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COMPOSITES AND PRIMES AMONG POWERS OF FIBONACCI NUMBERS, INCREASED OR DECREASED BY ONE

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It is well known that, among the Fibonacci numbers F_n, given by

$$F_1 = 1 = F_2, \qquad F_{n+1} = F_n + F_{n-1},$$

$F_n + 1$ is composite for each $n \geqslant 4$, while $F_n - 1$ is composite for $n \geqslant 7$. It is easily shown that $F_n^2 \pm 1$ is also composite for any n, since

$$F_n^2 \pm 1 = F_{n-2}F_{n+2}, \qquad F_n^2 \mp 1 = F_{n+1}F_{n-1}.$$

Here, we raise the question of when $F_k^m \pm 1$ is composite.

First, if $k \not\equiv 0 \pmod 3$, then F_k is odd, F_k^m is odd, and $F_k^m \pm 1$ is even and hence composite. Now, suppose we deal with $F_{3k}^m \pm 1$. Since $A^n - B^n$ always has $(A - B)$ as a factor, we see that $F_{3k}^m - 1^m$ is composite except when $(A - B) = 1$; that is, for $k = 1$. Thus,

Theorem 1. $F_k^m - 1$ is composite, $k \neq 3$.

We return to $F_{3k}^m + 1$. For m odd, then $A^m + B^m$ is known to have the factor $(A + B)$, so that $F_{3k}^m + 1^m$ has the factor $(F_{3k} + 1)$, and hence is composite. If m is even, every even m except powers of 2 can be written in the form $(2j + 1)2^i = m$, so that

$$F_{3k}^m + 1^m = (F_{3k}^{2^i})^{2j+1} + (1^{2^i})^{2j+1}$$

which, from the known factors of $A^m + B^m$, m odd, must have $(F_{3k}^{2^i} + 1)$ as a factor, and hence, $F_{3k}^m + 1$ is composite.

This leaves only the case $F_{3k}^m + 1$, where $m = 2^i$. When $k = 1$, we have the Fermat primes $2^{2^i} + 1$, prime for i = 0, 1, 2, 3, 4 but composite for i = 5, 6. It is an unsolved problem whether or not $2^{2^i} + 1$ has other prime values. We note in passing that, when $k = 2$, $F_6 = 8 = 2^3$, and $8^m \pm 1 = (2^3)^m \pm 1 = (2^m)^3 \pm 1$ is always composite, since $A^3 \pm B^3$ is always factorable. It is thought that $F_9^4 + 1$ is a prime.

Since $F_{3k} \equiv 0 \pmod{10}$, $k \equiv 0 \pmod 5$, $F_{15k'}^{2^i} + 1 = 10^{2^i} \cdot t + 1$.

Since $F_{3k}^{2^i} \equiv 6 \pmod{10}$, $i \geqslant 2$, $k \not\equiv 0 \pmod 5$, $F_{3k}^{2^i} + 1$ has the form $10t + 7$, $k \neq 0 \pmod 5$. We can summarize these remarks as

Theorem 2. $F_k^m + 1$ is composite, $k \neq 3s$, $F_{3k}^m + 1$ is composite, $m \neq 2^i$.

It is worthwhile to note the actual factors in at least one case. Since

$$F_{k+2}F_{k-2} - F_k^2 = (-1)^{k+1}$$
$$F_{k+1}F_{k-1} - F_k^2 = (-1)^k$$

moving F_k^2 to the right-hand side and then multiplying yields

$$F_{k-2}F_{k-1}F_{k+1}F_{k+2} = F_k^4 - 1.$$

We now note that

$$F_k^5 - F_k = F_{k-2}F_{k-1}F_kF_{k+1}F_{k+2}$$

which causes one to ask if this is divisible by 5!. The answer is yes, if $k \not\equiv 3 \pmod 6$, but if $k \equiv 3 \pmod 6$, then only 30 can be guaranteed as a divisor.

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