DOI 10.1007/s10958-023-06299-z
Journal of Mathematical Sciences, Vol. 269, No. 4, January, 2023

COMPLEXITY OF THE LAMBEK CALCULUS WITH ONE DIVISION AND A NEGATIVE-POLARITY MODALITY FOR WEAKENING

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Abstract. In this paper, we consider a variant of the Lambek calculus allowing empty antecedents. This variant uses two connectives: the left division and a unary modality that occurs only with negative polarity and allows weakening in antecedents of sequents. We define the notion of a proof net for this calculus, which is similar to those for the ordinary Lambek calculus and multiplicative linear logic. We prove that a sequent is derivable in the calculus under consideration if and only if there exists a proof net for it. We present a polynomial-time algorithm for deciding whether an arbitrary given sequent is derivable in this calculus.

Introduction

The aim of this paper is to find an efficient algorithm for deciding whether an arbitrary given sequent is derivable in a variant of the Lambek calculus allowing empty antecedents. This variant uses two connecives: the left division and a unary modality that occurs only with negative polarity and allows weakening in antecedents of sequents. We prove that there is a fast algorithm (its running time is bounded by a polynomial in the length of the assertion).

In Sec. 1, we introduce the calculus $L^*(\backslash,!)$, which is a variant of the Lambek calculus allowing empty antecedents. The only connectives of $L^*(\backslash,!)$ are the left division and a modality for weakening, which occurs only with negative polarity. In Sec. 2, we prove a derivability criterion for the calculus $L^*(\backslash,!)$. The criterion is based on graphs of a special form. In Sec. 3, we show how to verify the condition of this criterion in ploynomial time.

1. The Calculus

Lambek calculus was introduced in [1]. We consider the Lambek calculus allowing empty antecedents; it is denoted by L^*. We add a unary modality for weakening, denoted by !. We consider the fragment where only the left division and the modality ! are allowed. The *types* of this calculus are built of primitive types $p_1, p_2, \dots$, the binary connective $\backslash$, and the unary connective !. We agree that the conncetive ! has higher priority, i.e., $!A\backslash B$ stands for $(!A)\backslash B$, and not for $!(A\backslash B)$.

The set of all types thus built is denoted by $\mathrm{Tp}(\backslash,!)$. Capital letters $A, B, \dots$ range over types, whereas capital Greek letters range over finite sequences of types. The set of all finite sequences of types is denoted by $\mathrm{Tp}(\backslash,!)^*$. The empty sequence is denoted by Λ.

If $\Gamma = A_1 \dots A_n$, then by $!\Gamma$ we denote the sequence $(!A_1) \dots (!A_n)$. By $!^n A$ we denote the type $\underbrace{!\dots!}_{n\text{ times}} A$.

Derivable objects of the Lambek calculus with one division and a modality for weakening allowing empty antecedents are *sequents* of the form $\Gamma \to A$. (Here Γ is the *antecedent* and A is the *succedent* of the sequent.) The axioms are of the form $p_i \to p_i$. Derivations are built using the following rules:

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 23, No. 4, pp. 143–162, 2021.

$$\frac{A\Pi \to B}{\Pi \to A\backslash B}\ (\to \backslash), \qquad \frac{\Pi \to A \quad \Gamma B \Delta \to C}{\Gamma \Pi (A\backslash B)\Delta \to C}\ (\backslash \to),$$
$$\frac{\Gamma A \Delta \to B}{\Gamma !A \Delta \to B}\ (! \to), \qquad \frac{\Gamma \Delta \to B}{\Gamma !A \Delta \to B}\ (\mathrm{W}! \to).$$

We shall denote this calculus by $\mathrm{L}^*(\backslash, !)$.

Example 1.1. The derivation

$$\dfrac{\dfrac{p_1 \to p_1}{\to p_1\backslash p_1}\ (\to \backslash) \quad \dfrac{p_2 \to p_2}{!p_2 \to p_2}\ (! \to)}{(p_1\backslash p_1)\backslash !p_2 \to p_2}\ (\backslash \to)$$

shows that $\mathrm{L}^*(\backslash, !) \vdash (p_1\backslash p_1)\backslash !p_2 \to p_2$.

Example 1.2. The derivation

$$\dfrac{\dfrac{p_2 \to p_2}{!(p_1\backslash p_1)\, p_2 \to p_2}\ (\mathrm{W}! \to)}{p_2 \to !(p_1\backslash p_1)\backslash p_2}\ (\to \backslash)$$

shows that $\mathrm{L}^*(\backslash, !) \vdash p_2 \to !(p_1\backslash p_1)\backslash p_2$.

We shall formulate a derivability criterion for $\mathrm{L}^*(\backslash, !)$, analogous to the criteria for $\mathrm{L}(\backslash)$ and $\mathrm{L}^*(\backslash)$, proved by Yury Savateev (cf. [4, 5]), and the criterion for $\mathrm{LM}^*(\backslash)$ from [2].

2. Proof Nets

We found flaws in our paper [3], where a derivability criterion was proposed for the calculus $\mathrm{L}^*(\backslash, !)$. The statements of Theorem 2.7 and Lemma 2.17 in [3] are not valid. Thus, we had to make the definition of a proof net more complex (and add a binary relation R).

Let us now prove the derivability criterion for the calculus $\mathrm{L}^*(\backslash, !)$ in a corrected form. The criterion is based on graphs of a special form. Following the tradition of linear logic, we call such a graph a *proof net*. At the end of this section, we shall prove that a sequent is derivale in $\mathrm{L}^*(\backslash, !)$ if and only if there exists a proof net correspoding to the sequent.

Formally speaking, a proof net consists of a directed tree $\langle V, O\rangle$ with the set of leafs W, a labeling of leafs label: $W \to \{p_1, p_2, \ldots\}$, a labeling of edges op: $O \to \{-1, 1, 2\}$, a strict linear order $<$ on the set V, a set $U \subseteq V$, and two binary relations $S \subseteq V \times V$ and $R \subseteq V \times V$. We shall denote the root of the tree $\langle V, O\rangle$ by $\overset{0}{g}$. Here the tree $\langle V, O\rangle$ is given by the set of vertices V and the set of edges O, where the edges are directed towards the root, i.e., the binary relation O is the graph of a function from $V \setminus \{\overset{0}{g}\}$ to V (this function will also be denoted by O) and for any element $a \in V$, there is a natural number n such that

$$\underbrace{O(\ldots O}_{n \text{ times}}(a)\ldots) = \overset{0}{g}.$$

Note that the function label need not be injective.

The first four components (the tree $\langle V, O\rangle$, the two labelings, and the linear order $<$) form a *proof frame*. A proof frame is constructed for a given sequent in a unique manner as follows. A proof frame for a sequent

$$A_1 \ldots A_{n-1}\, A_n \to B$$

coincides with a proof frame for the type

$$A_n\backslash(A_{n-1}\backslash \ldots \backslash(A_1\backslash B)\ldots),$$

whereas proof frames for types are constructed by induction over types. A primitive type has a proof frame with one vertex (which is labeled by this primitive type).

The proof frame for a type of the form $!A$ is obtained from the proof frame for A by adding a new root and an edge from the root of the proof frame for A to the new root (the label of this edge is 2). The linear order is the same as in the proof frame for A, but the new root is added as the mininal element.

The proof frame for $A\backslash B$ is obtained from the disjoint union of the proof frames for A and B by adding a new root and two edges leading from the root of the proof frame for A to the new root (the label of this edge is -1) and from the root of the proof frame for B to the new root (the label of this edge is 1). The linear order is constructed as the sum of the reverse order from the proof frame for A, the trivial order on the root (as a singleton set), and the (non-inverted) order from the proof frame for B.

This completes the inductive construction of the labeled tree and the linear order for any type. Before formulating the conditions for the remaining two components of a proof net, we introduce some notation and prove some lemmas.

The number of edges labeled -1 on the path from a vertex a to the root will be denoted by $\mathrm{d}(a)$. For example, $\mathrm{d}(\overset{0}{g}) = 0$.

By $\le$ we denote the reflexive closure of the binary relation $<$. By O^+ we denote the transitive closure of O. By O^* we denote the reflexive transitive closure of O.

For each vertex a, by $H(a)$ we denote the (unique) leaf from which to a there is a path without negative edge labels. It is obvious from the definition of d that $\mathrm{d}\big(H(a)\big) = \mathrm{d}(a)$.

Lemma 2.1. *If* $\mathrm{d}(a) = 0$*, then* $H(a) = H(\overset{0}{g})$.

It is easy to prove (by induction on types) that in any proof net the following statements hold:

(D1) If $a \le b \le c$, $a\, O^*\, d$, and $c\, O^*\, d$, then $b\, O^*\, d$.
(D2) If $a\, O^*\, b$ and $\mathrm{d}(b)$ is even, then $a \le H(b)$.
(D3) If $a\, O^*\, b$ and $\mathrm{d}(b)$ is odd, then $H(b) \le a$.
(D4) If $a\, O^*\, b$ and $\mathrm{d}(a) \ne \mathrm{d}(b)$, and $\mathrm{d}(b)$ is even, then $a < H(b)$.
(D5) If $a\, O^*\, b$ and $\mathrm{d}(a) \ne \mathrm{d}(b)$, and $\mathrm{d}(b)$ is odd, then $H(b) < a$.

We shall denote by $\mathrm{pr}_1(S)$ the first projection of the relation S. Formally,

$$\mathrm{pr}_1(X) = \{a \mid \exists b\ (a,b) \in X\}.$$

As the last three components of a proof net one can take any sets $U \subseteq V$, $S \subseteq V \times V$, and $R \subseteq V \times V$, satisfying the conditions listed below. For brevity, we shall denote $T = S \cup R \cup R^{-1}$.

(PN1) If $(a,b) \in S$, then $(b,a) \in S$ (i.e., the relation S is symmetric).
(PN2) If $(a,b) \in S$ and $(a,c) \in S$, then $b = c$ (i.e., the relation S is functional).
(PN3) If $(a,b) \in T$, $(c,d) \in T$, and $a < c < b$, then $a \le d \le b$.
(PN4) If $(a,b) \in S$, then $\mathrm{label}(a) = \mathrm{label}(b)$.
(PN5) If $(a,b) \in T$ and $a < b$, then $\mathrm{d}(a) = \mathrm{d}(b) + 1$.
(PN6) If $a\, O^*\, c$, $b\, T\, H(c)$, and $a < b < H(c)$, then there is an element d such that $d\, T\, H(c)$ and either $d < b$, or $H(c) < d$.
(PN7) $S \ne \varnothing$.
(PN8) $\mathrm{pr}_1(S) = W \cap U$.
(PN9) If $a\, O\, b$ and $a \in U$, then $b \in U$.
(PN10) If $a\, O\, b$, $b \in U$, and $\mathrm{op}(a,b) \ne 2$, then $a \in U$.
(PN11) If $a\, O\, b$ and $\mathrm{op}(a,b) = 2$, then $\mathrm{d}(b)$ is odd.
(PN12) $\mathrm{pr}_1(R) = \{b \in U \mid \exists a\, (a \notin U \wedge a\, O\, b)\}$.
(PN13) If $(a,b) \in R$, then $b \in \mathrm{pr}_1(S)$.
(PN14) If $(a,b) \in R$ and $(a,c) \in R$, then $b = c$ (i.e., the relation R is functional).

Note that from PN1 it follows that the relation T is symmetric.

Lemma 2.2. *If* $S(a)$ *is defined, then* $H\big(S(a)\big) = S(a)$.

Lemma 2.3. *If* $\mathrm{d}(a)$ *is even and* $a \in U$*, then* $H(a) \in U$.

Proof. Suppose to the contrary that $H(a) \notin U$. In view of (PN11), for all vertices b on the path from $H(a)$ to a we have $\mathrm{d}(b) = 2n$. In view of (PN11), none of these vertices belongs to U. □

Lemma 2.4. *If $a \in \mathrm{pr}_1(R)$, then $\mathrm{d}(a)$ is odd.*

Proof. Let $b \in \mathrm{pr}_1(R)$. In view of (PN12) we obtain $a \notin U$ and $O(a) = b \in U$. In view of (PN10) we obtain $\mathrm{op}(a, b) = 2$. From (PN11) it follows that $\mathrm{d}(a)$ is odd. □

Lemma 2.5. *If $b\ T\ a$, $b\ T\ c$, and $a \neq c$, then $\mathrm{d}(b)$ is even.*

Lemma 2.6. *If $a \in \mathrm{pr}_1(T)$, then $a \in U$.*

Proof. Let $a\ T\ b$. If $a\ S\ b$ or $b\ R\ a$, then from (PN8) and (PN13) we obtain $a \in U$. If $a\ R\ b$, then $a \in U$ follows from (PN12). □

Lemma 2.7. *If $a \in U$ and there is no element $b \in U \cap W$ such that $b\ O^*\ a$, then there is an element c such that $c\ O\ a$ and $\mathrm{op}(c, a) = 2$.*

In view of (PN2), the relation S is the graph of a partial function. We shall also denote this partial function by S.

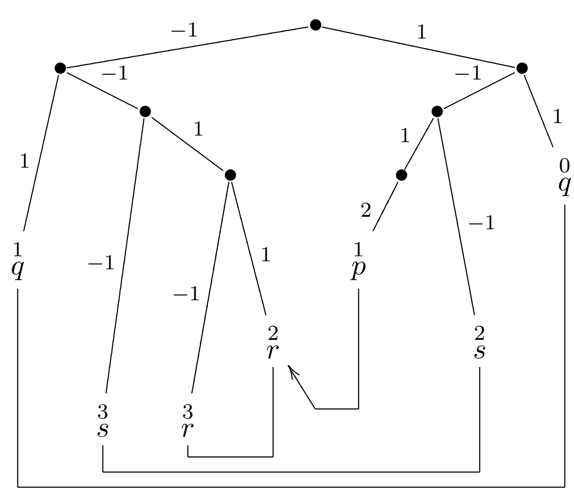
For minima, maxima, open intervals, closed intervals, and left-closed, right-open intervals with respect to the linear order $<$ we shall use the notation min, max, $(a; b)$, $[a; b]$, and $[a; b)$. If $a \in V$ and $\beta \subseteq V$, then the formula $a < \beta$ means that for all $b \in \beta$ we have $a < b$.

In diagrams, we shall locate the vertices of the tree on the plane so that moving from left to right corresponds to the order $<$ (if $a < b$, then a is to the left of b) whereas the edges lead upwards. In a diagram, a vertex a is represented by label(a) (with the number $\mathrm{d}(a)$ above it). The binary relation O is shown by straight arrows (labeled with the values of op), whereas the relations S and R are shown by polygonal paths each of which consists of three segments (with an arrowhead in the case of R).

Example 2.8. The sequent $s\backslash !p\,\big(s\backslash(r\backslash r)\big)\backslash q \to q$ is derivable.

$$\dfrac{\dfrac{\dfrac{s \to s \qquad \dfrac{\dfrac{r \to r}{r\ !p \to r}\,(\mathrm{W}! \to)}{!p \to r\backslash r}\,(\to \backslash)}{s\,(s\backslash !p) \to r\backslash r}\,(\backslash \to)}{s\backslash !p \to s\backslash(r\backslash r)}\,(\to \backslash) \qquad q \to q}{s\backslash !p\,\big(s\backslash(r\backslash r)\big)\backslash q \to q}\,(\backslash \to)$$

One of its possible proof nets is shown below.



Lemma 2.9. *Every proof net for a sequent of the form $\Pi \to A\backslash B$ is also a proof net for the sequent $A\,\Pi \to B$, and vice versa.*

Theorem 2.10. *Let $\mathrm{L}^*(\backslash, !) \vdash \Phi \to D$. Then there exists a proof net corresponding to the sequent $\Phi \to D$.*

Proof. Induction on derivation length. In the induction step, let us consider the case of the rule

$$\frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Pi\, A \backslash B \Delta \to C}\ (\backslash \to).$$

Assume that for $\Pi \to A$ the induction hypothesis gives a set U' and binary relations S' and R', whereas for $\Gamma B \Delta \to C$ the induction hypothesis gives a set U'' and binary relations S'' and R''. If the root of the subtree corresponding to the type B belongs to U'', then we construct the set U and the relations S and R for $\Gamma\Pi(A\backslash B)\Delta \to C$ as the natural images of U'' and the relations S'' and R''; otherwise we construct the set U as the union of the natural images of the sets U' and U'', the relation S as the union of the natural images of the relations S' and S'', and the relation R as the union of the natural images of the relations R' and R''.

The other cases in the induction step are easy to prove. □

Theorem 2.11. *A sequent $\Phi \to D$ is derivable in the calculus* $\mathrm{L}^*(\backslash, !)$ *if and only if there exists a proof net corresponding to the sequent $\Phi \to D$.*

The "only if" part of Theorem 2.11 has already been proved above. We shall prove the "if" part by induction on the total number of connectives in Φ and D. The rest of this section is devoted to the proof of the induction step. According to Lemma 2.9, it suffices to consider the sequents where the succedent D is a primitive type. Let us consider a sequent $\Phi \to p_i$ and a corresponding proof net $\langle V, O, \mathrm{label}, \mathrm{op}, <, U, S, R\rangle$ and assume that for all sequents with less connectives the theorem has already been proved.

Lemma 2.12. *If* $\mathrm{d}(a) > 0$*, then* $a < H(\overset{0}{g})$*.*

Proof. In view of (D4) from $a\ O^*\ \overset{0}{g}$ and $\mathrm{d}(a) > 0 = \mathrm{d}(\overset{0}{g})$ we obtain $a < H(\overset{0}{g})$. □

Lemma 2.13. $\overset{0}{g} \in U$*.*

Lemma 2.14. $H(\overset{0}{g}) \in \mathrm{pr}_1(S)$*.*

Proof. Suppose to the contrary that $H(\overset{0}{g}) \notin \mathrm{pr}_1(S)$. Evidently, $H(\overset{0}{g}) \notin W$. In view of (PN8), $H(\overset{0}{g}) \notin U$. In view of Lemma 2.3, we obtain $\overset{0}{g} \notin U$. From (PN9) by induction on the path length we obtain that no element of the set V belongs to U. In view of (PN8), $\mathrm{pr}_1(S) = \varnothing$. This contradicts Lemma 2.13. □

Lemma 2.15. *If $a\ T\ H(\overset{0}{g})$, then* $\mathrm{d}(a) = 1$*.*

Let us denote

$$\delta = \big\{a \mid \mathrm{d}(a) = 2 \wedge a \in U \wedge \mathrm{op}\big(a, O(a)\big) = -1\big\}.$$

Lemma 2.16. *Let the set δ be empty. Then the original sequent $\Phi \to p_i$ is of the form $!\Theta\, !^n p_i\, !\Xi \to p_i$.*

Proof. Obviously, $\mathrm{label}\big(H(\overset{0}{g})\big) = p_i$. In view of Lemma 2.14 the vertex $H(\overset{0}{g})$ is in relation S with some vertex $S\big(H(\overset{0}{g})\big)$. In view of condition (PN4) and Lemma 2.15 we have $\mathrm{label}\big(S\big(H(\overset{0}{g})\big)\big) = p_i$ and $\mathrm{d}\big(S\big(H(\overset{0}{g})\big)\big) = 1$. We denote by D the type in the antecedent that contains the vertex $S\big(H(\overset{0}{g})\big)$. Let us consider the case where D contains division and thus is of the form $!^n(A\backslash B)$. We denote by a the root of the subtree corresponding to A. Then $O(a)$ is the root of the subtree corresponding to $A\backslash B$. Obviously, $\mathrm{d}\big(O(a)\big) = 1$ and $\mathrm{op}\big(a, O(a)\big) = -1$, whence $\mathrm{d}(a) = 2$. Since $\mathrm{d}\big(H\big(O(a)\big)\big) = \mathrm{d}\big(O(a)\big) = 1$ and in the subtree corresponding to $A\backslash B$ there is only one leaf with 1 as the value of d, we obtain $H\big(O(a)\big) = S\big(H(\overset{0}{g})\big)$. Thus, $a \in \delta$, which contradicts the condition of the lemma.

It remains to consider the case where D does not contain division and thus is of the form $!^n p_i$. The only leaf in D is $S\big(H(\overset{0}{g})\big)$. If $U \cap W = \big\{H(\overset{0}{g}), S\big(H(\overset{0}{g})\big)\big\}$, then according to Lemma 2.7 all types in the antecedent except D begin with !, whence in this case the lemma is proved. Otherwise, we put

$b = \min\bigl((U \cap W) \diagdown \{H(\overset{0}{g}), S(H(\overset{0}{g}))\}\bigr)$. We denote by c the element satisfying the conditions $b\ O^*\ c$ and $H\bigl(O(c)\bigr) = H(\overset{0}{g})$. In view of (PN9), from $b \in U$ we obtain $c \in U$.

First, let us consider the case where $H(c) \in U$. Evidently, $\mathrm{d}\bigl(H(c)\bigr) = \mathrm{d}(c) = 1$. In view of (PN8), the value $S\bigl(H(c)\bigr)$ is defined. If $S\bigl(H(c)\bigr) < H(c)$, then $S\bigl(H(c)\bigr)\ O^*\ c$ is impossible, whence $S\bigl(H(c)\bigr)$ belongs to the subtree of some other type and this type cannot be D, but this contradicts the choice of c. Let $H(c) < S\bigl(H(c)\bigr)$. Then $\mathrm{d}\bigl(S\bigl(H(c)\bigr)\bigr) = 0$. In view of Lemma 2.2, we have $S\bigl(H(c)\bigr) = H\bigl(S\bigl(H(c)\bigr)\bigr) = H(\overset{0}{g})$. In view of the symmetricity of S, we have $H(c) = S\bigl(H(\overset{0}{g})\bigr)$. Then $H(c)$ belongs to the subtree corresponding to D, which contradicts the choice of c.

Now we shall consider the case where $H(c) \notin U$. Then the type in the antecedent corresponding to c is of the form $A\backslash B$. We denote by a the root of the subtree corresponding to A. From $\mathrm{d}(c) = 1$ we obtain $\mathrm{d}(a) = 2$. In view of (PN10), we have $a \in U$. From $H\bigl(O(a)\bigr) = H(c) \notin U$ and $\mathrm{op}\bigl(a, O(a)\bigr) = -1$ we obtain $a \in \delta$, which contradicts the condition of the lemma. □

It remains to consider the case where $\delta \neq \varnothing$. Let us denote

$$\overset{2}{e} = \max\bigl\{a \mid H\bigl(O(a)\bigr) = \min\bigl\{H\bigl(O(b)\bigr) \mid b \in \delta\bigr\}\bigr\}.$$

In the following lemmas, we assume that $\delta \neq \varnothing$.

Lemma 2.17. $\mathrm{d}(\overset{2}{e}) = 2$.

Proof. This is evident from the definition of δ. □

Let us denote

$$\alpha = \{a \mid a\ O^*\ \overset{2}{e}\}, \quad \varphi = \bigl\{b \mid H(\overset{2}{e}) < b \wedge \bigl(b\ T\ H(\overset{0}{g}) \vee \exists c\,(b\ T\ c \wedge c < \alpha)\bigr)\bigr\}.$$

Lemma 2.18. *Let* $S\bigl(H(\overset{0}{g})\bigr) < H(\overset{2}{e})$. *Then* $S\bigl(H(\overset{0}{g})\bigr) < \alpha$.

Proof. Suppose to the contrary that $a \le S\bigl(H(\overset{0}{g})\bigr)$. The condition $a \in \alpha$, means that $a\ O^*\ \overset{2}{e}$. Since $a \le S\bigl(H(\overset{0}{g})\bigr) \le \overset{2}{e}$, from D1 we obtain that $S\bigl(H(\overset{0}{g})\bigr)\ O^*\ \overset{2}{e}$. This contradicts the inequality $\mathrm{d}\bigl(S\bigl(H(\overset{0}{g})\bigr)\bigr) = 1 < 2 = \mathrm{d}(\overset{2}{e})$. □

Lemma 2.19. *The set* φ *is not empty.*

Proof. If $H(\overset{2}{e}) < S\bigl(H(\overset{0}{g})\bigr)$, then $S\bigl(H(\overset{0}{g})\bigr) \in \varphi$.

The case $H(\overset{2}{e}) = S\bigl(H(\overset{0}{g})\bigr)$ is impossible in view of PN5, since $\mathrm{d}\bigl(H(\overset{2}{e})\bigr) = 2$, but $\mathrm{d}\bigl(H(\overset{0}{g})\bigr) = 0$.

Let now $S\bigl(H(\overset{0}{g})\bigr) < H(\overset{2}{e})$. We are going to prove that $H(\overset{0}{g}) \in \varphi$. From Lemmas 2.17 and 2.12 we obtain that $H(\overset{2}{e}) < H(\overset{0}{g})$. In view of Lemma 2.18, we have $S\bigl(H(\overset{0}{g})\bigr) < \alpha$. Consequently, $H(\overset{0}{g}) \in \varphi$. □

Let us denote

$$f = \min \varphi, \quad \pi = (H(\overset{2}{e}); f).$$

Lemma 2.20. *Let* $\mathrm{d}(\overset{2}{c}) = 2$, $\mathrm{d}\bigl(O(\overset{2}{c})\bigr) = 1$, $H(\overset{2}{c}) < H(\overset{2}{e})$, $H(\overset{2}{c}) < a$, $h\ O^*\ \overset{2}{c}$, $h\ T\ a$. *Then there is an element* d *such that* $H(\overset{2}{c})\ T\ d$ *and* $H(\overset{2}{e}) < d$.

Proof. From (D2) we obtain $h \le H(\overset{2}{c})$.

In view of Lemma 2.6, from $h\ O^*\ \overset{2}{c}$ and $h\ T\ a$. we obtain $\overset{2}{c} \in U$. In view of Lemma 2.3, we obtain $H(\overset{2}{c}) \in U$. Consequently, $H(\overset{2}{c}) \in W \cap U$. According to (PN8) the set $\{d \mid H(\overset{2}{c})\ T\ d\}$ is not empty. Let us denote by b the least element of this set. From $h < H(\overset{2}{c}) < a$ according to (PN3) we obtain $h \le b$.

Let us prove that there is an element d such that $H(\overset{2}{c})\ T\ d$ and $H(\overset{2}{c}) < d$. If $H(\overset{2}{c}) < b$, then we can take b as d; otherwise we consider two cases: $h = b$ and $h < b$.

Let $h = b$. According to (PN5) we obtain $\mathrm{d}(b) = 3$. From $a\ T\ b$ in view of (PN5) we obtain $\mathrm{d}(a) = 2$. If $b \notin W$, then from $\overset{2}{c}\ T\ b$ and $a\ T\ b$ according to (PN8) and (PN13) we obtain $b\ R\ \overset{2}{c}$ and $b\ R\ a$, which contradicts (PN14). Let now $b \in W$. Then $b\ R\ \overset{2}{c}$ and $b\ R\ a$ are impossible in view of (PN12), whereas $\overset{2}{c}\ R\ b$ and $a\ R\ b$ are impossible in view of Lemma 2.4. There remains the case $\overset{2}{c}\ S\ b$ and $a\ S\ b$, but this contradicts (PN2).

Now let $h < b$. According to (PN6) there is an element d such that $d\ T\ H(c)$ and either $d < b$ or $H(c) < d$. The case $d < b$ is impossible due to the minimality of b.

We have shown that there is an element d such that $H(\overset{2}{c})\ T\ d$ and $H(\overset{2}{c}) < d$. From (PN5) we obtain $\mathrm{d}(d) = 1$. According to the construction of $\overset{2}{e}$ from $H\big(O(\overset{2}{c})\big) \leq H\big(O(\overset{2}{e})\big)$ we obtain $H\big(O(\overset{2}{c})\big) = H\big(O(\overset{2}{e})\big)$. In view of $\mathrm{d}(d) = 1$, the element d cannot be between $H(\overset{2}{c})$ and $H(\overset{2}{e})$. Consequently, $H(\overset{2}{e}) < d$. □

Lemma 2.21. $\mathrm{d}(f) \leq 1$.

Proof. According to the construction of φ, from $f \in \varphi$ we obtain that $f\ T\ H(\overset{0}{g})$ or there exists an element $h < \alpha$ such that $f\ T\ h$. In the first case $\mathrm{d}(f) = 1$ in view of Lemma 2.15.

It remains to consider the case $h < \alpha$. Then from $h < H(\overset{2}{e})$ and $H(\overset{2}{e}) < f$ we obtain $h < f$.

Suppose to the contrary that $\mathrm{d}(f) \geq 2$. In view of (PN5), we have $\mathrm{d}(h) \geq 3$. Let us denote by $\overset{2}{c}$ the element satisfying the conditions $h\ O^+\ \overset{2}{c}$, $\mathrm{d}(\overset{2}{c}) = 2$ and $\mathrm{d}\big(O(\overset{2}{c})\big) = 1$. From $h < \alpha$, $h\ O^+\ \overset{2}{c}$, and $\mathrm{d}(\overset{2}{c}) = \mathrm{d}(\overset{2}{e})$ we obtain that $H(\overset{2}{c}) \leq H(\overset{2}{e})$. From $h \notin \alpha$, we obtain that $H(\overset{2}{c}) \neq H(\overset{2}{e})$. Thus, $H(\overset{2}{c}) < H(\overset{2}{e})$ and, moreover, $H(\overset{2}{c}) < \alpha$. In view of transitivity, $H(\overset{2}{c}) < f$. In view of Lemma 2.20 we find an element d such that $H(\overset{2}{c})\ T\ d$ and $H(\overset{2}{e}) < d$.

We obtain $d \in \varphi$, whence $f \leq d$. From $h < H(\overset{2}{c}) < f$ according to (PN3) we obtain $d \leq f$. Thus, $d = f$, but then $\mathrm{d}(f) = 1$. □

Lemma 2.22. *Let $a \in \alpha \cup \pi$. Then $a < f$.*

Proof. The case $a \in \pi$ is obvious. Let $a \in \alpha$, i.e., $a\ O^*\ \overset{2}{e}$. In view of Lemma 2.17, according to (D2) we obtain that $a \leq H(\overset{2}{e})$. From $f \in \varphi$ we obtain $H(\overset{2}{e}) < f$, whence $a < f$. □

Lemma 2.23. *Let $a \leq b \leq c$, $a \in \alpha \cup \pi$, and $c \in \alpha \cup \pi$. Then $b \in \alpha \cup \pi$.*

Proof. First, we consider the case $b \leq H(\overset{2}{e})$. Since $a \notin \pi$, we have $a \in \alpha$. In view of (D1), from $a\ O^*\ \overset{2}{e}$ and $H(\overset{2}{e})\ O^*\ \overset{2}{e}$ we obtain that $b\ O^*\ \overset{2}{e}$, i.e., $b \in \alpha$.

Now we consider the case $H(\overset{2}{e}) < b$. In view of Lemma 2.22, we have $c < f$. We see that $H(\overset{2}{e}) < b < f$, i.e., $b \in \pi$. □

Lemma 2.24. *Let $a\ T\ b$ and $a \in \alpha \cup \pi$. Then $b \in \alpha \cup \pi$.*

Proof. According to Lemma 2.22 we have $a < f$.

First, we consider the case $b < a$. Suppose to the contrary that $b \notin \alpha \cup \pi$. According to Lemma 2.23 we have $b < \alpha$. If $H(\overset{2}{e}) < a$, then $a \in \varphi$, whence $f \leq a$, but this contradicts Lemma 2.22. It remains to consider the case $a \leq H(\overset{2}{e})$. Evidently, $a \in \alpha$, i.e., $a\ O^*\ \overset{2}{e}$. In view of Lemma 2.17, we have $\mathrm{d}(a) \geq 2$. From (PN5) we obtain that $\mathrm{d}(b) \geq 3$. Let us denote by $\overset{2}{c}$ the element satisfying the conditions $b\ O^+\ \overset{2}{c}$, $\mathrm{d}(\overset{2}{c}) = 2$, and $\mathrm{d}\big(O(\overset{2}{c})\big) = 1$. If $a \leq H(\overset{2}{c})$, then according to (D1) we obtain $H(\overset{2}{c})\ O^*\ \overset{2}{e}$, whence $\overset{2}{c}\ O^*\ \overset{2}{e}$ and $b\ O^*\ \overset{2}{e}$, which contradicts $b \notin \alpha$. Consequently, $H(\overset{2}{c}) < a$. We apply Lemma 2.20, taking b as h. We obtain an element d such that $H(\overset{2}{c})\ T\ d$ and $H(\overset{2}{e}) < d$. According to (PN3) from $b < H(\overset{2}{c}) < a$ we obtain that $d \leq a$. Consequently, $H(\overset{2}{e}) < a$, but this contradicts (D2).

Now we consider the case $a < b < f$. Note that $\max\{c \mid c < f\} \in \alpha \cup \pi$. From $a \leq b \leq \max\{c \mid c < f\}$, according to Lemma 2.23 we obtain that $b \in \alpha \cup \pi$.

It remains to consider the case $f \leq b$. From $f \in \varphi$ it follows that there exists an element $h\ T\ f$ such that $h = H(\overset{0}{g})$ or $h < \alpha$. If $f < b$, then in view of (PN3) we have $a \leq h \leq b$, which contradicts both $h = H(\overset{0}{g})$ and $h < \alpha$. Consequently, $f = b$. In view of Lemma 2.5, we obtain that $\mathrm{d}(f)$ is even. From Lemma 2.21 it follows that $\mathrm{d}(f) = 0$, whence we obtain $f = H(\overset{0}{g})$. According to (PN5) we have $\mathrm{d}(a) = 1$. Thus, $a \notin \alpha$. Consequently, $a \in \pi$. We obtain $a \in \varphi$, which contradicts the inequality $a < f$. □

Lemma 2.25. *Let $b\ O^+\ a$ and $\mathrm{d}(a) = 1$. Then the conditions $a \in \pi$ and $b \in \pi$ are equivalent.*

Proof. In view of (D5), we have $a < b$.

First, suppose to the contrary that $a \in \pi$, but $b \notin \pi$. Evidently, $f \leq b$. The conditions of the lemma yield $\mathrm{d}(b) \neq 0$, i.e., $b \neq H(\overset{0}{g})$. Thus, $b < H(\overset{0}{g})$. We obtain $f \neq H(\overset{0}{g})$, i.e., $\mathrm{d}(f) \neq 0$. In view of Lemma 2.21, we have $\mathrm{d}(f) = 1$. From $f \leq b$, $b\ O^+\ a$ and $\mathrm{d}(f) = \mathrm{d}(a)$ according to (D1) we deduce that $f \leq a$, which contradicts the condition $a \in \pi$.

Now suppose to the contrary that $b \in \pi$, but $a \notin \pi$. Evidently, $a \leq H(\overset{2}{e}) < b$. From (D1) we obtain that $H(\overset{2}{e})\ O^*\ a$. From Lemma 2.17 and $\mathrm{d}(a) = 1$ we obtain that $a = O(\overset{2}{e}) = S\big(H(\overset{0}{g})\big)$. Let us denote by c the element satisfying the conditions $b\ O^*\ c$ and $O(c) = a$. Evidently, $c \in \delta$. Thus, $c \leq H(\overset{2}{e}) < b$. Since $\mathrm{d}(c) = 2$, we see that $c < b$ contradicts (D2). □

Lemma 2.26. *The restriction of the proof net $\langle V, O, \mathrm{label}, \mathrm{op}, <, U, S, R\rangle$ to the set $V \diagdown (\alpha \cup \pi)$ is a proof net for some sequent.*

Proof. By the construction, the vertex $O(\overset{2}{e})$ is the root of the subtree corresponding to some type of the form $A\backslash B$ in the antecedent. Here, the set α corresponds to the type A. In view of Lemma 2.25, the original sequent has the form $\Gamma\,\Pi\,(A\backslash B)\,\Delta \to p_i$, where the set π corresponds to the sequence Π. Thus, the set $V \diagdown (\alpha \cup \pi)$ corresponds to the sequent $\Gamma\, B\, \Delta \to p_i$. For the restriction of the proof net $\langle V, O, \mathrm{label}, \mathrm{op}, <, U, S, R\rangle$ to the set $V \diagdown (\alpha \cup \pi)$ it is easy to check the conditions (PN1)–(PN14). □

Lemma 2.27. *There exist types A and B, sequences Γ and Π, and a natural number n such that*

$$\Phi = \Gamma\,\Pi\,!^n(A\backslash B)\,\Delta$$

and there exist proof nets for $\Pi \to A$ and $\Gamma\, B\, \Delta \to p_i$.

Proof. We define A and Π as in the proof of Lemma 2.26, where a proof net for the sequent $\Gamma\, B\, \Delta \to p_i$ was constructed. A proof net for the sequent $\Pi \to A$ can be obtained from the restriction of the proof net $\langle V, O, \mathrm{label}, \mathrm{op}, <, U, S, R\rangle$ to the set $\alpha \cup \pi$ using the following obvious transformations: we swap the two parts (α and π) in the sense of the order $<$ and make the vertex $\overset{2}{e}$ the root of the whole tree, whence the depth of every vertex from α will decrease by 2.

Formally, assume that the sequent $\Pi \to A$ corresponds to the tree $\langle V', O'\rangle$, the labelings label' and op', and the linear order $<'$. In the tree $\langle V', O'\rangle$, the number of edges labeled -1 on the path from a vertex a to the root will be denoted by $\mathrm{d}'(a)$. It is easy to prove that there exists a bijection $\mathfrak{h}\colon V' \to \alpha \cup \pi$ with the following properties:

$$\begin{aligned}
&a\ O'\ b \Leftrightarrow (\mathfrak{h}(a)\ O\ \mathfrak{h}(b) \wedge \mathfrak{h}(a) \in \alpha \wedge \mathfrak{h}(b) \in \alpha)\\
&\quad \vee (\mathfrak{h}(a)\ O\ \mathfrak{h}(b) \wedge \mathfrak{h}(a) \in \pi \wedge \mathfrak{h}(b) \in \pi) \vee (\mathfrak{h}(a)\ O\ H(\overset{0}{g}) \wedge \mathfrak{h}(b) = \overset{2}{e}),\\
&\mathrm{label}'(a) = \mathrm{label}\big(\mathfrak{h}(a)\big),\\
&\mathrm{op}'(a, b) = \mathrm{op}\big(\mathfrak{h}(a), \mathfrak{h}(b)\big),\\
&a <' b \Leftrightarrow (\mathfrak{h}(a) < \mathfrak{h}(b) \wedge \mathfrak{h}(a) \in \alpha \wedge \mathfrak{h}(b) \in \alpha)\\
&\quad \vee (\mathfrak{h}(a) < \mathfrak{h}(b) \wedge \mathfrak{h}(a) \in \pi \wedge \mathfrak{h}(b) \in \pi) \vee (\mathfrak{h}(a) \in \pi \wedge \mathfrak{h}(b) \in \alpha),\\
&\mathrm{d}'(a) = \begin{cases} \mathrm{d}\big(\mathfrak{h}(a)\big) - 2 & \text{if } \mathfrak{h}(a) \in \alpha,\\ \mathrm{d}\big(\mathfrak{h}(a)\big) & \text{if } \mathfrak{h}(a) \in \pi. \end{cases}
\end{aligned}$$

The desired proof net for the sequent $\Pi \to A$ is defined by the set

$$U' = \{a \mid \mathfrak{h}(a) \in U\}$$

and the binary relations

$$S' = \big\{(a,b) \mid \big(\mathfrak{h}(a), \mathfrak{h}(b)\big) \in S\big\},$$
$$R' = \big\{(a,b) \mid \big(\mathfrak{h}(a), \mathfrak{h}(b)\big) \in R\big\}.$$

It remains to check conditions (PN1)–(PN14) for U', S', and R'.

Let us prove (PN9) for U'. Let $a \in U'$. If $\mathfrak{h}(a) \in \alpha$ and $\mathfrak{h}\big(O'(a)\big) \in \alpha$, then $O'(a) \in U'$ follows from condition (PN9) for U. Similarly, if $\mathfrak{h}(a) \in \pi$ and $\mathfrak{h}\big(O'(a)\big) \in \pi$, then $O'(a) \in U'$ follows from condition (PN9) for U. If $\mathfrak{h}(a) \in \pi$ and $\mathfrak{h}\big(O'(a)\big) \in \alpha$, then $\mathfrak{h}\big(O'(a)\big) = \overset{2}{e} \in \mathrm{pr}_1(S)$, whence $O'(a) \in U'$.

Condition (PN7), i.e., $S' \neq \varnothing$, follows from $\overset{2}{e} \in \alpha$ and $\overset{2}{e} \in \mathrm{pr}_1(S)$.

Other conditions from the definition are easy to check. □

In order to complete the proof of Theorem 2.11, it suffices to note that the induction step follows from Lemmas 2.16 and 2.27. In the first case, the original sequent $\Phi \to D$ can be derived from an axiom by applying the rules (! →) and (W! →) several times; in the second case, the sequent $\Phi \to D$ is obtained from two shorter sequents, using the rules (! →) and (\ →), and the induction hypothesis can be applied to these shorter sequents. Theorem 2.11 is proved.

3. The Algorithm

The aim of this section is to show that the condition of the derivability criterion formulated in Sec. 2 can be checked using dynamic programming.

Given a sequent, we construct the tree $\langle V, O\rangle$ with the set of leafs W, the labeling of leafs label, the labeling of edges op, and the strict linear order $<$ as in the definition of a proof net. If (PN11) does not hold, then the sequent is not derivable. Further, we assue that the condition (PN11) holds.

A set I is called an interval on the set V, if $I \subseteq V$ and for all $a \in I$ and $b \in I$ we have $(a; b) \subseteq I$.

We define a quaternary predicate P on the set of all subsets of V. Let $P(I, E, F, G)$ be true if and only if I is an interval on V, $G \subseteq I \cap W$, $|G| \leq 1$, $(\forall a \in G)\, \mathrm{d}(a) \,\vdots\, 2$, $E \subseteq G$, $F \subseteq G$, and there are binary relations S and R on the set I satisfying conditions (F1)–(F14). For brevity, we shall denote $T = S \cup R \cup R^{-1}$.

- (F1) If $(a,b) \in S$, then $(b,a) \in S$.
- (F2) If $(a,b) \in S$ and $(a,c) \in S$, then $b = c$.
- (F3) If $(a,b) \in T$, $(c,d) \in T$ and $a < c < b$, then $a \leq d \leq b$.
- (F4) If $(a,b) \in S$, then $\mathrm{label}(a) = \mathrm{label}(b)$.
- (F5) If $(a,b) \in T$ and $a < b$, then $\mathrm{d}(a) = \mathrm{d}(b) + 1$.
- (F6) If $a\, O^*\, c$, $b\, T\, H(c)$, $a < b < H(c)$, and $H(c) \notin F$, then there is an element d such that $d\, T\, H(c)$ and either $d < b$ or $H(c) < d$.
- (F7) If $a\, O^*\, b$ and $b \in \mathrm{pr}_1(T)$, then $a \in I$.
- (F8) If $b \in \mathrm{pr}_1(T)$ and $a\, O^+\, b$, then $a \notin \mathrm{pr}_1(T)$.
- (F9) If $a \in W \cap I$, then there is an element $b \in \mathrm{pr}_1(T)$ such that $a\, O^*\, b$.
- (F10) $G \setminus E \subseteq \mathrm{pr}_1(S) \subseteq W \setminus E$.
- (F11) If $b \in \mathrm{pr}_1(R)$, then there is an element a such that $a\, O\, b$ and $\mathrm{op}(a,b) = 2$.
- (F12) If $(a,b) \in R$, then $b \in \mathrm{pr}_1(S) \cup E$.
- (F13) If $(a,b) \in R$ and $(a,c) \in R$, then $b = c$.
- (F14) If $(a,c) \in T$ and $a < b < c$, then $b \notin G$.

Informally, if the set E is non-empty, then its only element must be in relation S with some element of the interval I in order to avoid violating condition (PN13). If the set F is non-empty, then its only element must be in relation T with some element of the interval I in order to avoid violating condition (PN6). The elements of $I \setminus G$ will not be connected to elements outside the interval I by the relation T.

Lemma 3.1. *Let $P([c;h], E, F, G)$. Then $D(c)$ or $c \in W$.*

Lemma 3.2. *The formula $P(V, \varnothing, \varnothing, \varnothing)$ is true if and only if there are sets $U \subseteq V$, $S \subseteq V \times V$, and $R \subseteq V \times V$ satisfying conditions* (PN1)–(PN14).

Proof. First, we prove the "if" part. Assume that sets U, S, and R satisfy conditions (PN1)–(PN14). It suffices to verify conditions (F1)–(F14) for the same sets S and R.

Let us prove (F8). Let $b \in \mathrm{pr}_1(T)$ and $a\ O^+\ b$. Then $b \notin W$. According to (PN8) we have $b \notin \mathrm{pr}_1(S)$. Consequently, $b \in \mathrm{pr}_1(R)$. From (PN12) we obtain that $b \in U$ and there is a vertex c such that $c \notin U$ and $c\ O\ b$. According to (PN10) we have $\mathrm{op}(c, b) = 2$. Then only one edge leads to b, whence $a\ O^*\ c$. According to (PN9), from $c \notin U$ we obtain $a \notin U$. Applying Lemma 2.6, we obtain $a \notin \mathrm{pr}_1(T)$.

Let us prove (F9). Let $a \in W$. If $a \in U$, then we apply (PN8). Let $a \notin U$. Lemma 2.13 states that $\overset{0}{g} \in U$. Thus, on the path from the leaf a to the root $\overset{0}{g}$ there is a vertex c such that that $c \notin U$ and $O(c) \in U$. Let $b = O(c)$. Obviously, $a\ O^*\ b$. According to (PN12), we obtain $b \in \mathrm{pr}_1(R)$.

The other twelve conditions are easy to verify.

Now we prove the "only if" part. Assume that sets S and R satisfy conditions (F1)–(F14) for $I = V$ and $E = F = G = \varnothing$. Let $U = \{a \in V \mid \exists\, b\, (b\ O^*\ a \wedge b \in \mathrm{pr}_1(T))\}$. We shall verify conditions (PN1)–(PN14) for the same sets S and R.

Let us prove (PN7). Since $W \neq \varnothing$, according to (F9) we obtain $T \neq \varnothing$, i.e., $S \neq \varnothing$ or $R \neq \varnothing$. According to (F12) from $R \neq \varnothing$ there follows $S \neq \varnothing$.

Let us prove (PN8). According to (F11) the sets $\mathrm{pr}_1(R)$ and W are disjoint. In view of (F10) it suffices to show that for any a from W the conditions $a \in \mathrm{pr}_1(S)$ and $a \in U$ are equivalent. The condition $a \in U$ means that there exists an element b such that $b\ O^*\ a$ and $b \in \mathrm{pr}_1(T)$. In view of $a \in W$, the condition $b\ O^*\ a$ is equivalent to $b = a$. Thus, $a \in U$ is equivalent to $a \in \mathrm{pr}_1(T)$, which is equivalent to the existence of c such that $a\ T\ c$. The cases $a\ S\ c$ and $c\ T\ a$ imply $a \in \mathrm{pr}_1(S)$, whereas the case $a\ T\ c$ is impossible, since $a \in W$.

Let us prove (PN12). First, we verify the inclusion

$$\mathrm{pr}_1(R) \supseteq \{b \in U \mid \exists\, a\, (a \notin U \wedge a\ O\ b)\}.$$

Let $b \in U$, $a \notin U$, and $a\ O\ b$. The condition $b \in U$ means that there exists an element c such that $c\ O^*\ b$ and $c \in \mathrm{pr}_1(T)$. There exists an element $d \in W$ such that $d\ O^*\ a$. According to (F9) there exists an element $e \in \mathrm{pr}_1(T)$ such that $d\ O^*\ e$. From $d\ O^*\ a$ and $d\ O^*\ e$ we obtain that $e\ O^*\ a$ or $a\ O^*\ e$. The case $e\ O^*\ a$ is impossible in view of $a \notin U$. Consequently, $a\ O^+\ e$, i.e., $b\ O^*\ e$, whence we obtain $c\ O^*\ e$. The case $c\ O^+\ e$ contradicts condition (F8). It remains to consider the case $c = e$. From the antisymmetricity of the relation O^* we obtain $b = c$. From $b \in \mathrm{pr}_1(T)$ we obtain $b \in \mathrm{pr}_1(S)$ or $b \in \mathrm{pr}_1(R)$. The case $b \in \mathrm{pr}_1(S)$ is impossible, since $b \notin W$ in view of (F10). Consequently, $b \in \mathrm{pr}_1(R)$.

Now we verify the inclusion

$$\mathrm{pr}_1(R) \subseteq \{b \in U \mid \exists\, a\, (a \notin U \wedge a\ O\ b)\}.$$

Let $b \in \mathrm{pr}_1(R)$. According to (F11) there is an element a such that $a\ O\ b$ and $\mathrm{op}(a, b) = 2$. It suffices to prove that $b \in U$ and $a \notin U$. The property $b \in U$ follows from $b \in \mathrm{pr}_1(T)$. In order to prove $a \notin U$, we assume to the contrary that $a \in U$. This means that there is an element c such that $c\ O^*\ a$ and $c \in \mathrm{pr}_1(T)$. Then $c\ O^+\ b$, and we obtain a contradiction with (F8).

Let us prove (PN10). Suppose to the contrary that $a\ O\ b$, $b \in U$, $\mathrm{op}(a, b) \neq 2$, but $a \notin U$. From (PN12), which was proved above, we obtain $b \in \mathrm{pr}_1(R)$. According to (F11) there is an element c such that $c\ O\ b$ and $\mathrm{op}(c, b) = 2$. Then only one edge leads to b, whence $a = c$. We obtain a contradiction between $\mathrm{op}(a, b) \neq 2$ and $\mathrm{op}(c, b) = 2$.

Condition (PN11) is assumed at the beginning of this section. The other nine conditions are easy to verify. □

Let us denote by L_1, L_2, and M the binary predicates on V defined as follows:

$$L_1(a,b) \leftrightarrow \mathrm{d}(a) = \mathrm{d}(b) + 1,$$
$$L_2(a,b) \leftrightarrow (a \in W \wedge b \in W \wedge \mathrm{label}(a) = \mathrm{label}(b) \wedge \mathrm{d}(a) = \mathrm{d}(b) + 1),$$
$$M(a,b) \leftrightarrow \exists\, e\, \big(e < a \wedge H\big(O(e)\big) = b \wedge a\; O^*\; \big(O(e)\big)\big).$$

Let us denote by D the unary predicate on V defined as follows:

$$D(a) \leftrightarrow \exists\, b\, \exists\, c\, (b\; O\; a \wedge c\; O\; a \wedge b \neq c).$$

Lemma 3.3. *The formula $P(I,E,F,G)$ is true if and only if I is an interval on V, $G \subseteq I \cap W$, $|G| \le 1$, $(\forall\, a \in G)\, \mathrm{d}(a) \vdots 2$, $E \subseteq G$, $F \subseteq G$ and at least one of the following formulas holds*:

$$G = \varnothing \wedge (\forall\, a \in I)\, D(a), \tag{1}$$
$$G = \varnothing \wedge \exists\, e_1\, P(I, \varnothing, \varnothing, \{e_1\}), \tag{2}$$
$$\exists\, c\, \exists\, d\, \exists\, h\, \big(c < d \le h \wedge I = [c;h] \wedge P([c;d), E, F, G) \wedge P([d;h], \varnothing, \varnothing, \varnothing)\big), \tag{3}$$
$$\exists\, c\, \exists\, d\, \exists\, h\, \big(c < d \le h \wedge I = [c;h] \wedge P([c;d), \varnothing, \varnothing, \varnothing) \wedge P([d;h], E, F, G)\big), \tag{4}$$
$$\exists\, f\, \exists\, h\, \big(f < h \wedge I = [f;h] \wedge P\big((f;h), \varnothing, \varnothing, \varnothing\big) \wedge L_2(f,h) \wedge (G = \{f\} \vee G = \{h\}) \wedge E = \varnothing \wedge \big(F = G \leftrightarrow M(f,h)\big), \big) \tag{5}$$
$$\exists\, f\, \exists\, h_1\, \exists\, h_2\, \exists\, h\, \big(f < h_1 \wedge I = [f;h] \wedge P\big((f;h_1), \varnothing, \varnothing, \varnothing\big) \wedge L_1(f,h) \wedge h_2\; O\; h \wedge \mathrm{op}(h_2, h) = 2 \wedge \{a \in V \mid a\; O^*\; h\} = [h_1; h] \wedge E = G = \{f\} \wedge F = \varnothing\big), \tag{6}$$
$$\exists\, f_1\, \exists\, f_2\, \exists\, f\, \exists\, h\, \big(f < h \wedge I = [f_1;h] \wedge P\big((f;h), \varnothing, \varnothing, \varnothing\big) \wedge L_1(f,h) \wedge f_2\; O\; f \wedge \mathrm{op}(f_2, f) = 2 \wedge \{a \in V \mid a\; O^*\; f\} = [f_1; f] \wedge E = G = \{h\} \wedge \big(F = G \leftrightarrow M(f,h)\big)\big), \tag{7}$$
$$\exists\, e\, \exists\, f\, \exists\, h\, \exists\, F'\, \big(I = [e;h] \wedge P([e;h), G, F', G) \wedge L_2(f,h) \wedge G = \{f\} \wedge E = F = \varnothing\big), \tag{8}$$
$$\exists\, f\, \exists\, h\, \exists\, F'\, \big(I = [f;h] \wedge P\big((f;h], G, F', G\big) \wedge L_2(f,h) \wedge G = \{h\} \wedge E = \varnothing \wedge \big(F = G \leftrightarrow M(f,h)\big)\big), \tag{9}$$
$$\exists\, e\, \exists\, f\, \exists\, h_1\, \exists\, h_2\, \exists\, h\, \exists\, F'\, \big(I = [e;h] \wedge P([e;h_1), E, F', G) \wedge L_1(f,h) \wedge h_2\; O\; h \wedge \mathrm{op}(h_2, h) = 2 \wedge \{a \in V \mid a\; O^*\; h\} = [h_1; h] \wedge G = \{f\} \wedge F = \varnothing\big), \tag{10}$$
$$\exists\, f_1\, \exists\, f_2\, \exists\, f\, \exists\, h\, \exists\, F'\, \big(I = [f_1;h] \wedge P\big((f;h], E, F', G\big) \wedge L_1(f,h) \wedge f_2\; O\; f \wedge \mathrm{op}(f_2, f) = 2 \wedge \{a \in V \mid a\; O^*\; f\} = [f_1; f] \wedge G = \{h\} \wedge \big(F = G \leftrightarrow M(f,h)\big)\big). \tag{11}$$

Proof. First, we prove that the disjunction of formulas (1)–(11) follows from $P(I,E,F,G)$. Let $P(I,E,F,G)$. If $|I| \le 1$, then formula (1) holds.

Let $|I| \ge 2$. Let us denote $c = \min I$ and $h = \max I$. If $D(c)$, then one can prove that $P(\{c\}, \varnothing, \varnothing, \varnothing)$, $P\big((c;h], E, F, G\big)$ and there holds formula (3). If $D(h)$, then one can prove that $P([c;h), E, F, G)$, $P(\{h\}, \varnothing, \varnothing, \varnothing)$ and there holds formula (3).

Further, we assume that the formulas $D(c)$ and $D(h)$ are false. Then $h \in \mathrm{pr}_1(T)$.

Let us denote

$$\chi = \{a \mid a\; T\; h\}, \quad f = \min \chi, \quad \varphi = \{a \mid a\; T\; f\},$$
$$h_1 = \min\{a \mid a\; O^*\; h\}, \quad f_1 = \min\{a \mid a\; O^*\; f\}, \quad e = \min(\varphi \cup \{f_1\}).$$

If $G = \varnothing$, then we put $G' = \{a \in \{f,h\} \mid \mathrm{d}(a) \vdots 2\}$. Then $P(I, \varnothing, \varnothing, G')$ and there holds formula (2).

Futher, we assume that $G \neq \varnothing$.

Let us consider the case $c \neq e$. There exists an element d such that $d < e$ and $(d;e) = \varnothing$. If $G \subseteq (d;h]$, then one can prove that $P([c;d), \varnothing, \varnothing, \varnothing)$, $P\big((d;h], E, F, G\big)$ and there holds formula (4). If $G \nsubseteq (d;h]$, then one can prove that $P([c;d), E, F, G)$, $P\big((d;h], \varnothing, \varnothing, \varnothing\big)$ and there holds formula (3).

Further, we assume that $c = e$. If $f \notin W$, then we put $f_2 = \max\{a \mid a\; O^+\; f\}$. Similarly, if $h \notin W$, then we put $h_2 = \max\{a \mid a\; O^+\; h\}$.

Let us consider the case $h \notin W$, $|\varphi| = 1$. Then $P\big((f;h_1), \varnothing, \varnothing, \varnothing\big)$ and there holds formula (6).

Now we consider the case where $h \notin W$ and $|\varphi| > 1$. Let us denote

$$F' = \begin{cases} G & \text{if } \varphi < f \text{ and } M(\min \varphi, f), \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $P([e;h_1), E, F', G)$ and there holds formula (10).

Let us consider the case $h \in W$, $|\chi| = 1$, $f \notin W$. Then $P\big((f;h), \varnothing, \varnothing, \varnothing\big)$ and there holds formula (7).

Let us consider the case $h \in W$, $|\chi| = 1$, $f \in W$, $|\varphi| = 1$. Then $P\big((f;h), \varnothing, \varnothing, \varnothing\big)$ and there holds formula (5).

Let us consider the case $h \in W$, $|\chi| = 1$, $f \in W$, $|\varphi| > 1$. Let us denote

$$F' = \begin{cases} G & \text{if } \varphi < f \text{ and } M(\min \varphi, f), \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $P([e;h), G, F', G)$ and there holds formula (8).

Let us consider the case $h \in W$, $|\chi| > 1$, $f \in W$. Let us denote

$$F' = \begin{cases} G & \text{if } (\exists\, a \in \chi \cap I) M(a,h), \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $P\big((f;h], G, F', G\big)$ and there holds formula (9).

It remains to consider the case $h \in W$, $|\chi| > 1$, $f \notin W$. Let us denote

$$F' = \begin{cases} G & \text{if } (\exists\, a \in \chi \cap I) M(a,h), \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $P\big((f;h], E, F', G\big)$ and there holds formula (11).

Now we are going to prove that $P(I, E, F, G)$ follows from the disjunction of formulas (1)–(11). Let us consider eleven cases, corresponding to each of the formulas.

Assume (1). Then $E = F = G = \varnothing$. Let $S = R = \varnothing$.

The case (2) is obvious (the relations S and R do not change).

Now assume (3). Let us consider some relations S' and R' satisfying conditions (F1)–(F14) for the quadruple consisting of $[c;d)$, E, F, and G, and some relations S'' and R'' satisfying conditions (F1)–(F14) for the quadruple consisting of $(d;h]$, $\varnothing$, $\varnothing$, and $\varnothing$. We put $S = S' \cup S''$ and $R = R' \cup R''$ and check conditions (F1)–(F14) for the quadruple consisting of $[c;h]$, E, F, and G.

We prove condition (F8). Let $b \in \mathrm{pr}_1(T)$ and $a\ O^+\ b$. If $b \in [c;d)$, then according to (F7) we obtain $a \in [c;d)$ and it remains to apply (F8) for $[c;d)$. Similarly for $b \in (d;h]$.

We prove condition (F9). Let $a \in W \cap I$. If $a < d$, then we apply (F9) for $[c;d)$. If $a > d$, then we apply (F9) for $(d;h]$. The case $a = d$ is impossible, since in view of Lemma 3.1 we have $\min(d;h] \in W$, which contradicts $d \in W$.

The other conditions are easy to verify. We have proved that (3) implies $P(I, E, F, G)$. The case (4) can be treated in a similar fashion.

Now assume (5). Let us consider some relations S' and R' satisfying conditions (F1)–(F14) for the quadruple consisting of $(f;h)$, $\varnothing$, $\varnothing$, and $\varnothing$. We put $S = S' \cup \{(f,h),(h,f)\}$ and $R = R'$ and check conditions (F1)–(F14) for the quadruple consisting of $[f;h]$, E, F, and G. Let us denote $T' = S' \cup R' \cup R'^{-1}$.

We prove condition (F6). Let $a\ O^*\ c$, $b\ T\ H(c)$, $a < b < H(c)$ and $H(c) \notin F$. If $H(c) \in (f;h)$ or $b \in (f;h)$, then $b\ T'\ H(c)$ and we can apply (F6) for $(f;h)$. It remains to consider the case where $b = f$ and $H(c) = h$. In view of (D1) we have $b\ O^*\ c$. In view of (D3) the number $\mathrm{d}(c)$ is even. Consequently, $\mathrm{d}\big(H(c)\big)$ is even. Thus, $G = \{h\}$. Let us prove $M(f,h)$. Since $a\ O^+\ c$, there is an element e such that $a\ O^*\ e$ and $e\ O\ c$. It is easy to see that $e < c$, $\mathrm{op}(e,c) = -1$, and the number $\mathrm{d}(e)$ is odd. Thus, $e \le a < f$. On the other hand, $H\big(O(e)\big) = H(c) = h$. From $a \le f \le c$ according to (D1) we obtain $f\ O^*\ c$, i.e.,

$f\ O^*\ \big(O(e)\big)$. We have established $M(f,h)$. Consequently, $F = G = \{h\} = \{H(c)\}$, which contradicts $H(c) \notin F$.

We prove condition (F7). Let $a\ O^*\ b$ and $b \in \mathrm{pr}_1(T)$. If $b \in (f;h)$, then we can apply (F7) for $(f;h)$. It remains to consider the case where $b = f$ or $b = h$. From $L_2(f,h)$ we obtain $b \in W$. Consequently, $a = b$, whence we obtain $a \in [f;h]$.

Condition (F8) is treated similarly to condition (F7). If $b = f$ or $b = h$, then $b \in W$, which contradicts $a\ O^+\ b$.

We prove condition (F9). Let $a \in W \cap I$. It remains to consider the case where $a = f$ or $a = h$. Let $b = a$ and we obtain $b \in \mathrm{pr}_1(T)$ and $a\ O^*\ b$.

The other conditions from the definition of the predicate P are easy to verify.

Now assume (6). Let us consider some relations S' and R' satisfying conditions (F1)–(F14) for the quadruple consisting of $(f;h_1)$, $\varnothing$, $\varnothing$, and $\varnothing$. We put $S = S'$ and $R = R' \cup \{(h,f)\}$ and check conditions (F1)–(F14) for the quadruple consisting of $[f;h]$, $\{f\}$, $\varnothing$, and $\{f\}$. Let us denote $T' = S' \cup R' \cup R'^{-1}$.

We prove condition (F6). Let $a\ O^*\ c$, $b\ T\ H(c)$, $a < b < H(c)$ and $H(c) \notin F$. The case where $b = f$ and $H(c) = h$, is impossible, since $\mathrm{d}\big(H(c)\big)$ is even, whereas from $\mathrm{op}(h_2,h) = 2$ according to (PN11) it follows that $\mathrm{d}(h)$ is odd.

We prove condition (F7). Let $a\ O^*\ b$ and $b \in \mathrm{pr}_1(T)$. If $b \in (f;h)$ or $b = f$, then the proof is similar to the case of formula (5). Let $b = h$. From $a\ O^*\ h$ we obtain $a \in [h_1;h] \subseteq [f;h]$.

We prove condition (F8). Let $a\ O^+\ b$ and $b \in \mathrm{pr}_1(T)$. If $b = h$, then $a \in [h_1;h)$. Applying (F7) for $(f;h_1)$, we obtain $\mathrm{pr}_1(T') \subseteq (f;h_1)$. Thus, $a \notin \mathrm{pr}_1(T') \cup \{f,h\} = \mathrm{pr}_1(T)$.

We prove condition (F9). Let $a \in W \cap [f;h]$. If $a = f$, then we put $b = f$. If $a \in [h_1;h]$, then we put $b = h$. In both cases, we obtain $b \in \mathrm{pr}_1(T)$ and $a\ O^*\ b$.

The other conditions from the definition of the predicate P are easy to verify.

Now assume (7). Let us consider some relations S' and R' satisfying conditions (F1)–(F14) for the quadruple consisting of $(f;h)$, $\varnothing$, $\varnothing$, and $\varnothing$. We put $S = S'$ and $R = R' \cup \{(f,h)\}$ and check conditions (F1)–(F14) for the quadruple consisting of $[f_1;h]$, E, F, and G. Condition (F6) is proved similarly to the case where (5) holds. The other conditions can be proved similarly to the case of formula (6).

Now assume (8). Let us consider some relations S' and R' satisfying conditions (F1)–(F14) for the quadruple consisting of $[e;h)$, $\{f\}$, F', and $\{f\}$. We put $S = S' \cup \{(f,h),(h,f)\}$ and $R = R'$ and check conditions (F1)–(F14) for the quadruple consisting of $[e;h]$, $\varnothing$, $\varnothing$, and $\{f\}$.

We prove condition (F2). Let $(a,b) \in S$ and $(a,c) \in S$. If $a = h$, then $b = f$ and $c = f$. If $a = f$, then in view of condition (F10) for $[e;h)$ we obtain $a \notin \mathrm{pr}_1(T')$, whence $b = h$ and $c = h$. If $a \notin f$ and $a \notin h$, then it remains to apply (F2) for $[e;h)$.

We prove condition (F3). Let $(a,b) \in T$, $(c,d) \in T$ and $a < c < b$. The case where $d = h$ and $c = f$, is impossible in consequence of condition (F14) for $[e;h)$. The case $c = h$ is impossible in consequence of $c < b$. Consequently, $(c,d) \in T'$.

The case $a = h$ is impossible in consequence of $a < c$. Let us consider the case where $b = h$ and $a = f$. From $d \in \mathrm{pr}_1(T)$ we obtain $d \le h$. Suppose to the contrary that $d < a$. We obtain a contradiction with (F14) for $[e;h)$. Consequently, $(a,b) \in T'$ and we can apply (F3) for $[e;h)$.

We prove condition (F6). Let $a\ O^*\ c$, $b\ T\ H(c)$, $a < b < H(c)$, and $H(c) \notin F$. If $H(c) \notin \{f,h\}$, then $b\ T'\ H(c)$ and we can apply (F6) for $[e;h)$, since $H(c) \notin F'$ in view of $F' \subseteq \{f\} \subseteq \{f,h\}$. If $H(c) = f$, then we can take h as d. It remains to consider the case where $H(c) = h$. Then $b = f$. It can be proved that $\mathrm{d}\big(H(c)\big)$ is even and $\mathrm{d}(b)$ is odd. Thus, $b \in G$ yields a contradiction with $(\forall\, a \in G)\, \mathrm{d}(a) \,\vdots\, 2$.

The other conditions can be proved similarly to the case of formula (5).

Now assume (9). Let us consider some relations S' and R' satisfying conditions (F1)–(F14) for the quadruple consisting of $(f;h]$, $\varnothing$, F', and $\{h\}$. We put $S = S' \cup \{(f,h),(h,f)\}$ and $R = R'$ and check conditions (F1)–(F14) for the quadruple consisting of $[f;h]$, $\{h\}$, F, and $\{h\}$. Conditions (F2) and (F3) are proved similarly to the case where (8) holds. The other conditions can be proved similarly to the case of formula (5).

Now assume (10). Let us consider some relations S' and R' satisfying conditions (F1)–(F14) for the quadruple consisting of $[e;h_1)$, E, F', and $\{f\}$. We put $S=S'$ and $R=R'\cup\{(h,f)\}$ and check conditions (F1)–(F14) for the quadruple consisting of $[e;h]$, E, $\varnothing$, and $\{f\}$.

Condition (F10) follows from the same condition for $[e;h_1)$.

We prove condition (F12). Let $(a,b)\in R$. If $(a,b)\in R'$, then we can apply (F12) for $[e;h_1)$. It remains to consider the case where $a=h$ and $b=f$. If $E=\{f\}$, then $b=f\in \mathrm{pr}_1(S)\cup E$. Let now $E=\varnothing$. Using the first inclusion from (F10) for $[e;h_1)$, we obtain $\{f\}\diagdown\varnothing\subseteq \mathrm{pr}_1(S')=\mathrm{pr}_1(S)$. Consequently, $b=f\in \mathrm{pr}_1(S)\cup E$.

The other conditions can be proved similarly to the case of formula (6).

Now assume (11). Let us consider some relations S' and R' satisfying conditions (F1)–(F14) for the quadruple consisting of $(f;h]$, E, F', and $\{h\}$. We put $S=S'$ and $R=R'\cup\{(f,h)\}$ and check conditions (F1)–(F14) for the quadruple consisting of $[f_1;h]$, E, F, and $\{h\}$. Conditions (F10) and (F12) are proved similarly to the case where (10) holds. The other conditions can be proved similarly to the case of formula (7). □

Theorem 3.4. *There is a polynomial-time deterministic algorithm for deciding whether an arbitrary given sequent is derivable in the calculus* $\mathrm{L}^*(\backslash,!)$. *Moreover, there is a polynomial-time deterministic algorithm for constructing a derivation in the sequent calculus* $\mathrm{L}^*(\backslash,!)$ *for any given derivable sequent.*

Proof. The algorithm for deciding whether an arbitrary given sequent is derivable in the calculus $\mathrm{L}^*(\backslash,!)$ works as follows. We evaluate $P(I,E,F,G)$ for all values of the arguments where $E\cup F\subseteq G\subseteq I$, $|G|\le 1$, and I is an interval on V. For doing this, we use an outer cycle where the value $2|I|-|G|$ increases from 0 to $2|V|$ and inner cycles where I, E, F, and G range over all possible values. It is easy to see that for evaluating $P(I,E,F,G)$ using the criterion from Lemma 3.3 we spend polynomial time for each quadruple I, E, F, G. At the end of the algorithm, we use the criterion from Lemma 3.2 to find out whether there exists a proof net for the given sequent.

According to Theorem 2.11, the above algorithm answers the question about derivability in the calculus $\mathrm{L}^*(\backslash,!)$.

An algorithm for constructing a derivation in the sequent calculus $\mathrm{L}^*(\backslash,!)$ for any given derivable sequent is obtained from the constructive "if" part of the proof of Theorem 2.11. □

This work is supported by the Russian Science Foundation under grant 18-11-00100.

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