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COMPONENT COST ANALYSIS OF LARGE-SCALE SYSTEMS

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I. INTRODUCTION

One of the most fundamental concepts in systems theory is the basic definition of a dynamic system. A dynamic system may be defined as an interconnection of entities (which we

shall call "components") causally related in time. It seems equally natural and basic, therefore, to characterize the system's behavior in terms of contributions from each of the system's building blocks--"components." The performance of the dynamic system is quite often evaluated in terms of some performance metric we choose to call the "cost function V. The cost function might represent the system energy or a norm of the output errors over some interval of time. Concerning the physical or mathematical components of the system, it is only natural then to ask question CC: "What fraction of the overall system cost V is due to each component of the system?"

This chapter is devoted to a precise answer to question CC and to several applications of the mathematical machinery developed for answering the question. Such an analysis will be called *component cost analysis* (CCA). Conceptually, it is easy to imagine several uses for CCA.

(a) Knowledge of the magnitude of each component's contribution to the system performance can be used to suggest which components might be *redesigned* if better performance is needed. By redesigning so as to reduce the cost associated with these "critical" components (those with larger contributions to system performance), one is following a "cost-balancing" strategy for system design. *Thus, CCA can be useful in system design strategies.*

(b) Knowledge of the magnitude of each component's contribution to the system performance can be used to predict the performance degradation in the event of a *failure* of any component. *Thus, CCA can be useful in failure mode analysis.*

(c) Knowledge of the magnitude of each component's contribution to the performance of a higher order model of the system can be used to decide which components to delete from the model to produce lower order models. *Thus, CCA can be useful in model reduction.*

(d) Alternately, if one defines the components to include each dynamical element of a linear feedback controller, the knowledge of the magnitude of each component's contribution to the closed-loop system performance can be used to determine which dynamical elements of the controller to delete so as to cause the smallest change in performance which respect to the performance of the high-order controller. *Thus, CCA can be useful in the design of low-order controllers* that meet on-line controller software limitations.

This chapter will focus on possibility (c) in some detail.

This notion of using a performance metric is basic in the most well-developed and simplest problem of optimal control: the linear quadratic problem. However, one of the fundamental deficiencies of modern control theory is its absolute reliance on the fidelity of the mathematical model of the underlying physical system, which is essentially infinte dimensional. Many "failures" of modern control applications are due to modeling errors. Thus, theories that can more systematically relate the modeling problem and the control problem are sorely needed since these two problems are not truly separable, although most practice and theory presently treat them as separable. This chapter presents one such unifying theory and can be viewed as an *application* chapter in the sense that it is concerned with making the linear quadratic theory more

practical. Thus, the proposed insights into the behavior of dynamic systems are available within the standard mathematical tools of linear quadratic and linear quadratic Gaussian (LQG) theories [6]. Hence, the contributions of CCA lie not in the development of new mathematical theories, but in the presentation of cost decomposition procedures that readily reveal the "price" of system components. A similar notion of "pricing" of system components is a common strategy in operations research and mathematical programming problems such as Dantzig-Wolfe decomposition and the dual algorithm by Benders [1,2]. However, such useful notions of pricing seem not to have found their way into common control practice. This paper calls attention to the manner in which such notions can be used in dynamic systems. The mathematical details are quite different from the pricing of the static models of operations research, but the concepts are similar.

The concepts of CCA evolved in a series of presentations [3-5]. However, these introductory papers left unanswered the most important questions of stability, the best choice of coordinates, and development of the theory of minimal realizations with respect to quadratic performance metrics. This chapter, therefore, presents a complete theory for CCA and, in addition, develops the theory of minimal realizations with respect to quadratic performance metrics.

II. COMPONENT DESCRIPTIONS

The entities that compose dynamic systems are herein labeled "components." To illustrate the flexibility in the definition of components consider example 1.

Example 1. Let the vertical motion of a throttlable rocket be described by

$$m\dot{v} = f - mg, \tag{1a}$$

where m is the assumed constant mass, g is the gravitational constant, and f is the rocket thrust that is regulated by a fuel valve with the dynamics

$$\dot{f} = af + u \tag{1b}$$

for a given command u. Thus, for the system

$$\begin{pmatrix}\dot{v}\\ \dot{f}\end{pmatrix} = \begin{bmatrix}0 & 1/m\\ 0 & a\end{bmatrix}\begin{pmatrix}v\\ f\end{pmatrix} + \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}g\\ u\end{pmatrix}, \tag{2}$$

the *vehicle dynamics* (1a) with state v might be chosen as one system component, and the *valve dynamics* (1b) with state f might be chosen as another component. In this case one might wish to ascertain the relative contribution of the dynamics of the vehicle and the dynamics of the valve in the overall system performance metric

$$V = \frac{1}{T}\left\{\int_0^T f^2(t)dt + [v(T) - \bar{v}]^2\right\}, \tag{3}$$

where T is thc terminal time at which the velocity $v(T) = \bar{v}$ is desired.

Alternatively, one may define components of (2) in *any* transformed set of coordinates of (2). For example, one might wish to know the relative contribution in (3) of the *modal* coordinates of (2), in which case the system components are q_1 and q_2 described by

$$\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\end{pmatrix} = \begin{bmatrix}0 & 0\\ 0 & a\end{bmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix} + \begin{bmatrix}-1 & -1/ma\\ 0 & 1/ma\end{bmatrix}\begin{pmatrix}g\\ u\end{pmatrix}. \tag{4}$$

As described in the Introduction, "component cost analysis" (CCA), is the procedure developed for answering question CC for *any* choice of component definitions. In the special case where the components are *modal* coordinates, the procedure is called "modal cost analysis" (MCA) [4]. It is possible to use CCA with any choice of component definitions including the "balanced" coordinates of Moore [7], the "output-decoupled" coordinates used in Tse *et al.* [8], etc. For any choice of

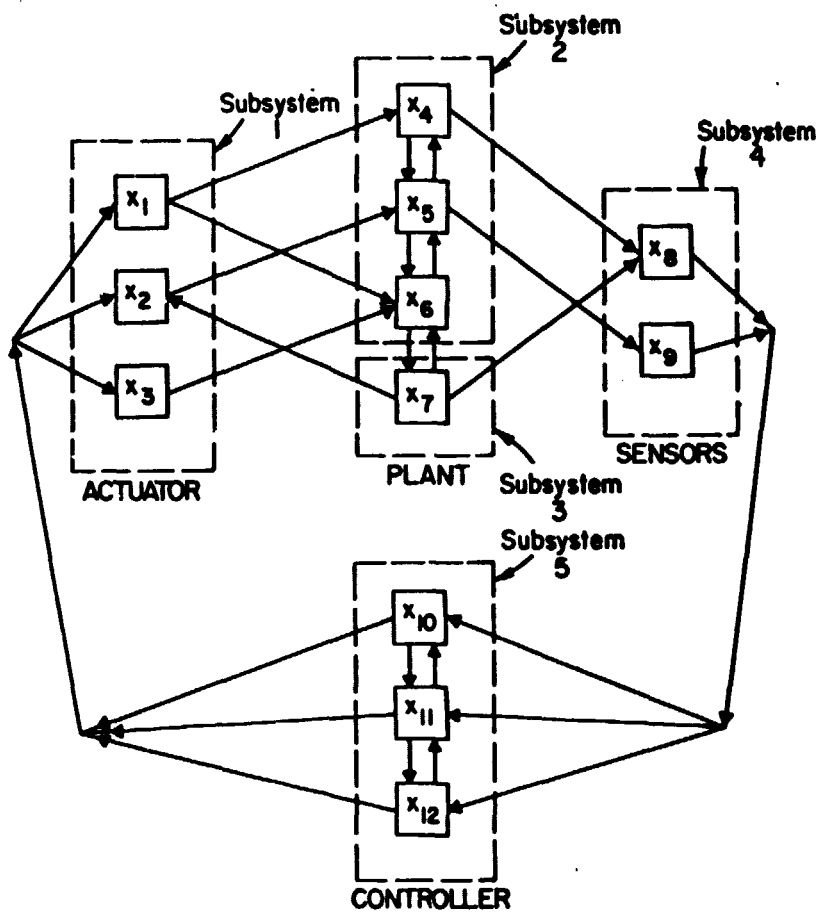

Fig. 1. Component definitions.

coordinates the n components may be described in the form

$$\dot{x}_i = \sum_{j=1}^{n} A_{ij}x_j + D_i w, \quad x_i \in R^{n_i},$$
$$y = \sum_{j=1}^{n} C_j x_j. \tag{5}$$

For notational convenience, we shall later need to differentiate between the definitions of coordinates, components, and subsystems. These distinctions can best be introduced via example. A certain system has state x. Let the symbols x_i, χ_i, X_i all represent partitions of the state vector to various levels of detail. The scalars χ_i, i = 1,..., N, will be called coordinates. The vectors $x_i \in R^{n_i}$, i = 1,..., n, will be called the states of the components and $X_i \in R^{N_i}$, i = 1,..., s, will be called the states of the subsystems. Then for n = 3, s = 2,

$$x = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_{n_1} \\ \hline \vdots \\ \hline \\ \chi_N \end{bmatrix} = \begin{bmatrix} x_1 \\ \hline x_2 \\ \\ x_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ \hline \\ X_2 \end{bmatrix}, \qquad \left(N = \sum_{i=1}^{n} n_i = \sum_{i=1}^{s} N_i \right). \tag{6}$$

N coordinates — n components — s subsystems

As an example of component definitions, consider Fig. 1, where dynamic elements x_i, i = 1,... 12, and their interconnections are described. Each of these dynamic elements

(selected *a priori* by the analyst) with state x_i, $i = 1, \ldots, 12$, is called a component of the system. However, each component may have additional dynamical variables $\chi_1, \chi_2, \ldots$, called coordinates. This coordinate view of the system is the *microscopic* view of the system, whereas, the view of certain collections of components, called subsystems is the *macroscopic* view of the system. For the example in Fig. 1, see from (6) that

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad X_2 = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix}, \quad X_3 = (\dot{x}_7),$$

$$X_4 = \begin{pmatrix} x_8 \\ x_9 \end{pmatrix}, \quad X_5 = \begin{pmatrix} x_{10} \\ x_{11} \\ x_{12} \end{pmatrix}. \tag{7}$$

Of course, when the analyst chooses $n_i = 1$ and $N_i = 1$, there is no distinction between coordinate, component, and subsystem.

III. CONCEPTS OF COST DECOMPOSITION

In our preliminary discussions, we presume that *the linear system model*

$$\dot{x} = A(t)x + D(t)w, \quad x \in R^n \tag{8a}$$

$$y = C(t)x, \quad y \in R^k, \; w \in R^d \tag{8b}$$

having components $x_i \in R^{n_i}$ *exists for the purpose of accurately modeling the outputs* $y(t)$ *over the interval* $0 \leq t \leq T$. To make this notion more precise, we construct the performance metric

$$V(T) = \frac{1}{T} E\left\{ \int_0^T Y(t)dt + Y(T) \right\},$$

$$Y(t) \triangleq \|y(t)\|^2_{Q(t)} \triangleq y^T(t)Q(t)y(t), \tag{9}$$

where Q(t) and Q(T) are positive definite and symmetric, and where the expected value operator E is needed if either the initial condition x(0) or the disturbance w is random. The basic idea of component cost decomposition is illustrated by the following example.

Example 2. The quadratic function of $x \in R^2$ given by

$$V \triangleq x^T Q x = x_1^2 Q_{11} + x_1 x_2 Q_{12} + x_2 x_1 Q_{12} + x_2^2 Q_{22} \tag{10}$$

may be decomposed into costs due to components x_1 and x_2 by defining the component cost by

$$V_1 \triangleq \frac{1}{2} \frac{\partial V}{\partial x_1} x_1 = x_1^2 Q_{11} + x_1 x_2 Q_{12}, \tag{11a}$$

$$V_2 \triangleq \frac{1}{2} \frac{\partial V}{\partial x_2} x_2 = x_2^2 Q_{22} + x_2 x_1 Q_{12}. \tag{11b}$$

Hence, the total cost is the sum of the component costs

$$V = \sum_{i=1}^{n} V_i, \quad V_i = \frac{1}{2} \frac{\partial V}{\partial x_i} x_i \tag{12}$$

where n = 2 in this example.

To extend this component cost concept to the systems (8) and (9), we must first specify the character of the excitations of (8). The situation is now separately described for deterministic and stochastic inputs.

A. COMPONENT COSTS FOR STOCHASTIC SYSTEMS

Let any inputs w(t) that are correlated with time or state be described by a Gauss-Markov model.

We will assume, however, in order to simplify notation, that w(t) in (8) is a zero-mean white noise process with intensity $W(t) > 0$ and that x(0) has covariance $x(0) \geq 0$.

The first definition follows the lead provided by (10)-(12).

Definition 2. The component cost V_i for the ith component of (5) and (8) with respect to the performance metric (9) is defined by

$$V_i(T) \triangleq \frac{1}{2T} E\left\{\int_0^T \frac{\partial Y(t)}{\partial x_i(t)} x_i(t)dt + \frac{\partial Y(T)}{\partial x_i(T)} x_i(T)\right\}. \tag{13}$$

Two important properties of the component costs $V_i(T)$ are

(a) the superposition property of component costs

$$V(T) = \sum_{i=1}^{n} V_i(T), \tag{14}$$

(b) the component cost formula

$$V_i(T) = \frac{1}{T} \operatorname{tr}\left\{\int_0^T XC^TQCdt + X(T)C^T(T)Q(T)C(T)\right\}_{ii}, \tag{15}$$

where X is the state covariance satisfying

$$\dot{X} = AX + XA^T + DWD^T, \quad X(0) = X_0, \tag{16}$$

and $\{\cdot\}_{ii}$ denotes the $n_i \times n_i$ matrix corresponding to the position of x_i in x.

These results allow one to examine individual component contributions in a variety of situations involving (i) *specified times* T, (ii) *specified time intervals* $t \in [0, T]$, and (iii) *time-varying systems*. Examples of situation (i) includes circumstances in which the system goes through "critical" times T, and at this time it is required to have more precise knowledge about the dynamical interactions of the system components than at other times. Some critical times in engineering problems include

(i1) spacecraft reentry time T,

(i2) time of rendezvous T of two spacecraft,

(i3) critical times T in a nuclear reactor,

(i4) switch-in or switch-out time T of a power substation in a larger power network, and

(i5) time T of maximum dynamic pressure of an aircraft or rocket.

Examples of situation (ii) include finite-time control problems:

(ii1) air-to-air missile intercept; quidance of a rocket to orbital insertion,

(ii2) rapid repositioning of a flexible space vehicle,

(ii3) a finite time industrial process.

Examples of the time-varying situation (iii) are common and will not be enumerated.

For time-invariant systems with $T \to \infty$, (9) and (15) simplify to

$$V(\infty) = \lim_{t\to\infty} EY(t) = \mathrm{tr}\, X(\infty)C^TQC, \tag{17}$$

$$V_i(\infty) = \frac{1}{2}\lim_{t\to\infty} E\frac{\partial Y(t)}{\partial x_i(t)}x_i(t) \tag{18a}$$

$$= \mathrm{tr}[X(\infty)C^TQC]_{ii}, \tag{18b}$$

where $X(\infty)$ exists if and only if the disturbable (controllable) modes of (A, D) are stable, and $X(\infty)$ is the positive definite solution of

$$0 = AX(\infty) + X(\infty)A^T + DWD^T \tag{19}$$

if (A, D) is a disturbable pair [6].

B. COMPONENT COSTS FOR DETERMINISTIC SYSTEMS

If all disturbances are written in differential equation form (8) *without* the noise w, and with specified initial conditions, then we may simplify the form of (8)

and (9) to

$$\dot{x} = Ax, \quad x(0) = x_0, \quad y = Cx, \tag{20}$$

$$V(T) = \frac{1}{T}\left\{\int_0^T Y(t)dt + Y(T)\right\}, \tag{21}$$

and (16) becomes

$$V_i(T) = \frac{1}{2T}\left\{\int_0^T \frac{\partial Y(t)}{\partial x_i(t)}\, x_i(t)dt + \frac{\partial Y(T)}{\partial x_i(T)}\, x_i(T)\right\}, \tag{22}$$

where (15) still holds except that (16) is now replaced by

$$\dot{X} = AX + XA^T, \quad X(0) \triangleq x(0)x^T(0), \tag{23}$$

which has nontrivial solutions $X(t)$, $t \in [0, T]$, for finite T.

Example 3. A finite-time deterministic problem.

For the example (2), find the component costs for vehicle and actuator components $v(t)$ and $f(t)$, if $T = 1000$, $m = 1$, $a = -1$, $\bar{v} = 100$, $v(0) = 0$, $f(0) = 0$, $g = 9.8$, and $u = 1$ is a step input. The deterministic model of the inputs augmented to (2) yields (20), where

$$A = \begin{bmatrix} 0 & 1/m & -1 \\ 0 & a & 1/9.8 \\ 0 & 0 & 0 \end{bmatrix}, \quad x = \begin{pmatrix} v - \bar{v} \\ f \\ g \end{pmatrix}, \quad x(0) = \begin{pmatrix} -100 \\ 0 \\ 9.8 \end{pmatrix}.$$

Putting (3) into the form (20) leads to

$$C(t) = [0 \quad 1 \quad 0], \quad Q(t) = 1, \quad 0 \le t < T$$

$$C(T) = [1 \quad 0 \quad 0], \quad Q(T) = 1.$$

Solving (15), subject to (23), yields for (22),

$$V_1(T) = 7.92 \times 10^4, \quad V_2(T) = 1.00, \quad V_3(T) = 0,$$

where $V_1(T)/V(T) = 0.9999$ is the fraction of the cost associated with vehicle dynamics, $V_2(T)/V(T) = 1.27 \times 10^{-5}$ is the fraction of the cost associated with actuator dynamics, and

$V_3(T)/V(T) = 0$ is the fraction of the cost due to the biases in the system (gravity). Clearly the vehicle dynamics dominate the performance.

C. *INFINITE-TIME DETERMINISTIC PROBLEMS*

In the limit $T \to \infty$, (21) yields $V(T) = 0$ if A is asymptotically stable. Hence, a *different* performance metric and component cost definition is required for the special case of *infinite-time deterministic problems*. An appropriate cost function for this case is

$$V_D(\infty) \triangleq \int_0^\infty Y(t)dt, \tag{24}$$

which leads to

$$V_D(\infty) = x_0^T K x_0, \quad 0 = KA + A^T K + C^T Q C, \tag{25}$$

where K exists if and only if the observable modes of (A, C) are stable, and K is positive definite if (A, C) is observable. Here the component cost V_{Di} associated with component i is defined as the net effect of the excitation of the ith component state x_i. Hence, in this case the excitation is $x_i(0)$ and the component cost is defined by

$$\begin{aligned} V_{Di}(\infty) = \frac{1}{2}\frac{\partial V_D(\infty)}{\partial x_i(0)} x_i(0) &= \sum_{j=1}^{n} x_i^T(0) K_{ij} x_j(0) \\ &= \mathrm{tr}[Kx(0)x^T(0)]_{ii}, \end{aligned} \tag{26}$$

where K satisfies (28).

The remainder of this chapter will focus on the *stochastic* problem rather than the deterministic problem of Section III.C. This means that the "output-induced" component costs (18) will be of interest, rather than the "input-induced" definitions of (26). The reader can find details of an input-induced

component cost for stochastic systems in [5], which, for deterministic problems, is based upon (26), and for the stochastic problem

$$V = E\int_0^\infty Y(t)\,dt, \quad \dot{x} = Ax, \quad Ex(0)x^T(0) = X(0), \tag{27}$$

[5] utilizes the stochastic version of the component cost definition (26) whose calculation is

$$V_i = \mathrm{tr}[KX(0)]_{ii}. \tag{28}$$

For the stochastic problem

$$V = \lim_{t\to\infty} EY(t), \quad \dot{x} = Ax + Dw, \quad Ew(t)w^T(\tau) = W\delta(t-\tau)$$

$$y = Cx, \quad Y \triangleq \|y\|_Q^2 \tag{29}$$

[5] utilizes the input-induced component cost definition

$$V_i \triangleq \frac{1}{2}\lim_{t\to\infty} E\frac{\partial Y}{\partial w_i}w_i, \quad w_i \triangleq D_i w \tag{30}$$

whose calculation is

$$V_i = \mathrm{tr}[KDWD^T]_{ii}. \tag{31}$$

The *input-induced* definitions (30) of component costs V_i represent the effect in V of excitations of component i, whereas the *output-induced* definitions (13) and (18) represent the total contributions of x_i in V in the presence of all excitations. The latter and more recent definition seems to be a much more complete notion of the contribution of component x_i in the system cost while the system is subject to all its natural environmental disturbances. Hence, this chapter will present a theory only for output-induced definitions of component cost, although the same procedures could be used to work out a corresponding theory for the input-induced case. To further simplify the presentation, only time-invariant

systems with infinite terminal time T will be treated in detail. The application of the concepts to the finite-time case will be straightforward.

IV. BASIC THEOREMS OF COMPONENT COST ANALYSIS FOR MODEL REDUCTION

Given the time-invariant linear system

$$\dot{x} = Ax + Dw, \quad y = Cx, \tag{32}$$

$$V = \lim_{t\to\infty} E\|y(t)\|_Q^2, \quad Q > 0 \tag{33}$$

with components described by (5), and with zero-mean white noise disturbances w(t) with intensity W, then the value of component i whose state is $x_i \in R^{n_i}$ has been shown in previous sections to be

$$V_i = \mathrm{tr}[XC^TQC]_{ii}, \quad 0 = AX + XA^T + DWD^T \tag{34}$$

and V_i is called the ith component cost. The fractional part of the ith component's contribution to V is V_i/V, where $V = \Sigma_{j=1}^{n} V_i$. This component cost information (34) might be useful to guide system redesigns, failure mode analysis, and model reductions as mentioned in the Introduction. In the context of model reduction there may be considerable freedom in the selection of coordinates before model reduction begins. That is to say the definition of components is up to the analyst. For any selected componert definition, the model reduction scheme proposed is simply to discard (truncate) some of the component equations (5). Suppose the component index i belongs to the set R ($i \in R$) corresponding to the *retained* components x_i, and $i \in \mathcal{T}$ denotes the set of indices associated

with the truncated (deleted) component equations. The reduced model is

$$\dot{\hat{x}}_R = A_R\hat{x}_R + D_R w, \quad \hat{x}_R \in R^{n_r}$$
$$\hat{y} = C_R\hat{x}_R, \quad n_r = \sum_{i=1}^{r} n_i \tag{35}$$

where A_R is composed of the set $\{i \in R\}$ of columns and rows of A, D_R is composed of the set $\{i \in R\}$ of rows of D, and C_R is composed of the set $\{i \in R\}$ of columns of C. The set R is determined by those r integers (here denoted generically by 1, 2,..., r) associated with the r largest component costs

$$V_1 \geq V_2 \geq V_3 \geq \cdots \geq V_r \geq V_{r+1} \geq \cdots \geq V_n. \tag{36}$$

The CCA algorithm for model reduction is therefore characterized by these two basic steps:

The Basic CCA model reduction algorithm

I. Compute component costs V_i by (34) and rank according to (36).

II. Delete the n - r components associated with the n - r smallest component costs. The resulting model is (35).

The remainder of the chapter seeks to characterize various mathematical properties of this CCA algorithm. This is clearly necessary since it is not apparent at this point whether the CCA algorithm produces "good" reduced models. To address this question of model error, we shall define a model error index Q in Section VI. But first a brief review of modal coordinates is in order.

V. MODAL COST ANALYSIS (MCA)

There is an important case in which the input- and output-induced definition of component costs yield the same result, and this case is summarized below.

Proposition 1. Consider system (29) where x_i is the ith modal coordinate and assume for convenience that A has distinct eigenvalues. Hence, A is diagonal. Then the CCA algorithm will produce the same reduced model, whether the output-induced or the input-induced modal cost definitions, (18a) or (30), respectively, are used.

Proof. To prove this result we must show that the component costs as computed by (18b) and (31) are identical if A_{ij} in (5) has the property $A_{ij} = \lambda_i \delta_{ij}$. First we shall show that for all real λ_i,

$$V_i = [XC^*QC]_{ii} = [KDWD^*]_{ii} \quad \text{for all } i = 1,\ldots, n, \tag{37}$$

where X and K satisfy

$$0 = XA^* + AX + DWD^*, \quad D^* = \text{complex conjugate transpose} = (\bar{D})^T \tag{38}$$

$$0 = KA + A^*K + C^*QC, \tag{39}$$

when $A_{ij} = \lambda_i \delta_{ij}$. The complex notation * is required due to the complex matrices A, D, and C. It is well known [6] that the total cost is the same by either calculation $V = \text{tr } XC^*QC$ or $V = \text{tr } KDWD^*$, but the issue here is whether *each* modal cost (37) is the same. Denoting the ith row of D by d_i^* and the ith column of C by c_i, the solutions of (38) and (39), respectively, are

$$X_{ij} = -d_i^* W d_j / (\lambda_i + \bar{\lambda}_j), \tag{40}$$

$$K_{ij} = -c_i^* Q c_j / (\bar{\lambda}_i + \lambda_j) \qquad (41)$$

We also remark that mode i is observable (disturbable) if and only if c_i (d_i) is not zero. Use (40) and (41) to obtain, respectively,

$$[XC^*QC]_{ii} = -d_i^* W \sum_{j=1}^{n} \left[\frac{d_j c_j^*}{\lambda_i + \bar{\lambda}_j} \right] Q c_i \qquad (42)$$

$$[KDWD^*]_{ii} = -c_i^* Q \sum_{j=1}^{n} \left[\frac{c_j d_j^*}{\bar{\lambda}_i + \lambda_j} \right] W d_i . \qquad (43)$$

Since the complex number on the right-hand side of (42) is the conjugate of the complex number on the right-hand side of (43), (37) is therefore verified for the special case of real eigenvalues of A. For a particular complex eigenvalue λ_i, let $\lambda_{i+1} = \bar{\lambda}_i$. Equations (42) and (43) show that the component cost of any x_i associated with a complex eigenvalue λ_i will be a complex number and that the component cost of x_{i+1} corresponding to the eigenvalue $\lambda_{i+1} = \bar{\lambda}_i$ will be the complex conjugate of V_i. That is $V_{i+1} = \bar{V}_i$ and V_i have the same norm. Hence, replacing V_i by $|V_i|$ in the CCA (presently MCA) truncation rule (36), the MCA model reduction algorithm would always truncate modal components so that complex conjugate *pairs* of eigenvalues are truncated. Note also that in the case of proposition 1, $n_i = 1$ and for a complex conjugate pair $\lambda_{i+1} = \bar{\lambda}_i$, it is true that

$$V_{i+1} + V_i = 2\mathrm{Re}V_i . \qquad (44)$$

Hence, the total cost V is real, and the sum of the modal cost of any two modal components associated with complex conjugate

pairs of eigenvalues will be real. The proof is concluded by noting that complex conjugates are truncated in pairs and from the fact

$$\overline{[XC^*QC]}_{i+1,i+1} = [XC^*QC]_{ii} = \overline{[KDWD^*]}_{ii}, \tag{45}$$

we conclude that for a complex pair

$$\begin{aligned} V_i + V_{i+1} &= [XC^*QC]_{ii} + [XC^*QC]_{i+1,i+1} \\ &= [KDWD^*]_{ii} + [KDWD^*]_{i+1,i+1}. \end{aligned} \tag{46}$$

Hence the same modes will be truncated by either definition of modal cost. #

Under special conditions the modal costs (42) and (43) simplify greatly.

Proposition 2. If either (a), (b), or (c) holds:

(a) $d_i^*Wd_j = 0, \quad i \neq j$ (disturbance decoupled modes);

(b) $c_i^*Qc_j = 0, \quad i \neq j$ (output decoupled modes);

(c) $(\mathrm{Re}\lambda_i/\mathrm{Im}\lambda_i)$ arbitrarily small, and $\omega_i \neq \omega_j$ (lightly damped modes);

then the modal costs of a linear system are given by

$$V_{ci} \triangleq V_i + \overline{V}_i = -\frac{c_i^*Qc_i d_i^*Wd_i}{\mathrm{Re}\lambda_i} = -\tau_i \|c_i\|_Q^2 \|d_i\|_W^2, \tag{47}$$

which holds for either the input-induced definition (30) or the output-induced definition of modal cost and where V is defined by (29). If λ_i is real, then the ith modal cost is $V_{ci} \triangleq V_i$.

The proof of parts (a) and (b) follow immediately from (42) and (43). The proof of part (c) is given in [5]. #

Examples of case (c) in proposition 2 appear in [5] and [17] where MCA is applied to systems of order up to 200. It should be noted that since the MCA formulas (47) are explicit [hence, the linear matrix Eq. (38) does not have to be solved numerically], the MCA algorithm may be applied to any system for which modal data are available. It will subsequently be shown that modal coordinates might *not* be the best coordinates in which to perform model truncation. However, much insight is available from (47) indicating that modal costs are composed of the product of three properties of a mode: (1) time constant, (2) observability norm, and (3) disturbability norm.

VI. MODEL ERROR INDICES

Having a reduced model (35), we now turn our attention to the definition and calculation of a convenient measure of "model error" when comparing the reduced model (35) with the evaluation model (32).

Definition 2. The errors associated with model (35) produced by the CCA algorithm are measured by the model error index

$$Q \triangleq \frac{1}{V}|(V - V_R)|, \tag{48}$$

where V_R is the performance metric associated with (35). If the disturbable modes of (A_R, D_R) are stable, then

$$V_R = \text{tr } X_R C_R^T Q C_R, \quad 0 = A_R X_R + X_R A_R^T + D_R W D_R^T, \tag{49}$$

and V is the performance metric associated with the "evaluation" model (32), as given by (33).

Of course, V_R can be computed only after model reduction. The information available *a priori* will be called the *predicted model error index* $\hat{Q}$.

Definition 3. The predicted model error index is defined by

$$\hat{Q} \triangleq \frac{1}{V}|(V - \hat{V}_R)|, \tag{50}$$

where

$$\hat{V}_R \triangleq \Sigma V_i, \quad i \in R. \tag{51}$$

From (36), (50), and (51), it follows also that

$$\hat{Q} = \frac{1}{V}|\hat{V}_T|, \quad \hat{V}_T \triangleq \Sigma V_i, \quad i \in T, \tag{52}$$

and

$$V = \hat{V}_R + \hat{V}_T. \tag{53}$$

When applying the model error index Q to the reduction of the *closed*-loop system (to reduce controllers rather than models), Q plays a role similar to the "suboptimality index" of Siljak [14]. Note also that the Q chosen here (48) is the difference in the norms of y and $\hat{y}$, whereas the model error index chosen in [5] is the norm of the difference $y - \hat{y}$. This choice (48) is primarily motivated by the controller reduction problem where V_R represents the performance using the reduced controller. In *that* problem $V_R \geq V$ since V_R represents the suboptimal controller perfromance. Since V_R is minimized if Q is minimized, the difference of norms represented by (48) is a more logical choice for controller reduction. This paper now focuses on the prerequisite problem of model reduction where all the essential mathematical results are derived for subsequent application to controller reduction.

For the *model reduction* problem, the model error index (48) would be a meaningless index if the parameters of the reduced model (A_R, D_R, C_R) were arbitrary, since in this case parameters can always be found to make $Q = 0$. Reasonableness is

added to the problem, however, by the fact that the search for small Q is subjected to the parameters (A_R, D_R, C_R), which are *constrained* to be a transformed subset of the original system parameters (A, D, C). We now continue with this model reduction problem.

The questions that naturally arise and are to be answered in the sequel are

(QI) Under what conditions is the predicted model error index $\hat{Q}$ exact ($\hat{Q} = Q$)?

(QII) Under what conditions is the model error index Q zero?

(QIII) Under what conditions is the model error index Q minimized by the CCA algorithm?

(QIV) Given that A is stable, under what conditions is the reduced model produced by CCA stable?

VII. COST-EQUIVALENT REALIZATIONS AND MINIMAL REALIZATIONS WITH RESPECT TO COST

Toward the development of the mathematical machinery required to answer questions (QI)-(QIV), we introduce the following definitions.

Definition 4. Cost-equivalent realizations

Let $\{A_R, D_R, C_R, X_R(0), W_R\}$ characterize the partial realization (35) and let {A, D, C, X(0), W} characterize the evaluation model ("complete" realization) (32). The partial realization is said to be cost-equivalent if $Q = 0$.

Definition 5. Minimal cost-equivalent realizations

With respect to the given components (5), the partial realization (35) is said to be a minimal cost-equivalent realization if r is the smallest integer for which $Q = 0$.

To simplify our bookkeeping, let us assume that the components (5) are arranged in order of their component costs and define

$$x_R^T \triangleq (\ldots, x_i^T, \ldots), \quad i \in R, \tag{54a}$$

$$x_T^T \triangleq (\ldots, x_i^T, \ldots), \quad i \in T. \tag{54b}$$

Then (32) may be written in the form

$$\begin{aligned} \begin{pmatrix} \dot{x}_R \\ \dot{x}_T \end{pmatrix} &= \begin{bmatrix} A_R & A_{RT} \\ A_{TR} & A_T \end{bmatrix} \begin{pmatrix} x_R \\ x_T \end{pmatrix} + \begin{bmatrix} D_R \\ D_T \end{bmatrix} w, \\ y &= [C_R \quad C_T] \begin{pmatrix} x_R \\ x_T \end{pmatrix}. \end{aligned} \tag{55}$$

Let X as defined by (34) be likewise partitioned in the manner

$$X = \begin{bmatrix} \hat{X}_R & \hat{X}_{RT} \\ \hat{X}_{RT}^T & \hat{X}_T \end{bmatrix}. \tag{56}$$

Due to symmetry of X, the partitioned form of the linear equation (34) using (55) and (56) yields three linear equations of smaller dimensions. Two of these equations are

$$0 = A_R\hat{X}_{RT} + \hat{X}_{RT}A_T^T + A_{RT}\hat{X}_T + \hat{X}_R A_{TR}^T + D_R W D_T^T, \tag{57a}$$

$$0 = A_T\hat{X}_T + \hat{X}_T A_T^T + A_{TR}\hat{X}_{RT} + \hat{X}_{RT}^T A_{TR}^T + D_T W D_T^T. \tag{57b}$$

The remaining equation in $\hat{X}_R$ is subtracted from (49) to yield

$$0 = A_R\tilde{X}_R + \tilde{X}_R A_R^T + A_{RT}\hat{X}_{RT}^T + \hat{X}_{RT}A_{RT}^T, \tag{57c}$$

where $\tilde{X}_R \triangleq \hat{X}_R - X_R$.

A. ANSWERS TO QUESTIONS QI, QII, AND QIII

Using the above symbols, question QI can now be answered.

Proposition 3. The predicted model error index $\hat{Q}$ is exact in the sense $\hat{Q} = Q$ under any of the following conditions:

(a) if $\operatorname{tr}\left(\tilde{X}_R C_R^T Q C_R + \hat{X}_{RT} C_T^T Q C_R\right) = 0$ and $V \geq V_R$; (58a)

(b) if $\operatorname{tr} \tilde{X}_R C_R^T Q C_R + \operatorname{tr} 2\hat{X}_T C_T^T Q C_T + \operatorname{tr} 3\hat{X}_{RT} C_T^T Q C_R$

$= 0$ and $V < V_R$; (58b)

(c) if $\hat{X}_{RT} = 0$;

(d) if x_T is unobservable;

(e) if x_T is undisturbable.

Proof. Noting (49) and (50), it follows that the proof requires that $\hat{V}_R = V_R$ if $V \geq V_R$ and requires that $2V = V_R + \hat{V}_R$ if $V < V_R$. To show that $V_R = \hat{V}_R$ when (58a) holds, we first write from (34), using (55) and (56),

$$\hat{V}_R = \sum_i \operatorname{tr}[XC^T QC]_{ii} = \operatorname{tr}\left(\hat{X}_R C_R^T Q C_R + \hat{X}_{RT} C_T^T Q C_R\right), \quad i \in R$$

$$= \operatorname{tr}\left\{(\tilde{X}_R + X_R) C_R^T Q C_R + \hat{X}_{RT} C_T^T Q C_R\right\}. \tag{59}$$

Now subtract (49) from (59) to obtain (58a) directly. To prove (58b), write, using (34), (55), and (56),

$$V = \operatorname{tr} \hat{X}_R C_R^T Q C_R + 2 \operatorname{tr} \hat{X}_{RT} C_T^T Q C_R + \operatorname{tr} \hat{X}_T C_T^T Q C_T. \tag{60}$$

Substitute (60) into $2V = V_R + \hat{V}_R$, using (49) and (59) to get

$$2\left(\operatorname{tr} \hat{X}_R C_R^T Q C_R + \operatorname{tr} 2\hat{X}_{RT} C_T^T Q C_R + \operatorname{tr} \hat{X}_T C_T^T Q C_T\right)$$

$$= \operatorname{tr} X_R C_R^T Q C_R + \operatorname{tr} \hat{X}_R C_R^T Q C_R + \operatorname{tr} \hat{X}_{RT} C_T^T Q C_R, \tag{61}$$

which reduces to (58b). To prove (c), set $\hat{X}_{RT} = 0$ in (57c) to obtain $\tilde{X}_R = 0$. Furthermore since $\hat{X}_{RT} = 0$, we have from (60) and (49)

$$V - V_R = \text{tr } \hat{X}_R C_R^T Q C_R + \text{tr } \hat{X}_T C_T^T Q C_T - \text{tr } X_R C_R^T Q C_R$$

$$= \text{tr } \tilde{X}_R C_R^T Q C_R + \text{tr } \hat{X}_T C_T^T Q C_T = \text{tr } \hat{X}_T C_T^T Q C_T$$

Now, since the state covariance X is at least positive semi-definite [6], $\hat{X}_T \geq 0$ and hence $V \geq V_R$. Hence, (58a) is applicable, and this proves (c). To prove (d), one may without loss of generality assume $A_{RT} = 0$ and $C_T = 0$ since x_T is unobservable. This yields from (57c) $\tilde{X}_R = 0$, which immediately leads to (58), since $C_T = 0$. To prove (e), assume x_T is undisturbable (i.e., set $A_{TR} = 0$, $D_T = 0$). This yields from (57a) and (57b) $\hat{X}_T = 0$, $\hat{X}_{RT} = 0$, and (57c) yields $\tilde{X}_R = 0$. Hence, condition (58) is again satisfied. #

It may be comforting to know that the predicted model error index is accurate, but the initial issue of the "best" choice of coordinates and components is still unresolved. That is, some choice of coordinates may lead to smaller model error indices than other choices, even though the predicted model error index may be exact for each choice. Before we try to resolve the question of the best set of coordinates, we shall define the limiting case where the reduced model is "perfect." Thus, the following result answers question QII.

Proposition 4. The partial realization (35) is a cost-equivalent realization of (32) under either of these conditions:

(a) if and only if

$$\text{tr } \tilde{X}_R C_R^T Q C_R + \text{tr } 2\hat{X}_{RT} C_T^T Q C_R + \text{tr } \hat{X}_T C_T^T Q C_T = 0; \qquad (62)$$

(b) if x_T is unobservable;

(c) if x_T is undisturbable.

Proof. From (58), it follows that proof of (a) relies upon a proof that $V = V_R$ if (62) holds. Subtract (49) from (60) to get (62) directly. To prove (b) we rely on the proof of theorem 3, which showed that $\tilde{X}_R = 0$ if x_T is unobservable and that $C_T = 0$ may be assumed. The conditions $\tilde{X}_R = 0$, $C_T = 0$ lead to satisfaction of (62). To prove (c), note from the proof of theorem 3 that $\tilde{X}_R = 0$, $\hat{X}_T = 0$, $\hat{X}_{RT} = 0$ if x_T is undisturbable. These substitutions in (62) conclude the proof. #

Having answered questions QI and QII, it is now possible to provide an answer to QIII. This answer is summarized by proposition 5.

Proposition 5. Given a specified r and the components (5), which satisfy proposition 3 ($\hat{Q} = Q$), the model error index Q is minimized by the CCA algorithm.

Proof. Since $\hat{Q} = Q$ the model error index is given by (52). Among the set of $\{V_i, i = 1, 2, \ldots n\}$, the $\hat{V}_T$ in (52) is composed (*by definition*) of the n - r smallest subset of V_is, according to (35). Hence, $\hat{Q}$ cannot be decreased by any other choice of r components from the given set of n components (5). #

We must not read too much into proposition 5. It only guarantees that there are not better r choices of the *given* n components. The *a priori choices of component definitions* that can be made are infinite. In any model truncation problem these three factors are important:

(a) choice of coordinates,

(b) choice of a truncation criterion, and

(c) choice of an evaluation criterion for the reduced model.

In CCA, the best choice (a) has not yet been determined, choice (b) is given by (36) and (52), and choice (c) is given by (48). One suggestion for choice (a) is introduced in the next section. It should be noted, however, that depending upon the question being addressed, the analyst may *not* have a choice of coordinates. In this case the results of Section VII.A apply, but the coordinate transformation of Section VII.B will not be permitted.

B. COST-DECOUPLED COMPONENTS

The previous section describes CCA for any given choice of components, and this flexibility is important for the analysis of component costs using *physical* components. However, in model reduction the analyst may be free to *choose* the reference coordinates and may *not* be restricted to the analysis of physical components. The component costs for some choices of components (i.e., choices associated with the underlying coordinate transformations) are more convenient to interpret than others. As an example of possible confusion, note that even though the *sum* of component costs V_i is positive (14), an individual V_i defined by (15) or (34) can be negative. All theorems of previous sections are still valid, but one might obtain better reduced models by using absolute value signs around each V_i in (36). Clearly, such issues need not be of concern if all V_i are proven to be nonnegative. The cost-decoupled components to be defined in this section will prove to have this property.

It may also be observed from the basic ideas of (11), from the general component cost formula (15), and from the steady-state cases of (18) and (34), that the component costs V_i and V_j are not generally independent. That is, the ith component cost V_i is influenced by component j. This presents no problem for the *in situ* component cost analysis for purposes other than model reduction. But for model reduction such dependence between V_i and V_j leads to errors in the predicted model quality index $\hat{Q}$, since in this case the deletion of component j also modifies the cost of the retained component i. This nuisance can be removed by choosing components that have independent costs. Thus, the motivation for such component choices is to gain the property $\hat{Q} = Q$ of proposition 3. For the purposes of this section, define the *components* $x_i \in R^1$ to be *each coordinate* of the cost-decoupled state x_2. Hence $n_i = 1$ for all i in this case. From part (c) of proposition 3, it is clear that uncorrelated components (i.e., $X_{ij} = 0$, $i \neq j$) yield the property $\hat{Q} = Q$. An additional property is added to obtain the "cost-decoupled" coordinates defined as follows.

Definition 6. The "cost-decoupled" coordinates of a linear system are any coordinates for which the covariance X and the state weighing C^TQC are both diagonal matrices.

A convenient choice of cost-decoupled coordinates may be computed as follows. Let $\{x^\circ, X^\circ, C^\circ, A^\circ, D^\circ\}$ represent an original set of coordinates and data, and let $\{x, X, C, A, D\}$ represent the transformed data according to the transformation

$$x^\circ = \Theta x, \quad 0 = X^\circ A^{\circ T} + A^\circ X^\circ + D^\circ W D^{\circ T}, \tag{63a}$$

where $\theta = \theta_x\theta_y$ and θ_x, θ_y satisfy

$$X^\circ = \theta_x\theta_x^T \tag{63b}$$

$$\theta_y \triangleq E_y\Omega_y, \quad \theta_x^T C^{\circ T} Q C^\circ \theta_x = E_y \Lambda_y^2 E_y^T. \tag{63c}$$

Note that θ_x is the square root of the covariance matrix X°. The nonsingular diagonal matrix Ω_y is arbitrary. The orthonormal matrix of eigenvectors of $\theta_x^T C^T Q C \theta_x$ is E_y and the corresponding eigenvalues (which are also singular values [9] since the matrix is symmetric) are elements of the diagonal matrix Λ_y^2.

In cost-decoupled coordinates, the system (32) is transformed by

$$A = \theta_y^{-1}\theta_x^{-1}A^\circ\theta_x\theta_y, \tag{64a}$$

$$D = \theta_y^{-1}\theta_x^{-1}D^\circ, \tag{64b}$$

$$C = C^\circ\theta_x\theta_y. \tag{64c}$$

The calculation of the steady-state covariance matrix of a stable system in cost-decoupled coordinates reveals that

$$X = \Omega_y^{-2} \tag{65}$$

and the *state* weighting matrix $[C^T \; QC]$ in the performance metric

$$V = \lim_{t\to\infty} E\|y\|_Q^2 = \text{tr } XC^TQC$$

is

$$C^TQC = \Omega_y^2\Lambda_y^2. \tag{66}$$

Hence from (65) and (66),

$$V = \text{tr } XC^TQC = \text{tr } \Lambda_y^2 = \sum_{i=1}^{k} \lambda_i\left[\theta_x^TC^TQC\theta_x\right], \tag{67a}$$

where $\lambda_i[\cdot]$ denotes eigenvalue of $[\cdot]$, and the summation is only up to k, since there are only k nonzero eigenvalues of $\theta_x^T C^T Q C \theta_x$, since rank C = k. Note that the component costs in cost-decoupled coordinates are

$$V_i = \lambda_i\left[\theta_x^T C^T Q C \theta_x\right], \tag{67b}$$

which leads to this simple interpretation of cost-decoupled coordinates and component costs: In coordinates (components) that are uncorrelated ($X_{ij} = 0$) and output decoupled ($[C^T Q C]_{ij} = 0$), the component costs are the eigenvalues of the state-weighting matrix. In view of (67a) which holds for any Ω_y, there seems to be no disadvantage in the choice $\Omega_y = I$, although a different choice for Ω_y will be chosen in Section VII.D for convenient comparisons with the work of others. Temporarily, we choose $\Omega_y = I$.

The useful properties of cost-decoupled coordinates are now summarized in the following proposition.

Proposition 6. In cost-decoupled coordinates, the full-order model (55) has the following properties:

(1) $V_i \geq 0$ (the component costs are all nonnegative);

(2) A_R has no eigenvalue in the open right half plane;

(3) A_R is asymptotically stable if and only if the pair (A_R, D_R) is disturbable.

Proof. Claim (1) follows immediately from (67b) since

$$V_i = \lambda_i\left[\theta_x^T C^T Q C \theta_x\right] \geq 0. \tag{68}$$

To prove claim (2) and (3), partition (65) (with $\Omega_y = I$) as

$$X = \begin{bmatrix} \hat{X}_R & \hat{X}_{RT} \\ \hat{X}_{RT}^T & \hat{X}_T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{69a}$$

This reveals that $\hat{X}_{RT} = 0$. Hence, writing the partioned form of (34), partitioned compatibly with (55), yields

$$0 = A_R^T + A_R + D_R W D_R^T, \tag{69b}$$

$$0 = A_{TR}^T + A_{RT} + D_R W D_T^T, \tag{69c}$$

$$0 = A_T^T + A_T + D_T W D_T^T. \tag{69d}$$

Either (A_R, D_R) is disturbable or not. If (A_R, D_R) is disturbable, then the state covariance of (35) from (69b) is

$$X_R = \int_0^\infty e^{A_R^T} D_R W D_R^T e^{A_R^T} dt = I \tag{69e}$$

and the finiteness of X_R guarantees asymptotic stability of A_R (X_R would not be bounded for unstable A_R under the disturbability assumption). This proves the "if" part of claim (3). If (A_R, D_R) is not disturbable then there exists an orthonormal transformation

$$\chi = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T^T x_R = \begin{bmatrix} T_1^T \\ T_2^T \end{bmatrix} x_R \tag{70a}$$

to take the system (35) to the controllable conical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ 0 \end{bmatrix} w, \tag{70b}$$

$$y_R = [C_1 \quad C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Where (A_{11}, D_1) is completely disturbable. Now (69b) becomes

$$0 = A_{11}^T + A_{11} + D_1 W D_1^T, \tag{71a}$$

$$0 = A_{12}, \tag{71b}$$

$$0 = A_{22}^T + A_{22}. \tag{71c}$$

The eigenvalues of A_R are those of A_{11} and A_{22}. Since (A_{11}, D_1) is disturbable and $\int_0^\infty e^{A_{11}t} D_1 W D_1^T e^{A_{11}^T t} dt = I < \infty$, the eigenvalues of A_{11} must lie in the open left-hand half plane by reasons mentioned above. The eigenvalues of A_{22} must lie on the imaginary axis since A_{22} is skew-symmetric ($A_{22} = -A_{22}^T$). Hence, no eigenvalues of A_R can lie in the right-hand half plane but there are eigenvalues with zero real parts. This proves claim (2). Moreover this proves that A_R is not asymptotically stable if (A_R, D_R) is not disturbable, the "only if" part of claim (3). #

Proposition 7. If the CCA algorithm using *cost-decoupled coordinates* produces a disturbable pair (A_R, D_R) then the following properties hold:

(1) $\hat{Q} = Q$ (the predicted model error index is exact);

(2) Q is minimized for a given r;

(3) $Q = 0$ if $r \geq k$ (the CCA algorithm produces a minimal cost-equivalent realization of order k = rank C).

Proof. Claim (1) is proven by showing that (58a) holds. By virtue of the fact that $\hat{X}_{RT} = 0$ (since X in definition 6 is diagonal) it follows from (57c) that $\tilde{X}_R = 0$. Hence (58a) is

satisfied if $V \geq V_R$. To show that $V \geq V_R$ note from (67a) that

$$V_R = \sum_i \lambda_i \left[\theta_x^T C^T Q C \theta_x\right], \quad i \in R.$$

Hence, since $\lambda_i \geq 0$ for all i,

$V = V_R$ if $r \geq k$, $k = [\text{rank } C]$

$V > V_R$ if $r < k$

and (1) is proven. The proof of claim (2) follows from (1) and theorem 5. Claim (3) follows from claim (1) together with (67b) and the fact that $\left[\theta_x^T C^T Q C \theta_x\right]$ can have no more than k $\triangleq$ rank C nonzero eigenvalues. # It may be readily verified that proposition 7 holds for the general cost-decoupled coordinates in definition 6, and proposition 7 is not restricted to the special choice of cost-decoupled coordinates given by (63). Furthermore, claim (3) of proposition 7 shows that the CCA algorithm using cost-decoupled coordinates yields a cost-equivalent realization of (32) if $r \geq k$ and if the reduced-order model is disturbable. These are only sufficient conditions. We shall now present the precise conditions in which such cost-equivalent realizations are obtained.

Proposition 8. The CCA algorithm using *cost-decoupled coordinates* yields *cost-equivalent realizations* of (32) if and only if (a) $r \geq k$ and (b) the undisturbable subspace of (A_R, D_R) is unobservable.

Proof. For any pair (A_R, D_R), the transformation defined in (70) exists. (If (A_R, D_R) is disturbable then $I = T = T_1$, $A_{11} = A_R$, $D_1 = D_R$, and $C_1 = C_R$.) Then from equations (70) and (71), it can be seen that A_R is not asymptotically stable. Those eigenvalues of A_R which are not asymptotically stable are contained in the set of eigenvalues of A_{22},

and this corresponds to the undisturbable part of (A_R, D_R). Hence, the undisturbable modes are the only ones that are not asymptotically stable. Since the unstable (and undisturbable) part of A_R does not contribute to the cost V_R [6], the model (35) can be further reduced to yield

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + D_1 w, \\ y_1 &= C_1 x_1, \end{aligned} \tag{72}$$

such that

$$V_R = \lim_{t\to\infty} E\|y_R(t)\|_Q^2 = \lim_{t\to\infty} E\|y_1(t)\|_Q^2.$$

Now from (71a) and (49) we have

$$\begin{aligned} V_R &= \text{tr } C_1^T Q C_1 = \text{tr } T_1^T C_R^T Q C_R T_1 \\ &= \text{tr } C_R^T Q C_R - \text{tr } T_2^T C_R^T Q C_R T_2, \end{aligned} \tag{73a}$$

where the orthonormal property of $T(T_1 T_1^T + T_2 T_2^T = I)$ is used. From (66), (67a), and the partitioning of C in (55), it can be seen that

$$\text{tr } C_R^T Q C_R = \sum_{i=1}^{r} \lambda_i \left[\theta_X^T C^T Q C \theta_X\right].$$

Hence,

$$V_R = \sum_{i=1}^{r} \lambda_i \left[\theta_X^T C^T Q C \theta_X\right] - \text{tr } C_2^T Q C_2 \tag{73b}$$

where $C_2 \triangleq C_R T_2$. Now, since $A_{12} = 0$ from (71b), considering (70b) to be in observable canonical form [6], it can be said that $C_2 = 0$ if and only if condition (b) holds. Furthermore, since the columns of T_2 span the undisturbable subspace of

(A_R, D_R) [6], we have the following:

(i) if condition (b) holds (equivalently if $C_2 = 0$) then

$$V_R = \begin{cases} \sum_{i=1}^{r} \lambda_i\left[\theta_X^T C^T Q C \theta_X\right] = V & \text{if } r \geq k \quad (74a) \\ \sum_{i=1}^{r} \lambda_i\left[\theta_x^T C^T Q C \theta_X\right] < V & \text{if } r < k \quad (74b) \end{cases}$$

(ii) if condition (b) does not hold (i.e., $C_2 \neq 0$), then

$$V_R = \sum_{i=1}^{r} \lambda_i\left[\theta_X^T C^T Q C \theta_X\right] - \alpha < V, \quad (74c)$$

where $\alpha \triangleq \text{tr } C_2^T Q C_2 > 0$. Obviously $\alpha = 0$ if (A_R, D_R) is disturbable since $[T_1\ T_2] \to [T_1] = I$ and $[C_1] \to [C_R]$ implying $C_2 \to 0$. Note, therefore, from (74) that if condition (a) does not hold, then $V_R < V$ and (35) is not a cost-equivalent realization. #

One obvious conclusion from proposition 8 is that the order of the minimal CER is never less than k, the number of independent outputs. It is of interest to classify those systems whose minimal CER is of order greater than k.

Proposition 9. For all systems (32) whose first Markov Parameters is zero (CD = 0) the order of the minimal CER is greater than k.

Proof. Let the system (55) be in cost-decoupled coordinates and let r = k. Hence, assuming $\Omega_y^2 = I$, from (66) we have

$$C^T Q C = \begin{bmatrix} C_R^T Q C_R & C_R^T Q C_T \\ C_T^T Q C_R & C_T^T Q C_T \end{bmatrix} = \begin{bmatrix} \Lambda^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad (75a)$$

where $\Lambda^2 \triangleq \text{diag}\{\lambda_1[C^TQC], \lambda_2[C^TQC], \ldots, \lambda_k[C^TQC]\}$. Now since rank C = k, $\lambda_i[C^TQC] \neq 0$, i = 1, 2,..., k. Hence, equating

$$C_R^TQC_R = \Lambda^2, \tag{75b}$$

$$C_T^TQC_T = 0, \tag{75c}$$

and recognizing that Q > 0, it can then be claimed that C_R (of dimension k × k) is square and of full rank and that $C_T = 0$. Now, since Markov parameters are invariant under similarity transformation, we have

$$CD = 0 \leftrightarrow [C_R \; 0]\begin{bmatrix} D_R \\ D_T \end{bmatrix} = C_RD_R = 0. \tag{76}$$

Equation (76) is satisfied if and only if $D_R = 0$, since C_R is square and of full rank. In this event, the pair (A_R, D_R) is obviously undisturbable. Furthermore, due to full rank of C_R, the pair (C_R, A_R) is completely observable. Therefore, the undisturbable subspace of (A_R, D_R) cannot be also unobservable. This violates condition (b) of proposition 8. Hence the minimal CER cannot be of order k. #

Nevertheless, a minimal CER of order r > k, can be constructed for the systems defined in proposition 9, by increasing r until condition (b) of proposition 8 is satisfied.

C. THE ALGORITHM FOR COST-EQUIVALENT REALIZATIONS (CER)

Cost-equivalent realizations (CERs) are provided by the CCA algorithm using cost-decoupled coordinates and the CERs have all the properties of propositions 6 and 7. The two steps of the basic CCA algorithm are described in Section IV

and the cost-decoupled coordinates are described in Section VII.B. Combining these two ingredients leads to the following CER algorithm.

The CER algorithm

Step 1. Given the model and performance objectives of (32) and (33):

(A, D, C, Q, W) where $Q > 0$, $W > 0$, A stable.

(Choose $\Omega_y = I$.)

Step 2. Compute covariance X from[1]

$$0 = XA^T + AX + DWD^T. \tag{77}$$

Step 3. Compute θ_x the square root of X^2

$$X = \theta_x \theta_x^T. \tag{78}$$

Step 4. Compute θ_y, the orthonormal modal matrix of $\theta_x^T C^T Q C \theta_x$.[3] The component costs are

$$\theta_y^T \theta_x^T C^T Q C \theta_x \theta_y = \text{diag}\{V_1, V_2, \ldots, V_k, 0, \ldots, 0\}, \tag{79}$$

where the number of nonzero component costs are $k = \text{rank } [C]$.

Step 5. Rearrange the columns of θ_y so that the V_i appear in order

$$V_1 \geq V_2 \geq \cdots \geq V_k. \quad \text{Set } r = k = \text{rank } C. \tag{80}$$

Step 6. Then define θ_R by

$$\theta_y = [\theta_R, \theta_T], \quad \theta_R \in R^{n \times r}. \tag{81}$$

[1] *For efficient solution of the linear Liapunov equation, use the algorithm in [11].*

[2] *For efficient calculation of θ_x, see the computer codes in [12].*

[3] *For this task use singular value decomposition [9] or use an eigenvalue/eigenvector program specialized for symmetric matrices.*

Step 7. Compute

$$\left.\begin{aligned} A_R &= \theta_R^T \theta_X^{-1} A \theta_X \theta_R \\ D_R &= \theta_R^T \theta_X^{-1} D \\ C_R &= C \theta_X \theta_R \end{aligned}\right\} \text{CER.} \qquad (82)$$

Step 8. Compute modal data for A_R:

$$A_R e_i = \lambda_i e_i, \quad i = 1, 2, \ldots, r$$

If $\|C_R e_i\| > 0$ for any i such that $R_e \lambda_i = 0$, where $R_e(\cdot)$ denotes "real part" of $(\cdot)$, set $r = r + 1$ and go to step 6. Otherwise stop.

Remark: The product $C_R e_i$ is defined as the *observability vector* associated with mode i [5] (mode i in a nondefective system is unobservable if and only if its observability vector is zero). Hence, the purpose of step 8 is to check if the unstable mode ($R_e \lambda_i = 0$) is observable. Since in cost-decoupled coordinates the unstable modes of A_R are also undisturbable, step 8 amounts to checking if the condition (b) of proposition 8 holds.

This algorithm guarantees the construction of a CER. However, the construction of a *minimal* CER is guaranteed only if the algorithm converges within the first two iterations, in which case the CER is of order $r = k$ or $r = k + 1$. For the minimal CER of order k the triple (A_R, D_R, C_R) is both disturbable and observable and asymptotically stable. For any other CER produced by the algorithm, the disturbable, observable spectrum of (A_R, D_R, C_R) is asymptotically stable. *After* the first iteration of the algorithm, the selection of the best sequence of eigenvector calculations in step 5 has not been determined and is under investigation.

If the CER algorithm yields a CER of order r with an unstable (and undisturbable and unobservable) spectrum, then the CER can be further reduced, as shown in (70) and (72), to yield a realization of order less than r. This smaller realization is still a CER as is proved following proposition 8.

D. *RELATIONSHIPS BETWEEN THE COST-DECOUPLED COORDINATES AND THE BALANCED COORDINATES OF MOORE [7]*

The balanced coordinates of Moore [7] are defined by the transformation that diagonalizes the controllability and observability matrices (X and K in this paper). Singular value analysis provides the efficient tools to compute the balanced coordinates. As mentioned in the introduction, CCA can be applied to any choice of coordinates, including balanced coordinates. The most powerful results from CCA are obtained with the use of the cost-decoupled coordinates defined in the last section using the CER algorithm. Moore [7] introduced balanced coordinates to reduce numerical ill-conditioning, thereby making data more manageable in the computer. On the other hand, the primary goal of CCA is specifically to tailor the reduced model to the control or output response objectives (36). It would be of interest to know whether there are circumstances under which balanced coordinates of Moore are cost-decoupled.

To obtain *balanced* coordinates, a coordinate transformation is selected so that the new (balanced) coordinates have the properties [see Eqs. (37) and (38) in [7]],

$$X = K = \Sigma^2 = \text{diag}, \tag{83}$$

where X is the disturbability matrix

$$X \triangleq \int_0^\infty e^{At} DWD^T e^{A^T t}\, dt, \quad W > 0 \tag{84}$$

satisfying

$$0 = XA^T + AX + DWD^T, \tag{85}$$

and K is the observability matrix

$$K \triangleq \int_0^\infty e^{A^T t} C^T Q C e^{At}\, dt, \quad Q > 0 \tag{86}$$

satisfying

$$0 = KA + A^T K + C^T Q C. \tag{87}$$

To obtain the *cost-decoupled* coordinates of Section VII.B, a coordinate transformation is selected so that the new (cost-decoupled) coordinates have the properties from definition 6,

$$X = \text{diag}, \quad C^T Q C = \text{diag}, \tag{88}$$

To summarize these results from (83) and (88), proposition 10 specifies the condition under which cost-equivalent realizations can be obtained from balanced coordinates.

Proposition 10. If in balanced coordinates the state weighting $C^T Q C$ happens to be diagonal, then balanced coordinates are cost-decoupled and hence have all the properties of proposition 6.

Proof. Cost-decoupled coordinates are defined by (88) and balanced coordinates satisfy (83). The comparison of (83) and (88) concludes the proof. #

VIII. SHOULD REDUCED MODELS DEPEND UPON THE WEIGHTS IN THE QUADRATIC COST?

The reader should be reminded of the fact that the state weighting $[C^TQC]$ in the performance metric

$$V = \lim_{t\to\infty} E\|y\|_Q^2 = \lim_{t\to\infty} Ex^T[C^TQC]x$$

often contains parameters chosen in an *ad hoc* fashion. Why then, one might ask, should one adopt a model reduction strategy in which the reduced models depend upon the weight C^TQC? This question is briefly answered as follows. The selection of a performance metric V reflects, to the best of one's ability, the objective of the model analysis (to describe accurately specific outputs y). Thus, it is important to keep in mind that there are many problems in which the entire state weighting matrix C^TQC is *not* arbitrary, but only the output weighting Q might be free to be manipulated. This notion of penalizing only specific physical variables represented by y allows the number of free parameters in the $n \times n$ state weighting C^TQC to be reduced from $n(n + 1)$ to $k(k + 1)$, the free parameters in Q. Thus, C^TQC contains important information by its very structure. For example, a certain spacecraft may have a mission to keep optical line-of-sight errors small in a space telescope. These error variables, collected in the vector herein labeled y, make up only a small subset of all the state variables $y = Cx$. Alternatively, the same spacecraft may have a communications mission where one is interested in the RMS deflections over the entire surface of a flexible antenna. These two problems have entirely *different* modeling (and control) objectives and *it is precisely the weights* C^TQC *that distinguish between the two objectives.* That is, the

reduced-order model that is best for the analysis (estimation, control) of optical errors is different from the model that is best for analysis of errors in the parabolic shape of the antenna. To ignore these weights C^TQC is to force a complete and artificial *separation* between the control problem and the modeling problem, a state of affairs which the authors believe is not realistic. The authors' opinion is that one's ability to evaluate the quality of *any* reduced model (obtained by any method) is no better and no worse than his ability to choose a precise performance metric. Of course, if one has no physical objective to motivate the choice of specific output variables $y = Cx$ and if he instead arbitrarily chooses an *equal* weighting on all *balanced* coordinates ($C^TQC = I$), *then* the CCA algorithm produces the same partial realization as balanced coordinate methods of model reduction. A primary goal of this chapter is therefore to promote a systematic beginning for the integration of the modeling and the estimation/control problems, to allow modeling decisions to be influenced by specific quadratic control or estimation objectives, without relying upon nonlinear programming methods.

It should also be mentioned that for *scalar* input-output systems the reduced models produced by the CCA algorithm are *independent* of the choices of the output weighting Q and the noise intensity W. This can be readily verified by noting that Q and W in (34) and also in (47) are scalars that are common factors in every component cost V_i, and cannot therefore influence the cost ordering (36).

IX. STABILITY CONSIDERATIONS

Stability may or may not be an important feature of a reduced model. In fact, several schemes for "improving" reduced models upon which state estimators are based include the *intentional destabilization* of the model as a means to reduce or eliminate Kalman filter divergence. This means of improving models is discussed in [15] and its references. Also note that the technique of guaranteeing stability margins in linear regulator problems by multiplying the state weight in the quadratic cost function by $\exp(2\alpha t)$ causes the closed loop eigenvalues to lie to the left of the line $-\alpha$ in the complex plane [16]. This method is also equivalent to *intentionally destabilizing* a stable plant model by replacing A by $A + \alpha I$ in lieu of multiplying the state weight by $\exp(2\alpha t)$. It is not our purpose to recommend necessarily such methods for estimator or control design, but merely to point out that stability is neither a necessary nor sufficient qualification for a reduced model to be a "good" model of a stable system.

The model error index Q is finite if the observable modes of (A, C) and the observable modes of (A_R, C_R) are stable. Hence, stability is a sufficient but not a necessary condition for the existence of Q. If stability is an overriding concern in the selection of a partial realization, then one may choose special coordinates for which the CCA algorithm guarantees stability.

Presently, if the order of the partial realization is fixed *a priori*, the only coordinates for which asymptotic stability of the partial realizations produced by CCA has been guaranteed is modal coordinates. The modal cost analysis (MCA)

of Section V produces stable models since the eigenvalues of the reduced model are a subset of the eigenvalues of the original (stable) system. However, since other coordinate choices (such as the cost-equivalent coordinates of Section VII) may produce better models, it is suggested that the CER be found first and examined for stability. If stability of the reduced model is required and not obtained from the CER, then obtain the reduced model by application of MCA, which guarantees stability. Note, however, that if both realizations (from CER and MCA) of order r are stable, the authors have not found a single example in which the CER failed to yield a smaller model error index Q.

Furthermore, if the order of the partial realization is *not* fixed *a priori*, then the CER algorithm always yields a CER that is asymptotically stable.

X. CER EXAMPLES

The concepts are best illustrated with simple problems. We begin with a second-order example.

Example 1. The CER for the system (32) with parameters

$$A = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & -10 \end{array}\right], \quad D = \left[\begin{array}{c|c} 1 & 1 \\ \hline 70 & 1 \end{array}\right], \quad Q = 1$$

$$C = [1, -0.2], \quad W = I$$

with transfer functions

$$y(s) = G(s)w(s),$$

$$G(s) = [(s + 1)(s + 10)]^{-1}[-13s - 4, 0.8s + 9.8]$$

is

$$A_R = -10.318, \quad C_R = -2.867, \quad D_R = [4.534, -0.279],$$

which has the transfer function

$$\hat{y}(s) = G_R(s)w(s), \quad G_R(s) = [-13,\ 0.8][(s + 10.33)]^{-1}.$$

The reduced-order model has an eigenvalue near the fast mode (-10) of the original system as a consequence of the fact that this mode is highly disturbable from w(t).

Example 2. Several authors on model reduction have cited the fact that there seems to be no simple way to say that

$$G(s) = \frac{(s + 1.1)}{(s + 1)(s + 10)} \approx \frac{1}{s + 10}.$$

We consider a little more general situation: We find that for

$$G(s) = \frac{s + \alpha}{(s + 1)(s + 10)}$$

the minimal CER is

$$G_R(s) = \frac{1}{s + \dfrac{110}{10 + \alpha^2}}.$$

Table I provides the results for a variety of choices of α, and the corresponding CER. The table illustrates (for α = 1, 1.1, 10) the proper use of zero information in a near

Table I.

α	*Example G(s)*	$G_R(s)$ *of CER*
1	$\frac{s + 1}{(s + 1)(s + 10)}$	$\frac{1}{s + 10}$
1.1	$\frac{s + 1.1}{(s + 1)(s + 10)}$	$\frac{1}{s + 9.8}$
10	$\frac{s + 10}{(s + 1)(s + 10}$	$\frac{1}{s + 1}$
-10	$\frac{s - 10}{(s + 1)(s + 10)}$	$\frac{1}{s + 1}$
- 1	$\frac{s - 1}{(s + 1)(s + 10)}$	$\frac{1}{s + 10}$
0	$\frac{s}{(s + 1)(s + 10)}$	$\frac{1}{s + 11}$

pole-zero cancellation situation--a situation that frustrates many model reduction schemes. The reader is reminded that for scalar input-output systems, the CER parameters (A_R, D_R, C_R) are *independent* of the noise intensity $W > 0$ and output weighting $Q > 0$.

Example 3. Consider the following system whose first Markov parameter is zero (i.e., $CD = 0$)

$$\dot{x} = Ax + Dw, \quad w \sim N(0, 1)$$

$$y = Cx$$

where

$$A = \begin{bmatrix} -10 & 1 & 0 \\ -5 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = [1\ 0\ 0]$$

From proposition 9 a minimal CER of order 1 does not exist for this system. However a CER of order 2 exists and is given by

$$\dot{\hat{x}}_R = A_R\hat{x}_R + D_R w$$

$$y_R = C_R\hat{x}_R,$$

where

$$A_R = \begin{bmatrix} 0 & 0.7384 \\ -0.7384 & -8.166 \end{bmatrix}, \quad D_R = \begin{bmatrix} 0 \\ 4.0413 \end{bmatrix}$$

$$C_R = [0.335,\ 0].$$

This CER is asymptotically stable, disturbable and observable.

XI. APPLICATION OF CCA TO CONTROLLER REDUCTION

Given a model of high-order $n > n_c$, the traditional approach to designing a linear controller of specified order n_c is first to use model reduction methods to reduce the model to order n_c and then design a controller that is perhaps optimal for the reduced model. There are at least two

objections to this strategy. The first disadvantage is that most model reduction techniques ignore the effect of the (yet to be determined) control inputs, and it is well known that the inputs (whether they be functions of time or state) can have a drastic effect on the quality of the reduced model. The second disadvantage is that optimal control theory applied to a poor model can certainly yield poor results, often destabilizing the actual system to which the low-order "optimal" controller is applied.

The design strategy suggested for obtaining a controller of order n_c given a model of order $n >> n_c$ is as follows:

A controller-reduction algorithm

1. Apply CCA to reduce the *model* to order $N_R > n_c$, where N_R is the largest dimension of a Riccati equation that can be reliably solved on the local computer.

2. Solve for the optimal controller of order N_R, using the reduced model of order N_R.

3. Apply CCA to reduce the *controller* to order $n_c < N_R$

The purpose of this section is to show how to accomplish step 3. The intended advantage of this algorithm over the traditional approach (which skips step 3 and sets $N_R = n_c$) is that more information about the higher order system and its would-be optimal controller is made available for the design of the reduced-order controller.

The controller reduction can be presented as a restricted model reduction problem as follows: Consider the plant,

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw, \\ y &= Cx, \\ z &= Mx + v, \end{aligned} \tag{89}$$

$x \in R^n$, $u \in R^m$, $w \in R^d$,

$y \in R^k$, $z \in R^\ell$, $rk[B] = m \leq n$,

where w(t) and v(t) are uncorrelated zero-mean white noise processes with intensities $W > 0$ and $V > 0$, respectively. The measurement is z, y is the output to be controlled, and u is the control chosen to minimize

$$V = \lim_{t \to \infty} E\left(\|y\|_Q^2 + \|u\|_R^2\right), \quad Q > 0, \; R > 0. \tag{90}$$

Under the assumptions that (A, B) and (A, D) are controllable, (A, C) and (A, M) observable, the optimal controller for (89) takes the form

$$\begin{aligned} \dot{x}_c &= A_c x_c + Fz, \quad x_c \in R^n, \\ u &= Gx_c, \end{aligned} \tag{91a}$$

where

$$A_c \triangleq A + BG - FM, \tag{91b}$$

$$G = -R^{-1}B^T K, \quad KA + A^T K - KBR^{-1}B^T K + C^T QC = 0, \tag{92}$$

$$F = PM^T V^{-1}, \quad PA^T + AP - PM^T V^{-1} MP + DWD^T = 0. \tag{93}$$

Augmenting the plant (89) and the controller (91) yields the closed-loop system.

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A & BG \\ FM & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{94}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix}.$$

The cost V can be expressed as

$$V = \operatorname{tr} \hat{X}_{11} C^T QC + \operatorname{tr} \hat{X}_{22} G^T RG, \tag{95}$$

where

$$\begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}_{12}^T & \hat{X}_{22} \end{bmatrix}\begin{bmatrix} A^T & M^T F^T \\ G^T B^T & A_c^T \end{bmatrix} + \begin{bmatrix} A & BG \\ FM & A_c \end{bmatrix}\begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}_{12}^T & \hat{X}_{22} \end{bmatrix} + \begin{bmatrix} DWD^T & 0 \\ 0 & FVF^T \end{bmatrix} = 0. \tag{96}$$

Now if the two "components" of (90) are defined as the plant (with state x) and the controller (with state x_c), then the component costs for x and x_c are denoted V° and V^c, respectively, where

$$V = V^\circ + V^c \tag{97}$$

and

$$V^\circ \triangleq \text{tr } \hat{X}_{11} C^T Q C, \tag{98}$$

$$V^c \triangleq \text{tr } \hat{X}_{22} G^T R G. \tag{99}$$

Since we desire to reduce the dimension of the controller and not the plant, we further decompose V^c into individual component costs associated with controller states.

$$V = V^\circ + \sum_{i=1}^{n} V_i^c, \tag{100}$$

where

$$V_i^c \triangleq \left(\hat{X}_{22} G^T R G\right)_{ii}. \tag{101}$$

Having defined the controller components, the controller reduction can be shown to be a special "model reduction" problem by simply interpreting (90) in the form of (32). That is,

substitute

$$x^T \to [x^T \; x_c^T], \quad y^T \to \begin{bmatrix} y^T & u^T \end{bmatrix}$$

$$A \to \begin{bmatrix} A & BG \\ FM & A_c \end{bmatrix}, \quad D \to \begin{bmatrix} D & 0 \\ 0 & F \end{bmatrix}, \quad W \to \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}$$

$$Q \to \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}, \quad C \to \begin{bmatrix} C & 0 \\ 0 & G \end{bmatrix}$$

Now with a very minor modification the standard CCA algorithm can be applied to obtain the reduced-order model of dimension $N_c = n + n_c$, where n_c is the dimension of the reduced-order controller desired. The minor restriction is that the plant component of dimension n is not to be truncated, regardless of the value of V°.

Motivated by the theory of cost-decoupled coordinates, CERs and definition 6, we desire to transform the controller coordinates so that both $\hat{X}_{22}$ and $G^T RG$ in (97) are diagonal.

The cost-decoupled controller (CDC) algorithm

Step 1. Given the model and performance objectives (A, B, D, C, M, W, V, Q, R).

Step 2. Compute the optimal controller (A_c, F, G) from (91)-(93) and the covariances $\hat{X}_{11}$ and $\hat{X}_{22}$ satisfying (96).

Step 3. Compute θ_1 the square root of $\hat{X}_{22}$

$$\hat{X}_{22} = \theta_1 \theta_1^T$$

Step 4. Compute θ_2 the orthonormal modal matrix of $\theta_1^T G^T RG \theta_1$. The controller component costs are

$$\theta_2^T \theta_1^T G^T RG \theta_1 \theta_2 = \text{diag}\{V_1^c, \quad V_2^c, \ldots, V_m^c, 0, \ldots 0\}$$

where the number of nonzero controller component costs are m = rank B.

Step 5. Rearrange the columns of θ_2 so that the V_i^c appear in order

$$V_1^c \geq V_2^c \geq \cdots \geq V_m^c.$$

Then define θ_R by

$$\theta_2 = [\theta_R, \theta_T], \quad \theta_R \in R^{n\times m}.$$

Step 6. The reduced CDC is

$$\dot{\hat{x}}_R = A_R\hat{x}_R + F_R z, \quad \hat{x}_R \in R^m,$$

$$u = G_R\hat{x}_R,$$

where

$$A_R \triangleq \theta_R^T\theta_1^{-1}A_c\theta_1\theta_R,$$

$$F_R \triangleq \theta_R^T\theta_1^{-1}F,$$

$$G_R \triangleq G\theta_1\theta_R.$$

Additional properties of the CDC must be explored in future investigations. Space limitations suggest this convenient stopping point in the presentation of the CER theory and its application to both model and controller reduction.

XII. CONCLUSIONS

A summary of the ideas of cost decomposition is given to aid in the determination of the relative cost (or "price") of each component of a linear dynamic system using quadratic performance criteria. In addition to the insights into system behavior that are afforded by such a component cost analysis (CCA), these CCA ideas naturally lead to a theory for cost-equivalent realizations.

Cost-equivalent realizations (CERs) of linear systems are defined, and an algorithm for their construction is given. The partial realizations of order r produced by this algorithm have these properties:

1. a minimized model error index;

2. the model error index is zero (i.e., the original system and the partial realization have the same value of the quadratic performance metric), if $r \geq k$, where k is the number of independent outputs;

3. the algorithm does not require the computation of modal data of the plant matrix A;

4. the method is applicable to large-scale systems, limited only by the necessity to solve a linear Liapunov-type algebraic equation;

5. the CER algorithm produces stable realizations of a stable system.

The algorithm is based upon component cost analysis (CCA), which is described for time-varying systems, for time-invariant systems, and for systems for which accurate modeling is of concern only over a finite interval of time. These component costs are shown herein to be useful in obtaining the above cost-equivalent realizations, but they are also useful in closed-loop applications where controllers, rather than models, are to be simplified.

Property 2 above reveals that cost-equivalent realizations can be *smaller* than Kalman's minimal realization, which is always of the dimension of the controllable, observable subspace.

Section VI is a point of departure for further research using different model error criteria. Instead of using the difference of norms as in (48), an error criterion using the norm of the differences can be studied much more extensively than done in [5], where only input-induced component costs were used. The model error criterion utilized herein, (48), is chosen for its appropriateness to the reduction of optimal controllers. Other uses of component cost analysis (CCA), which warrant further research include decentralized control, failure analysis, and system redesign strategies based upon "cost-balancing."

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