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Complexity Studies in Some Piecewise Continuous Dynamical Systems

ASHOK K CHITKARA[1], NEHA KUMRA[1*] AND L. M. SAHA[2]

[1]Department of Mathematics, Chitkara University, Himachal Pradesh, India

[2]IIIMIT, Shiv Nadar University, Gautam budh Nagar, Uttar Pradesh, India

[*]**Email: neha.kumra@chitkarauniversity.edu.in**

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Abstract The, *"Complex systems"*, stands as a broad term for many diverse disciplines of science and engineering including natural & medical sciences. Complexities appearing in various dynamical systems during evolution are now interesting subjects of studies. Chaos appearing in various dynamical systems can also be viewed as a form of complexity. For some cases nonlinearities within the systems and for other cases piecewise continuity property of the system are responsible for such complexity. Dynamical systems represented by mathematical models having piecewise continuous properties show strange complexity character during evolution. Interesting recent articles explain widely on complexities in various systems. Observable quantities for complexity are measurement of Lyapunov exponents (LCEs), topological entropies, correlation dimension etc.

The present article is related to study of complexity in systems having piecewise continuous properties. Some mathematical models are considered here in this regard including famous Lozi map, a discrete mathematical model and Chua circuit, a continuous model. Investigations have been carried forward to obtain various attractors of these maps appearing during evolution in diverse and interesting pattern for different set of values of parameters and for different initial conditions. Numerical investigations extended to obtain bifurcation diagrams, calculations of LCEs, topological entropies and correlation dimension together with their graphical representation.

Keywords: Chaos; Lyapunov Exponents; Topological Entropy; Bifurcation.

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1. INTRODUCTION

A nonlinear system, on the contrary to linear system, is a system which does not satisfy the principle of superposition. That is output of a nonlinear system is not directly proportional to the input. When we take into account real phenomena, we find all most all real systems are nonlinear. Thus, the nonlinear problems are of interest to engineers, physicists, biologists, mathematicians and many others because most of the systems are inherently nonlinear in nature. Parameters of nonlinear systems completely specify the property of nonlinearity of the system. When parameters of such systems vary in certain manner, we observe: Instability, bistability, bifurcations, chaos etc. Linearization of a nonlinear model representing a real system without proper justification often leads to complete wrong results. We can prove this fact for many nonlinear problems solved by linearizing the model e.g. damped and forced pendulum evolving chaotically, problems of viscous fluid flow etc.

A complex system can be viewed as a system composed of many components which may interact with each other. Complexity of different type has been explained extendedly through some important articles[6,13,20,21] A complex system exhibits some (and possibly all) of the following characteristics:

- Some degree of spontaneous order (spontaneous order is the spontaneous emergence of order out of seeming chaos).
- Robustness of the order (robustness is the ability of a computer system to cope with errors during execution).
- Sensitivity to initial condition (chaos). A System is chaotic if it possesses an strange attractor.
- Numerosity (broken symmetry and the nature of the hierarchical structure of science).

Actual motivation for this study is to see how complexities are arising for any piece wise continuous system even if it is not necessarily non-linear. Even a linear system can display complexity behavior during evolution if it is piecewise continuous.

2. MEASURES OF COMPLEXITIES

For any complex system, some of the measurements which justify complexity are as follows:

2.1 Lyapunov Characteristic Exponents (LCEs)

The motion be chaotic if the system exhibits sensitive dependence on initial conditions. That is two trajectories starting together with nearby positions, (initial conditions), will rapidly diverge from each other and have totally dissimilar features. The long-term prediction becomes impossible as the small qualms are amplified enormously fast. Lyapunov characteristic exponents[1,7-10,19,22], is very effective tool for identification of regular and chaotic motions since it measures the degree of sensitivity to initial condition in a system. Actually, Lyapunov exponents (λ) provide measure the exponential divergence of orbits originating nearby. If $\lambda > 0$, then it implies the system is evolving chaotically, (or chaos is observed), and if $\lambda < 0$ then it implies the evolution is regular, (or regularity or ordered motion).

For a smooth one dimensional map f and x_0 an initial point the Lyapunov Exponent be defined by

$$\lambda(x_0) = \lim_{k\to\infty}\left(\log|f'(x_0)| + \log|f'(x_1)| + \log|f'(x_2)| + \ldots + \log|f'(x_{k-1})|\right)$$

$$= \lim_{k\to\infty}\sum_{k=0}^{k-1}\log|f'(x_k)| \qquad (1)$$

where, $x_1, x_2, \ldots, x_{k-1}, \ldots$, are iterates of x_0 under f.

2.2 Topological Entropy

Topological entropy provides the measure of complexity. More topological entropy means more complex the system is. A topological entropy measures the exponential growth rate of the number of distinguishable orbits as time advances in the system [2].However positivity of its value does not justify the system be chaotic.

Consider a finite partition of a state space X denoted by $P = \{A_1, A_2, A_3, \ldots, A_N\}$. A measure μ on X with total measure $\mu = 1$ [21] defines the probability of a given reading as

$$p_i = \mu(A_i), i = 1, 2, \ldots, N.$$

Then the entropy of the partition be given by

$$H(p) = -\sum_{i=0}^{N} p_i \log p_i \qquad (2)$$

2.3 *Correlation Dimension*

Correlation dimension provides the dimensionality of the evolving system,[12]. It is a kind of fractal dimension and its numerical value is always non-integer. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. To determine correlation dimension one has to use statistical method, [16].

Consider an orbit O(x1) = $\{x_1, x_2, x_3, x_4, \ldots\}$, of a map f: U → U, where U is an open bounded set in R^n. To compute correlation dimension of $O(x_1)$, for a given positive real number r, we form the correlation integral,

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H(r - \|x_i - x_j\|) \quad ,i, j = 1, 2, 3, \ldots, n. \tag{3}$$

where $H(x) = \begin{cases} 0, x<0 \\ 1, x\geq 0 \end{cases}$

is the unit-step function, (Heaviside function). The summation indicates the number of pairs of vectors closer to r when $1 \leq i, j \leq n$ and $i \neq j$. C(r) measures the density of pair of distinct vectors x_i and x_j that are closer to r.

Steps are as follows: for a given small positive number r, first we construct the correlation integral C(r) defined by (3) where the summation counts: how many pairs of vectors are closer than r when $1 \leq i, j \leq n$ and $i \neq j$. Actually, C (r) measures the density of pair of distinct vectors x_i and x_j that are closer than r. Finally, the correlation dimension D_C is defined as

$$DC = \lim_{r\to 0} \frac{\log C(r)}{\log r} \tag{4}$$

3. PIECEWISE CONTINUOUS DYNAMICAL SYSTEMS

It is discrete or continuous time dynamical system whose phase space is partitioned by a switching boundary into different regions, each associated to a different functional form of the system vector field.

Mathematically, a piecewise-defined map is that which can be defined by multiple sub-function applied to a certain interval of the main function's domain (a sub-domain). For example, consider the system containing terms such as the absolute value function:

$$|x| = \begin{cases} -x, \text{if } x<0 \\ x, \text{if } x\geq 0 \end{cases} \tag{5}$$

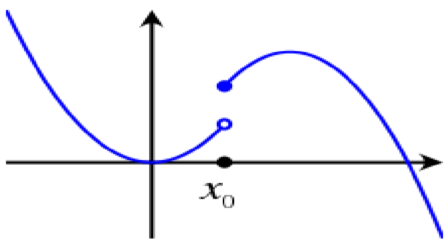

Figure 1: Piecewise continuous function.

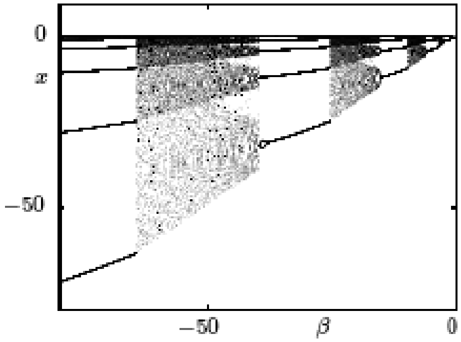

Figure 2: Bifurcation diagram of one dimensional piecewise continuous map [11] The above bifurcation diagram clearly indicates period doubling & chaos adding phenomena.

Or terms like | x + 1 |, | x – 1 |, Greatest Integer Function [21], Heaviside step function, Sign function etc.The pictured function, for example, is piecewise continuous throughout its sub domain, but is not continuous on the entire domain. The above function contains a jump discontinuity at x_0.

Even for a linear one dimensional piecewise continuous map, [11]

$F(x) = \begin{cases} \alpha x + \mu, \text{if } x \le 0 \\ \beta x + \mu, \text{if } x > 0 \end{cases}$ where α, β, μ are constants and for $\alpha > 0$, $\beta < 0$ and with $\mu = \pm 1$ one gets very interesting results for this map. For $\alpha = 0.4$,

$\mu = 1$ and $-80 \le \beta \le 0$ bifurcation diagram of this map is shown Figure 2. It is showing period adding and chaos doubling criteria.

4. COMPLEX DYNAMICS OF SOME PIECEWISE CONTINUOUS SYSTEMS

4.1 In our study, as an example, first we have considered the piecewise continuous map

$$f(x) = e^{-a|x|+b} \tag{6}$$

For a = 5.0, this map shows period doubling bifurcation leading to chaos for decreasing values of b from 0.4 and then, finally, it reduces to cycle one, Figure 3(a).

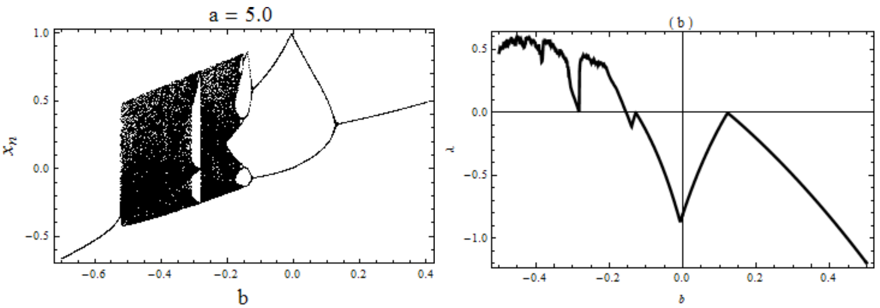

Figure 3(a): Bifurcation diagram for – 0.7 ≤ b ≤ 0.4 (left figure) and 3(b) LCE plot for – 0.5 ≤ b ≤ 0.5(right figure).

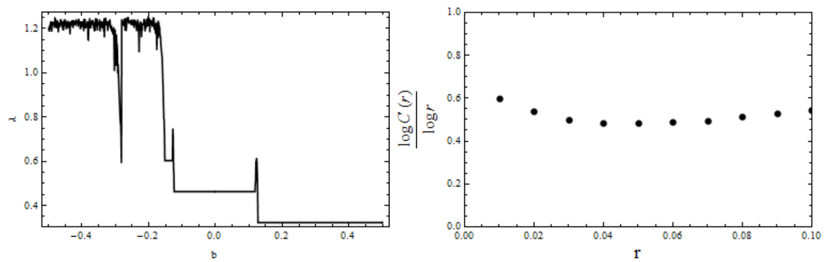

Figure 4: Plots of topological entropy (left) and correlation curve (right). Topological entropy plot be obtained for a = 0.5 and – 0.5 ≤ b ≤ 0.5 and correlation curve be obtained for a = 0.53 and b = – 0.3.

Plots of topological entropy & correlation dimension are shown in Figure 4: topological entropy for a=0.5 and –0.5≤ b ≤0.5 and correlation dimension curve for a=0.53 and b= –0.3.

Using linear fit to the correlation curve data one obtains

$$y = 0.52914 - 0.256885x \tag{7}$$

So, the correlation Dimension be obtained approximately as 0.53.

4.2 Dynamics of Lozi Map

Lozi map[14], is the subject of many recent articles [3-5,18] focused on its various properties. Quadratic term of Hénon map is replaced with a piecewise linear contribution in the former which produces some interesting chaotic attractors. The equations governing Lozi system can be written from Henon system

$$x_{n+1} = 1 + y - ax^2 \tag{8}$$

$$y_{n+1} = bx_n$$

By a piecewise linear contribution, |x|,

Lozi – map:

$$x_{n+1} = 1 + y - a|x|$$

$$y_{n+1} = bx_n \tag{9}$$

Lozi map shows abrupt change in behavior during transition from order-to disorder, (or chaos), and reveals interesting results. Attractors obtained during motion are also of very special type. Fixed Points of map (1) are: $P_1^* = \left(\frac{1}{1+a-b}, \frac{b}{1+a-b}\right), P_2^* = \left(\frac{1}{1-a-b}, \frac{b}{1-a-b}\right)$ and the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} -a\,sign(x) & 1 \\ b & 0 \end{bmatrix}$$

*Stability of fixed points P_1^* and P_2^****:** One observes, the fixed points P_1^* and P_2^*, are stable if |b|< 1.

The map (9) is different from many other 2–dimensional maps because of its piece wise continuous property and displays some special character during evolution, [18]:

- For, if we take b = –1 and a = 0.9, then $P_1^* \equiv$ (0.344828, –0.344828) and $P_2^* \equiv$ (0.909091, –0.909091)
- But the eigen values corresponding to P_1^* and P_2^* are same and be given by – 0.48 ± 0.893029 I ⇒ both P_1^* and P_2^* should be stable and orbits starting nearby these should be regular and not chaotic.
- for b ≤ 1, a ≤ b – 1, it has no fixed points and has a unique fixed point if b – 1 < a ≤ 1 – b. But, if a > 1 – b, the map has two fixed points. The parameters assumed a = 0.9, b = –1 fall in this category. b – 1 < a ≤ 1 – b, i. e. – 2 < a ≤ 2, and accordingly one gets only one fixed point, $P_1^* \equiv$ (0.344828, –0.344828) and it is a stable fixed point.

Characteristic Behavior For b = –1 and four values of a = 0.9, 1.5, 1.7, 1.99, phase plots are obtained starting closely to their corresponding fixed points and found all are periodic limit cycles: (Figure 5)

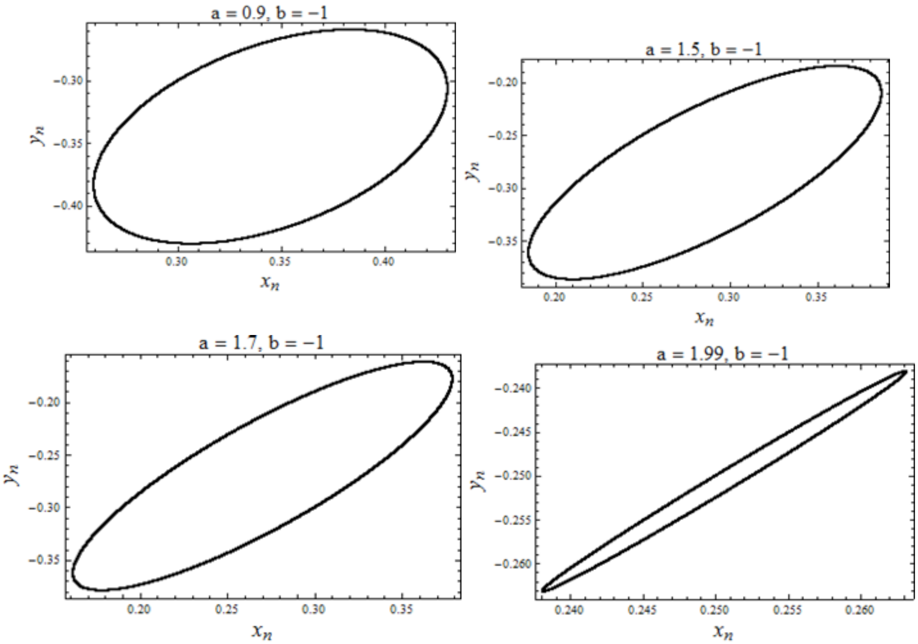

Figure 5: Plots are limit cycles for different set of values for a and b.

Taking b other than b = –1, but | b | < 1 and satisfying b – 1 < a ≤ 1 – b always lead to regularity.

Phase Plots nearby $P_1^*$and P_2^* fixed points have drawn. Nearby $P_1^*$and P_2^*, respectively, taken initial points as (0.35, –0.35) and (0.9, –0.9) we obtain, respectively, a stable limit cycle and an interesting type of strange attractor. Corresponding to these two initial conditions, plots of topological entropies for b = –1 and 0.8 ≤ a ≤ 1.1 are obtained and shown in the lower row of Figure 6. and (0.9, –0.9) and lower row represents the plots topological entropies for corresponding cases for parameter values b = –1 an d a = 0.9.

For b = 0.1 and 0.6 ≤ a ≤ 1.6 bifurcation diagrams, (upper row of Figure 7), are given below. It clearly indicates piecewise continuity while performing showing period doubling cascade. Another Similar diagram, (*lower row of Figure 7), is shown below for a = 1.5 and –0.6 ≤ b ≤ 1.6. Here we find bi-stability and overlapping criteria of bifurcation.

Plots of Lyapunov exponents (LCEs) for a = 0.9, b = –1 with two different initial conditions: (a) (0.35, –0.35) & (b) (0.9, –0.9) are given by Figure 8.

With b = –1 and values of a <1 – b, but very close to unity, map (1) displays interesting attractors. Some of such attractors are shown in Figure.

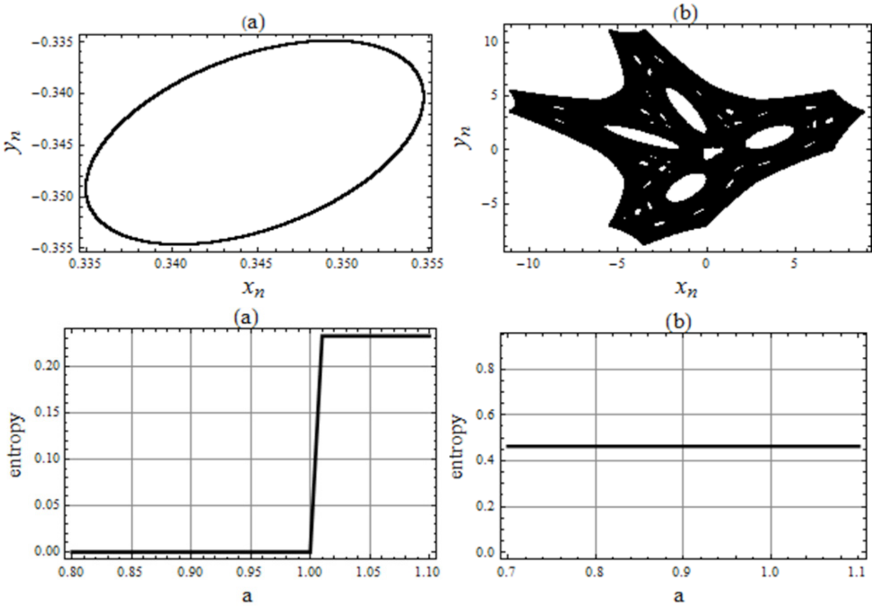

Figure 6: Upper row represent the plots of attractors when initial points are (0.35, –0.35).

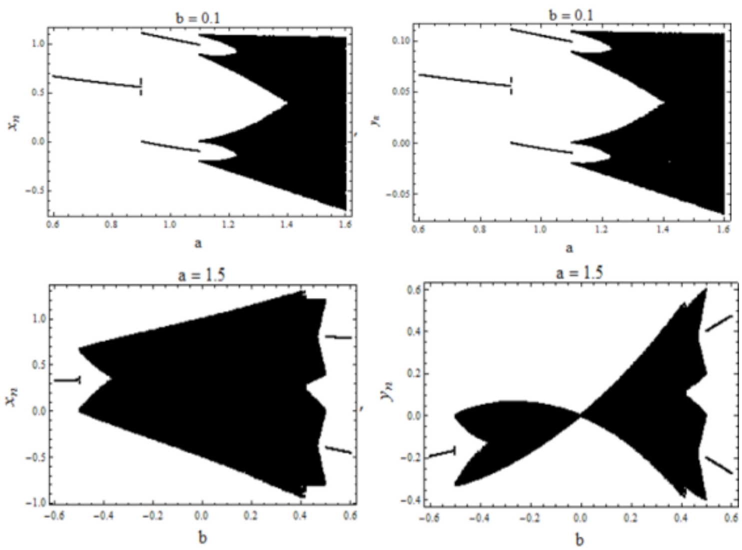

Figure 7: Bifurcation diagrams of Lozi map.

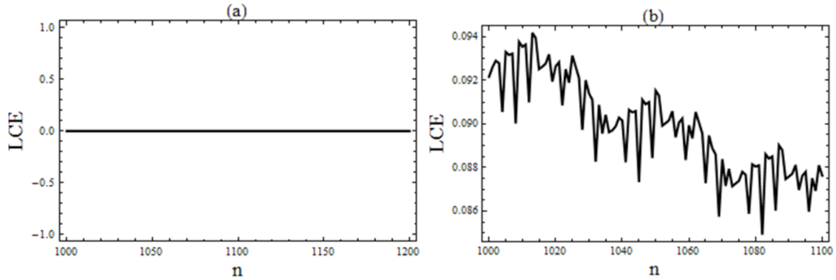

Figure 8: LCE plots for above two cases.

9. Attractors displayed are obtained when the system (1) started evolving at different initial points.

Next, let us study the cases when a > 1 – b i.e., when b = –1 and a > 2, then, we get points of cycles 2, 4, etc. For, a = 1.7, b = 0.5, one gets two fixed points given by P1*= (0.4545, 0.2273) and P2* = (–0.8333, –0.4167) approximately; for a = 2.8, b = –1, one gets 4 fixed points given by P1*= (–0.0676, –0.4054), P2*= (0.2083, –0.2083), P3*= (0.4054, 0.0676) and P4*= (–1.25, 1.25) approximately.

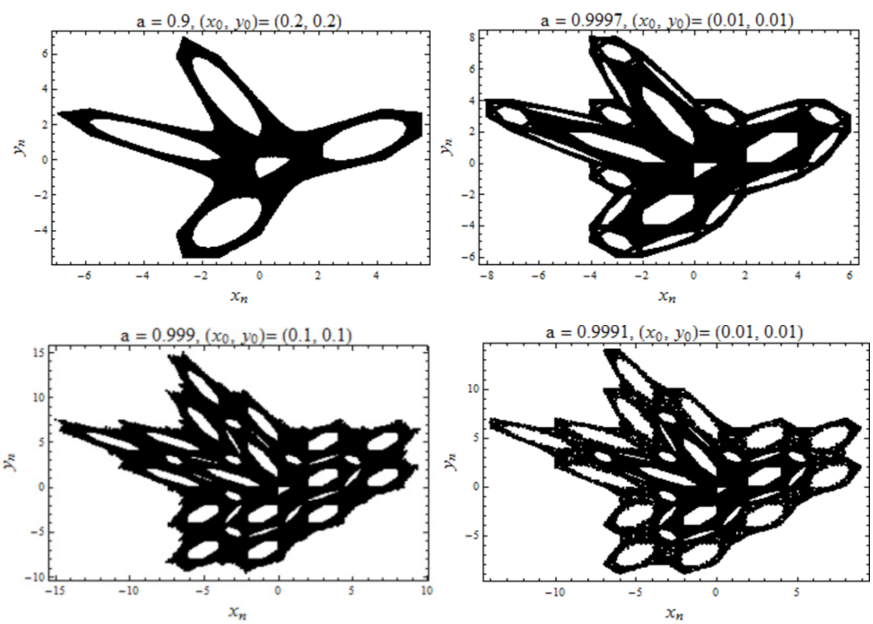

Figure 9: (Continued).

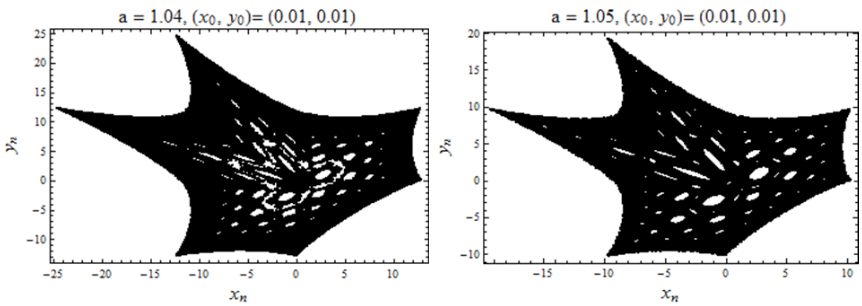

Figure 9: Some attractors of Lozi system when b = –1 and different values of a such that a < 1 – b, but closer to unity, and with different initial conditions.

For the case when a = 1.7, b = 0.5, taking initial points close to P1* and P2*as points (0.4, 0.2) and (–0.8, –0.4), time series and phase plots are obtained and shown in Figure 10 below. Both the orbits obtained show chaotic motion with strange attractors.

4.3 Chua's Circuit

Chua's map,[15],[17] ,is related to Chua's electric circuit theory that exhibits classic chaos theory behavior invented by Japanese *Prof. Leon O. Chua in 1983*. This means roughly that it is a "nonperiodic oscillator"; it produces an oscillating waveform that, unlike an ordinary electronic oscillator, never

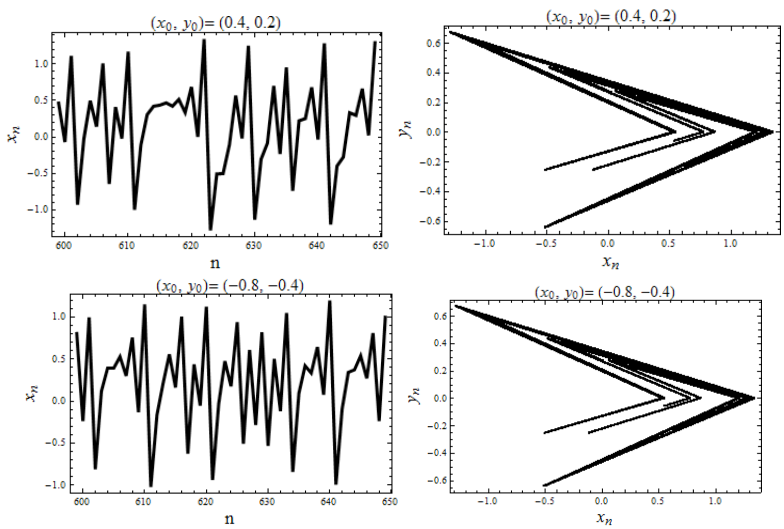

Figure 10: Plots of Lyapunov exponents, (left figure), and corresponding strange attractors, (right figure).

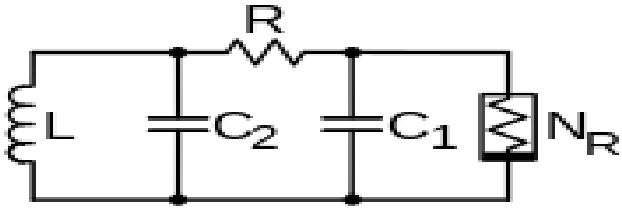

Figure 11: Diagram of Chua circuit.

"repeats". It is usually made of a circuit containing an amplifier with positive feedback.

- N_R is a nonlinear negative resistance called a Chua's diode.
- v_1 and v_2 and i are voltages across capacitors C_1 and C_2 and current through inductor L respectively

The circuit equations are given by

$$\frac{dv_1}{dt} = \frac{[G(v_2 - v_1) - f(v_1)]}{C_1}, \frac{dv_2}{dt} = \frac{[G(v_1 - v_2) + i]}{C_2}, \frac{di}{dt} = -\frac{v_2}{L}$$

where $f(v_1) = G_b v_1 + 0.5 (G_a - G_b)(|v_1 + B_p| - |v_1 - B_p|)$ be the characteristic of N_R .

In dimensionless form these equations can be written as

$$\frac{dx}{dt} = a[y - x - g(x)] \frac{dy}{dt} = x - y + z \frac{dz}{dt} = -by,$$

where a and b are dimensionless parameters and

$$g(x) = cx + \frac{1}{2}(d - c)(|x+1| - |x-1|)$$

where c & d are constants.

Above systems exhibit many interesting phenomena including period – doubling cascades to chaos.

For regular motion one takes: a = 10, b = 26.58, c= –5/7, d = –8/7; (x (0), y (0), z(0)) = (1.6, 0, 1.6) (Figure 12)

Chua circuit diaplays chaos for parameters a = 15, b=26.58, c = –5/7, d = –8/7; with initial condition (x(0),y(0),z(0)) = (1.6, 0, 1.6) motion is chaotic and confirmed through plots of time series (Figure 13)

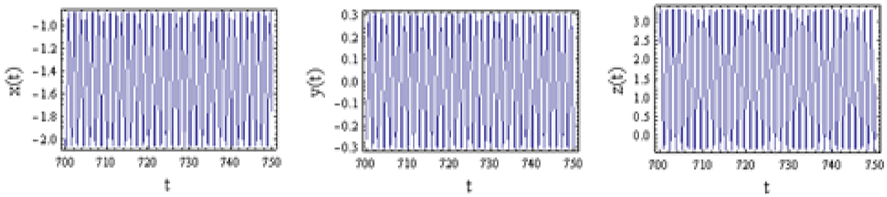

Figure 12: Plots of time series of Chua circuit for regular case when a = 10, b = 26.58, c = –5/7, d = –8/7; (x(0),y(0),z(0)) = (1.6, 0, 1.6).

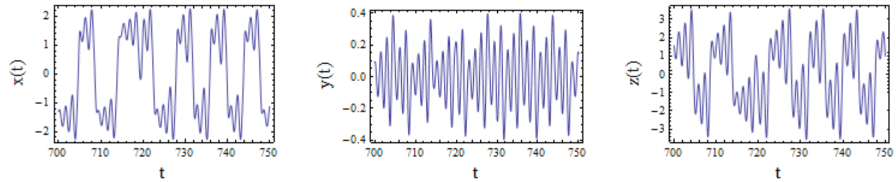

Figure 13: Time series plots chaotic Chua circuit.

For chaotic case of Chua circuit, we have numerically calculated Lyapunov exponents (LCEs) and shown in Figure 14

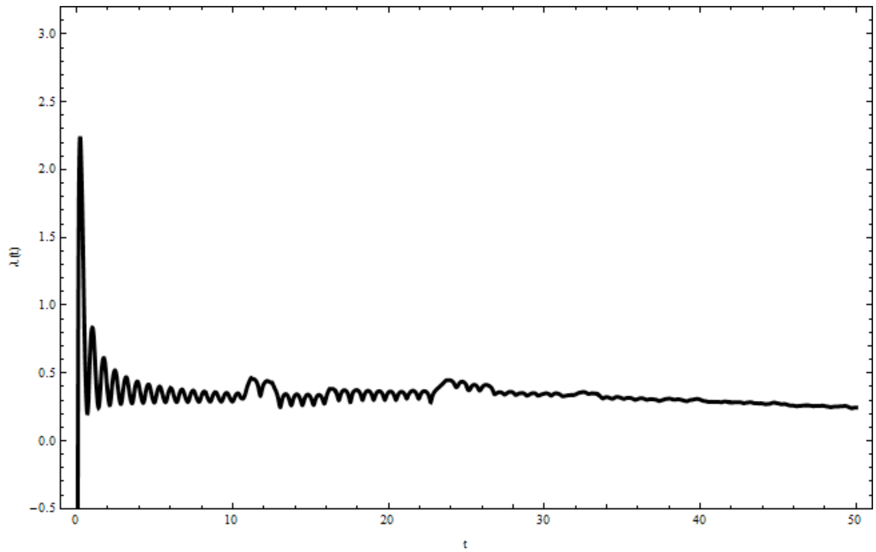

Figure 14: LCEs plot of Chua circuit. Barring some initial steps, (transient cases), LCEs are all positive.

DISCUSSION AND CONCLUSION

Chaotic evolutions are possible not only in nonlinear systems but also in some linear systems such as in 2-D Lozi map and also 1-D map stated in [11] provided the systems show piecewise continuity property. With this piecewise continuity in systems interesting chaotic attractors emerged through numerical calculations .Dynamics of Chua's circuit is widely studied and bifurcation analysis have been done by varying each parameters displaying interesting behavior. Also, different complexity measures have been carried by researchers. Here, also we have carried forward some studies on this map. In conclusion, it appears piecewise continuous dynamical systems are equally interesting and one should move forward in this direction.

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