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Components with the expected codimension in the moduli scheme of stable spin curves

ABSTRACT. Here we study the Brill–Noether theory of "extremal" Cornalba's theta-characteristics on stable curves C of genus g, where "extremal" means that they are line bundles on a quasi-stable model of C with $\sharp(\mathrm{Sing}(C))$ exceptional components.

1. Introduction. For any integer $g \geq 2$ let $\overline{M}_g$ denote the moduli space of stable curves of genus g over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. Fix any $Y \in \overline{M}_g$. The topological type (if $\mathbb{K} = \mathbb{C}$) or the equisingular type (for arbitrary $\mathbb{K}$) τ may be described in the following way. Fix an ordering $Y_1, \dots, Y_s$ of the irreducible components of Y. The type τ is uniquely determined by the string of integers listing the geometric genera of $Y_1, \dots, Y_s$, the integers $\sharp(\mathrm{Sing}(Y_i))$, $1 \leq i \leq s$, and the integers $\sharp(Y_i \cap Y_j)$, $1 \leq i < j \leq s$ (see [1], p. 99). Recently, the Brill–Noether theory of theta-characteristics of smooth curves had a big advances due to a solution by L. Benzo ([3]) of a conjecture of G. Farkas ([6], Conjecture 3.4). In this note we show that such a result may be used for the study of the Brill–Noether theory of Cornalba's theta-characteristics on $\overline{M}_g$. Indeed, we will check that for the extremal theta-characteristics we are looking for in this note the existence of such a theta-characteristic on Y with prescribed number of

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linearly independent sections, $r+1$, is equivalent to the existence of theta-characteristics $E_1, \dots, E_s$ on the normalizations $C_1, \dots, C_s$ of $Y_1, \dots, Y_s$ and with $\sum_{i=1}^{s} h^0(C_i, A_i) = r+1$.

Let $\mathcal{S}_g$, $g \ge 2$, be the set of all theta-characteristics on smooth genus g curves, i.e. the set of all pairs (C, L) with $C \in M_g$, $L \in \mathrm{Pic}(C)$ and $L^{\otimes 2} \cong \omega_C$. For all integers $r \ge -1$ set $\mathcal{S}_g^r := \{(C, L) \in \mathcal{S}_g : h^0(L) = r+1\}$. The set $\mathcal{S}_g^r$ is a locally closed subset of $\mathcal{S}_g$ and each point of it has codimension at most $\binom{r+1}{2}$ in $\mathcal{S}_g$ ([8], part (ii) of Theorem 1.10). Maurizio Cornalba proved the existence of a compactification $\overline{\mathcal{S}}_g$ of $\mathcal{S}_g$ equipped with a finite morphism $u_g : \overline{\mathcal{S}}_g \to \overline{M}_g$ such that each fiber of u_g has cardinality 2^{2g} ([5], Proposition 5.2 and first part of §3). There are many topological types for which the Brill–Noether theory of theta-characteristics with $r+1$ linearly independent sections never occurs in the expected codimension, i.e. in codimension $\binom{r+1}{2}$ (see [2] for a description of all theta-characteristics with g linearly independent sections). The claim of this note is that to study the Brill–Noether theorem of $\overline{\mathcal{S}}_g \setminus \mathcal{S}_g$ one needs to distinguish the quasi-stable model on which a Cornalba's theta-characteristic lives as a line bundle. In other compactifications of $\mathcal{S}_g$ (as in [9]) torsion-free sheaves are used; prescribing the non-locally free points of these sheaves on some $C \in \overline{M}_g$ is equivalent to prescribe the images in $\mathrm{Sing}(C)$ of the quasistable model of C on which a Cornalba's theta-characteristic "is" a line bundle (it is not quite a line bundle L, but a line bundle up-to inessential isomorphisms and we also need to prescribe the line bundle $L^{\otimes 2}$ ([5], Lemma 2.1 and first part of §3)). None of these problems affect the Brill–Noether theory for the theta-characteristics we will consider in this note (we call them the maximally singular ones). For these theta-characteristics the computation of h^0 is reduced to the computations of h^0 for theta-characteristics on the normalizations of all the irreducible components of the given $C \in \overline{M}_g$. Hence the existence part is reduced to an existence part on smooth curves for all genera up to g. There is a natural injective morphism from $\overline{\mathcal{S}}_g$ into Caporaso's compactification $\overline{P}_{g-1,g}$ ([4]) of the set of all degree $g-1$ line bundles on M_g ([7]). A Cornalba's theta-characteristic associated to a stable curve C is said to be *maximally singular* if it is a line bundle on the quasi-stable model C' of C obtained blowing up all singular points of C. A Cornalba's theta-characteristic on C is maximally singular if and only if it induces a theta-characteristic on the normalization of C ([5], Lemma 1.1). If C has compact type, then each theta-characteristic on C is maximally singular, because for each $S \subset \mathrm{Sing}(C)$, the quasi-projective curve $C \setminus S$ has $\sharp(S) + 1$ connected components.

Obviously $\binom{a}{2} = 0$ for $a = 0, 1$. Define the function $\alpha : \mathbb{N} \to \mathbb{N}$ in the following way. Set $\alpha(0) := 1$ and $\alpha(1) := 1$. For all integers $q \ge 2$ let $\alpha(q)$

be the maximal positive integer such that $\binom{\alpha(q)+1}{2} \le q$. We have $\alpha(2) = 1$ and $\alpha(3) = 2$.

Theorem 1. *Fix a type τ for genus g stable curves. Let $q_1, \dots, q_s$ be the geometric genera of the irreducible components of stable curves with type τ. Fix integers a_i, $1 \le i \le s$, such that $0 \le a_i \le \alpha(q_i)$ for all i and set $r := -1 + \sum_{i=1}^{s} a_i$. Then there is an irreducible component Γ of the set of all maximally singular Cornalba's theta-characteristics for stable curves with type τ with codimension $\sum_{i=1}^{s} \binom{a_i}{2}$ and such that for a general $(Y, L) \in \Gamma$ with $Y = Y_1 \cup \cdots \cup Y_s$, each Y_i of geometric genus q_i and $h^0(C_i, L|C_i) = a_i$ for all i, where C_i is the normalization of Y_i.*

In most cases no component satisfying the thesis of Theorem 1 may be smoothable, i.e., it is in the closure inside $\overline{\mathcal{S}}_g$ of an irreducible component of $\mathcal{S}_g^r$, just because r may be very high.

2. The proof.

Remark 1. Fix an integer $q \ge 0$ and a smooth genus q curve D. If $q \ge 3$, then assume that D is general in its moduli space. A corollary of Gieseker–Petri theorem (case $q \ge 3$) ([1], Proposition 21.6.7) or Riemann–Roch gives that every theta-characteristic A on D satisfies $h^0(D, A) \le 1$. We will only use the existence of theta-characteristics A, B on D such that $h^0(D, A) = 0$ and $h^0(D, B) = 1$.

Remark 2. Notice that $\mathcal{S}_3^1$ has codimension 1 in M_3, because the hyperelliptic locus of M_3 has dimension 5. By [6], Theorem 1.2, $\mathcal{S}_g^1$ has a component of the expected codimension, 1, for all $g \ge 3$.

Lemma 1. *Let Y be a reduced projective curve such that $Y = C \cup T$ such that $T \cong \mathbb{P}^1$, $\sharp(C \cap T) = 2$ and each point of $C \cap T$ is a nodal point of Y. Let R be any line bundle on Y such that $\deg(R|T) = 1$. Then $h^i(Y, R) = h^i(C, R|C)$, $i = 0, 1$.*

Proof. We have the Mayer–Vietoris exact sequence:

$$0 \to R \to R|C \oplus R|T \to R|C \cap T \to 0 \tag{1}$$

Since $\deg(C \cap T) = 2$, $\deg(R|T) = 1$ and R is a line bundle, the restriction map $H^0(T, R|T) \to H^0(C \cap T, R|C \cap T)$ is an isomorphism. Hence (1) gives $h^i(Y, R) = h^i(C, R|C)$, $i = 0, 1$. □

Proof of Theorem 1. Fix a stable curve $Y = Y_1 \cup \cdots \cup Y_s$ with each Y_i of geometric genus q_i. Let $C = C_1 \sqcup \cdots \sqcup C_s$ be the normalization of Y with C_i the normalization of Y_i. Assume for the moment the existence of a theta-characteristic A_i on C_i such that $h^0(C_i, A_i) = a_i$ and let A' be the line bundle on $C_1 \sqcup \cdots \sqcup C_s$ with $A'|C_i = A_i$ for all i. Let Y' be the quasi-stable curve with Y as its stable reduction and with $\sharp(\mathrm{Sing}(Y))$ exceptional components. Let A be any line bundle on Y' with A' as its pull-back to

C and $\deg(A|J) = 1$ for each exceptional component J of Y'. Applying $\sharp(\mathrm{Sing}(Y))$ times Lemma 1, we get $h^0(Y', A) = r+1$. A is a totally singular Cornalba's theta-characteristic. Now we count the parameters. By the definitions of the integers $\alpha(q_i)$ and a_i we have $q_i \ge \binom{a_i+1}{2}$ for all i if $a_i \ge 2$. By [3], Theorem 1.2, there is an irreducible component $\Gamma_i \subset \mathcal{S}_{q_i}^{a_i-1}$ if $a_i \ge 2$. For the case $a_i = 0$ use Remark 1. For the case $a_i = 1$ use Remark 2. Taking all (Y, A) coming from all $(C_i, A_i) \in \Gamma_i$, we get a family of curves Y with codimension $\sum_{i=1}^{s} \binom{a_i}{2}$ in the subset $M(\tau) \subset \overline{M}_g$ with type τ. This is a maximal family (i.e. an open subset of an irreducible component of $\overline{\mathcal{S}}_g^r$), because each Γ_i is a maximal family and for all $Y \in M(\tau)$ the fiber $u_g^{-1}(Y)$ has the same number of elements. $\square$

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