \tag{2.17}$$

If we plug in the correspondent values of $x$ in each region, we note that in the first case our function is again increasing in $\lambda$. Furthermore, since at $\lambda = \frac{\sqrt{2}u}{E'_\infty}$, $x_\lambda = u$, we are led to the following variational principle valid in both cases of (2.17):

$$\begin{aligned}
&\max_{\sqrt{2}u/E'_\infty \le \lambda \le -\sqrt{2}} \left\{ -u^2 + \lambda^2 - \frac{1}{\alpha^2}\big(\nu' u - \sqrt{2\nu''}\lambda\big)^2 - 2(k+1)I_1\big(|\lambda|\big) \right\} \\
&\quad = -\left(1 + \frac{\nu'^2}{\alpha^2}\right)u^2 + \max_{\sqrt{2}u/E'_\infty \le \lambda \le -\sqrt{2}} \left\{ \frac{2\nu'\sqrt{2\nu''}}{\alpha^2} u\lambda - \frac{\nu'' - \nu' + \nu'^2}{\alpha^2}\lambda^2 \right. \\
&\qquad\qquad \left. - 2(k+1)I_1\big(|\lambda|\big) \right\} \\
&\quad = -\left(1 + \frac{\nu'^2}{\alpha^2}\right)u^2 + \max_{\sqrt{2}u/E'_\infty \le \lambda \le -\sqrt{2}} \Psi(\lambda) = \max_{\sqrt{2}u/E'_\infty \le \lambda \le -\sqrt{2}} \Gamma(\lambda).
\end{aligned} \tag{2.18}$$

Note that $\Psi(\lambda)$ is a parabola $a\lambda^2 + b\lambda$, $a < 0$ plus an increasing function. The critical point of the parabola is given by

$$\lambda_c = \frac{\nu'\sqrt{2\nu''}u}{\nu'' - \nu' + \nu'^2} \ge -\sqrt{2} \quad \Longleftrightarrow \quad u \ge -E_\infty. \tag{2.19}$$

Therefore if $u \ge -E_\infty$, $\Psi$ is an increasing function in $\lambda$, so its maximum is attained at $\lambda = -\sqrt{2}$. This proves the theorem in the region $-E_\infty \le u \le -E'_\infty$.

If $u < -E_\infty$, equation (2.19) and the facts that $\Psi'(-\sqrt{2}) < 0$ and $\Psi'(\lambda_c) > 0$ imply that the maximum is taken in the interior of the interval $[\lambda_c, -\sqrt{2}]$ at $\lambda_k^*[u]$. This completes the proof of the theorem in the case $B = (-\infty, u)$.

Now, it is easy to extend it to any open set $B$. Let $u^*$ be the point that realizes the $\sup_{\{u \in B\}} \theta_{k,\nu}(u)$. From the continuity and uniqueness of a local maxima of $\theta_{k,\nu}$, it is clear that either $u^* = -E'_\infty$ or $u^*$ is in the boundary of $B$. Assume without loss of generality that there exists an increasing sequence $u_n$ in $B$ approaching $u^*$. Since $B$ is open, there exist $\varepsilon_n > 0$ such that

$$\begin{aligned}
\mathbb{E}\big(\mathrm{Crt}_{N,k}(-\infty, u_n) - \mathrm{Crt}_{N,k}(-\infty, u_n - \varepsilon_n)\big) &= \mathbb{E}\mathrm{Crt}_{N,k}(u_n - \varepsilon_n, u_n) \\
&\le \mathbb{E}\mathrm{Crt}_{N,k}(B) \\
&\le \mathbb{E}\mathrm{Crt}_{N,k}\big(-\infty, u^*\big).
\end{aligned}$$

But since $\theta_{k,\nu}$ is continuous and increasing for $u \le -E'_\infty$, the above equation implies

$$\theta_{k,\nu}(u_n) \le \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k}(B) \le \theta_{k,\nu}\big(u^*\big)$$

for all $n$, which proves Theorem 1.1 for any $B$ open. □

It remains to prove Proposition 4. We first need the following miraculous lemma.

LEMMA 2. *For all* $u < -E_\infty$,

$$\frac{\partial}{\partial \nu''}\theta_{0,\nu}(u) = 0.$$

PROOF. The proof relies on how we derived $\theta_{0,\nu}(u)$. When $u < -E_\infty$, $\theta_{0,\nu}(u)$ is the maximum over $\lambda$ of the functional $\Gamma$ (that depends on $\nu''$) given in (2.18). Its maximizer $\lambda^*(u)$ is the smallest root of a second degree polynomial that can be derived from (2.11). This second degree equation is given by $A + B\lambda + C\lambda^2 = 0$ where

$$\begin{aligned} A &= 2 + \frac{2u^2\nu'^2\nu''}{(\nu' - \nu'^2 + \nu'')^2}, \\ B &= -\frac{2\sqrt{2}u\nu'\sqrt{\nu''}((-1+\nu')\nu' + \nu'')}{(\nu' - \nu'^2 + \nu'')^2}, \\ C &= \frac{2((-1+\nu')^2\nu'^2 + \nu''^2)}{(\nu' - \nu'^2 + \nu'')^2}. \end{aligned} \tag{2.20}$$

Now the chain rule and the fact that $\lambda^*(u)$ is a maximum imply that $\frac{\partial}{\partial\nu''}\theta_{0,\nu}(u) = 0$ if and only if $\frac{\partial}{\partial\nu''}(\Gamma(\lambda^*(u))) = 0$, and this holds if and only if $(\frac{\partial}{\partial\nu''}\Gamma)(\lambda^*(u)) = 0$. The last condition can be written as a second degree equation of the form

$$\begin{aligned} &\frac{1}{2}\biggl(-\frac{u^2(-\nu' - \nu'')}{(\nu' - \nu'^2 + \nu'')^2} - \frac{u^2}{\nu' - \nu'^2 + \nu''} - \frac{2\sqrt{2}u\nu'\sqrt{\nu''}\lambda}{(\nu' - \nu'^2 + \nu'')^2} \\ &\qquad + \frac{\sqrt{2}u\nu'\lambda}{\sqrt{\nu''}(\nu' - \nu'^2 + \nu'')}\biggr) \\ &\quad + \frac{1}{2}\biggl(-\frac{\lambda^2}{\nu' - \nu'^2 + \nu''} + \frac{(-\nu' + \nu'^2 + \nu'')\lambda^2}{(\nu' - \nu'^2 + \nu'')^2}\biggr) + \frac{1}{2\nu''} = 0. \end{aligned} \tag{2.21}$$

Comparing the coefficients of (2.20) with (2.21) one sees that their ratios are constantly equal to $\frac{1}{4\nu''}$. This immediately implies that they share the same roots. So $\lambda^*(u)$ indeed satisfies $(\frac{\partial}{\partial\nu''}\Gamma)(\lambda^*(u)) = 0$, and the lemma is proven. □

PROOF OF PROPOSITION 4. From Lemma 2 we know that for $u < -E_\infty$, $\theta_{k,\nu}$ does not depend on $\nu''$. By choosing $\nu'' = \nu'^2 - \nu' + \varepsilon$ and taking $\varepsilon$ to zero we get the desired result. Indeed, when $\varepsilon$ goes to zero

$$\lambda^*(u) \to \frac{u\nu'}{\sqrt{2}\sqrt{(\nu'-1)\nu'}}, \qquad F\bigl(\lambda^*(u), u\bigr) \to \frac{-u^2(\nu'-2)}{4(\nu'-1)}.$$ □

2.2.2. *Proof of Theorem* 1.2. We want to prove that there are no critical values of index $k$ of $H_N$ above $-N(E_\infty^- - \varepsilon)$. The function $\theta_{k,\nu}$ is strictly decreasing on $(-E_\infty^-, \infty)$. Using Theorem 1.1, we have

$$\mathbb{E}[\mathrm{Crt}_{N,k}((-E_\infty^- + \varepsilon, \infty))] \le \exp\{N\theta_{k,\nu}(-E_\infty^- + \varepsilon) + o(N)\}.$$

The constant $-E_\infty^-$ is defined by $\theta_{k,\nu}(-E_\infty^-) = 0$ for all $k$. Therefore, $\theta_{k,p}(-E_k + \varepsilon) = c(k, \nu, \varepsilon) < 0$. An application of Markov's inequality as

$$\mathbb{P}(B_{N,k}(\varepsilon)) \le \mathbb{E}[\mathrm{Crt}_{N,k}(-E_\infty^- + \varepsilon, \infty)] \le e^{-Nc(k,\nu,\varepsilon)}$$

proves Theorem 1.2 for the event $B_{N,k}(\varepsilon)$. The proof for the event $A_{N,k}(\varepsilon)$ is analogous.

2.2.3. *Proof of Theorem* 1.3. The proof of Theorem 1.3 follows the same steps as the proof of Theorem 1.1. First by Lemma 3.5 of [2], for any $\varepsilon > 0$, there exists a constant $c = c(\gamma, \varepsilon) > 0$ such that

$$\mathbb{P}(|\lambda_k^N - s_\gamma| > \varepsilon) \le e^{-cN^2}.$$

Therefore if we use Proposition 3, (2.14) and the above statement we have that for any $\varepsilon > 0, \delta > 0$ there exists constants $c = c(\varepsilon), d = d(\varepsilon)$ such that for $N$ large enough

$$\begin{aligned}\mathbb{E}\mathrm{Crt}_{N,k}(B) &\le C_N \int_B e^{N/2(F(\lambda_k^N, y))} \mathbf{1}\{\lambda_k^N \in (s_\gamma - \varepsilon, s_\gamma + \varepsilon)\} + e^{dN} e^{-cN^2} \\ &\le C_N \int_B e^{N/2 \sup_{\lambda \in (s_\gamma - \varepsilon, s_\gamma + \varepsilon)}\{F(\lambda, y)\}} \,\mathrm{d}y + e^{dN} e^{-cN^2} \\ &\le C_N e^{N/2 \sup_{\lambda \in (s_\gamma - \varepsilon, s_\gamma + \varepsilon), y \in B}\{F(\lambda, y)\}} (1 + \delta) + e^{dN} e^{-cN^2}.\end{aligned}$$

On the other hand we have the lower bound

$$\begin{aligned}\mathbb{E}\mathrm{Crt}_{N,k}(B) &\ge C_N \int_B e^{N/2(F(\lambda_k^N, y))} \mathbf{1}\{\lambda_k^N \in (s_\gamma - \varepsilon, s_\gamma + \varepsilon)\} \\ &\ge C_N \int_B e^{N/2 \inf_{\lambda \in (s_\gamma - \varepsilon, s_\gamma + \varepsilon)}\{F(\lambda, y)\}} \,\mathrm{d}y \\ &\ge C_N e^{N/2 \inf_{\lambda \in (s_\gamma - \varepsilon, s_\gamma + \varepsilon)}\{\sup_{y \in B}\{F(\lambda, y)\}\}} (1 - \delta).\end{aligned}$$

Taking $\frac{1}{N}$ log on both bounds and taking $\varepsilon$ to zero afterward, we see that

$$\frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k}(B) = \sup_{y \in B}\{F(s_\gamma, y)\}.$$

2.2.4. *Proof of Theorem* 1.4. We now prove the asymptotic limit of the mean number of critical points at some level of energy.

Since the total number of critical points is greater than the number of critical points of index $k(N)$ with $k(N)$ satisfying (1.20) for $\gamma \in [0, 1]$ we clearly have the lower bound

$$\sup_{\gamma\in[0,1]}\sup_{u\in B}\theta_{\gamma,\nu}(u) \le \lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\mathrm{Crt}_N(B). \tag{2.22}$$

For $u \le -E'_\infty$, taking $\gamma = 0$ (i.e., considering the complexity of local minima) we get the right-hand side of (1.23). For $u \in (-E'_\infty, E'_\infty)$ the supremum on $\gamma$ of $\theta_{\gamma,\nu}(u)$ is attained at $\gamma \in (0, 1)$ such that $s_\gamma = \frac{\sqrt{2}u}{E'_\infty}$. Plugging this value back on the left-hand side of (2.22), we get the right-hand side of (1.23). Last, for $u \ge E_\infty$, one just needs to take the complexity of local maxima. This is enough to prove a lower bound.

To show a matching upper bound, we proceed as follows. A sum over $k$ in Proposition 3 gives us that

$$\begin{aligned}&\mathbb{E}[\mathrm{Crt}_N(B)]\\&\quad = 2N\sqrt{\frac{2}{\nu'}}\left(\frac{\nu''}{\nu'}\right)^{N/2}\int_B \mathbb{E}^N_{\mathrm{GOE}}\int \exp\{NF(z,y)\}\,\mathrm{d}y L_N(\mathrm{d}z),\end{aligned}$$

and $L_N$ is the empirical spectral measure of the GOE matrix. The constant in front the integral gives a constant term $C_\nu$ after the $\frac{1}{N}$ log limit. Furthermore,

$$\begin{aligned}&\int_B \mathbb{E}^N_{\mathrm{GOE}}\int \exp\{NF(z,y)\}\,\mathrm{d}y L_N(\mathrm{d}z)\\&\qquad \le N\int_B \sup_{z\in\mathbb{R}}\exp\{NF(z,y)\}\,\mathrm{d}y\\&\qquad \le N\int_B e^{-(N/2)(\nu''-\nu')/(\nu'^2-\nu'+\nu'')y^2}\,\mathrm{d}y.\end{aligned} \tag{2.23}$$

So if $B \cap (-E'_\infty, E'_\infty) \neq \varnothing$, this matches the right-hand side of (1.23). If $B \subseteq (-\infty, -E'_\infty)$, then we can estimate (2.23) with

$$N\int_B \mathbb{E}^N_{\mathrm{GOE}}\int \exp\{NF(\lambda_0, y)\}.$$

Applying log, dividing by $N$ and taking limits we get Theorem 1.4 from Theorem 1.1.

**3. Proof of Proposition 2 and Theorem 1.5.** In this section we prove Proposition 2 and Theorem 1.5.

PROOF OF PROPOSITION 2. We start with the following identity:

$$
\begin{aligned}
\mathbb{E}\chi(A_u) &= \sum_{k=0}^{N-1}(-1)^k \mathrm{Crt}_k\big(A(u)\big) \\
&= \sum_{k=0}^{N-1}(-1)^k \int_{S^{N-1}(\sqrt{N})} \mathbb{E}\big(\big|\det\nabla^2 H_N(\sigma)\big|\mathbf{1}_{\{i(\nabla^2 H_N(\sigma))=k\}}\mathbf{1}_{\{H_N(\sigma)\le Nu\}}\big| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \nabla H_N(\sigma)=0\big) \\
&\qquad\qquad \times \phi_{\nabla H_N}(0)\,\mathrm{d}\sigma \\
&= \big(2\nu'\pi\big)^{-(N-1)/2}\big|S^{N-1}(\sqrt{N})\big|\frac{1}{\sqrt{2\pi N}} \\
&\quad \times \sum_{k=0}^{N-1}\int_{-\infty}^{Nu}\mathbb{E}\big((-1)^k\big|\det\nabla^2 H_N(\sigma)\big|\mathbf{1}_{\{i(\nabla^2 H_N(\sigma))=k\}}\big| H_N(\sigma)=x\big) \\
&\qquad\qquad \times e^{-(1/2N)x^2}\,\mathrm{d}x \\
&= \big(2\nu'\pi\big)^{-(N-1)/2}\frac{2\pi^{N/2}}{\Gamma(N/2)}N^{(N-1)/2} \\
&\quad \times \frac{\sqrt{N}}{\sqrt{2\pi}}\int_{-\infty}^{u}\mathbb{E}\big(\det\nabla^2 H_N(\sigma)\big| H_N(\sigma)=Nx\big)e^{-(N/2)x^2}\,\mathrm{d}x \\
&= \nu'^{-(N-1)/2}2^{-(N-2)/2} \\
&\quad \times \frac{N^{N/2}}{\Gamma(N/2)}\int_{-\infty}^{u}\mathbb{E}\big(\det\nabla^2 H_N(\sigma)\big| H_N(\sigma)=Nx\big)e^{-(N/2)x^2}\,\mathrm{d}x.
\end{aligned}
$$

LEMMA 3. *If $M^N$ is a $N\times N$ GOE with variance $\mathbb{E}M_{ij}^2=\frac{1+\delta_{ij}}{2N}$, then for any $x\in\mathbb{R}$*

$$\mathbb{E}\det\big(M^N-xI\big)=2^{-N}N^{-N/2}(-1)^N h_N(\sqrt{N}x),$$

*where $h_N(x)$ is given in* (1.25).

PROOF. The proof, a straight-forward linear algebra exercise, can be found as Corollary 11.6.3 in [1]. □

Now by Lemma 1,

$$
\begin{aligned}
&\mathbb{E}\chi(A_u) = \nu'^{-(N-1)/2}2^{-(N-2)/2}\frac{N^{N/2}}{\Gamma(N/2)}\frac{\sqrt{N}}{\sqrt{2\pi}} \\
&\qquad \times \int_{-\infty}^{\infty}\int_{-\infty}^{u}\mathbb{E}\Big(\det\Big[\Big(\frac{N-1}{N}2\nu''\Big)^{1/2}M^{N-1}+\big(\alpha y-\nu' x\big)I\Big]\Big) \qquad (3.1)\\
&\qquad\qquad \times e^{-(N/2)x^2}e^{-(N/2)y^2}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

The double integral becomes

$$\left(\frac{N-1}{N}2\nu''\right)^{(N-1)/2} \times \int_{-\infty}^{\infty}\int_{-\infty}^{u} \mathbb{E}\left(\det\left[M^{N-1} + \left(\frac{N-1}{N}2\nu''\right)^{-1/2}(\alpha y - \nu' x)I\right]\right) \times e^{-N/2(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y,$$

which by Lemma 3 can be rewritten as

$$(-1)^{N-1}\left(\frac{\nu''}{2N}\right)^{(N-1)/2}\int_{-\infty}^{\infty}\int_{-\infty}^{u} h_{N-1}\left(\frac{\sqrt{N}(\nu' x - \alpha y)}{\sqrt{2\nu''}}\right) \times e^{-N/2(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.2}$$

Combining (3.1) and (3.2) we get Proposition 2. $\square$

We will need the following lemma to prove Theorem 1.5:

LEMMA 4. *Let $a, b$ be constants such that $a > 1/2$ and $b \geq 0$. Set*

$$I_N(M) = \int_M^{\infty} \phi_{N-1}(\sqrt{N}x)e^{ax^2+bx}\,\mathrm{d}x.$$

*As $N$ goes to infinity*:

(1) *If $\sqrt{2} \leq M$, then $I_N(M) = O(e^{-N(aM^2+bM+I_1(M))})$.*
(2) *If $-\sqrt{2} < M < \sqrt{2}$ and if we set $M = \sqrt{2}\cos\omega$ with $\varepsilon < \omega < \pi - \varepsilon$, then $I_N(M)$ is equal to*

$$\frac{2^{-3/4}\pi^{-1/2}e^{-N(aM^2+bM)}}{N^{5/4}|m'(2\iota(M))|(\sin\omega)^{1/2}}\sin\left[\left(\frac{N}{2}-\frac{1}{4}\right)(\sin 2\omega - 2\omega) + \frac{3\pi}{4} + \alpha(M)\right] \times \left(1 + O\left(N^{-1}\right)\right). \tag{3.3}$$

(3) *If $M \leq -\sqrt{2}$, then $I_N(M) = LN^{-1/2}e^{-N\lambda(a,b,M)}$ where $\lambda(a, b, M)$ is the minimum of $ax^2 + bx + I_1(-x)$ in $[M, -\sqrt{2}]$ and $L$ is a positive constant that depends on $a, b$ and $M$ as in* (3.19).

A few comments before the proof of the above lemma. First, under the assumption that $a > 1/2$ and $b > 0$ the major contribution to the integral in part (2) comes from a small neighborhood of M, instead of the minimum of $ax^2 + bx$. This is due to rapid oscillations of $\phi_{N-1}$ inside the "bulk" $(-\sqrt{2}, \sqrt{2})$. Second, in part (3), the condition that the minimizer of $ax^2 + bx + I_1(-x)$ lies inside $[M, -\sqrt{2}]$ is similar to the condition on (2.11). This will lead to the asymptotic Euler's characteristic in the region $u < -E'_{\infty}$.

The main tool to prove Lemma 4 is the following well-known formula for the asymptotics of the Hermite functions, first proved by Plancherel–Rotach [11]. Let

$$h(x) = \left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|^{1/4} + \left|\frac{x+\sqrt{2}}{x-\sqrt{2}}\right|^{1/4}.$$

LEMMA 5 (Plancherel–Rotach [11]). *There exists* $\delta_0 > 0$ *such that for any* $0 < \delta < \delta_0$, *the following asymptotics hold uniformly in each region*:

(1) *If* $x < -\sqrt{2} - \delta$,

$$\phi_{N-1}(\sqrt{N}x) = (-1)^{N-1} \frac{e^{-NI_1(-x)}}{\sqrt{4\pi\sqrt{2N}}} h(x)\bigl(1 + O(N^{-1})\bigr).$$

(2) *If* $-\sqrt{2} - \delta < x < -\sqrt{2} + \delta$,

$$\begin{aligned}
&\phi_{N-1}(\sqrt{N}x) \\
&\quad = \frac{(-1)^{N-1}}{(2N)^{1/4}} \biggl\{ \left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|^{1/4} \left|\frac{3N}{2} I_1(-x)\right|^{1/6} \mathrm{Ai}\biggl[\biggl(\frac{3N}{2} I_1(-x)\biggr)^{2/3} \varepsilon(x)\biggr] \\
&\qquad\qquad \times \bigl(1 + O(N^{-1})\bigr) \\
&\qquad\qquad - \left|\frac{x+\sqrt{2}}{x-\sqrt{2}}\right|^{1/4} \left|\frac{3N}{2} I_1(-x)\right|^{-1/6} \\
&\qquad\qquad\qquad \times \mathrm{Ai}'\biggl[\biggl(\frac{3N}{2} I_1(-x)\biggr)^{2/3} \varepsilon(x)\biggr]\bigl(1 + O(N^{-1})\bigr)\biggr\},
\end{aligned}$$

*where* $\mathrm{Ai}(x)$ *is the Airy function of first kind*, $\mathrm{Ai}(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} \cos(\frac{t^3}{3} + tx)\,\mathrm{d}t$, *and* $\varepsilon(x) = \frac{-x-\sqrt{2}}{|-x-\sqrt{2}|}$, $x \neq -\sqrt{2}$, $\varepsilon(-\sqrt{2}) = 0$ *and* $Ai'(x)$ *is the derivative of* $Ai(x)$.

(3) *If* $-\sqrt{2} + \delta < x < \sqrt{2} - \delta$ *and if we set* $x = \sqrt{2}\cos\omega$ *with* $\varepsilon < \omega < \pi - \varepsilon$, *then*

$$\begin{aligned}
\phi_{N-1}(\sqrt{N}x) &= \frac{2^{1/4}}{\pi^{1/2}N^{1/4}} \frac{1}{(\sin\omega)^{1/2}} \sin\biggl(\biggl(\frac{N}{2} + \frac{1}{4}\biggr)(\sin 2\omega - 2\omega) + \frac{3\pi}{4}\biggr) \\
&\quad \times \bigl(1 + O(N^{-1})\bigr).
\end{aligned}$$

(4) *If* $x > \sqrt{2} + \delta$,

$$\phi_{N-1}(\sqrt{N}x) = \frac{e^{-NI_1(x)}}{\sqrt{4\pi\sqrt{2N}}} h(x)\bigl(1 + O(N^{-1})\bigr).$$

PROOF OF LEMMA 4. *Part* (1): We can use the uniform asymptotics given by the exponential region (4) in Lemma 5. Precisely, by hypothesis, the function

$K(x) := ax^2 + bx + I_1(x)$ is increasing in $[M, \infty)$, and by Laplace's method,

$$\begin{aligned} I_N(M) &= \int_M^\infty \frac{e^{-N(ax^2+bx+I_1(x))}}{\sqrt{4\pi\sqrt{2N}}} h(x)\big(1+O(N^{-1})\big)\,\mathrm{d}x \\ &= \frac{e^{-NK(M)}}{N|K'(M)|\sqrt{4\pi\sqrt{2N}}} h(M)(1+O(N^{-1})). \end{aligned}$$

*Part* (2): Choose $\delta < \delta_0$ such that $-\sqrt{2} < M < \sqrt{2}-\delta$. We eqnarray the integral $I_N(M)$ into three parts,

$$I_N(M) = \left(\int_M^{\sqrt{2}-\delta} + \int_{\sqrt{2}-\delta}^{\sqrt{2}+\delta} + \int_{\sqrt{2}+\delta}^{\infty}\right) := I_1(M) + I_2 + I_3. \tag{3.4}$$

We will show that the main contribution in this case comes from $I_1(M)$. As in part (1), it is easy to see that

$$I_3 = O\big(e^{-NK(\sqrt{2})}\big). \tag{3.5}$$

Next since $|x|^{1/4}|\operatorname{Ai}(x)|$ and $|x|^{-1/4}|\operatorname{Ai}'(x)|$ are bounded functions on $\mathbb{R}$, a change of variables $z = I_1(-x)$ when using part (2) of Lemma 5 immediately implies that for any $\varepsilon > 0$,

$$I_2 = O\big(e^{-N(a(\sqrt{2}-\delta)^2 + b(\sqrt{2}-\delta)) + \varepsilon}\big). \tag{3.6}$$

Now we estimate $I_1(M)$. Using the uniform asymptotics of $\phi_{N-1}$ we need to evaluate

$$\begin{aligned} &\frac{2^{1/4}}{\pi^{1/2}N^{1/4}} \int_M^{\sqrt{2}-\delta} e^{-N(ax^2+bx)} \frac{1}{(\sin\omega)^{1/2}} \\ &\qquad \times \sin\left(\left(\frac{N}{2}-\frac{1}{4}\right)(\sin 2\omega - 2\omega) + \frac{3\pi}{4}\right)\mathrm{d}x. \end{aligned} \tag{3.7}$$

Performing the change of variables $x = \sqrt{2}\cos\omega, 0 < \omega < \pi$ the integral above becomes (for some different $\delta > 0$)

$$\begin{aligned} &\sqrt{2}\int_{\iota(M)}^{\pi-\delta} e^{-N(2a\cos^2\omega + \sqrt{2}b\cos\omega)} \\ &\qquad \times (\sin\omega)^{1/2} \sin\left(\left(\frac{N}{2}-\frac{1}{4}\right)(\sin 2\omega - 2\omega) + \frac{3\pi}{4}\right)\mathrm{d}\omega \end{aligned} \tag{3.8}$$

for $\iota(M) = \arccos(2^{1/2}M)$. We now rewrite $\cos^2\omega = \frac{1+\cos 2\omega}{2}$ and use the substitution $2\omega = z$ to obtain the integral

$$\begin{aligned} &\frac{1}{\sqrt{2}}\int_{2\iota(M)}^{2\pi-2\delta} e^{-N(a+a\cos z + (b/\sqrt{2})\cos(z/2))} \\ &\qquad \times \left(\sin\frac{z}{2}\right)^{1/2} \sin\left(\left(\frac{N}{2}-\frac{1}{4}\right)(\sin z - z) + \frac{3\pi}{4}\right)\mathrm{d}z. \end{aligned} \tag{3.9}$$

Last, we write

$$\sin\left(\left(\frac{N}{2}-\frac{1}{4}\right)(\sin z - z) + \frac{3\pi}{4}\right) = \frac{1}{2i}\left[e^{i(N/2)(\sin z - z)}e^{if_1(z)} - e^{-i(N/2)(\sin z - z)}e^{-if_1(z)}\right], \tag{3.10}$$

where $f_1(z) = -\frac{1}{4}(\sin z - z) + \frac{3\pi}{4}$.

Therefore, we just need to evaluate the asymptotics of

$$\int_{2\iota(M)}^{2\pi-2\delta} e^{-Nm(z)} j(z)\,\mathrm{d}z, \qquad \int_{2\iota(M)}^{2\pi-2\delta} e^{-Nn(z)} k(z)\,\mathrm{d}z, \tag{3.11}$$

where $m$ and $n$ are entire functions given by

$$m(z) = a + a\cos z + \frac{b}{\sqrt{2}}\cos\frac{z}{2} - \frac{i}{2}(\sin z - z), \tag{3.12}$$

$$n(x) = a + a\cos z + \frac{b}{\sqrt{2}}\cos\frac{z}{2} + \frac{i}{2}(\sin z - z) \tag{3.13}$$

and $j(z) = \sin(\frac{z}{2})^{1/2}e^{if_1(z)}$, $k(z) = \sin(\frac{z}{2})^{1/2}e^{-if_1(z)}$.

We will change our contour of integration and apply Laplace's integral in the appropriate integrals. Notice that the steepest descent paths are given by the equations

$$\operatorname{Im}\big(m(z)\big) = \sin x\left(a\sinh y + \frac{\cosh y}{2}\right) + \frac{b}{\sqrt{2}}\sin\frac{x}{2}\sinh\frac{y}{2} - \frac{x}{2} = \text{constant},$$

$$\operatorname{Im}\big(n(z)\big) = \sin x\left(a\sinh y - \frac{\cosh y}{2}\right) + \frac{b}{\sqrt{2}}\sin\frac{x}{2}\sinh\frac{y}{2} + \frac{x}{2} = \text{constant}.$$

The phase diagram for the steepest paths of $m$ is described as follows. First all lines $x = 2k\pi$, $k \in \mathbb{N}$ are steepest paths. Second, for every $t \in (0, 2\pi)$ the steepest path that passes through $t$ goes from $0 - i\infty$ to $\pi + i\infty$ if $b > 0$ and from $\pi - i\infty$ to $\pi + i\infty$ if $b = 0$. The real part of $m(z)$ is given by

$$\operatorname{Re}\big(m(z)\big) = \cos x\left(a\cosh y + \frac{1}{2}\sinh y\right) + a + \frac{b}{\sqrt{2}}\cos\frac{x}{2}\cosh y - \frac{y}{2},$$

$$\operatorname{Re}\big(n(z)\big) = \cos x\left(a\cosh y - \frac{1}{2}\sinh y\right) + a + \frac{b}{\sqrt{2}}\cos\frac{x}{2}\cosh y + \frac{y}{2}.$$

If we integrate $m(z)$ between two points $\alpha, \beta \in (0, 2\pi)$, we can deform our contour to be equal to the two steepest paths that connect $\alpha$ and $\beta$ to $z = 0 - i\infty$. Precisely, we deform our contour into three pieces: we first follow the steepest descent path from $\alpha$ to a point with imaginary part $y_0 < 0$, $|y_0|$ large. From there we go along the straight line $y = y_0$ until we reach the steepest path that passes through $\beta$, $\gamma_{y_0}$, and then we integrate on this steepest path back to $\beta$. We see that if we choose $|y_0|$ large enough, every point in the straight segment $y = y_0$ that we cross has real part $x$ sufficiently close to 0 so $\cos x > 0$. This together with $a > 1/2$ implies that $\mathrm{Re}(m(z))$ diverges to infinity as $y$ goes to negative infinity. The trivial bound

$$\left|\int_{\gamma_{y_0}} e^{-Nm(z)} j(z)\,\mathrm{d}z\right| \le \int_{\gamma_{y_0}} e^{-N\,\mathrm{Re}(m(z))}\,\mathrm{d}z \sup_{z\in\gamma_{y_0}} |j(z)| \tag{3.14}$$

combined with the bounded length of $\gamma_{y_0}$ show that the contribution of this part can be made as small as we want by choosing $y_0$ large enough.

In the two remaining paths the imaginary part of $m$ is constant and therefore we can apply Laplace's method to get the asymptotic behavior. Since we assumed that $M < \sqrt{2}$ the contribution at $2\pi - 2\delta$ is negligible compared to the one at $2\iota(M)$. Indeed, by formula (7.2.11) of [5],

$$\begin{aligned}\int_{2\iota(M)}^{2\pi-2\delta} & e^{-Nm(z)} j(z)\,\mathrm{d}z \\ &= \frac{e^{-Nm(2\iota(M))+i(\pi-\alpha(M))} j(2\iota(M))}{N|m'(2\iota(M))|}\bigl(1 + O(N^{-1})\bigr),\end{aligned} \tag{3.15}$$

where $\alpha(M)$ is the angle of the steepest descent path of $m$ at $z = 2\iota(M)$,

$$\alpha(M) = \arctan\biggl(\frac{1 - a\cos z}{2a\sin z + b\sin(x/2)/\sqrt{2}}\biggr). \tag{3.16}$$

The above argument adapted to the function $n$ implies

$$\begin{aligned}\int_{2\iota(M)}^{2\pi-2\delta} & e^{-Nn(z)} k(z)\,\mathrm{d}z \\ &= \frac{e^{-Nn(2\iota(M))+i(\pi-\alpha(M))} k(2\iota(M))}{N|n'(2\iota(M))|}\bigl(1 + O(N^{-1})\bigr).\end{aligned} \tag{3.17}$$

Noting that for any $x \in (0, 2\pi)$ $|n'(x)| = |m'(x)|$, we can combine (3.10), (3.15) and (3.17) to recover that $I_1(M)$ is asymptoticly equivalent to

$$\begin{aligned}& \frac{2^{1/4}}{\pi^{1/2}N^{5/4}} \frac{e^{-N(aM^2+bM)}}{2|m'(2\iota(M))|(\sin\omega)^{1/2}} \\ &\quad \times \sin\biggl[\biggl(\frac{N}{2} - \frac{1}{4}\biggr)(\sin 2\omega - 2\omega) + \frac{3\pi}{4} + \alpha(M)\biggr]\bigl(1 + O(N^{-1})\bigr).\end{aligned} \tag{3.18}$$

This ends the proof of part (2) of lemma. The proof of part (3) follows from the proof of part (2) and Laplace's method as in part (1) applied to the integral

$$\int_M^{-\sqrt{2}-\delta} e^{ax^2+bx+I_1(-x)} h(x)\,\mathrm{d}x = O\bigl(e^{-N\lambda(M,a,b)}\bigr).$$

In this case,

$$L(M,a,b) = \frac{\sqrt{2\pi}\,h(\lambda(M,a,b))}{2\lambda(M,a,b)+b+I_1'(\lambda(M,a,b))}. \tag{3.19}$$

We leave the details to the reader. □

We now turn to the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. We can rewrite (1.27) as

$$\begin{aligned}\mathbb{E}\chi(A_u) &= (-1)^{N-1}\left(\frac{\nu''}{\nu'}\right)^{(N-1)/2} c(N,\nu)\\ &\quad\times \int_{-\infty}^{\infty}\int_{-\infty}^{u/(\sqrt{2\nu''})} \phi_{N-1}\bigl(\sqrt{N}(\nu' x - \alpha y)\bigr)\\ &\qquad\times e^{-N\nu''(x^2+y^2)} e^{(N/2)(\nu' x-\alpha y)^2}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{3.20}$$

where

$$c(N,\nu) = 2\nu''\bigl([N-1]!\sqrt{\pi}\bigr)^{1/2}\frac{2^{-((N-1)/2)}N}{\sqrt{\pi}\,\Gamma(N/2)}. \tag{3.21}$$

For the case $\alpha \neq 0$, we can change variables $z = \nu' x - \alpha y$, $w = \alpha x + \nu' y$ to get

$$x = (\nu' z + \alpha w)\left(\frac{1}{\alpha^2 + \nu'^2}\right), \qquad y = (\nu' w - \alpha z)\left(\frac{1}{\alpha^2+\nu'^2}\right),$$

and the above double integral becomes (using $\alpha^2 = \nu'' + \nu' - \nu'^2$)

$$\frac{1}{\nu''+\nu'}\int\int_{\nu' z+\alpha w \le (\nu''+\nu')u/2\nu''} \phi_{N-1}(\sqrt{N}z) e^{-(N\nu''(z^2+w^2))/(\nu''+\nu')} e^{Nz^2/2}\,\mathrm{d}z\,\mathrm{d}w.$$

So we have to evaluate the asymptotic behavior of the following integral:

$$\begin{aligned}J &= \int_{-\infty}^{\infty} \phi_{N-1}(\sqrt{N}z) e^{N(\nu'-\nu'')z^2/(2(\nu''+\nu'))}\\ &\qquad\times \int_{-\infty}^{1/\alpha((\nu''+\nu')u/(\sqrt{2\nu''})-\nu' z)} e^{-N\nu'' w^2/(\nu'+\nu'')}\,\mathrm{d}w\,\mathrm{d}z.\end{aligned}$$

We write the outside integral $\int_{-\infty}^{\infty}\mathrm{d}z$ as $\int_{-\infty}^{M} + \int_{M}^{\infty}$ with

$$M = \frac{(\nu'+\nu'')u}{\sqrt{2\nu''}\nu'}.$$

The inside integral is just a Gaussian integral, and therefore after a straight-forward computation, the problem amounts to computing the asymptotics of the two following one-dimensional integrals:

$$J_1 = \int_M^\infty \phi_{N-1}(\sqrt{N}z)e^{-N((\nu'^2+\nu''-\nu')/(2(\nu''+\nu'-\nu'^2))z^2-\sqrt{2\nu''}\nu' u/(\nu''+\nu'-\nu'^2)z)}\,\mathrm{d}z,$$
$$J_2 = \int_M^\infty \phi_{N-1}(\sqrt{N}z)e^{-N(2(\nu'+\nu''))/(\nu''-\nu')z^2}\,\mathrm{d}z$$

as $J = (J_1 + J_2)(1 + O(N^{-1/2}))$ if $N$ is even and $J = (J_1 - J_2)(1 + O(N^{-1/2}))$ if $N$ is odd. Take $u \le 0$. We use Lemma 4 in both cases. Note that by (1.8),

$$a = \frac{\nu'^2 + \nu'' - \nu'}{2(\nu'' + \nu' - \nu'^2)} > \frac{1}{2} \quad \text{and} \quad b = -\frac{\sqrt{2\nu''}\nu' u}{\nu'' + \nu' - \nu'^2} \ge 0.$$

Now the condition $M \le -\sqrt{2}$ ($M > -\sqrt{2}$) is exactly the condition $u \le -E'_\infty$ ($u > -E'_\infty$). Applying the appropriate cases of Lemma 4 we see that the integral $J_2$ is negligible compared to $J_1$. A comparison with (1.22) and (2.18) gives the proof of part (1) and part (2) of the theorem with $a$ and $b$ as above,

$$\begin{aligned} \alpha(w) &= \arctan\left(\frac{1 - a\cos\omega}{2a\sin\omega + b\sin(\omega/2)/\sqrt{2}}\right), \\ f(\omega) &= \left(\left|m'(2\omega)\right|\sin^{1/2}\omega\right)^{-1} \end{aligned} \tag{3.22}$$

and

$$C(N, \nu, u) = \frac{1}{\nu'' + \nu'}c(N, \nu)L(M, a, b), \tag{3.23}$$

where $m$ is given in (3.12), $c(N, \nu)$ in (3.21) and $L(M, a, b)$ in (3.19).

If $\alpha = 0$, then the integral with respect to $y$ in (3.20) can be explicitly computed and the mean Euler characteristic is a single integral of the form given in Lemma 4. Applying part (1) and (2) of Lemma 4, we get Theorem 1.5 with

$$C(N, \nu, u) = \frac{1}{\sqrt{2\pi}\Theta'_\nu(u)}h\left(\frac{(\nu' + \nu'')u}{\sqrt{2\nu''}\nu'}\right).$$

Part (3) follows from symmetry of the Hamiltonian and (1.26). □

**4. Connection to mean field spin glasses.** In this section we discuss our main motivation to study the problems addressed in this manuscript. The function $H_N$ is the Hamiltonian of a classical model in statistical physics, the mixed spherical $p$-spin model [7]. The study of the landscape of these Hamiltonians is intimately related to the study of the most important question in these systems, the $N$ limit of the Gibbs measure

$$G_N(\boldsymbol{\sigma}) = \frac{1}{Z_N}e^{\beta H_N(\boldsymbol{\sigma})}.$$

These mean-field models, as well as other spin glass models, are well-known to be very challenging to analyze. It is believed (see [6] and the references therein) that a subset of the spherical models that we study here share the same interesting static and dynamical behavior as the famous Sherrington–Kirkpatrick model at low temperature.

The understanding of the landscape of these Hamiltonians might prove useful for the study of both static and dynamical questions of these models. First, the structure derived from Theorem 1.1 and described below may shed a light on the metastability of Langevin dynamics (in longer time scales than those studied in [4]). Second, it may provide an insight (discussed below) in a possible prediction for the structure of the Parisi measure, the functional order parameter of these models.

The complexity of critical points $\theta_{k,\nu}(u)$ of finite index has two pieces for negative values of $u$: one "with a branching" for $u \in (-\infty, -E'_\infty)$, another with a single curve, $u \in (-E'_\infty, 0)$; see Figure 1. This difference allows us to eqnarray the models of Gaussian smooth functions on the sphere in two classes that we describe now.

Let

$$G(\nu', \nu'') := \log \frac{\nu''}{\nu'} - \frac{(\nu'' - \nu')(\nu'' - \nu' + \nu'^2)}{\nu'' \nu'^2} = \theta_{0,\nu}(-E_\infty). \tag{4.1}$$

DEFINITION 4.1. A mixture $\nu$ is called a *pure-like* mixture if and only if $G(\nu', \nu'') > 0$. If $G(\nu', \nu'') < 0$, $\nu$ is called a *full mixture*. When $G(\nu', \nu'') = 0$, $\nu$ is called *critical*.

EXAMPLE 1. One can easily verify that all pure $p$-spins, $\nu(x) = x^p$, $p \geq 3$ are pure-like while the spherical SK model, $p = 2$, is critical.

EXAMPLE 2. Consider the case

$$\nu(t) = \mu t^2 + (1 - \mu) t^p, \tag{4.2}$$

where $\mu \in [0, 1]$. Then, if $p > 3$, then it is possible to show that there exists a $0 < \mu_c(p) < 1$ such that $\nu$ is *pure-like* if and only if $\mu \leq \mu_c(p)$. $\mu_c(p)$ is given as the unique zero in (0, 1) of

$$-\frac{(p^2 - 2p)(1 - \mu)(2(p^2 - p) - 3(p^2 - 2p)\mu + (p - 2)^2 \mu^2)}{2((p^2 - p)(1 - \mu) + 2\mu)(p + 2\mu - p\mu)^2} + \frac{1}{2} \log\left[1 + p - \frac{2p}{p + 2\mu - p\mu}\right];$$

see Figure 2. Remarkably, $p = 3$ in (4.2) is the only case where the mixture is a pure-like mixture for all values of $\mu$.

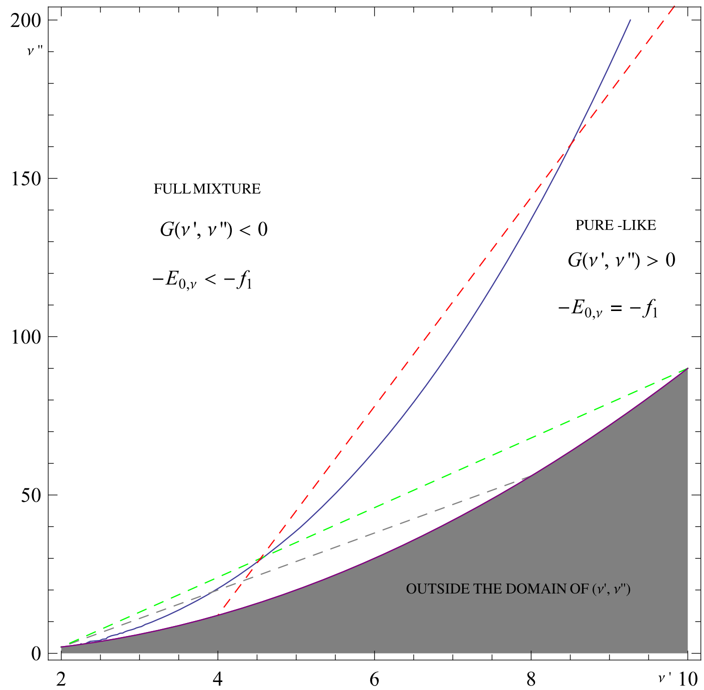


FIG. 2. *Graph of $\nu' \times \nu''$. In blue, the level set $G(\nu', \nu'') = 0$, that is, the case where $\nu$ is critical. Dotted lines are the possible values of $(\nu', \nu'')$ for the mixtures $2 + 6$, $2 + 10$ and $4 + 30$. The gray region is outside the domain of possible values for $(\nu', \nu'')$.*

It follows directly from the definition of pure-like and (1.18) that:

PROPOSITION 5. *If $\nu$ is a pure-like mixture, then the sequence $E_k(\nu)$ is strictly decreasing, and $E_k(\nu)$ converges to $E_\infty^+$ as $k$ goes to infinity.*

This proposition combined with Theorem 1.2 says if the mixture $\nu$ is *pure-like*, then the landscape of $\nu$ at low levels of energy is similar to the pure case as in [2]. In particular, the same interesting layered structure for the lowest critical values of the Hamiltonian $H_N$ holds. Namely, the lowest critical values above the ground state energy are (with an overwhelming probability) only local minima, this being true up to the value $-NE_1(\nu)$, and that in a layer above, $(-NE_1(\nu), -NE_2(\nu))$, one finds only critical values with index 0 (local minima) or saddle points with index 1, and above this layer one finds only critical values with index 0, 1 or 2, etc.

There is one curiosity about pure-like mixtures. Define

$$f_1 := \inf_{(a,b)\in[0,\infty)^2} \left\{ \frac{1}{2}\left( b + \nu' a + \frac{1}{b}\left( \log \frac{a+b}{a} \right) \right) \right\}. \tag{4.3}$$

PROPOSITION 6. *$\nu$ is pure-like or critical if and only if $f_1 = E_0(\nu)$.*

The curiosity is that $f_1$ can be interpreted as the zero-temperature limit of the 1-RSB Parisi functional in analogy to equation (5.11) in [2]. We refer the reader to [13] or Section 5 of [2] for a definition of this terminology. This leads us to the following question:

QUESTION 4.1. *Is it true that a mixture is* 1-*RSB at low temperature if and only if* $\nu$ *is pure-like?*

The question raised above is consistent with a picture proposed by physicists. In [6], it is claimed that a $2 + p$ spherical spin glass model with $p \geq 4$, at low temperature is either 1-RSB or its Parisi measure has an absolute continuous part (a Full RSB or a 1-Full RSB) depending on how much weight is assigned to the 2-spin model. The regions *pure-like* and *full mixture* seem to numerically agree and to extend (since we do not need the 2 spin component) the one proposed by [6].

We end this section with the following statement about full-mixtures. We first need the following result about the global minima of $H_N$ which is also of independent interest.

THEOREM 4.1. *The following limit exists almost surely*:

$$\lim_{N\to\infty} \frac{1}{N} \min_{\boldsymbol{\sigma}} H_N(\boldsymbol{\sigma}) := -f_\infty. \tag{4.4}$$

The following is now a corollary of Proposition 6 and Theorem 1.1.

COROLLARY 4.1. *If* $\nu$ *is a full mixture, then for any* $u \in (-E_0(\nu), -f_\infty)$, *the probability of having a critical value below* $u$ *goes to zero while the mean number of local minima is exponentially large in* $N$. *Namely for such* $u$ *there exist constants* $0 < C_1 < C_2$ *such that for* $N$ *sufficiently large*,

$$\mathbb{E}\mathrm{Crt}_{N,0}(-\infty, u) \geq e^{NC_1} \quad \textit{and} \quad \mathbb{P}\big(\mathrm{Crt}_N(-\infty, u) > e^{NC_1}\big) \leq e^{-NC_2}. \tag{4.5}$$

**5. Proofs from Section 4.** In this section we prove Propositions 5, 6 and Theorem 4.1. We start by proving Theorem 4.1. We will need to introduce some notation and the lemma below. Let $\boldsymbol{\sigma}^*$ be a point on the sphere such that $H_N(\boldsymbol{\sigma}^*) = \min_{\boldsymbol{\sigma}} H_N$, and let $d$ denote the geodesic distance on the sphere. For $\rho, \alpha, K > 0$, let

$$B_{N,\rho} \equiv \{\boldsymbol{\sigma} \in S_{N-1}(\sqrt{N}) : d(\boldsymbol{\sigma}, \boldsymbol{\sigma}^*) < \rho\}$$

and $A_{\varepsilon,\alpha,K}(N)$, be the event

$$A_{\varepsilon,\alpha,K}(N) \equiv \Big\{ \sup_{\boldsymbol{\sigma}\in B_{N,\sqrt{N}\varepsilon}} \big|H_N(\boldsymbol{\sigma}) - H_N(\boldsymbol{\sigma}^*)\big| \leq K N \varepsilon^\alpha \Big\}. \tag{5.1}$$

LEMMA 6. *For any* $0 < \alpha < 1$, *there exist constants* $K$, $K_1 > 0$ *so that for all* $\varepsilon > 0$ *and all* $N$ *sufficiently large*

$$\mathbb{P}(A_{\varepsilon,\alpha,K}(N)^c) < 2e^{-K_1 N}. \tag{5.2}$$

Note that this bound is independent of $\varepsilon$.

PROOF. Clearly,

$$A_{\varepsilon,K}(N) \supseteq \hat{A}_{\alpha,K}(N) \equiv \{\|H_N\|_\alpha \le K N^{1-\alpha/2}\},$$

where

$$\|H_N\|_\alpha = \sup_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \frac{|H_N(\boldsymbol{\sigma}) - H_N(\boldsymbol{\sigma}')|}{d(\boldsymbol{\sigma},\boldsymbol{\sigma}')^\alpha}. \tag{5.3}$$

Now consider the centered Gaussian process $\mathbf{X}_\alpha$ field on $S_{N-1}(\sqrt{N}) \times S_{N-1}(\sqrt{N})$ given by

$$\mathbf{X}_\alpha(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \begin{cases} \dfrac{H_N(\boldsymbol{\sigma}) - H_N(\boldsymbol{\sigma}')}{d(\boldsymbol{\sigma},\boldsymbol{\sigma}')^\alpha}, & \text{if } d(\boldsymbol{\sigma},\boldsymbol{\sigma}') > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

Since the Gaussian field $H_N$ is $C^1$ almost surely, then

$$\mathbb{P}(\hat{A}_{\alpha,K}(N)^c) = \mathbb{P}\Big(\sup_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} |X_\alpha(\boldsymbol{\sigma},\boldsymbol{\sigma}')| > K N^{1-\alpha/2}\Big). \tag{5.5}$$

But now a simple computation yields for $\boldsymbol{\sigma} \neq \boldsymbol{\sigma}'$,

$$\begin{aligned} \mathbb{E}\mathbf{X}_\alpha^2(\boldsymbol{\sigma},\boldsymbol{\sigma}') &= \frac{2N}{d(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_1')^{2\alpha}}\Big[1 - \nu\Big(\frac{1}{N}\langle\boldsymbol{\sigma},\boldsymbol{\sigma}'\rangle\Big)\Big] \\ &= \frac{2N}{(\sqrt{N}\theta)^{2\alpha}}[1 - \nu(\cos\theta)], \end{aligned} \tag{5.6}$$

where $\theta$ is the angle between $\boldsymbol{\sigma},\boldsymbol{\sigma}'$ in $\mathbb{R}^N$.

Therefore by the boundedness of $\nu'(x)$ in $[-1,1]$ there exists a constant $C$ independent of $N$ such that [if $\alpha < 1/2$ or $\alpha < 1$—using the boundedness of $\nu'(x)$ and $\nu''(x)$]

$$\sup_{(\boldsymbol{\sigma},\boldsymbol{\sigma}')} \mathbb{E}\mathbf{X}_\alpha^2(\boldsymbol{\sigma},\boldsymbol{\sigma}') \le C N^{1-\alpha}. \tag{5.7}$$

Now, by Borell's inequality, (see pages 50 and 51 of [1], where we take $u = KN^{1-\alpha/2}$, $\sigma_T \le CN^{1-\alpha}$) for all $\delta$, if $N$, $K$ is large enough

$$\begin{aligned} \mathbb{P}\Big(\sup_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \mathbf{X}_\alpha(\boldsymbol{\sigma},\boldsymbol{\sigma}') > KN^{1-\alpha/2}\Big) &\le e^{\delta K N^{1-\alpha/2}} e^{-K^2 N^{2(1-\alpha/2)}/(2CN^{1-\alpha})} \\ &\le e^{-K^2 N/(4C)}. \end{aligned} \tag{5.8}$$

Taking $K_1 = K^2/4C$ in the last equation, using (5.5) and symmetry of $\mathbf{X}_\alpha$ the lemma is proven. $\square$

PROOF OF THEOREM 4.1. Let $GS_N = \frac{1}{N}\min_{\boldsymbol{\sigma}} H_N(\boldsymbol{\sigma})$. We will show the existence of a constant $f_\infty$ so that for any $\delta > 0$ there exists $\varepsilon(\delta)$ such that if $N$ is large enough,

$$\mathbb{P}(|GS_N + f_\infty| > \delta) \le \mathbb{P}(A_{\varepsilon(\delta),\alpha,K}(N)^c). \tag{5.9}$$

The proof of Theorem 4.1 will then follow from (5.9) and Borel–Cantelli's lemma since for all $\delta > 0$ by Lemma 6,

$$\sum_{N=1}^{\infty} \mathbb{P}(|GS_N + f_\infty| > \delta) < \infty. \tag{5.10}$$

We will prove (5.9) by showing that for any $\delta > 0$ if $N$ is large enough $A_{\varepsilon,\alpha,K}(N) \subset \{|GS_N + f_\infty| < \delta\}$.

On $A_{\varepsilon,\alpha,K}(N)$,

$$\begin{aligned} Z_{N,\nu}(\beta) &:= \int_{S^{N-1}(\sqrt{N})} e^{-\beta H_N(\boldsymbol{\sigma})} \Lambda_N(\mathrm{d}\boldsymbol{\sigma}) \\ &\ge e^{-\beta N GS_N - K\beta N\varepsilon^\alpha} \Lambda_N(B_{N,\sqrt{N}\varepsilon}). \end{aligned} \tag{5.11}$$

Recall that $\Lambda_N(\mathrm{d}\boldsymbol{\sigma})$ is the surface measure of $S_N(\sqrt{N})$ normalized to be a probability measure. We trivially have the bound

$$\frac{1}{N}\log Z_{N,\nu}(\beta) \le -\beta GS_N. \tag{5.12}$$

Combining (5.11) and (5.12) we then have on $A_{\varepsilon,\alpha,K}(N)$,

$$\begin{aligned} -\frac{1}{N\beta}\log Z_{N,\nu}(\beta) - K\varepsilon^\alpha + \frac{1}{N\beta}\log \Lambda_N(B_{N,\sqrt{N}\varepsilon}) \\ \le GS_N \le -\frac{1}{N\beta}\log Z_{N,\nu}(\beta). \end{aligned} \tag{5.13}$$

Note that using spherical coordinates and the inequality $\frac{2\theta}{\pi} \le \sin\theta$ for $\theta \le \frac{\pi}{2}$, we have for $\varepsilon < \pi/2$,

$$\begin{aligned} \Lambda_N(B_{N,\sqrt{N}\varepsilon}) &= \left(\int_0^\varepsilon \sin^{N-2}(\phi)\,\mathrm{d}\phi\right)\left(\int_0^\pi \sin^{N-2}(\phi)\,\mathrm{d}\phi\right)^{-1} \\ &\ge \left(\frac{2\varepsilon}{\pi}\right)^{N-1} \frac{1}{\pi(N-1)}. \end{aligned} \tag{5.14}$$

So on $A_{\varepsilon,\alpha,K}(N)$, for some constant $C > 0$,

$$-\frac{1}{N\beta}\log Z_{N,\nu}(\beta) - K\varepsilon^\alpha + C\varepsilon \le GS_N \le -\frac{1}{N\beta}\log Z_{N,\nu}(\beta). \tag{5.15}$$

By Holder's inequality the function $\frac{1}{N}\mathbb{E}\log Z_{N,\nu}(\beta)$ is convex in $\beta$, therefore its limit that we denote by $F_\infty(\beta)$ is also convex. The existence of this limit is given by the famous Parisi formula [13], Theorem 1.1.

So $F(\beta)$ is convex, positive and grows at most linearly. This easily implies that

$$\lim_{\beta\to\infty}\frac{1}{\beta}F_\infty(\beta)=\sup_\beta\frac{1}{\beta}F_\infty(\beta):=f_\infty\in[0,\infty). \tag{5.16}$$

Therefore, for any $\delta_1>0$ one can take $N$ large enough so that

$$-\frac{F_\infty(\beta)}{\beta}-K\varepsilon^\alpha+C\varepsilon-\frac{\delta_1}{\beta}\le GS_N\le-\frac{F_\infty(\beta)}{\beta}+\frac{\delta_1}{\beta}. \tag{5.17}$$

By taking $\beta$ large enough, part (a) of this theorem and by choosing $\varepsilon$ sufficiently small, (5.9) is proven. □

We now prove Propositions 5.

PROOF OF PROPOSITION 5. If $\nu$ is pure-like, then $\theta_{k,\nu}(-E_\infty)>0$. Since $\theta_{k,\nu}(u)$ converges to negative infinity as $u$ goes to negative infinity, $E_k(\nu)$ are well defined. Furthermore, as $k$ goes to infinity, $\lambda_k^*(u)$ converges to $-\sqrt{2}$ for any $u\le -E_\infty$, implying that $\theta_{k,\nu}(u)$ converges to $F(-\sqrt{2},u)$ pointwise. Therefore, taking $u$ in a small neighborhood of $E_\infty^+$ and using the fact that $\theta_{k,\nu}$ are increasing in that neighborhood, we see that the zero of $\theta_{k,\nu}$ has to converge to the zero of $F(-\sqrt{2},u)$. Namely $E_k(\nu)$ converges to $E_\infty^+$. □

5.1. *Proof of Proposition* 6. We now provide a proof for Proposition 6. We will need a collection of calculus exercises.

LEMMA 7. *$f_1$ depends continuously on the first derivative $\nu'$.*

REMARK 5. Note that while the $k$-complexity function depends on the first two derivatives at 1 of the covariance function $\nu$, and $f_1$ depends only on the first derivative $\nu'$ and $E_0(\nu)=f_1$ for any pure-like mixture.

PROOF. By solving for the critical points of (4.3), we can get an expression for $f_1$ in terms of $\nu'$. Namely,

$$f_1=\frac{1}{2}\left(\frac{\nu'y^2-1}{\nu'y}+\frac{1}{y}+\frac{\nu'y}{\nu'y^2-1}\log(\nu'y^2)\right)=y+\frac{\nu'-1}{y\nu'},$$

where $y=y(\nu')$ is given by the unique solution of

$$\left(\frac{\nu'y^2-1}{\nu'y}\right)^2y+\frac{\nu'y^2-1}{\nu'y}=y\log(\nu'y^2),\qquad y>\nu'^{-1/2}.$$

In other words, $y = \frac{\sqrt{a}}{\sqrt{\nu'}}$ where $a$ is the unique solution of

$$a\log[a] - a + 1 - \frac{(a-1)^2}{\nu'} = 0, \qquad a > 1.$$

This immediately implies the proof of Lemma 7. $\square$

PROPOSITION 7. *A mixture $\nu$ is critical if and only if*

$$f_1 = E_\infty = E_{0,\nu} = \frac{\nu'' - \nu' + \nu'^2}{\nu'\sqrt{\nu''}}. \tag{5.18}$$

PROOF. If $\nu$ is critical, then $y = \frac{\sqrt{\nu''}}{\nu'}$ is the unique solution of (5.1) with $y > \frac{1}{\sqrt{\nu'}}$. Indeed,

$$1 - \frac{\nu''}{\nu'} + \frac{(-\nu' + \nu'')(-\nu' + \nu'^2 + \nu'')}{\nu'^3} - \frac{(-1 + \nu''/\nu')^2}{\nu'} = 0.$$

Plugging back the value of $y$ in (5.1) we get $f_1$. On the other hand, if $f_1 = \frac{\nu''-\nu'+\nu'^2}{\nu'\sqrt{\nu''}}$, then one solves equation (5.1) in $y$ to see that the only positive solution is $y = \frac{\sqrt{\nu''}}{\nu}$. By the definition of $y$ in (5.1) this immediately implies that $\nu$ is critical. And trivially, $\nu$ critical is precisely the case where $E_\infty = E_{0,\nu}$. $\square$

Now we analyze the case where $\nu$ is critical or a full mixture, that is, the case where $G(\nu', \nu'') \leq 0$. In this case, the zero of the complexity function can be explicitly computed and is given by

$$-E_{0,\nu} = -E_\infty^+,$$

where $E_\infty^+$ was defined in (1.13). Note that $E_{0,\nu}$ is a function of $\nu'$ and $\nu''$.

PROPOSITION 8. *If $G(\nu', \nu'') \leq 0$, then*

$$\frac{\partial}{\partial \nu''} E_{0,\nu} = 0 \quad \textit{if and only if} \quad G(\nu', \nu'') = 0.$$

PROOF. Let

$$A(\nu', \nu'') = \sqrt{(\nu'' - \nu'^2 + \nu')\Big((\nu' + \nu'')\log\Big[\frac{\nu''}{\nu'}\Big] - 2(\nu'' - \nu')\Big)}.$$

Calculating the derivative $\frac{\partial}{\partial \nu''} E_{0,\nu}$ one gets

$$\begin{aligned}
&\Big(\nu'^2\nu''(\nu' + \nu'')\log\Big[\frac{\nu''}{\nu'}\Big] + (\nu'' - \nu')\Big) \\
&\quad \times (\nu'^3 + \nu''^2 - \nu'^2(1 + 3\nu'') - 2\nu'\sqrt{\nu''}A(\nu', \nu'')) \\
&\quad \times (2\nu''(\nu' + \nu'')^2 A(\nu', \nu''))^{-1}.
\end{aligned} \tag{5.19}$$

Sufficiency comes from a simplification of the above formula. To get necessity we solve a second degree equation on the variable $M = \log[\frac{\nu''}{\nu'}]$ to see that this second degree equation has a unique zero given by

$$\frac{\nu'^2 - \nu'^3 - 2\nu'\nu'' + \nu'^2\nu'' + \nu''^2}{\nu'^2\nu''}.$$

This is precisely $G(\nu', \nu'') = 0$. $\square$

With the above propositions we now prove Proposition 6.

PROOF OF PROPOSITION 6. If $\nu$ is critical, Proposition 6 is Proposition 7. Now suppose that $\nu$ is pure-like. By Lemma 7 and (2.10), both $f_1(\nu) := f_1$ and $E_0(\nu)$ are independent of $\nu''$. Consider then another mixture $\mu$ such that $\mu' = \nu'$ and $\mu$ satisfies $G(\mu', \mu'') = 0$. Since $G$ is continuous on its domain, we have

$$f_1(\nu) = f_1(\mu) = E_0(\mu) = E_0(\nu).$$

On the other hand, if $\nu$ is a full-mixture, Proposition 8 combined with Lemma 7 shows that $f_1 \neq E_0(\nu)$. This ends the proof of Proposition 6. $\square$

**Acknowledgments.** We want to underline our debt to Michel Ledoux for his friendly help with the results of Section 5. We also would like to thank Jiri Cerny for a careful reading of this manuscript and Yan Fyodorov for pointing out that the method used in this paper is similar to [9] and [10]. We want to thank MSRI, IMPA, Université de Marseille for their hospitality and the Université de Nice where a mini-course based on these results were given. We are also in debt to an anonymous referee who helped us to improve the presentation of this manuscript.

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