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Composite Particles with Rotational Mass Levels

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In this letter we formulate the wave equation of particles with rotational mass levels and discuss their properties. This problem is originally due to Takabayasi,[1] who obtained a solution by using four-component spinor formulation and obtained the internal symmetry of two Lorentz groups. In this paper, we present a solution using sixteen-component spinor formulation for a composite system of particles with three internal Lorentz groups. The physical difference of the two methods is also evident.

We start with the mass formula for rotational levels:

$$m^2=\kappa_0{}^2+\kappa_1{}^2 J(J+1), \qquad (1)$$

where the κ's are constants and J is the spin of the particles. Introducing the momentum operator P_μ and the Lubanski operator W_μ, we have $P_\mu P^\mu=-m^2$, $W_\mu W^\mu=-P_\mu P^\mu J(J+1)$, where the metric is $(0, 1, 2, 3)=(- + + +)$. W_μ is given by $W_\mu=\frac{1}{2}\epsilon_{\mu\nu\kappa\lambda} M^{\kappa\lambda} P^\nu$, where $M^{\kappa\lambda}$ is the generator of the homogeneous Lorentz group (to be specified in more detail later). In this manner, we can express (1) as

$$(P_\mu P^\mu)^2+\kappa_0{}^2 P_\mu P^\mu-\kappa_1{}^2 W_\mu W^\mu=0. \qquad (2)^{*)}$$

To take the square root of (2), it is necessary and sufficient to introduce eight 16×16 matrices $\Gamma_\alpha(\alpha=0\sim7)$ which have the properties $\{\Gamma_\alpha, \Gamma_\beta\}=2g_{\alpha\beta}$.**) Using these Γ we obtain our fundamental equation:

$$(P_\mu P^\mu+i\kappa_0\Gamma_\mu P^\mu-\kappa_1\Gamma_{\mu+4}W^\mu)\psi=0,$$
$$\psi=\{\psi_\rho(X_\mu, n)\}, \qquad (3)$$

where the index ρ runs from 1 to 16. As stated explicitly below, ψ also carries the argument "n" which describes the internal degrees of freedom generating rotational excitation.

For the invariance of Eq. (3) under the Lorentz transformation $P^\mu\to a^\mu{}_\nu P^\nu$, $\psi(x, n)\to\psi'(x', n')=\Lambda\psi(x, n)$, where in the infinitesimal form, $a^\mu{}_\nu=\delta^\mu{}_\nu+\delta w^\mu{}_\nu$, $\Lambda=1+\frac{1}{2}S_\mu{}^\nu\,\delta w^\mu{}_\nu$, we find

$$S_{\mu\nu}=\tfrac{1}{4}[\Gamma_\mu, \Gamma_\nu]+\tfrac{1}{4}[\Gamma_{\mu+4}, \Gamma_{\nu+4}].$$

Each term of the right-hand side separately generates two Lorentz groups which are denoted by $O_s(3, 1)$ and $O_{s'}(3, 1)$ respectively. ψ undergoes transformation of the (Spin 1/2)×(Spin 1/2) representation of $O_s(3, 1)\times O_{s'}(3, 1)$ with $S_{\mu\nu}$ as generators. Note that Eq. (3) is also invariant under space inversion and the parity operator is $\pm i\Gamma_1\Gamma_2\Gamma_3\Gamma_4$.

*) Up to this point, the formulation of Takabayasi and the author are identical, but the differences arise when we take the square root of (2) by use of the new Γ-matrices instead of Dirac's γ-matrices in order to obtain (2) from (3). The method of Takabayasi which uses the four-component spinor formulation does not reproduce (1) except in the special case where $\kappa_0=\kappa_1/2$. The method presented here, using the sixteen-component spinor, reproduces (1) generally.

**) α, β, γ, etc. run from 0 to 7; the metric tensor in eight-dimensional space is : diag $(g_{\alpha\beta})=(- + + + - + + +)$.

In order to accommodate the rotational excitations, that is, to describe particles of arbitrary spin J, the internal symmetry group must contain at least the three-dimensional orthogonal group $O(3)$. As an example we propose to use the $O(4, 2)$ as the internal symmetry group. The motivation for this particular choice comes from the fact that in this case the electromagnetic form factors of the ground state behave preferably, i.e. like q^{-4} as $q^2 \to \infty$.

The internal variables are given by three integers: $\psi = \psi(X_\mu; n, n_1, n_2)$. Using parabolic coordinates, $|n, n_1, n_2\rangle$ is expressed as

$$|n, n_1, n_2\rangle = [n_1!(n_2+m)!n_2!(n_1+m)!]^{-1/2} \times (a_1^+)^{n_2+m}(a_2^+)^{n_1}(b_1^+)^{n_1+m}(b_2^+)^{n_2}|0\rangle,$$
$$n, n_1, n_2, m = 0, 1, 2, \cdots,$$

where a_i and b_i $(i=1, 2)$ are ladder operators which satisfy

$$[a_i, a_j^+] = \delta_{ij}, \quad [b_i, b_j^+] = \delta_{ij},$$

and all other commutators are zero. The $O(4, 2)$ generators are the same as those of Barut and Kleinert,[2] and Nambu.[3] The wave function ψ admits the transformation $P \times O_s(3, 1) \times O_{s'}(3, 1) \times O(4, 2)$, where P is the orbital Poincaré group = {translations} $\times O_{\rm orb}(3, 1)$ operating on X_μ. The internal group $O(4, 2)$ contains as its subgroup a Lorentz group denoted by $O_{\rm in}(3, 1)$. The *physical Lorentz group* is the "*extracted*" $O(3, 1)$ subgroup of the direct product $O_{\rm orb}(3, 1) \times O_s(3, 1) \times O_{s'}(3, 1) \times O_{\rm in}(3, 1)$.

Its generator is

$$M_{\mu\nu} = M_{\mu\nu}^{\rm orb} + S_{\mu\nu} + M_{\mu\nu}^{\rm in}.$$

Four Lorentz groups are coupled to each other by the wave equation (3).

$O_{\rm orb}(3, 1)$ does not contribute to W_μ; hence

$$W_\mu = W_\mu^s + W_\mu^{s'} + W_\mu^{\rm in}.$$

The physical spin J of the particle is the resultant of the three spins, i.e. 1/2 of $O_s(3, 1)$, 1/2 of $O_{s'}(3, 1)$ and j of $O_{\rm in}(3, 1)$, or, $J = j + \xi$, where $\xi = 0, \pm 1$. This is an analogue of the quark model in which the spin of the particle is composed of three spins 1/2, 1/2 and l (the relative angular momentum of a quark and an antiquark). Thus J contains three series due to three values of ξ. The internal $O(4, 2)$ describes the selfinteraction or the structure of the particle as noted by Barut et al.[2),3)] The form factors of the ground state ($\xi = 0$, $J = 1/2$) have been calculated along this line and were found to behave like q^{-4} asymptotically.

To conclude, the particle described by Eq. (3) is a composite system with internal symmetry (extracted from) $O_s(3, 1) \times O_{s'}(3, 1) \times O(4, 2)$, and the spin J is the resultant of three spins. Its mass is given by the formula (1). The electromagnetic form factors of the ground state preferably behave asymptotically.

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2) A. O. Barut and H. Kleinert, Fourth Coral Gables Conferences, 1967; Phys. Rev. **161** (1967), 1464.
3) Y. Nambu, Prog. Theor. Phys. Suppl. Nos. 37 & 38 (1966), 368.