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# Complexity of Stratifications of Semi-Pfaffian Sets*

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**Abstract.** An effective algorithm for a smooth (weak) stratification of a real semi-Pfaffian set is suggested, provided an oracle deciding consistency of a system of Pfaffian equations and inequalities is given. An explicit estimate of the complexity of the algorithm and of the resulting stratification is given, in terms of the parameters of the Pfaffian functions defining the original semi-Pfaffian set. The algorithm is applied to sets defined by sparse polynomials and exponential polynomials.

## 1. Introduction

In 1957 Whitney [19] proved that a real algebraic variety can be represented as a finite disjoint union of smooth manifolds which are semialgebraic sets. Łojasiewicz [11], [12] extended Whitney's theorem to the class of real semianalytic sets. His method explicitly involves the Weierstrass preparation theorem, that accounts for essential nonconstructiveness of the proof and for the impossibility of restricting the class of functions defining the smooth strata. In 1993 Gabrielov [3] showed (as a part of an elementary proof of his theorem [2] on projections of semianalytic sets) that the smooth strata of a semianalytic set $X$ can be defined by functions belonging to

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the smallest extension of the family defining $X$ which is closed under additions, multiplications, and taking partial derivatives. Apart from polynomials, important classes sharing this property consist of all Pfaffian functions and their special subclasses, such as exponential and sparse polynomials.

Pfaffian functions, introduced by Khovanskii [9], [10], define the semianalytic (semi-Pfaffian) sets which have important global finiteness properties similar to those of semialgebraic sets. Moreover, the characteristics that are finite (such as the number of isolated roots of a system of equations) can be effectively bounded from above, in terms of the format parameters of the defining functions.

Recently Gabrielov [4] estimated the multiplicity of intersections of Pfaffian varieties. The purpose of this paper is to show that the latter bound allows us to construct an algorithm which produces a smooth stratification for a semi-Pfaffian set and to estimate its complexity. Under stratification here we always mean a weak stratification, i.e., a subdivision into smooth nonintersecting pieces (strata) without any requirement on the boundary of a stratum to be a union of some other strata.

We always consider real semi-Pfaffian sets, although the main algorithm from Section 3 is applicable, without any change, to complex constructible Pfaffian sets, with the inequalities "greater than" and "less than" replaced by "not equal." The estimate in [4] is valid in the complex case, too.

We are interested in the bounds on the parameters of the output of the algorithm and on its computational complexity. We give more precise definitions and bounds in Section 4 below. Let us mention now that the complexity turns out to be a doubly exponential function in the number of variables $n$. For a fixed $n$, this function is singly exponential in the maximal *order* $r$ of Pfaffian functions involved and, for fixed $n$ and $r$, polynomial in all the other parameters.

Note that the known stratification algorithms for general semialgebraic sets ($r = 0$) have essentially the same complexity. They are based on a recursive application of a fast procedure for quantifier elimination in the first-order theory of reals (i.e., on an effective algorithmical version of the Tarski–Seidenberg principle) [8], [16]. The latter technique gives a much stronger result: a Whitney stratification of an arbitrary semialgebraic set [14]. Let us mention also that a singly exponential (in $n$) algorithm for the Whitney stratification is known for a rather broad class of real algebraic varieties [18].

On the other hand, our algorithm can handle semialgebraic sets defined by *fewnomials*, or sparse polynomials, with the size of output estimated in terms of the number of nonzero monomials, independent of their degrees. Besides, it represents strata in a more convenient form.

The content of this paper is as follows. The Pfaffian functions are defined (following Khovanskii [9], [10]) in Section 2, and certain parameters of these functions are introduced. We explain how the basic operations over the functions affect the parameters.

In Section 3 an algorithm for a stratification of an elementary semi-Pfaffian set is described. Section 4 contains a complexity estimate of this algorithm.

Section 5 describes a stratification procedure for arbitrary semi-Pfaffian sets, while Section 6 deals with special classes, defined by polynomials, fewnomials, and exponential polynomials (dense and sparse).

## 2. Pfaffian Functions

**Definition 1.** (See [9], [10], and [4].) A *Pfaffian chain* of the *order* $r \geq 0$ and *degree* $\alpha \geq 1$ in an open domain $G \subset \mathbf{R}^n$ is a sequence of real analytic functions $f_1, \dots, f_r$ in $G$ satisfying Pfaffian equations

$$df_j(x) = \sum_{i=1}^{n} g_{ij}(x, f_1(x), \dots, f_j(x))\, dx_i \qquad \text{for} \quad j = 1, \dots, r. \tag{1}$$

Here $g_{ij}(x, y)$ are polynomials in $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_j)$ of degree not exceeding $\alpha$. A function $f(x) = P(x, f_1(x), \dots, f_r(x))$ where $P(x, y_1, \dots, y_r)$ is a polynomial of degree not exceeding $\beta \geq 1$ is called a *Pfaffian function* of order $r$ and degree $(\alpha, \beta)$.

**Remark 1.** Note that our definition is more restrictive than the definitions in [10] and [4] where the Pfaffian chains were defined as sequences of nested integral manifolds of polynomial 1-forms. Our definition coincides with the definition of a special Pfaffian chain in [4]. Both definitions lead to essentially the same class of Pfaffian functions, although the orders and degrees of Pfaffian chains for the same Pfaffian function can be different according to these two definitions. We found our present definition to be more convenient to trace the behavior of parameters of Pfaffian functions under different operations. Also, it gives a better estimate for the multiplicity in [4].

**Examples.**

(a) Pfaffian functions of order 0 and degree $(1, \beta)$ are polynomials of degree not exceeding $\beta$.
(b) The exponential function $f(x) = e^{ax}$ is a Pfaffian function of order 1 and degree $(1, 1)$ in $\mathbf{R}$, due to the equation $df(x) = af(x)\, dx$.
(c) The function $f(x) = 1/x$ is a Pfaffian function of order 1 and degree $(2, 1)$ in the domain $x \neq 0$, due to the equation $df(x) = -f^2(x)\, dx$.
(d) The logarithmic function $f(x) = \ln(|x|)$ is a Pfaffian function of order 2 and degree $(2, 1)$ in the domain $x \neq 0$, due to the equations $df(x) = g(x)\, dx$ and $dg(x) = -g^2(x)\, dx$, with $g(x) = 1/x$.
(e) The polynomial $f(x) = x^m$ can be considered as a Pfaffian function of order 2 and degree $(2, 1)$ in the domain $x \neq 0$ (but not in $\mathbf{R}$), due to the equations $df(x) = mf(x)g(x)\, dx$ and $dg(x) = -g^2(x)\, dx$, with $g(x) = 1/x$. A better way to deal with it, however, is to change the variable $x = \exp(u)$ reducing this case to the exponential function.
(f) The function $f(x) = \tan(x)$ is a Pfaffian function of order 1 and degree $(2, 1)$ in the domain $x \neq \pi/2 + k\pi$, for all integer $k$, due to the equation $df(x) = (1 + f^2(x))\, dx$.
(g) The function $f(x) = \arctan(x)$ is a Pfaffian function in $\mathbf{R}$ of order 2 and degree $(3, 1)$, due to the equations $df(x) = g(x)\, dx$ and $dg(x) = -2xg^2(x)\, dx$, with $g(x) = (x^2 + 1)^{-1}$.

(h) The function $\cos(x)$ is Pfaffian of order 2 and degree $(2, 1)$ in the domain $x \neq \pi + 2k\pi$, for all integer $k$, due to the equations $\cos(x) = 2f(x) - 1$, $df(x) = -f(x)g(x)\,dx$, and $dg(x) = \frac{1}{2}(1 + g^2(x))\,dx$, with $f(x) = \cos^2(x/2)$ and $g(x) = \tan(x/2)$. Also, since $\cos(x)$ is a polynomial of degree $m$ of $\cos(x/m)$, the function $\cos(x)$ is Pfaffian of order 2 and degree $(2, m)$ in the domain $x \neq m\pi + 2km\pi$, for all integer $k$. The same is true, of course, for any shift of the above domain by a multiple of $\pi$. However, $\cos(x)$ is not Pfaffian in the whole real line.

The following lemmas (see [10]) provide additional means for the construction of Pfaffian functions.

**Lemma 1.** *The sum (resp. product) of two Pfaffian functions, $f_1$ and $f_2$, of orders $r_1$ and $r_2$ and degrees $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$, is a Pfaffian function of order $r_1 + r_2$ and degree $(\alpha, \max(\beta_1, \beta_2))$ (resp. $(\alpha, \beta_1 + \beta_2)$) where $\alpha = \max(\alpha_1, \alpha_2)$. If the two Pfaffian functions are defined by the same Pfaffian chain of order $r$, the order of the sum and product is also $r$.*

*Proof.* We can combine the Pfaffian chains for the functions $f_1$ and $f_2$ into a Pfaffian chain for $f_1 + f_2$ and $f_1 f_2$. If a Pfaffian chain is common for the two functions, it is also a Pfaffian chain for their sum and product. □

**Lemma 2.** *A partial derivative of a Pfaffian function of order $r$ and degree $(\alpha, \beta)$ is a Pfaffian function of order $r$ and degree $(\alpha, \alpha + \beta - 1)$.*

*Proof.* Let $f(x) = (f_1(x), \ldots, f_r(x))$ be a Pfaffian chain of order $r$ and degree $\alpha$, and let $P(x, y)$ be a polynomial of degree $\beta$. The statement follows from the differentiation formula

$$\frac{\partial P(x, f(x))}{\partial x_i} = \left.\frac{\partial P(x, y)}{\partial x_i}\right|_{y=f(x)} + \sum_{j=1}^{r} \left.\frac{\partial P(x, y)}{\partial y_j}\right|_{y=f(x)} \frac{\partial f_j(x)}{\partial x_i}$$

after substitution of $\partial f_j(x)/\partial x_i = g_{ij}(x, f_1(x), \ldots, f_j(x))$ from (1). □

**Lemma 3.** *Let $z = (z_1, \ldots, z_l)$ and let $f(x, z)$ be a Pfaffian function of order $r_1$ and degree $(\alpha_1, \beta_1)$ in a domain $G_1 \subset \mathbf{R}^{n+l}$. Let $h(x) = (h_1(x), \ldots, h_l(x))$ be an $l$-tuple of Pfaffian functions of order $r_2$ and degree $(\alpha_2, \beta_2)$, with a common Pfaffian chain, defined in a domain $G_2 \subset \mathbf{R}^n$ such that $(x, h(x)) \in G_1$, for all $x \in G_2$. Then $f(x, h(x))$ is a Pfaffian function in $G_2$ of order $r_1 + r_2$ and degree $(\alpha_1 \beta_2 + \alpha_2 + \beta_2 - 1, \beta_1)$.*

*Proof.* The Pfaffian chain for functions $h$ can be extended to a Pfaffian chain of order $r_1 + r_2$ by adding functions $f_j(x, h(x))$ where $f_j(x, y)$ constitute the Pfaffian chain for $f$. The statement follows from the differentiation formula

$$\frac{\partial f_j(x,h(x))}{\partial x_i} = \left.\frac{\partial f_j(x,z)}{\partial x_i}\right|_{z=h(x)} + \sum_{\nu=1}^{l} \left.\frac{\partial f_j(x,z)}{\partial z_\nu}\right|_{z=h(x)} \frac{\partial h_\nu(x)}{\partial x_i}$$

after substitution of the partial derivatives from the corresponding Pfaffian chains. □

**Definition 2.** For a set of differentiable functions $\mathbf{h} = (h_1, \dots, h_k)$, a set of distinct indices $\mathbf{i} = (i_1, \dots, i_k)$ with $1 \le i_\nu \le n$, and an index $j$, $1 \le j \le n$, different from all $i_\nu$, we define partial differential operator

$$\partial_{\mathbf{h},\mathbf{i},j} = \det \begin{pmatrix} \frac{\partial h_1}{\partial x_{i_1}} & \cdots & \frac{\partial h_1}{\partial x_{i_k}} & \frac{\partial h_1}{\partial x_j} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_k}{\partial x_{i_1}} & \cdots & \frac{\partial h_k}{\partial x_{i_k}} & \frac{\partial h_k}{\partial x_j} \\ \frac{\partial}{\partial x_{i_1}} & \cdots & \frac{\partial}{\partial x_{i_k}} & \frac{\partial}{\partial x_j} \end{pmatrix}.$$

When $k = 0$, the corresponding operator is simply $\partial_j = \partial/\partial x_j$. We define $\partial^m_{\mathbf{h},\mathbf{i},j}$ (resp. $\partial_j^m$) as the $m$th iteration of $\partial_{\mathbf{h},\mathbf{i},j}$ (resp. $\partial_j$).

**Lemma 4.** *For a Pfaffian function $f$ of order $r$ and degree $(\alpha, \beta)$, for a set $\mathbf{h} = (h_1, \dots, h_k)$ of Pfaffian functions of order $r$ and degrees $(\alpha, \beta_1), \dots, (\alpha, \beta_k)$ defined by the same Pfaffian chain as $f$, and for a set of distinct indices $\{\mathbf{i} = (i_1, \dots, i_k), j\}$, the function $\partial^m_{\mathbf{h},\mathbf{i},j} f(x)$ is a Pfaffian function of order $r$ and degree $(\alpha, \beta')$ where $\beta' = \beta + m[(\alpha - 1)(k+1) + \beta_1 + \cdots + \beta_k]$, defined by the same Pfaffian chain as $f$.*

*Proof.* The statement follows from the Lemmas 1 and 2.

**Proposition 1.** *Let $\mathbf{i} = (i_1, \dots, i_k)$ be a set of distinct indices, $1 \le i_\nu \le n$. Let $f$ be a Pfaffian function of order $r$ and degree $(\alpha, \beta)$ in an open neighborhood $G$ of $x \in \mathbf{R}^n$, and let $\mathbf{h} = (h_1, \dots, h_k)$ be a set of Pfaffian functions of order $r$ and degrees $(\alpha, \beta_1), \dots, (\alpha, \beta_k)$ defined in $G$ by the same Pfaffian chain as $f$, such that $h_1(x) = \cdots = h_k(x) = 0$,*

$$\det \begin{pmatrix} \frac{\partial h_1}{\partial x_{i_1}} & \cdots & \frac{\partial h_1}{\partial x_{i_k}} \\ \cdots & \cdots & \cdots \\ \frac{\partial h_k}{\partial x_{i_1}} & \cdots & \frac{\partial h_k}{\partial x_{i_k}} \end{pmatrix}(x) \neq 0. \tag{2}$$

*Let*

$$M = M(k, r, \alpha, \beta, \beta_1, \dots, \beta_k)$$
$$= 2^{r(r-1)/2}\, \beta\beta_1 \cdots \beta_k (\min\{r, k+1\}\alpha - k + \beta + \beta_1 + \cdots + \beta_k + 1)^r. \quad (3)$$

*Suppose that*

$$\partial^{m_1}_{\mathbf{h},\mathbf{i},1} \cdots \partial^{m_n}_{\mathbf{h},\mathbf{i},n} f(x) = 0 \quad \text{for} \quad 0 \le m_1 + \cdots + m_n \le M, \quad m_{i_1} = \cdots = m_{i_k} = 0. \quad (4)$$

*Then the function $f$ vanishes identically on $Y = \{h_1 = \cdots = h_k = 0\}$ in the neighborhood of $x$.*

*Proof.* First, we want to reduce the problem to the case $n = k + 1$. Suppose that $f|_Y \not\equiv 0$ in a neighborhood of $x$. As $f|_Y$ is analytic, the Taylor expansion of $f|_Y$ at $x$ starts with terms of degree $\kappa < \infty$. Let $K \subset T_x Y$ be the set of zeros of the initial form of $f|_Y$ at $x$ of degree $\kappa$, i.e., the tangent cone to $\{f|_Y = 0\}$ at $x$. Let $L$ be a linear subspace of dimension $k + 1$ through $x$ such that $\gamma = L \cap Y$ is one-dimensional and the tangent vector to $\gamma$ at $x$ does not belong to $K$. Then the order of $f|_\gamma$ at $x$ is equal to $\kappa$. In particular, $f|_\gamma \not\equiv 0$ in the neighborhood of $x$. Replacing $\mathbf{R}^n$ by $L$ and $Y$ by $\gamma$, we can reduce the problem to $n = k + 1$.

Renumerating coordinates reduces the problem to the case $(i_1, \dots, i_k) = (1, \dots, k)$, and (4) becomes

$$\partial^{m_n}_{\mathbf{h},\mathbf{i},n} f(x) = 0 \quad \text{for} \quad 0 \le m_n \le M.$$

The set $Y$ is one-dimensional, and the operator $\partial_{\mathbf{h},\mathbf{i},n}$ is a differentiation along a vector field tangent to $Y$. Due to (2) this vector field is nonzero at $x$, and its integral curve passing through $x$ contains a neighborhood of $x$ in the set $Y$. If $f$ does not vanish identically in the neighborhood of $x$ in $Y$, the multiplicity at $x$ of the Pfaffian intersection $\{f = h_1 = \cdots = h_k = 0\}$ is greater than $M$. This contradicts the bound on the multiplicities of the Pfaffian intersections in [4]. □

**Definition 3.** Let $f_1, \dots, f_l$ be a family of Pfaffian functions defined in an open domain $G \subset \mathbf{R}^n$. The *number of consistent sign assignments* for this family is the number of all consistent (having a solution in $G$) systems of equations and strict inequalities of the kind

$$f_{i_1} = \cdots = f_{i_{l_1}} = 0, \quad f_{j_1} > 0, \dots, f_{j_{l_2}} > 0, \quad f_{k_1} < 0, \dots, f_{k_{l_3}} < 0,$$

where $\{i_1, \dots, i_{l_1}, j_1, \dots, j_{l_2}, k_1, \dots, k_{l_3}\} = \{1, \dots, l\}$. A (nonempty) set defined by such a system is called a *cell*. Note that any two cells have an empty intersection.

Obviously, the number of all consistent sign assignments does not exceed $3^l$. However, if the domain $G = \mathbf{R}^n$, we can obtain a less trivial bound.

**Proposition 2** (see [13]). *Let $f_1, \dots, f_l$ be a family of Pfaffian functions of order $r$ and degrees $(\alpha, \beta_1), \dots, (\alpha, \beta_l)$ defined in $\mathbf{R}^n$ by the same Pfaffian chain. Then the number of consistent sign assignments for this family does not exceed*

$$\min\left\{3^l, 4^{r^2+n+r} n^r(\alpha + \beta_1 + \cdots + \beta_l)^{n+r}\right\}.$$

*Proof.* The bound $3^l$ is trivial.

Choose one arbitrary point in each cell and obtain a finite set of points $\mathscr{X}$. A positive $\varepsilon \in \mathbf{R}$ exists such that for every $x \in \mathscr{X}$ and every $i$, $1 \le i \le l$, the inequality $f_i(x) > 0$ implies $f_i(x) > \varepsilon$, and $f_i(x) < 0$ implies $f_i(x) < -\varepsilon$.

Introduce a Pfaffian function, defined in $\mathbf{R}^n$:

$$g = \prod_{1 \le i \le l} (f_i + \varepsilon)^2 (f_i - \varepsilon)^2$$

of order $r$ and degree $(\alpha, \beta)$ where $\beta = 4 \sum_{1 \le i \le l} \beta_i$ (due to Lemma 1). Let us prove that the points

$$x^{(1)} = \sigma^{(1)} \cap \mathscr{X}, \qquad x^{(2)} = \sigma^{(2)} \cap \mathscr{X},$$

for two different cells, $\sigma^{(1)}$ and $\sigma^{(2)}$, belong to different connected components of $\{g > 0\}$ (according to the definition of $\varepsilon$, neither $x^{(1)}$ nor $x^{(2)}$ belong to $\{g = 0\}$).

Suppose that, contrary to our claim, $x^{(1)}$ and $x^{(2)}$ belong to the same connected component. It follows that there is a connected curve $\Gamma$ containing $x^{(1)}$ and $x^{(2)}$ and belonging to this connected component.

Since $x^{(1)}, x^{(2)}$ belong to different cells, at least one function $f_{i_0}$ $(1 \le i_0 \le l)$ having different signs at $x^{(1)}$ and $x^{(2)}$ exists.

Let, for instance, $f_{i_0}(x^{(1)}) > 0$ and $f_{i_0}(x^{(2)}) = 0$. Then $f_{i_0}(x^{(1)}) > \varepsilon$, so there is a point $x \in \Gamma$ such that $f_{i_0}(x) = \varepsilon$. Hence, $g(x) = 0$, which contradicts the definition of $\Gamma$.

All other combinations of signs of $f_{i_0}$ at $x^{(1)}$ and $x^{(2)}$ can be treated analogously. Therefore, the number of cells (consistent sign assignments) does not exceed the number $K$ of connected components of $\{g > 0\}$. An estimate

$$K \le 2^{r^2} \beta^n (r\alpha + n\beta)^r < 4^{r^2} n^r (\alpha + \beta)^{n+r}$$

follows from a more general result of Khovanskii [10]. □

**Definition 4.** An *elementary semi-Pfaffian set* is defined by a system of equations and inequalities

$$f_1(x) = \cdots = f_I(x) = 0, \qquad g_1(x) > 0, \dots, g_J(x) > 0, \tag{5}$$

where $f_i$, $g_j$ are Pfaffian functions with a common Pfaffian chain defined in an open domain $G \subset \mathbf{R}^n$. A *semi-Pfaffian set* is a finite union of elementary semi-Pfaffian sets with a common Pfaffian chain. Thus, a semi-Pfaffian set can be defined by a Boolean formula (in a disjunctive normal form) with atomic subformulas of the kind

$f_i = 0$ and $g_j > 0$. If a formula consists of $N$ disjunctions and all functions $f_i$, $g_j$ $(i = 1, \dots, I; j = 1, \dots, J)$ are Pfaffian of order $r$ and degree $(\alpha, \beta)$, defined by a common Pfaffian chain, then the 6-tuple $(N, I, J, r, \alpha, \beta)$ is called the *format* of the formula. For a system (5), define the format as $(I, J, r, \alpha, \beta)$.

**Definition 5.** A *weak stratification* of a semi-Pfaffian set $X$ is a subdivision of $X$ into a disjoint union of smooth, not necessarily connected (or even having a finite number of connected components), possibly empty semi-Pfaffian subsets $X_\alpha$, called *strata*. A stratification is *elementary* if all strata are elementary semi-Pfaffian sets. The system of equalities and inequalities for each stratum $X_\alpha$ of codimension $k$ includes a set of $k$ Pfaffian functions $h_{\alpha,1}, \dots, h_{\alpha,k}$ such that $h_{\alpha,j}|x_\alpha \equiv 0$, for $j = 1, \dots, k$, and $dh_{\alpha,1} \wedge \cdots \wedge dh_{\alpha,k} \neq 0$ at every point of $X_\alpha$. Note that for the algebraic case exactly the same kind of strata (under the name *algebraic partial manifolds*) were considered by Whitney [19]. We do not require the boundary of a stratum to be a union of some other strata.

## 3. Algorithm

The following algorithm for a weak stratification of an elementary semi-Pfaffian set is a modification of the algorithm suggested in [3]. It is based on the Whitney [19] approach to stratification of real semialgebraic sets.

Let $X \subset \mathbf{R}^n$ be an elementary semi-Pfaffian subset (5) of the format $(I, J, r, \alpha, \beta)$. Let $M_0=1$, $M_1=M(0, r, \alpha, \beta)$ (see (3) for the definition of $M(k, r, \alpha, \beta, \beta_1, \dots, \beta_k)$). For $1 \le k < n$, we define consecutively $\beta_k = \beta + (M_k - 1) \cdot [(\alpha - 1)k + \beta_1 + \cdots + \beta_{k-1}]$ and $M_{k+1} = M(k, r, \alpha, \beta_k, \beta_1, \dots, \beta_k)$.

For $1 \le k \le n$, consider a sequence $(\mathbf{i}, \mathbf{j}, \mathbf{m}^1, \dots, \mathbf{m}^k)$ where $\mathbf{i} = (i_1, \dots, i_k)$, $\mathbf{j} = (j_1, \dots, j_k)$, and $\mathbf{m}^\mu = (m_1^\mu, \dots, m_{i_\mu}^\mu)$, for $1 \le \mu \le k$, with the following properties:

$$1 \le i_1 < \cdots < i_k \le n, \qquad 1 \le j_\mu \le I, \tag{6}$$

$$0 \le m_1^\mu + \cdots + m_{i_1}^\mu \le M_1, \dots, 0 \le m_{i_{\mu-1}+1}^\mu + \cdots + m_{i_\mu}^\mu \le M_\mu, \qquad 0 < m_{i_\mu}^\mu, \tag{7}$$

$$(m_{i_\nu}^\mu, \dots, m_1^\nu, j_\mu) \prec (m_{i_\nu}^\nu, \dots, m_1^\nu, j_\nu) \qquad \text{for} \quad 1 \le \nu \le \mu. \tag{8}$$

Here $\prec$ is the lexicographic order.

For $1 \le i \le i_1$, we define $\hat{\partial}_i = \partial_i$. Let

$$h_1 = \hat{\partial}_{i_1}^{m_{i_1}^1 - 1} \hat{\partial}_{i_1-1}^{m_{i_1-1}^1} \cdots \hat{\partial}_1^{m_1^1} f_{j_1}, \qquad \hat{\partial}_i = \partial_{h_1, i_1, i} \quad \text{for} \quad i_1 < i \le i_2,$$

$$h_2 = \hat{\partial}_{i_2}^{m_{i_2}^2 - 1} \hat{\partial}_{i_2-1}^{m_{i_2-1}^2} \cdots \hat{\partial}_1^{m_1^2} f_{j_2}, \qquad \hat{\partial}_i = \partial_{h_1, h_2, i_1, i_2, i} \quad \text{for} \quad i_2 < i \le i_3,$$

and so on till

$$h_k = \hat{\partial}_{i_k}^{m^k_{i_k}-1}\, \hat{\partial}_{i_k-1}^{m^k_{i_k-1}} \cdots \hat{\partial}_1^{m^k_1} f_{j_k}, \qquad \hat{\partial}_i = \partial_{h_1,\dots,h_k,i_1,\dots,i_k,i} \quad \text{for} \quad i_k < i \le n.$$

Let $X^0 = \{x \in X;\, \hat{\partial}_n^{q_n} \cdots \hat{\partial}_1^{q_1} f_j(x) = 0, \text{ for } 1 \le j \le I,\, 0 \le q_1 + \cdots + q_n \le M_1\}$ and

$$\begin{aligned} X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\dots,\mathbf{m}^k} = \Big\{ & x \in X,\, \hat{\partial}_n^{q_n} \cdots \hat{\partial}_1^{q_1} f_j(x) = 0, \\ & \text{for } 0 \le q_1 + \cdots + q_{i_1} \le M_1, \dots, 0 \le q_{i_{k-1}+1} + \cdots + q_{i_k} \le M_k, \\ & 0 \le q_{i_k+1} + \cdots + q_n \le M_{k+1},\, 1 \le j \le I, \\ & (q_{i_1},\dots,q_1,j) \prec \left(m^1_{i_1},\dots,m^1_1,j_1\right),\dots,(q_{i_k},\dots,q_1,j) \\ & \prec \left(m^k_{i_k},\dots,m^k_1,j_k\right),\, \hat{\partial}_{i_1} h_1(x) \ne 0,\dots,\hat{\partial}_{i_k} h_k \ne 0 \Big\}, \end{aligned}$$

for $1 \le k \le n$.

**Theorem 1.** *Each set $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\dots,\mathbf{m}^k}$ is either empty or nonsingular of codimension $k$, with*

$$\Delta = \det \begin{pmatrix} \dfrac{\partial h_1}{\partial x_{i_1}} & \cdots & \dfrac{\partial h_1}{\partial x_{i_k}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial h_k}{\partial x_{i_1}} & \cdots & \dfrac{\partial h_k}{\partial x_{i_k}} \end{pmatrix} \ne 0$$

*at each point of $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\dots,\mathbf{m}^k}$. The set $X$ is a disjoint union of the sets $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\dots,\mathbf{m}^k}$ over all sequences $(\mathbf{i},\mathbf{j},\mathbf{m}^1,\dots,\mathbf{m}^k)$ satisfying* (6)–(8), *for* $0 \le k \le n$.

*Proof.* The sets $X^k_{\mathbf{i},\mathbf{j},\mathbf{m}^1,\dots,\mathbf{m}^k}$ with $\mathbf{i},\mathbf{j},\mathbf{m}^1,\dots,\mathbf{m}^k$ satisfying conditions (6)–(8) can be consecutively defined as follows. We consider all the partial derivatives $\partial_n^{q_n} \cdots \partial_1^{q_1} f_j$ with $q_1 + \cdots + q_n \le M_1$, ordered lexicographically in $(q_n,\dots,q_1,j)$. We consider either the set $X^0 \subset \mathbf{R}^n$ where all these derivatives vanish, or a set $Z^1_{i_1,j_1,\mathbf{m}^1}$ of those $x \in X$ where all the derivatives in the list preceding $h_1' = \partial_{i_1}^{m^1_{i_1}} \cdots \partial_1^{m^1_1} f_{j_1}$ vanish, while $h_1'(x) \ne 0$. Obviously, each $x \in X$ belongs either to $X^0$ or to one of the sets $Z^1_{i_1,j_1,\mathbf{m}^1}$, with $m^1_1 + \cdots + m^1_{i_1} \le M_1$, $m^1_{i_1} > 0$, and all these sets are disjoint.

In the first case, due to Proposition 1, all the functions $f_j$ are identically zero in a neighborhood of each $x \in X^0$, hence $X^0$, if nonempty, is a nonsingular open set in $\mathbf{R}^n$. In the second case, we define $h_1 = \partial_{i_1}^{m^1_{i_1}-1}\, \partial_{i_1-1}^{m^1_{i_1-1}} \cdots \partial_1^{m^1_1} f_{j_1}$, so that $h_1' = \partial_{i_1} h_1$, and consider a nonsingular submanifold $Y^1 = \{x \in X,\ h_1(x) = 0,\ h_1'(x) \ne 0\} \supseteq Z^1_{i_1,j_1,\mathbf{m}^1}$. Due to Lemma 4, the formats of all the Pfaffian functions that appear in the equations defining $Z^1_{i_1,j_1,\mathbf{m}^1}$ do not exceed $(r, \alpha, \beta_1)$. Let us denote the set of all these functions as $F^1$. Note that, for $i < i_1$, all the functions from $F^1$, including $h_1$, do not depend on $x_i$, due to Proposition 1.

We now consider the partial derivatives

$$\hat{\partial}_n^{q_n} \cdots \hat{\partial}_{i_1+1}^{q_{i_1+1}} = \partial_{h_1, i_1, n}^{q_n} \cdots \partial_{h_1, i_1, i_1+1}^{q_{i_1+1}}$$

of the functions $f_\nu \in F^1$, $\nu = (q_{i_1}, \dots, q_1, j)$, along $Y^1 \cap \{x_i = \text{const, for } i < i_1\}$ with $q_{i_1+1} + \cdots + q_n \le M_2$, ordered lexicographically in $(q_n, \dots, q_{i_1+1}, \nu)$. We consider either the set $X^1_{i_1, j_1, \mathbf{m}^1}$ where all these derivatives vanish, or a set $Z^2_{i_1, i_2, j_1, j_2, \mathbf{m}^1, \mathbf{m}^2}$ of those $x \in Z^1_{i_1, j_1, \mathbf{m}^1}$ where all the derivatives in the list preceding $h'_2 = \hat{\partial}_{i_2}^{m^2_{i_2}} \cdots \hat{\partial}_1^{m^2_1} f_{j_2}$ vanish, while $h'_2(x) \ne 0$.

Again, each $x \in X$ belongs either to $X^0$, or to one of the sets $X^1_{i_1, j_1, \mathbf{m}^1}$, or to one of the sets $Z^2_{i_1, i_2, j_1, j_2, \mathbf{m}^1, \mathbf{m}^2}$, with $m^1_1 + \cdots + m^1_{i_1} \le M_1$, $m^1_{i_1} > 0$,

$$m^2_{i_1+1} + \cdots + m^2_{i_2} \le M_2, \qquad m^2_{i_2} > 0, \qquad \left(m^2_{i_1}, \dots, m^2_1, j_2\right) \prec \left(m^1_{i_1}, \dots, m^1_1, j_1\right),$$

and all these sets are disjoint.

We apply Proposition 1 to show that $X^1_{i_1, j_1, \mathbf{m}^1} \cap \{x_i = \text{const, for } i < i_1\}$ is open in $Y^1 \cap \{x_i = \text{const, for } i < i_1\}$. As all the functions in the equations defining $X^1_{i_1, j_1, \mathbf{m}^1}$ and $Y^1$ do not depend on $x_i$, for $i < i_1$, this implies that $X^1_{i_1, j_1, \mathbf{m}^1}$ is open in $Y^1$ in the neighborhood of each $x \in X^1_{i_1, j_1, \mathbf{m}^1}$. Hence $X^1_{i_1, j_1, \mathbf{m}^1}$ is nonsingular of codimension 1.

The same arguments as before show that $Z^2_{i_1, i_2, j_1, j_2, \mathbf{m}^1, \mathbf{m}^2}$ belongs to

$$Y^2 = \Big\{x \in X, h_1(x) = h_2(x) = 0, h'_2(x) = \hat{\partial}_{i_2} h_2(x)$$
$$= \partial_{i_1} h_1(x)\, \partial_{i_2} h_2(x) - \partial_{i_2} h_1(x)\, \partial_{i_1} h_2(x) \ne 0\Big\},$$

with $h_2 = \hat{\partial}_{i_2}^{m^2_{i_2}-1} \hat{\partial}_{i_2-1}^{m^2_{i_2-1}} \cdots \hat{\partial}_1^{m^2_1} f_{j_2}$, which is nonsingular of codimension 2.

The continuation of this procedure leads to the consecutive definition of the sets $X^k_{\mathbf{i}, \mathbf{j}, \mathbf{m}^1, \dots, \mathbf{m}^k}$, for $k = 2, \dots, n$, with $(\mathbf{i}, \mathbf{j}, \mathbf{m}^1, \dots, \mathbf{m}^k)$ satisfying conditions (6)–(8). The same arguments as above show that all these sets are disjoint, nonsingular, and the union of all these sets is equal to $X$. □

The algorithm looks through all the sequences of the kind $(\mathbf{i}, \mathbf{j}, \mathbf{m}^1, \dots, \mathbf{m}^k)$ satisfying (6)–(8), and for each of them computes recursively the corresponding functions $\hat{\partial}_n^{q_n} \cdots \hat{\partial}_1^{q_1} f_j(x)$, $h_\mu$. Each recursion step consists of computing a determinant of a Jacobian matrix (see Definition 2) whose elements are polynomials in variables $x_1, \dots, x_n$ and in at most $r$ symbols of function belonging to the Pfaffian chain for the input. This computation can be done effectively by a version of a Gauss algorithm over the ring $\mathbf{R}[x_1, \dots, x_n, u_1, \dots, u_r]$ (see, e.g., [7]). Thus, the algorithm outputs the system of equations and inequalities defining the set $X^k_{\mathbf{i}, \mathbf{j}, \mathbf{m}^1, \dots, \mathbf{m}^k}$.

This concludes the description of the algorithm.

## 4. Complexity of the Algorithm

The computation protocol of the algorithm is a sequence of arithmetic operations over polynomials in variables $x_1, \dots, x_n$ and in symbols of functions occurring in the

Pfaffian chain for $f_1, \dots, f_l$, actually over real coefficients of these polynomials. An arithmetic operation over two reals, occurring in the sequence, is considered as an *elementary step* of the algorithm.

By the *complexity* (*running time*) of the algorithm we mean the number of its elementary steps (in the worst case) as a function of the format of the input system of inequalities.

For the complexity estimates we need the following lemma.

**Lemma 5.** *For $0 < k \le n$, the values $M_k = M(k-1, r, \alpha, \beta_{k-1}, \beta_1, \dots, \beta_{k-1})$ and $\beta_k$ are less than*

$$(2^{r^2}\beta(\alpha+\beta))^{2^{3k}(2r+3)^k}.$$

*Proof.* Proceed by induction on $k$. For $k = 1$, due to (3),

$$M(0, r, \alpha, \beta) \le 2^{r(r-1)/2}\beta(\alpha+\beta+1)^r \le 2^{r(r-1)/2}\beta(2\alpha+2\beta)^r \le 2^{r^2}\beta(\alpha+\beta)^r,$$

and, by the definition of symbol $\beta_1$, $\beta_1 \le \beta + M(0, r, \alpha, \beta)(\alpha - 1) \le \beta + 2^{r^2}\beta(\alpha+\beta)^{r+1}$.

Suppose, now, that the bound is proved for $\beta_k, M_k$ $(1 \le k < n)$. Observe that according to the definition of $\beta_i$, the values $\beta_i$ increase and $\beta_i \ge i$ $(1 \le i \le k)$. Because of this property and according to (3),

$$\begin{aligned} M_{k+1} &= M(k, r, \alpha, \beta_k, \beta_1, \dots, \beta_k) \\ &= 2^{r(r-1)/2}\beta_k\beta_1 \cdots \beta_k(\min\{r, k+1\}\alpha + \beta_k + \beta_1 + \cdots + \beta_k - k + 1)^r \\ &\le 2^{r^2}\beta_k\beta_1 \cdots \beta_k((k+1)\alpha + (k+1)\beta_k)^r \\ &= 2^{r^2}\beta_k\beta_1 \cdots \beta_k(k+1)^r(\alpha+\beta_k)^r \le 2^{r^2}\beta\beta_1 \cdots \beta_k(\alpha+\beta_k)^{2r+1}. \end{aligned}$$

Hence, due to the definition of symbols $\beta_i$,

$$\begin{aligned} \beta_{k+1} &\le \beta + M_{k+1}[(\alpha-1)(k+1) + \beta_1 + \cdots + \beta_k] \\ &\le 2^{2r^2}\beta\beta_1 \cdots \beta_k(\alpha+\beta_k)^{2r+1}(\alpha+\beta_k)(k+1) \le 2^{2r^2}\beta\beta_1 \cdots \beta_k(\alpha+\beta_k)^{2r+3}. \end{aligned}$$

Using the bounds for $\beta_i$ $(0 \le i \le k)$ from the inductive hypothesis, we get

$$M_{k+1} \le \beta_{k+1} \le 2^{2r^2}(2^{r^2}\beta(\alpha+\beta))^p\left(\alpha + (2^{r^2}\beta(\alpha+\beta))^{2^{3k}(2r+3)^k}\right)^{2r+3},$$

where, according to a formula for the sum of the first $k+1$ terms of geometric progression with the multiplier $2^3(2r+3)$, the power

$$p = \frac{2^{3(k+1)}(2r+3)^{k-1} - 1}{2^3(2r+3) - 1} < \frac{2^{3k+3}(2r+3)^{k+1}}{2^2} = 2^{3k+1}(2r+3)^{k+1}.$$

Hence,

$$M_{k+1} \le \beta_{k+1} \le 2^{2r^2}(2^{r^2}\beta(\alpha+\beta))^{2^{3k+1}(2r+3)^k}(2^{2r^2}\beta(\alpha+\beta))^{2^{3k}(2r+3)^{k+1}}$$
$$= 2^{2r^2+r^2 2^{3k+1}(2r+3)^k + r^2 2^{3k+1}(2r+3)^{k+1}}\beta^{2^{3k+1}(2r+3)^k+2^{3k}(2r+3)^{k+1}}$$
$$\times(\alpha+\beta)^{2^{3k+1}(2r+3)^k+2^{3k}(2r+3)^{k+1}}$$
$$\le 2^{r^2 2^{3k+3}(2r+3)^{k+1}}\beta^{2^{3k+3}(2r+3)^{k+1}}(\alpha+\beta)^{2^{3k+3}(2r+3)^{k+1}}$$
$$= (2^{r^2}\beta(\alpha+\beta))^{2^{3(k+1)}(2r+3)^{k+1}}.$$

The lemma is proved. □

For an arbitrary $c \in \mathbf{R}$, let

$$B(c) = (\alpha+\beta+1)^{(r+2)^{cn}}. \tag{9}$$

Lemma 5 implies that $M_n < B(c_1)$, for a positive constant $c_1$.

The algorithm considers successively less than $I^n 2^n M_n^{n^2} < I^n B(c_2)$ (for a constant $c_2 > 0$) sequences of the kind $(\mathbf{i}, \mathbf{j}, \mathbf{m}_1, \dots, \mathbf{m}_k)$, $1 \le k \le n$, each of which computes the functions $h_1, \dots, h_k$, and less than $IM_n^n$ functions of the kind $\hat{\partial}_n^{m_n} \cdots \hat{\partial}_1^{m_1} f_j$.

Each of these functions is obtained as a result of a successive application of differential operators of the kind $\partial_{\mathbf{h},\mathbf{p},q}$ (see Definition 2), i.e., of successively computing the determinants of appropriate Jacobian matrices with elements polynomial in variables $x_1, \dots, x_n$ and in symbols of functions in the Pfaffian chain for $f_1, \dots, f_I$. Due to the polynomial-time complexity of the Gauss algorithm from [7] and bounds on the formats of elements of the Jacobian matrices from Lemma 5, each function $h_\mu\, \hat{\partial}_n^{m_n} \cdots \hat{\partial}_1^{m_1} f_j$ is computed with complexity less that $B(c_3)$ for a constant $c_3 > 0$.

It follows that the total complexity of the algorithm is bounded from above by the value $I^{n+1}B(c_4)$ for a constant $c_4 > 0$.

We summarize the results proved in Sections 3 and 4 in the following theorem.

**Theorem 2.** *There is an algorithm which for an elementary semi-Pfaffian set $X$ produces a finite elementary stratification of $X$. The number of strata is less than $I^n B(c_2)$. Each stratum $X_i$ of codimension $k$ is an elementary semi-Pfaffian set defined by a system of equations and strict inequalities, including the input system (5), a system $h_{i1} = \cdots = h_{ik} = 0$ such that $h_{ij} \equiv 0$ on $X_i$, for $j = 1, \dots, k$, $dh_{i1} \wedge \cdots \wedge dh_{ik} \neq 0$ at every point of $X_i$, and possibly some other Pfaffian equations and inequalities. The format of the system defining $X_i$ is componentwise bounded from above by 5-tuple $(IB(c_4), J + 2^n, r, \alpha, B(c_1))$. All functions of the system have the same Pfaffian chain as the input functions. The running time of the algorithm is less than $I^{n+1}B(c_4)$. Here $c_1, \dots, c_4$ are positive constants, and $B(c)$ is defined by (9).*

**Remark 2.** Observe that the algorithm from Theorem 1 does not involve computations with the functions $g_1, \dots, g_J$. Thus, the functions need not be Pfaffian or even analytic. Observe, also, that we can modify the algorithm by replacing from the start the functions $f_1, \dots, f_I$ by the sum of their squares. In this case, $I = 1$ and all the bounds in Theorem 2 will not depend on the parameter $I$.

## 5. Arbitrary Semi-Pfaffian Sets

We can extend the algorithm to an arbitrary semi-Pfaffian set, i.e., finite union of (not necessarily disjoint) elementary semi-Pfaffian sets (see Definition 4). The idea is to represent the set as a *disjoint* union of elementary sets and then to apply Theorem 2 to each member of this union.

**Corollary 1.** *There is an algorithm which, for an arbitrary semi-Pfaffian set Y defined by a Boolean formula in a disjunctive normal form (DNF):*

$$Y = \bigcup_{1 \le l \le N} \{f_{l1} = \cdots = f_{lI_l} = 0, g_{l1} > 0, \dots, g_{lJ_l} > 0\} \tag{10}$$

*with* $\sum_{1 \le l \le N} I_l = I$, $\sum_{1 \le l \le N} J_l = J$, *with format* $(N, I, J, r, \alpha, \beta)$, *produces a finite elementary stratification of Y. The number of strata is less than* $3^{I+J}(I+J)^n B(c_5)$, *the format of each formula defining a stratum is bounded by the 5-tuple*

$$((I+J)B(c_5), I+J+2^n, r, \alpha, B(c_1)).$$

*All functions in a formula have the same Pfaffian chain as the input functions. The running time of the algorithm is less than*

$$3^{I+J}(I+J)^{n+1}B(c_5).$$

*Here* $c_5$ *is a positive constant and* $B(c)$ *is defined by (9).*

*Proof.* The algorithm considers all (not necessarily consistent) sign assignments for the family of functions $f_{li}, g_{lj}$, i.e., all $3^{I+J}$ systems of equations and strict inequalities that can be constructed using these functions.

The set $Y$ is the disjoint union of elementary semi-Pfaffian sets, defined by all sign assignments. For each elementary semi-Pfaffian set the algorithm applies the procedure from Theorem 2. □

The number of strata produced by the procedure from Corollary 1 and its complexity depend on the term $3^{I+J}$ which did not appear in the bounds of Theorem 2. We can avoid this term by taking the input functions with the whole space $\mathbf{R}^n$ as the domain, and using an *oracle* $\mathcal{O}$ for deciding whether a system of Pfaffian equations and inequalities is consistent.

An oracle is a subroutine which can be used by the algorithm any time it needs to check the consistency. We assume that this subroutine always gives the answer though we do not specify how it actually works. In fact, it is even unclear whether the problem of consistency for an arbitrary Pfaffian system is algorithmically decidable. However, for some classes of Pfaffian functions, closed under differentiation and arithmetic operations, the problem is definitely decidable. Apart from polynomials, such a class forms, for instance, terms of the kind $P(e^h, x_1, \dots, x_n)$ where $h$ is a fixed polynomial in $x_1, \dots, x_n$ and $P$ is an arbitrary polynomial in $u, x_1, \dots, x_n$ (see [17]). For such classes the oracle can be replaced by a deciding procedure, and we get an algorithm in the usual sense.

Denote the (possibly unknown) complexity of the oracle $\mathcal{O}$ by $D$. Thus, $D$ is a function of the parameters of the system to which the oracle is applied, i.e., we assume that each oracle call requires $D(\mathcal{F})$ *elementary oracle steps*, where $\mathcal{F} = (N', I', J', r', \alpha', \beta')$ is a format of a corresponding system of inequalities.

Thus, we assume that an *algorithm with an oracle* can have elementary steps of two sorts: arithmetic operations over reals and elementary oracle steps. The complexity of an algorithm with an oracle is the number of its elementary steps (in the worst case) as a function of the format of the input Boolean formula.

**Lemma 6** (see [5]). *Given an oracle $\mathcal{O}$ of complexity $D$, there is an algorithm which, for an arbitrary Boolean formula in DNF of a format $\mathcal{F} = (N, I, J, r, \alpha, \beta)$, defining a semi-Pfaffian set $Y$, as in* (10), *with functions $f_{li}, g_{lj}$ defined in $G = \mathbf{R}^n$, produces another Boolean formula in DNF, defining the same set $Y$, and such that the disjunction members define a disjoint family of elementary semi-Pfaffian sets. The number of these sets does not exceed*

$$\mathcal{B}(\mathcal{F}) = 4^{r^2+n+r} n^r (\alpha + (I+J)\beta)^{n+r}.$$

*Each elementary set uses the same family of atomic functions as $Y$ and the defining system of equations and strict inequalities with a format bounded by a 5-tuple $(I+J, I+J, r, \alpha, \beta)$. The running time of the algorithm is less than*

$$(I+J)\mathcal{B}(\mathcal{F}) D(I+J, I+J, r, \alpha, \beta).$$

*Proof.* The algorithm works recursively, building a tree $\mathcal{T}$ of height not exceeding $I+J$. The vertices of $\mathcal{T}$ are some consistent systems of Pfaffian inequalities and each vertex has a number of sons not exceeding three.

The root (the vertex of level zero) of $\mathcal{T}$ is an identically zero function. Suppose that the algorithm had constructed a system $K$ which is a vertex of $\mathcal{T}$ of level $i < I+J$. The algorithm chooses the $(i+1)$th function $f$ from the list $f_{11}, \dots, f_{NI_N}, g_{11}, \dots, g_{NJ_N}$ and decides the consistency of systems $K \& (f = 0)$, $K \& (f > 0)$, and $K \& (f < 0)$ with the help of the oracle $\mathcal{O}$. Every consistent system among them is a son of $K$, a vertex of level $i+1$. The process of constructing $\mathcal{T}$ terminates when $i = I+J$.

Observe that the family of sets defined by all terminal vertices of $\mathcal{T}$ coincides with the family of all cells for $f_{11}, \dots, f_{NI_N}, g_{11}, \dots, g_{NJ_N}$. Using $\mathcal{O}$ select all the cells contained in $Y$. Then the disjunction of all selected terminal vertices is a desired output of the algorithm.

From the Proposition 2 it follows that the number of cells does not exceed $\mathscr{B}(\mathscr{T})$. Therefore, the total number of vertices in $\mathscr{T}$ is less than $(I+J)\mathscr{B}(\mathscr{T})$ and the complexity of the algorithm is bounded by

$$(I+J)\mathscr{B}(\mathscr{T})D(I+J, I+J, r, \alpha, \beta). \qquad \square$$

**Theorem 3.** *Given an oracle $\mathscr{O}$ of complexity D, there is an algorithm which, for an arbitrary semi-Pfaffian set Y defined by a Boolean formula in DNF of a format $(N, I, J, r, \alpha, \beta)$, so that* (10) *holds, with functions $f_{li}, g_{lj}$ defined in $G = \mathbf{R}^n$, produces a finite elementary stratification of Y. The number of strata is less than $(I + J)^{n+r}B(c_6)$. The format of each formula defining a stratum is bounded by a 5-tuple*

$$((I+J)B(c_6), I+J+2^n, r, \alpha, B(c_6)).$$

*All functions in a formula have the same Pfaffian chain as the input functions. The running time of the algorithm is less than*

$$(I+J)^{n+r}B(c_6)D(I+J, I+J, r, \alpha, \beta).$$

*Here $c_6$ is a positive constant and $B(c)$ is defined by* (6).

*Proof.* First the algorithm uses the procedure from Lemma 6 to represent $Y$ as a union of disjoint elementary semi-Pfaffian sets. After that it applies the method from Theorem 2 to stratify each of these sets. The family of all the strata produced forms a stratification of $Y$.

The complexity analysis is straightforward. $\square$

## 6. Fewnomials and Exponential Polynomials

Generalizing Examples (a) and (e) from Section 2, we can consider a polynomial $f \in \mathbf{R}[x_1, \ldots, x_n]$ as a Pfaffian function of two different formats.

1. (Sparse representation). Each monomial

$$f_{i_1 \cdots i_n} = a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

of $f$ with $a_{i_1 \cdots i_n} \neq 0$ is a Pfaffian function in the domain $G = \{x_1 \cdots x_n \neq 0\} \subset \mathbf{R}^n$, of order $n+1$ and degree (2, 1), due to the equations

$$df_{i_1 \ldots i_n} = \sum_{1 \le j \le n} i_j f_{i_1 \ldots i_n} g_j \, dx_j,$$

$$dg_j = -g_j^2 \, dx_j,$$

with $g_j = 1/x_j$.

According to Lemma 1, a polynomial $f$ is a Pfaffian function in $G$ of degree (2, 1) and order $n + m$, where $m$ is the number of all monomials in $f$ with nonzero coefficients.

Let $\mathscr{K}$ be a set of all monomials of $f$. In the sparse setting $f$ is called a *fewnomial* or *sparse polynomial* with support $\mathscr{K}$.

A polynomial $F = P(x_1, \dots, x_n, u_1, \dots, u_m)$ of degree $\beta$ in variables $x_i$ and monomials $u_j \in \mathscr{K}$ is called a *sparse polynomial of pseudodegree* $\beta$ with support $\mathscr{K}$. Obviously $F$ is a Pfaffian function of degree $(2, \beta)$ and of order $n + m$. Note that $\beta$ may be not equal to the degree $d$ of the polynomial $P$ after substitution of monomials $u_j$. We call $d$ the *degree* of $F$.

2. (Dense representation). On the other hand (see Example (a), Section 2), polynomial $f$ of degree $d$ can be considered as "dense," i.e., as a Pfaffian function (in $\mathbf{R}^n$) of order 0 and degree $(\alpha, d)$, where $\alpha$ is arbitrary.

Consider a semialgebraic set $Y$ defined by formula (10), where the degrees of the polynomials $f_{li}, g_{lj} \in \mathbf{R}[x_1, \dots, x_n]$ are less than $d$ and the total number of monomials with nonzero coefficients in the polynomials $f_{li}$ is $m$.

In the sparse representation (i.e., $f_{li}$, are considered as fewnomials with a common support $\mathscr{K}$, card$(\mathscr{K}) = m$) the format of (10) is $(N, I, J, n + m, 2, 1)$. In the dense representation the format can be, e.g., $(N, I, J, 0, 0, d)$.

Note that in the sparse representation the functions $f_{li}$ and $g_{lj}$ are defined only in the domain $G$.

Applying Theorem 3 to formula (10) in the dense representation, we get an algorithm for a stratification of $Y$. In this case, we replace the oracle $\mathscr{O}$ by a genuine effective procedure for deciding the consistency of semialgebraic sets [6], [15], [1], [20].

**Corollary 2** (Dense Stratification of Semi-Algebraic Sets). *There is an algorithm which for a semi-algebraic set $Y$ of with a format $(N, I, J, 0, 0, d)$ defined by (10) produces a finite elementary stratification for $Y$. The number of strata is less than $(I + J)^n B'(c_7)$. Each stratum $Y_0$ is represented by a system of (dense) polynomial equations and strict inequalities with a format bounded by a 5-tuple $((I + J)B'(c_7), I + J + 2^n, 0, 0, B'(c_7))$. The running time of the algorithm is less than*

$$(I + J)^{c_7 n} d^{2^{c_7 n}}.$$

*Here $c_7$ is a positive constant,*

$$B'(c) = d^{2^{cn}}$$

*for arbitrary $c \in \mathbf{R}$.*

*Proof.* Use the procedures and their complexity bounds from [6] and [15]. □

**Remark 3.** Using the procedures from [6] and [15] the algorithm can also select all nonempty strata from among the one produced with the complexity bound $(I+J)^{c_8 n} d^{2^{c_8 n}}$ for a positive $c_8$. This, appended, algorithm proves a special case of a known theorem (see, e.g., [14], [1], and [20]) stating that a *Whitney stratification* of a semialgebraic set $Y$ can be produced in time $(I+J)^{cn} d^{2^{cn}}$. However, the known proofs of this theorem are specifically algebraic (involving resultants, etc.) and very cumbersome. Also, they do not produce, as the algorithm from Corollary 2, for each stratum, a system of equations having the Jacobian matrix of the maximal rank at every point of the stratum.

Now consider the case of $Y$ defined by (10) in the sparse representation.

**Corollary 3** (Sparse Stratification of Semi-Algebraic Sets). *There is an algorithm which, for a semi-algebraic set $Y$ defined by* (10), *of the format* $(N, I, J, n+m, 2, 1)$, *having atomic fewnomials with a common support* $\mathscr{K}$, $\mathrm{card}(\mathscr{K}) = m$, *and degrees less than* $d$, *produces a finite elementary stratification for* $Y$. *The number of strata is less than* $(I+J)^{2n+m} B''(c_8)$. *Each stratum* $Y_0$ *is represented by a system of sparse polynomial equations and inequalities of pseudodegree* $B''(c_8)$ *with support* $\mathscr{K}$ *of a format bounded by* $\mathscr{F} = ((I+J)B''(c_8), I+J+2^n, n+m, 2, B''(c_8))$. *The running time is less than*

$$(I+J)^{c_8 n + m} B''(c_8) d^{c_8 n}.$$

*Here* $c_8$ *is a positive constant,*

$$B''(c) = 2^{(n+m)^{cn}}$$

*for arbitrary* $c \in \mathbf{R}$.

*Proof*. The estimates of the number of strata and the format of a stratum are straightforward, taking into the account that a common Pfaffian chain for all $f_{li}, g_{lj}$ has the order $n+m$. Note that the bounds depend only on the format of the input (and do not depend on the degree $d$). The bound on the running time includes, however, the estimate of the complexity of deciding consistency of systems of polynomial inequalities. According to Theorem 3, the running time is less than $(I+J)^{2n+m} B''(c_9) D(I+J, I+J, r, \alpha, \beta, d)$, where $d$ is an upper bound for the degrees of the input polynomials, considered as dense, and real $c_9 > 0$.

Using the decision procedure and complexity bounds from [6] and [15], we can take for $D(I+J, I+J, r, \alpha, \beta, d)$ the value

$$((I+J)d)^{c_{10} n},$$

for some $c_{10} > 0$. Thus the total running time of the sparse stratification algorithm is less than

$$(I+J)^{c_8 n + m} B''(c_8) d^{c_8 n}$$

for a positive constant $c_8$, and the corollary is proved. □

**Remark 4.** As in the case of the dense stratification (Remark 3), the algorithm can use the procedures from [6] and [15] to select all nonempty strata from among the one produced with the complexity bound

$$(I+J)^{2n+m}B''(c_{11})D(\mathcal{F},\Delta),$$

where $\Delta$ is an upper bound for the degrees of the *output* polynomials considered as dense. Let us compute $\Delta$.

First observe that for a polynomial $f$ of degree $\gamma$ and a set $\mathbf{h}=(h_1,\dots,h_k)$ of polynomials of degrees $\gamma_1,\dots,\gamma_k$, respectively, for a set of distinct indices $\{\mathbf{i}=(i_1,\dots,i_k),j\}$ and for $m>0$ the polynomial $\partial^m_{h,\mathbf{i},j}(f)$ is of degree $\gamma'=\gamma+m(\gamma_1+\cdots+\gamma_k)$. It follows that the degrees of (dense) polynomials $h_k(x)$, $\hat{\partial}^{m_n}_n\cdots\hat{\partial}^{m_1}_1 f_j(x)$, appearing on the recursive steps $k$ of the computation, do not exceed $d_k=d+M_k(d_1+\cdots+d_{k-1})$ and $d'_{k+1}=d+M_{k+1}(d_1+\cdots+d_{k-1}+2d_k)$, respectively, where $d_1,\dots,d_{k-1}$ are the upper bounds for the degrees of the corresponding polynomials $h_1,\dots,h_{k-1}$ and $M_k=M(k-1,r,\alpha,\beta_{k-1},\beta_1,\dots,\beta_{k-1})$. Since the sequence of integers $M_i$, $1\le i\le n$, increases (see (3)), we have, by induction,

$$d_k<2^{k-1}dM_k^{k-2},$$

so, due to Lemma 5, $\Delta$ can be taken to be less than

$$2^{n-1}dM_n^{n-2}<dB''(c_{12})$$

for a positive constant $c_{12}$.

Using the decision procedure and complexity bounds from [6] and [15], we can take for $D(\mathcal{F},\Delta)$ the value

$$((I+J)d)^{c_{13}n}B''(c_{13}),$$

and the total running time of the appended algorithm will be

$$(I+J)^{c_{14}n+m}B''(c_{14})d^{c_{14}n}$$

for a positive $c_{14}$.

Generalizing the case of a semialgebraic set, consider $Y\subset\mathbf{R}^n$, defined by (10), in which

$$f_{li}=P_{f_{li}}(e^{h(x_1,\dots,x_n)},x_1,\dots,x_n),$$

$$g_{lj}=P_{g_{lj}}(e^{h(x_1,\dots,x_n)},x_1,\dots,x_n),$$

where $h(x_1,\dots,x_n)$, $P_{f_{li}}(u,x_1,\dots,x_n)$, and $P_{g_{lj}}(u,x_1,\dots,x_n)$ are polynomials of degree less than $d$.

Suppose that the number of monomials (with nonzero coefficients) in polynomial $h$ is $t$ and the total number of monomials in polynomials $P_{f_{li}}$ and $P_{g_{lj}}$ is bounded by $m$. As in the case of polynomials, we can assign at least two different formats to the functions $f_{li}$, $g_{lj}$.

Let $P$ be either $P_{f_{li}}$ or $P_{g_{lj}}$. In the *dense* setting,

$$f = P(e^h, x_1, \ldots, x_n) \tag{11}$$

is a Pfaffian function of order 1 and degree $(d, d)$ in $\mathbf{R}^n$, and the Pfaffian chain consists of the unique function $e^h$.

On the other hand (*sparse* setting), $f$ is a Pfaffian function of order $n + t + m$ and degree $(3, 1)$ in $G = \{x_1 \cdots x_n \neq 0\}$. Indeed, let

$$f_{i_0 i_1 \ldots i_n} = a_{i_0 i_1 \ldots i_n} e^{i_0 h} x_1^{i_1} \cdots x_n^{i_n}$$

be a "monomial" of $f$. Then

$$\frac{\partial f_{i_0 i_1 \ldots i_n}}{\partial x_j} = i_0 \frac{\partial h}{\partial x_j} f_{i_0 i_1 \ldots i_n} + i_j f_{i_0 i_1 \ldots i_n} g_j,$$

with $g_j = 1/x_j$. Substituting here $\partial h / \partial x_j$ from the equations for the sparse representation of $h$ as a Pfaffian function of order $n + t$ and degree $(2, 1)$ (see the beginning of this section) we represent $\partial f_{i_0 i_1 \ldots i_n} / \partial x_j$ as a polynomial of degree 3 in $f_{i_0 i_1 \ldots i_n}$ and the $n + t$ elements of the Pfaffian chain for $h$. Therefore, a Pfaffian chain for $f$ consists of the Pfaffian chain for $h$, plus the $m$ functions $f_{i_0 i_1 \ldots i_n}$. Hence $f$ is a Pfaffian function with support $\mathscr{K}$ of order $n + t + m$ and degree $(3, 1)$ (Lemma 3, though applicable, gives a weaker bound).

Let $\mathscr{K}$ be a set of all monomials of $f$ (i.e., expressions of the kind $f_{i_0 i_1 \ldots i_n}$). In the sparse setting $f$ is a *sparse expression with support* $\mathscr{K}$.

The expression $F = Q(x_1, \ldots, x_n, u_1, \ldots, u_m)$ where $Q$ is a polynomial of degree $\beta$ in both $x_i$ and $u_j \in \mathscr{K}$ is called a *sparse expression of pseudodegree* $\beta$ with support $\mathscr{K}$.

Thus, $F$ is Pfaffian of degree $(3, \beta)$ and order $n + m + t$. Here $\beta$ may not be equal to the degree of $Q$ in $x_1, \ldots, x_n, e^h$ after substitution of monomials $u_j$.

**Corollary 4.**

1. (*Dense stratification*). *Let $Y$ be a set defined by* (10) *with $f_{ij}, g_{ij}$ being dense expressions of the form $P(e^h, x_1, \ldots, x_n)$. There is an algorithm which produces a finite stratification of $Y$. The number of strata is less than $(I + J)^n B'(c_{15})$. Each stratum $Y_0$ is represented by a system of equations and strict inequalities with atomic functions of the kind* (11) *and of a format bounded by $((I + J)B'(c_{15})$, $I + J + 2^n, 1, d, B'(c_{15}))$. The running time is less than*

$$(I + J)^{c_{15} n} d^{2^{c_{15} n}}.$$

*Here $c_{15}$ is a positive constant.*

2. (*Sparse stratification*). *Let $Y$ be a set defined by* (10) *with $f_{ij}, g_{ij}$ being sparse expressions of the form $P(e^h, x_1, \ldots, x_n)$. There is an algorithm which produces a finite stratification of $Y$. The number of strata is less than $B^{(3)}(c_{16})$. Each stratum $Y_0$ is represented by a system of equations and strict inequalities of the kind*

(11) *of pseudodegree* $B^{(3)}(c_{16})$ *with support* $\mathscr{K}$ *of a format bounded by* $((I+J)B^{(3)}(c_{16}), I+J+2^n, n+t+m, 3, B^{(3)}(c_{16}))$. *The running time is less than*

$$(I+J)^{c_{16}n+m}B^{(3)}(c_{16})d^{c_{16}n}.$$

*Here* $c_{16}$ *is some positive constant,*

$$B^{(3)}(c) = 2^{(n+t+m)^{cn}}$$

*for arbitrary* $c \in \mathbf{R}$.

*Proof.* The proof is analogous to proofs of Corollaries 2 and 3 (and statements from Remarks 3 and 4), except that here we use the procedure for deciding the consistency of systems of exponential polynomial inequalities and its complexity estimate from [17]. □

**Remark 5.** By *coefficients* of a Pfaffian function $f$ we mean the coefficients of all polynomials $g_{ij}$ and $P$ from Definition 1. Let, for Pfaffian functions in (10), their coefficients be integral with absolute values less than $2^M$ for a positive integer $M$. Then a straightforward computation shows that the bounds on the bit sizes of coefficients of atomic functions in formulas, defining smooth strata, from Theorems 2 and 3, Corollaries 2–4, and Remarks 3 and 4, depend polynomially on $M$. Taking into account the size of the coefficients, it is natural to take bit operations over integers as elementary steps of the algorithm (see the beginning of Section 4). In these terms the complexities of the algorithms from Corollaries 2–4 (and Remarks 3 and 4) depend polynomially on $M$.

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