COMPLICATED COLORINGS, REVISITED

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Abstract. In a paper from 1997, Shelah asked whether $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$ holds for every inaccessible cardinal λ. Here, we prove that an affirmative answer follows from $\square(\lambda^+)$. Furthermore, we establish that for every pair $\chi<\kappa$ of regular uncountable cardinals, $\square(\kappa)$ implies $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$.

1. Introduction

The subject matter of this paper is the following two anti-Ramsey coloring principles:

Definition 1.1 (Shelah, [She88]). $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $\sigma<\chi$, every pairwise disjoint subfamily $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every $\tau<\theta$, there is $(a,b)\in[\mathcal{A}]^2$ such that $c[a\times b]=\{\tau\}$.

Definition 1.2 (Lambie-Hanson and Rinot, [LHR18]). $\mathrm{U}(\kappa,\mu,\theta,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $\sigma<\chi$, every pairwise disjoint subfamily $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every $\tau<\theta$, there exists $\mathcal{B}\in[\mathcal{A}]^\mu$ such that, for every $(a,b)\in[\mathcal{B}]^2$, $\min(c[a\times b])\geq\tau$.[1]

The importance of this line of study — especially in proving instances of $\mathrm{Pr}_1(\dots)$ and $\mathrm{U}(\dots)$ with a large value of the 4^{th} parameter — is explained in details in the introductions to [Rin14a, Rin14b, LHR18]. In what follows, we survey a few milestone results, depending on the identity of κ.

► At the level of the first uncountable cardinal $\kappa=\aleph_1$, the picture is complete: In his seminal paper [Tod87], Todorčević proved that $\mathrm{Pr}_1(\aleph_1,\aleph_1,\aleph_1,2)$ holds, improving upon a classic result of Sierpiński [Sie33] asserting that $\mathrm{Pr}_1(\aleph_1,\aleph_1,2,2)$ holds. In 1980, Galvin [Gal80] proved that $\mathrm{Pr}_1(\aleph_1,\aleph_1,\theta,\aleph_0)$ is independent of ZFC for any cardinal $\theta\in[2,\aleph_1]$. Finally, a few years ago, by pushing further ideas of Moore [Moo06], Peng and Wu [PW18] proved that $\mathrm{Pr}_1(\aleph_1,\aleph_1,\aleph_1,\chi)$ holds for every $\chi\in[2,\aleph_0)$. As for the other coloring principle, and in contrast with Galvin's result, by [LHR18], $\mathrm{U}(\aleph_1,\aleph_1,\theta,\aleph_0)$ holds for any cardinal $\theta\in[2,\aleph_1]$.

► At the level of the second uncountable cardinal, $\kappa=\aleph_2$, a celebrated result of Shelah [She97] asserts that $\mathrm{Pr}_1(\aleph_2,\aleph_2,\aleph_2,\aleph_0)$ is a theorem of ZFC. Ever since, the following problem remained open:

Open problem (Shelah, [She97, She19]).
1. Does $\mathrm{Pr}_1(\aleph_2,\aleph_2,\aleph_2,\aleph_1)$ hold?
2. Does $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$ hold for λ inaccessible?

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[1]Note that $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ implies $\mathrm{U}(\kappa,2,\theta,\chi)$. However, by [LHR21a, Theorem 3.3], it does not imply $\mathrm{U}(\kappa,\kappa,\theta,\chi)$.

In comparison, by [LHR18], $\mathrm{U}(\lambda^+,\lambda^+,\theta,\lambda)$ is a theorem of ZFC for every infinite regular cardinal λ and every cardinal $\theta\in[2,\lambda^+]$.

▶ At the level of $\kappa=\lambda^+$ for λ a singular cardinal, the main problem left open has to do with the 3^{rd} parameter of $\mathrm{Pr}_1(\ldots)$ rather than the 4^{th} (see [She94a, ES05, ES09, Eis10, Eis13a, Eis13b]). This is a consequence of three findings. First, by the main result of [Rin12], for every singular cardinal λ and every cardinal $\theta\le\lambda^+$, $\mathrm{Pr}_1(\lambda^+,\lambda^+,\theta,2)$ implies $\mathrm{Pr}_1(\lambda^+,\lambda^+,\theta,\mathrm{cf}(\lambda))$. Second, by [RZ21a, §2], if λ is the singular limit of strongly compact cardinals, then $\mathrm{Pr}_1(\lambda^+,\lambda^+,2,(\mathrm{cf}(\lambda))^+)$ fails, meaning the the first result cannot be improved. Third, by [RZ21a, §2], $\mathrm{Pr}_1(\lambda^+,\lambda^+,2,\lambda)$ outright fails for every singular cardinal λ.

The situation with $\mathrm{U}(\ldots)$ is slightly better. An analog of the first result may be found as [LHR18, Lemma 2.5 and Theorem 4.21(3)]. An analog of the second result may be found as [LHR18, Theorem 2.14]. In contrast, by [LHR18, Corollary 4.15], it is in fact consistent that $\mathrm{U}(\lambda^+,\lambda^+,\theta,\lambda)$ holds for every singular cardinal λ and every cardinal $\theta\in[2,\lambda^+]$.

▶ At the level of a Mahlo cardinal κ, by [She94b, Conclusion 4.8(2)], the existence of a stationary subset of $E^\kappa_{\ge\chi}$ that does not reflect at inaccessibles entails that $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ holds for all $\theta<\kappa$. By [RZ21a, §5], the existence of nonreflecting a stationary subset of $\mathrm{Reg}(\kappa)$ on which $\diamondsuit$ holds entails that $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\kappa)$ holds.

The situation with $\mathrm{U}(\ldots)$ is analogous: By [LHR18, Theorem 4.23], the existence of a stationary subset of $E^\kappa_{\ge\chi}$ that does not reflect at inaccessibles entails that $\mathrm{U}(\kappa,\kappa,\theta,\chi)$ holds for all $\theta<\kappa$. By [LHR21b, §2], the existence of nonreflecting a stationary subset of $\mathrm{Reg}(\kappa)$ entails that $\mathrm{U}(\kappa,\kappa,\theta,\kappa)$ holds for all $\theta\le\kappa$.

▶ At the level of an abstract regular cardinal $\kappa\ge\aleph_2$, we mention two key results. First, by [Rin14b], for every regular cardinal $\kappa\ge\aleph_2$ and every $\chi\in\mathrm{Reg}(\kappa)$ such that $\chi^+<\kappa$, the existence of a nonreflecting stationary subset of $E^\kappa_{\ge\chi}$ entails that $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$ holds (this is optimal, by [LHR21a, Theorem 3.4], it is consistent that for some inaccessible cardinal κ, E^κ_χ admits a nonreflecting stationary set, and yet, $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi^+)$ fails). Second, by [Rin14a], for every regular cardinal $\kappa\ge\aleph_2$ and every $\chi\in\mathrm{Reg}(\kappa)$ such that $\chi^+<\kappa$, $\square(\kappa)$ entails that $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$ holds.

Here, the situation with $\mathrm{U}(\ldots)$ is again better. By [LHR18, Corollaries 4.12 and 4.15] and [LHR21b, §4], the analogs of the two results are true even without requiring "$\chi^+<\kappa$"!

After many years without progress on the above mentioned Open Problem, in the last few years, there have been a few breakthroughs. In an unpublished note from 2017, Todorčević proved that CH implies a weak form of $\mathrm{Pr}_1(\aleph_2,\aleph_2,\aleph_2,\aleph_1)$, strong enough to entail one of its intended applications (the existence of a σ-complete $\aleph_2$-cc partial order whose square does not satisfy that $\aleph_2$-cc). Next, in [RZ21a, §6], the authors obtained a full lifting of Galvin's strong coloring theorem, proving that for every infinite regular cardinal λ, $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$ holds assuming the stick principle $⫯(\lambda^+)$. In particular, an affirmative answer to (1) follows from $2^{\aleph_1}=\aleph_2$. Then, very recently, in [She21], Shelah proved that for every regular uncountable cardinal λ, $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$ holds assuming the existence of a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$. So, by a standard fact from inner model theory, a negative answer to (1) implies that $\aleph_2$ is a Mahlo cardinal in Gödel's constructible universe.

The main result of this paper reads as follows:

Theorem A. *For every regular uncountable cardinal* λ*, if* $\square(\lambda^+)$ *holds, then so does* $\Pr_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$*. In particular, a negative answer to (1) implies that* $\aleph_2$ *is a weakly compact cardinal in Gödel's constructible universe.*

Thanks to the preceding theorem, we can now waive the hypothesis "$\chi^+<\kappa$" from [Rin14a, Theorem B], altogether getting a clear picture:

Theorem A'. *For every pair* $\chi<\kappa$ *of regular uncountable cardinals,* $\square(\kappa)$ *implies* $\Pr_1(\kappa,\kappa,\kappa,\chi)$*.*

Now, let us say a few words about the proof. As made clear by the earlier discussion, in the case that $\kappa=\chi^+$, it is easier to prove $\mathrm{U}(\kappa,\kappa,\theta,\chi)$ than proving $\Pr_1(\kappa,\kappa,\theta,\chi)$. Therefore, we consider the following slight strengthening of $\mathrm{U}(\ldots)$:

Definition 1.3. $\mathrm{U}_1(\kappa,\mu,\theta,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $\sigma<\chi$, every pairwise disjoint subfamily $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every $\epsilon<\theta$, there exists $\mathcal{B}\in[\mathcal{A}]^\mu$ such that, for every $(a,b)\in[\mathcal{B}]^2$, there exists $\tau>\epsilon$ such that $c[a\times b]=\{\tau\}$.

Shelah's proof from [She21] can be described as utilizing the hypothesis of his theorem twice: first to get $\mathrm{U}_1(\lambda^+,2,\lambda^+,\lambda)$, and then to derive $\Pr_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$ from the latter. Here, we shall follow a similar path, building on the progress made in [RZ21b, §5] with respect to walking along well-chosen $\square(\kappa)$-sequences. We shall also present a couple of propositions translating $\mathrm{U}_1(\ldots)$ to $\Pr_1(\ldots)$ and vice versa, demonstrating that $\mathrm{U}_1(\kappa,\mu,\theta,\chi)$ is of interest also with $\theta<\kappa$. For instance, it will be proved that for every regular uncountable cardinal λ that admits a stationary set not reflecting at inaccessibles (e.g., $\lambda=\aleph_1$), $\mathrm{U}_1(\lambda^+,2,\lambda,\lambda)$ iff $\Pr_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$. Thus, the core contribution of this paper reads as follows.

Theorem B. *Suppose that* $\chi\leq\theta\leq\kappa$ *are infinite regular cardinals such that* $\max\{\chi,\aleph_1\}<\kappa$*. If* $\square(\kappa)$ *holds, then so does* $\mathrm{U}_1(\kappa,2,\theta,\chi)$*.*

2. Preliminaries

In what follows, $\chi<\kappa$ denotes a pair of infinite regular cardinals. $\mathrm{Reg}(\kappa)$ stands for the set of all infinite and regular cardinals below κ. Let $E^\kappa_\chi:=\{\alpha<\kappa\mid \mathrm{cf}(\alpha)=\chi\}$, and define $E^\kappa_{\leq\chi}$, $E^\kappa_{<\chi}$, $E^\kappa_{\geq\chi}$, $E^\kappa_{>\chi}$, $E^\kappa_{\neq\chi}$ analogously. A stationary subset $S\subseteq\kappa$ is *nonreflecting* (resp. *nonreflecting at inaccessibles*) iff there exists no $\alpha\in E^\kappa_{>\omega}$ (resp. α a regular limit uncountable cardinal) such that $S\cap\alpha$ is stationary in α. For a set of ordinals a, we write $\mathrm{ssup}(a):=\sup\{\alpha+1\mid\alpha\in a\}$, $\mathrm{acc}^+(a):=\{\alpha<\mathrm{ssup}(a)\mid \sup(a\cap\alpha)=\alpha>0\}$, $\mathrm{acc}(a):=a\cap\mathrm{acc}^+(a)$ and $\mathrm{nacc}(a):=a\setminus\mathrm{acc}(a)$. For sets of ordinals that are not ordinals, a and b, we write $a<b$ to express that $\alpha<\beta$ for all $\alpha\in a$ and $\beta\in b$. For an ordinal σ and a set of ordinals A, we write $[A]^\sigma$ for $\{B\subseteq A\mid \mathrm{otp}(B)=\sigma\}$. In the special case that $\sigma=2$ and $\mathcal{A}$ is either an ordinal or a collection of sets of ordinals, we interpret $[\mathcal{A}]^2$ as the collection of *ordered* pairs $\{(a,b)\in\mathcal{A}\times\mathcal{A}\mid a<b\}$. In particular, $[\kappa]^2=\{(\alpha,\beta)\mid \alpha<\beta<\kappa\}$.

For the rest of this section, let us fix a *C-sequence* $\vec{C}=\langle C_\alpha\mid\alpha<\kappa\rangle$ over κ, i.e., for every $\alpha<\kappa$, C_α is a closed subset of α with $\sup(C_\alpha)=\sup(\alpha)$. The next definition is due to Todorčević; see [Tod07] for a comprehensive treatment.

Definition 2.1 (Todorčević)**.** From $\vec{C}$, derive maps $\mathrm{Tr}:[\kappa]^2\rightarrow{}^\omega\kappa$, $\rho_2:[\kappa]^2\rightarrow\omega$, $\mathrm{tr}:[\kappa]^2\rightarrow{}^{<\omega}\kappa$ and $\lambda:[\kappa]^2\rightarrow\kappa$, as follows. Let $(\alpha,\beta)\in[\kappa]^2$ be arbitrary.

- $\mathrm{Tr}(\alpha,\beta):\omega\rightarrow\kappa$ is defined by recursion on $n<\omega$:

$$\mathrm{Tr}(\alpha,\beta)(n) := \begin{cases}\beta, & n=0\\ \min(C_{\mathrm{Tr}(\alpha,\beta)(n-1)}\setminus\alpha), & n>0\ \&\ \mathrm{Tr}(\alpha,\beta)(n-1)>\alpha\\ \alpha, & \text{otherwise}\end{cases}$$

- $\rho_2(\alpha,\beta) := \min\{n<\omega \mid \mathrm{Tr}(\alpha,\beta)(n)=\alpha\}$;
- $\mathrm{tr}(\alpha,\beta) := \mathrm{Tr}(\alpha,\beta)\restriction\rho_2(\alpha,\beta)$;
- $\lambda(\alpha,\beta) := \max\{\sup(C_{\mathrm{Tr}(\alpha,\beta)(i)}\cap\alpha) \mid i<\rho_2(\alpha,\beta)\}$.

Convention 2.2. From any coloring $h:\kappa\rightarrow\kappa$, derive a function $\mathrm{tr}_h:[\kappa]^2\rightarrow{}^{<\omega}\kappa$ via

$$\mathrm{tr}_h(\alpha,\beta) := \langle h(\mathrm{Tr}(\alpha,\beta)(i)) \mid i<\rho_2(\alpha,\beta)\rangle.$$

The next fact is quite elementary. See, e.g., [Rin14b, Claim 3.1.2] for a proof.

Fact 2.3. *Whenever $\lambda(\gamma,\beta)<\alpha<\gamma<\beta<\kappa$, $\mathrm{tr}(\alpha,\beta)=\mathrm{tr}(\gamma,\beta){}^\frown\mathrm{tr}(\alpha,\gamma)$.*

We now recall the characteristic $\lambda_2(\cdot,\cdot)$, a variation of $\lambda(\cdot,\cdot)$ having the property that $\lambda_2(\gamma,\beta)<\gamma$ whenever $0<\gamma<\beta<\kappa$.

Definition 2.4 ([Rin14a])**.** Define $\lambda_2:[\kappa]^2\rightarrow\kappa$ via

$$\lambda_2(\alpha,\beta) := \sup(\alpha\cap\{\sup(C_\delta\cap\alpha) \mid \delta\in\mathrm{Im}(\mathrm{tr}(\alpha,\beta))\}).$$

Fact 2.5 ([LHR18, Lemma 4.7])**.** *Suppose that $\lambda_2(\gamma,\beta)<\alpha<\gamma<\beta<\kappa$. Then $\mathrm{tr}(\alpha,\beta)$ end-extends $\mathrm{tr}(\gamma,\beta)$, and one of the following cases holds:*

1. $\gamma\in\mathrm{Im}(\mathrm{tr}(\alpha,\beta))$*; or*
2. $\gamma\in\mathrm{acc}(C_\delta)$ *for* $\delta:=\min(\mathrm{Im}(\mathrm{tr}(\gamma,\beta)))$.

Convention 2.6 ([RZ21b])**.** For every ordinal $\eta<\kappa$ and a pair $(\alpha,\beta)\in[\kappa]^2$, let

$$\eta_{\alpha,\beta} := \min\{n<\omega \mid \eta\in C_{\mathrm{Tr}(\alpha,\beta)(n)} \text{ or } n=\rho_2(\alpha,\beta)\}+1.$$

Definition 2.7 ([RZ21a, §3])**.** $\chi_1(\vec{C})$ stands for the supremum of $\sigma+1$ over all $\sigma<\kappa$ satisfying the following. For every pairwise disjoint subfamily $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, there are a stationary set $\Delta\subseteq\kappa$ and an ordinal $\eta<\kappa$ such that, for every $\delta\in\Delta$, there exist κ many $b\in\mathcal{A}$ such that, for every $\beta\in b$, $\lambda(\delta,\beta)=\eta$ and $\rho_2(\delta,\beta)=\eta_{\delta,\beta}$.

Fact 2.8 ([RZ21a, §3])**.** *If the two hold:*

- (ℵ) *for all $\alpha<\kappa$ and $\delta\in\mathrm{acc}(C_\alpha)$, $C_\delta=C_\alpha\cap\delta$;*
- (ℶ) *for every club $D\subseteq\kappa$, there exists $\gamma>0$ with $\sup(\mathrm{nacc}(C_\gamma)\cap D)=\gamma$,*

then $\chi_1(\vec{C})=\sup(\mathrm{Reg}(\kappa))$.

Definition 2.9 (Todorčević, [Tod87])**.** For a cardinal $\mu\leq\kappa$, $\square(\kappa,{<}\mu)$ asserts the existence of a sequence $\vec{\mathcal{C}}=\langle\mathcal{C}_\alpha \mid \alpha<\kappa\rangle$ such that

1. for every $\alpha<\kappa$, $\mathcal{C}_\alpha$ is nonempty collection of less than μ many closed subsets C of α with $\sup(C)=\sup(\alpha)$;
2. for all $\alpha<\kappa$, $C\in\mathcal{C}_\alpha$ and $\delta\in\mathrm{acc}(C)$, $C\cap\delta\in\mathcal{C}_\delta$;
3. there exists no club C in κ such that $C\cap\alpha\in\mathcal{C}_\alpha$ for all $\alpha\in\mathrm{acc}(C)$.

The special case of $\square(\kappa,{<}\mu)$ with $\mu=2$ is denoted by $\square(\kappa)$.

Fact 2.10 (Hayut and Lambie-Hanson, [HLH17, Lemma 2.4])**.** *Clause (3) of Definition 2.9 is preserved in any κ-cc forcing extension, provided that $\mu<\kappa$.*

3. Theorem B

Theorem 3.1. *Suppose that $\chi \leq \theta \leq \kappa$ are infinite regular cardinals such that $\max\{\chi, \aleph_1\} < \kappa$. If $\square(\kappa)$ holds, then so does $\mathrm{U}_1(\kappa, 2, \theta, \chi)$.*

Proof. Suppose that $\square(\kappa)$ holds. Then, by [RZ21b, Lemma 5.1], we may fix a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ satisfying the following:

(1) $C_{\alpha+1} = \{0, \alpha\}$ for every $\alpha < \kappa$;
(2) for every club $D \subseteq \kappa$, there exists $\gamma > 0$ with $\sup(\mathrm{nacc}(C_\gamma) \cap D) = \gamma$;
(3) for every $\alpha \in \mathrm{acc}(\kappa)$ and $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
(4) for every $i < \kappa$, $\{\alpha < \kappa \mid \min(C_\alpha) = i\}$ is stationary.

Note that, by Fact 2.8, $\chi_1(\vec{C}) = \sup(\mathrm{Reg}(\kappa))$. If $\theta < \kappa$, then let $\mu := \theta$; otherwise, let $\mu := \chi$. Derive a coloring $h : \kappa \to \mu$ via

$$h(\alpha) := \begin{cases} \min(C_\alpha), & \text{if } \min(C_\alpha) < \mu; \\ 0, & \text{otherwise.} \end{cases}$$

We shall walk along $\vec{C}$. Define a coloring $d : [\kappa]^2 \to \mu$ via

$$d(\alpha, \beta) := \max(\mathrm{Im}(\mathrm{tr}_h(\alpha, \beta))).$$

Claim 3.1.1. *Suppose that α, β, γ are ordinals, and $\lambda_2(\gamma, \beta) < \alpha < \gamma < \beta < \kappa$.*

Then $\mathrm{Im}(\mathrm{tr}_h(\alpha, \beta)) = \mathrm{Im}(\mathrm{tr}_h(\alpha, \gamma)) \cup \mathrm{Im}(\mathrm{tr}_h(\gamma, \beta))$. *In particular,* $d(\alpha, \beta) = \max\{d(\alpha, \gamma), d(\gamma, \beta)\}$.

Proof. By Fact 2.5, one of the following cases holds:

- ▶ $\gamma \in \mathrm{Im}(\mathrm{tr}(\alpha, \beta))$. In this case, $\mathrm{tr}(\alpha, \beta) = \mathrm{tr}(\gamma, \beta)^\frown \mathrm{tr}(\alpha, \gamma)$, so we done.
- ▶ $\gamma \in \mathrm{acc}(C_\delta)$ for $\delta := \min(\mathrm{Im}(\mathrm{tr}(\gamma, \beta)))$. In this case, $\mathrm{tr}(\alpha, \beta) = \mathrm{tr}(\delta, \beta)^\frown \mathrm{tr}(\alpha, \delta)$, so that $\mathrm{Im}(\mathrm{tr}_h(\alpha, \beta)) = \mathrm{Im}(\mathrm{tr}_h(\alpha, \delta)) \cup \mathrm{Im}(\mathrm{tr}_h(\delta, \beta))$. Since $\gamma \in \mathrm{acc}(C_\delta)$, Clause (3) above and the definition of the function h together imply that $\mathrm{tr}_h(\alpha, \delta) = \mathrm{tr}_h(\alpha, \gamma)$. In addition, $\mathrm{tr}(\gamma, \beta) = \mathrm{tr}(\delta, \beta)^\frown\langle\delta\rangle$, so that $\mathrm{Im}(\mathrm{tr}_h(\gamma, \beta)) = \mathrm{Im}(\mathrm{tr}_h(\delta, \beta)) \cup \{h(\delta)\}$. Since $h(\delta) \in \mathrm{Im}(\mathrm{tr}_h(\alpha, \delta))$, altogether,
$$\mathrm{Im}(\mathrm{tr}_h(\alpha, \gamma)) \cup \mathrm{Im}(\mathrm{tr}_h(\gamma, \beta)) = \mathrm{Im}(\mathrm{tr}_h(\alpha, \delta)) \cup \mathrm{Im}(\mathrm{tr}_h(\delta, \beta)). \qquad \square$$

We are now ready to define the sought coloring c. If $\mu = \theta$, then let $c := d$, and otherwise define $c : [\kappa]^2 \to \theta$ via

$$c(\alpha, \beta) := \max\{\xi \in \mathrm{Im}(\mathrm{tr}(\alpha, \beta)) \mid h(\xi) = d(\alpha, \beta)\}.$$

To see that c witnesses $\mathrm{U}_1(\kappa, 2, \theta, \chi)$, suppose that we are given $\epsilon < \theta$, $\sigma < \chi$ and a κ-sized pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$; we need to find $\tau > \epsilon$ and $(a, b) \in [\mathcal{A}]^2$ such that $c[a \times b] = \{\tau\}$. As $\sigma < \chi \leq \chi_1(\vec{C})$, we may fix a stationary subset $\Delta \subseteq \kappa$ and an ordinal $\eta < \kappa$ such that, for every $\delta \in \Delta$, there exists $b \in \mathcal{A}$ with $\min(b) > \delta$ such that $\lambda(\delta, \beta) = \eta$ for every $\beta \in b$. Set $\eta' := \max\{\eta, \epsilon\}$.

Consider the club $C := \{\gamma < \kappa \mid \sup\{\min(a) \mid a \in \mathcal{A} \cap \mathcal{P}(\gamma)\} = \gamma\}$. For all $\gamma \in C$ and $\varepsilon < \gamma$, fix $a^\gamma_\varepsilon \in \mathcal{A} \cap \mathcal{P}(\gamma)$ with $\min(a^\gamma_\varepsilon) > \varepsilon$; as $|a^\gamma_\varepsilon| < \mu$, $\tau^\gamma_\varepsilon := \sup\{d(\alpha, \gamma) \mid \alpha \in a^\gamma_\varepsilon\}$ is $< \mu$. Fix some stationary $\Gamma \subseteq C \cap E^\kappa_{\neq\mu}$ along with $\tau_0 < \mu$ such that, for every $\gamma \in \Gamma$, $\sup\{\varepsilon < \gamma \mid \tau^\gamma_\varepsilon \leq \tau_0\} = \gamma$.

By Clause (4), for each $i < \mu$, $H_i := \{\alpha < \kappa \mid h(\alpha) = i\}$ is stationary, so, fix $\delta \in \Delta \cap \bigcap_{i<\mu} \mathrm{acc}^+(H_i \cap \mathrm{acc}^+(\Gamma \setminus \eta'))$. Pick $b \in \mathcal{A}$ with $\min(b) > \delta$ such that $\lambda(\delta, \beta) = \eta$ for every $\beta \in b$. As $|b| < \mu$, $\tau_1 := \sup\{d(\delta, \beta) \mid \beta \in b\}$ is $< \mu$. If $\epsilon < \mu$, then pick

$\zeta \in H_{\tau_0+\tau_1+\epsilon+1} \cap \mathrm{acc}^+(\Gamma \setminus \eta')$; otherwise, pick $\zeta \in H_{\tau_0+\tau_1+1} \cap \mathrm{acc}^+(\Gamma \setminus \eta')$. Next, pick $\gamma \in \Gamma$ above $\max\{\lambda_2(\zeta,\delta),\eta'\}$. Finally, pick $\varepsilon < \gamma$ above $\max\{\lambda_2(\gamma,\zeta),\lambda_2(\zeta,\delta),\eta'\}$ such that $\tau^\gamma_\varepsilon \leq \tau_0$, and then set $a := a^\gamma_\varepsilon$.

Claim 3.1.2. *Let $\alpha \in a$ and $\beta \in b$. Then:*

(i) $\max\{\lambda_2(\gamma,\zeta),\lambda_2(\zeta,\delta),\lambda(\delta,\beta),\epsilon\} < \varepsilon < \alpha < \gamma < \zeta < \delta < \beta$;
(ii) $c(\alpha,\beta) = c(\gamma,\delta) > \epsilon$.

Proof. (i) This is clear, recalling that $\eta' = \max\{\lambda(\delta,\beta),\epsilon\}$.

(ii) From $\lambda(\delta,\beta) < \alpha < \delta < \beta$ and Fact 2.3, we infer that $\mathrm{tr}(\alpha,\beta) = \mathrm{tr}(\delta,\beta)^\frown \mathrm{tr}(\alpha,\delta)$, so that $d(\alpha,\beta) = \max\{d(\delta,\beta), d(\alpha,\delta)\}$. By Clause (i) and Claim 3.1.1,

$$d(\alpha,\delta) = \max\{d(\alpha,\zeta), d(\zeta,\delta)\} \geq h(\zeta) > \tau_1 \geq d(\delta,\beta).$$

Consequently, $d(\alpha,\beta) = d(\alpha,\delta)$ and $c(\alpha,\beta) = c(\alpha,\delta)$. By Clause (i) and Claim 3.1.1, $\mathrm{Im}(\mathrm{tr}_h(\alpha,\delta)) = \mathrm{Im}(\mathrm{tr}_h(\alpha,\gamma) \cup \mathrm{Im}(\mathrm{tr}_h(\gamma,\delta))$. As $d(\alpha,\gamma) \leq \tau_0 < h(\zeta) \leq d(\gamma,\delta)$, it follows that $d(\alpha,\delta) = d(\gamma,\delta)$ and $c(\alpha,\delta) = d(\gamma,\delta)$. Altogether, $c(\alpha,\beta) = c(\gamma,\delta)$.

Now, if $\theta < \kappa$, then $\epsilon < \theta = \mu$ and $c = d$, so that $c(\alpha,\beta) = d(\gamma,\delta) \geq h(\zeta) > \epsilon$. Otherwise, $c(\alpha,\beta) \geq \min(\mathrm{Im}(\mathrm{tr}(\alpha,\beta))) > \alpha > \epsilon$. □

Set $\tau := c(\gamma,\delta)$. Then $\tau > \epsilon$ and $c[a \times b] = \{\tau\}$, as sought. □

Remark 3.2. The preceding proof makes it clear that the auxiliary coloring d witnesses $\mathrm{U}_1(\kappa,2,\mu,\chi)$. By Fact 2.5, the coloring d is moreover *closed* in the sense that, for all $\beta < \kappa$ and $i < \theta$, the set $\{\alpha < \beta \mid c(\alpha,\beta) \leq i\}$ is closed below β. So, by [LHR18, Lemma 4.2], d witnesses $\mathrm{U}(\kappa,\kappa,\mu,\chi)$, as well.

4. Connecting U_1 with Pr_1

Throughout this section, $\chi < \kappa$ is a pair of infinite regular cardinals, and θ is a regular cardinal $\leq \kappa$. Let $\mathbb{A}^\kappa_\chi$ denote the collection of all pairwise disjoint subfamilies $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{A}| = \kappa$ and $\sup\{|a| \mid a \in \mathcal{A}\} < \chi$. Given a coloring $c : [\kappa]^2 \to \theta$, for every $\mathcal{A} \subseteq \mathcal{P}(\kappa)$, let $T_c(\mathcal{A})$ be the set of all $\tau < \theta$ such that, for some $(a,b) \in [\mathcal{A}]^2$, $c[a \times b] = \{\tau\}$. The next definition appears (with a slightly different notation) in Stage B in the proof of [She21, Theorem 1.1]:

Definition 4.1. For every coloring $c : [\kappa]^2 \to \theta$, let

$$F_{c,\chi} := \{T \subseteq \theta \mid \exists \mathcal{A} \in \mathbb{A}^\kappa_\chi\, [T_c(\mathcal{A}) \subseteq T]\}.$$

Proposition 4.2. *Suppose that a coloring $c : [\kappa]^2 \to \theta$ witnesses $\mathrm{U}_1(\kappa,2,\theta,\chi)$, and λ is some cardinal. Then:*

1. *$F_{c,\chi}$ is a χ-complete uniform filter on θ;*
2. *If every χ-complete uniform filter on θ is not weakly λ-saturated, then $\mathrm{Pr}_1(\kappa,\kappa,\lambda,\chi)$ holds.*

Proof. (1) It is clear that $F_{c,\chi}$ is upward-closed. To see that it is χ-complete, suppose that we are given a sequence $\langle X_i \mid i < \delta\rangle$ of elements of $F_{c,\chi}$, for some $\delta < \chi$. For each $i < \delta$, fix $\mathcal{A}_i \in \mathbb{A}^\kappa_\chi$ such that $T_c(\mathcal{A}_i) \subseteq X_i$. Pick $\mathcal{A} \in \mathbb{A}^\kappa_\chi$ such that, for every $a \in \mathcal{A}$, there is a sequence $\langle a_i \mid i < \delta\rangle \in \prod_{i<\delta} \mathcal{A}_i$ such that $a = \bigcup_{i<\delta} a_i$. Then, $T_c(\mathcal{A}) \subseteq \bigcap_{i<\delta} T_c(\mathcal{A}_i) \subseteq \bigcap_{i<\delta} X_i$ and hence the latter is in $F_{c,\chi}$. Finally, since c witnesses $\mathrm{U}_1(\kappa,2,\theta,\chi)$, for every $\mathcal{A} \in \mathbb{A}^\kappa_\chi$ and every $\epsilon < \theta$, $T_c(\mathcal{A}) \setminus \epsilon$ is nonempty. So $F_{c,\chi}$ consists of cofinal subset of θ. Since θ is regular, $F_{c,\chi}$ is uniform.

(2) Suppose that no χ-complete uniform filter on θ is weakly λ-saturated. In particular, by Clause (1), we may pick a map $\psi : \theta \to \lambda$ such that that the preimage of any singleton is $F_{c,\theta}$-positive. Then $\psi \circ c$ witnesses $\mathrm{Pr}_1(\kappa, \kappa, \lambda, \chi)$. $\square$

Corollary 4.3. *Suppose that λ is a regular uncountable cardinal.*

If λ admits a stationary set that does not reflect at regulars or if $\square(\lambda, {<}\mu)$ holds for some cardinal $\mu < \lambda$, then the following are equivalent:

1. $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$;
2. $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \lambda)$;
3. $\mathrm{U}_1(\lambda^+, 2, \lambda, \lambda)$.

Proof. The implication (1) $\implies$ (2) $\implies$ (3) is trivial, and the fact that (2) $\implies$ (1) is well-known (see, for instance, [KRS21, §6]). By the preceding proposition, to see that (3) $\implies$ (2), it suffices to prove that under our hypothesis on λ, no λ-complete uniform filter on λ is weakly λ-saturated. Now, if λ is a successor cardinal, then this follows from Ulam's theorem [Ula30], and if λ is an inaccessible cardinal admitting a stationary set that does not reflect at regulars, then this follows from a theorem of Hajnal [Haj69]. Finally, if $\square(\lambda, {<}\mu)$ holds for some cardinal $\mu < \lambda$, then this follows from [IR22, Theorem A]. $\square$

Lemma 4.4. *Suppose that λ is a regular uncountable cardinal and $\square(\lambda^+, {<}\lambda)$ holds. Then every λ-complete uniform filter on λ^+ is not weakly λ-saturated.*

Proof. Fix a $\square(\lambda^+, {<}\lambda)$-sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$. For each $\alpha < \lambda^+$, fix an injective enumeration $\langle C_{\alpha,i} \mid i < |\mathcal{C}_\alpha| \rangle$ of $\mathcal{C}_\alpha$.

Towards a contradiction, suppose that F is a λ-complete uniform filter on λ^+ that is weakly λ-saturated. Since F is λ-complete, F is moreover λ-saturated. Hence, $\mathcal{P}(\lambda^+)/F$ is a λ-cc notion of forcing.

Let G be $\mathcal{P}(\lambda^+)/F$-generic over V. Then G is a uniform V-ultrafilter over λ^+ extending F. By [For10, Propositions 2.9 and 2.14], $\mathrm{Ult}(V, G)$ is well-founded and $j : V \to M \simeq \mathrm{Ult}(V, G)$ satisfies $\mathrm{crit}(j) = \lambda$.

Now, work in $V[G]$. Denote $j(\vec{\mathcal{C}})$ by $\langle \mathcal{D}_\alpha \mid \alpha < j(\lambda^+) \rangle$. For every $\alpha < \lambda^+$, since $\mathrm{crit}(j) = \lambda > |\mathcal{C}_\alpha|$, it is the case that $\mathcal{D}_{j(\alpha)} = j(\mathcal{C}_\alpha) = j``\mathcal{C}_\alpha$. Since $G$ is uniform, $\gamma := \sup(j``\lambda^+)$ is $< j(\lambda^+)$, as witnessed by the identity map $\mathrm{id} : \lambda^+ \to \lambda^+$. As $V[G]$ is a λ-cc forcing extension of V, $\mathrm{cf}^V(\gamma) = \mathrm{cf}^{V[G]}(\gamma) = \lambda^+$, so that $\mathrm{cf}^M(\gamma) \geq \lambda^+$. Pick $D \in \mathcal{D}_\gamma$.

Claim 4.4.1. *$A := j^{-1}[\mathrm{acc}(D)]$ is a cofinal subset of λ^+.*

Proof. Given $\epsilon < \lambda^+$, we recursively define (in $V[G]$) an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ of ordinals below λ^+ such that:

1. $\epsilon = \alpha$, and
2. for all $n < \omega$, $(j(\alpha_n), j(\alpha_{n+1})] \cap D \neq \emptyset$.

Consider $\alpha^* := \sup_{n<\omega} \alpha_n$. Notice that $\mathrm{cf}^V(\alpha^*) < \lambda$, since if $\mathrm{cf}^V(\alpha^*) \geq \lambda$, then by the fact that $V[G]$ is a λ-cc forcing extension of V we have $\omega = \mathrm{cf}^{V[G]}(\alpha^*) \geq \lambda$ which is impossible. As a result, $\sup j``\alpha^* = j(\alpha^*) \in \mathrm{acc}(D)$, which implies that α^* is an element of A above ϵ. $\square$

For each $\alpha \in A$, $D \cap j(\alpha) \in \mathcal{D}_{j(\alpha)} = j``\mathcal{C}_\alpha$, so we may pick some $i_\alpha < \lambda$ such that $D \cap j(\alpha) = j(C_{\alpha,i_\alpha})$. Fix some $i < \lambda$ for which $A' := \{\alpha \in A \mid i_\alpha = i\}$ is cofinal in λ^+. For every $(\alpha, \beta) \in [A']^2$, $j(C_{\alpha,i}) = D \cap j(\alpha)$ and $j(C_{\beta,i}) = D \cap j(\beta)$, so, by elementarity, $C_{\alpha,i} = C_{\beta,i} \cap \alpha$. As A' is cofinal in λ^+, it follows that $C := \bigcup\{C_{\alpha,i} \mid \alpha \in A\}$ is a club in λ^+. Evidently, $C \cap \alpha \in \mathcal{C}_\alpha$ for every $\alpha \in \operatorname{acc}(C)$. However, $V[G]$ is a λ-cc forcing extension of V, contradicting Fact 2.10. $\square$

We are now ready to prove Theorem A:

Corollary 4.5. *Suppose that λ is a regular uncountable cardinal, and $\square(\lambda^+)$ holds. Then $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$ holds, as well.*

Proof. By Theorem 3.1, using $(\kappa, \theta, \chi) := (\lambda^+, \lambda^+, \lambda)$, $\mathrm{U}_1(\lambda^+, 2, \lambda^+, \lambda)$ holds. So, by Proposition 4.2 (using $\theta := \lambda^+$) and Lemma 4.4, $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \lambda)$ holds. Then, again by [KRS21, §6], $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$ holds, as well. $\square$

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