Working Paper
Series

FINRISK
National Centre of Competence in Research
Financial Valuation and Risk Management

National Centre of Competence in Research
Financial Valuation and Risk Management

Working Paper No. 409

Composition of Time-Consistent Dynamic Monetary Risk Measures in Discrete Time

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First version: July 2005
Current version: May 2006

This research has been carried out within the NCCR FINRISK project on "Mathematical Methods in Financial Risk Management"

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Composition of Time-Consistent Dynamic Monetary Risk Measures in Discrete Time

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first version: July 2005; this version: May 2006

Abstract

In discrete time, every time-consistent dynamic monetary risk measure can be written as a composition of one-step risk measures. We exploit this structure to give new dual representation results for time-consistent convex monetary risk measures in terms of one-step penalty functions. We first study risk measures for random variables modelling financial positions at a fixed future time. Then we consider the more general case of risk measures that depend on stochastic processes describing the evolution of financial positions or cumulated cash flows. In both cases the new representations allow for a simple composition of one-step risk measures in the dual. We discuss several explicit examples and provide connections to the recently introduced class of dynamic variational preferences.

Key words: Dynamic monetary risk measures, time-consistency, dual representations.

1 Introduction

Following the introduction of coherent, convex and monetary risk measures in [1, 2, 17, 18, 19], different dynamic extensions were proposed. This has led to the study of conditional representations and time-consistency properties of dynamic risk measures in various setups. We refer to [3, 29, 28, 30, 33, 12, 9, 31, 6, 23, 26, 16] for the discrete time case and [20, 11, 28, 4, 5, 26] for risk measures in continuous time; see also [14] and [27] for related results for dynamic preferences in discrete time.

In this paper we provide representations of time-consistent dynamic monetary risk measures in discrete time that are similar in spirit to the continuous-time representations of [20], [28] and [4, 5]. Rather than looking at general dynamic monetary risk measures and trying to establish conditions for time-consistency, we here only consider time-consistent ones and view them as compositions of one-step risk measures. For time-consistent dynamic convex monetary risk measures, we exploit this structure to derive new dual representations in terms of the penalty functions of the one-step risk measures. These representations permit a simple construction of time-consistent dynamic convex monetary risk measures by composing one-step risk measures in the dual.

*Supported by NSF grant DMS-0505932
†Supported by the Swiss National Science Foundation

The structure of the paper is as follows: Section 2 explains the notation. In Section 3, we study time-consistent dynamic monetary risk measures for random variables. They can be written as simple concatenations of one-step risk measures. In Theorem 3.4, Lemma 3.7 and Corollary 3.8 we give dual representations for time-consistent dynamic convex monetary risk measures for random variables. The time-consistency is reflected by an additive structure in the dual. We illustrate this with examples and provide connections to the dynamic variational preferences of [27]. In Section 4, we consider dynamic monetary risk measures that depend on stochastic processes describing the evolution of financial positions over time. In this case, the composition of one-step risk measures involves the aggregation of current and future risk. For time-consistent dynamic convex monetary risk measures, we translate this structure into a dual representation in terms of supermartingales; see Theorem 4.4, Lemma 4.8 and Corollary 4.9. We conclude by introducing a special class of one-step aggregators of composed form and discussing several related examples of risk measures that depend on the whole path of a stochastic process.

2 Notation

We fix a finite time horizon $T \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ be a filtered probability space such that $\mathbb{P}[A] \in \{0,1\}$ for all $A \in \mathcal{F}_0$. $\mathbb{P}$ is not necessarily understood as a physical probability measure. We use it as reference measure that specifies the negligible events. Equalities and inequalities between random variables or stochastic processes are understood in the $\mathbb{P}$-almost sure sense. For instance, $X \geq Y$ for two stochastic processes X and Y means $X_t \geq Y_t$ $\mathbb{P}$-almost surely for all $t = 0, \dots, T$. For $p \in [1, \infty]$ and $t \in \{0, \dots, T\}$, $L^p(\mathcal{F}_t)$ is the space of all (equivalence classes of) $\mathcal{F}_t$-measurable random variables with finite L^p-norm. $\mathcal{R}^\infty$ denotes the space of (equivalence classes of) adapted stochastic processes X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ such that $\|X\|_{\mathcal{R}^\infty} := \operatorname{ess\,inf} \{ m \in \mathbb{R} \mid \sup_{0 \leq t \leq T} |X_t| \leq m \} < \infty$. Finally, $\mathcal{P}$ consists of all probability measures which are absolutely continuous with respect to $\mathbb{P}$.

3 Dynamic monetary risk measures for random variables

In this section the risky objects are financial positions at time T modelled by the set $L^\infty(\mathcal{F}_T)$. We assume that there exists a money market account and use it as numeraire, that is, money at later times is expressed in multiples of the value of one dollar put into the money market account at time 0. A risk measure at time t is a mapping $\rho_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$. $\rho_t(X)$ is interpreted as a capital requirement at time t for the financial position X conditional on the information given by $\mathcal{F}_t$. For the study of dynamic risk measures, it is more convenient to work with the negative $\phi_t = -\rho_t$ of a monetary risk measure. We call ϕ_t a monetary utility function. Alternative names are risk adjusted valuation ([3]) or acceptability measure ([30]).

Definition 3.1 *Let $t \in \{0, \dots, T\}$. We call a mapping $\phi_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ a monetary utility function at time t, if it has the following properties:*

(N) **Normalization:** $\phi_t(0) = 0$

(M) **Monotonicity:** $\phi_t(X) \geq \phi_t(Y)$ *for all* $X, Y \in L^\infty(\mathcal{F}_T)$ *such that* $X \geq Y$

(T) **Translation property:** $\phi_t(X + m) = \phi_t(X) + m$ *for all* $X \in L^\infty(\mathcal{F}_T)$ *and* $m \in L^\infty(\mathcal{F}_t)$

We call ϕ_t a concave monetary utility functions at time t, if it also satisfies

(C) $\mathcal{F}_t$**-concavity:** $\phi_t(\lambda X + (1 - \lambda)Y) \geq \lambda \phi_t(X) + (1 - \lambda)\phi_t(Y)$

for all $X, Y \in L^\infty(\mathcal{F}_t)$ and $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \le \lambda \le 1$.

A dynamic monetary utility function is a family of monetary utility functions $(\phi_t)_{t=0}^T$. If all ϕ_t are concave, then we call $(\phi_t)_{t=0}^T$ a dynamic concave monetary utility function.

The normalization property (N) is convenient for the study of time-consistency questions. Every function $\phi_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ satisfying (M) and (T) can readily be normalized by passing to $\phi_t(.) - \phi_t(0)$. It satisfies (N), (M), (T) and induces the same conditional preference order on $L(\mathcal{F}_T)$ as ϕ_t. The properties (M) and (T) imply

(LP) **Local property:** $1_A \phi_t(X) = 1_A \phi_t(1_A X)$ for all $X \in L^\infty(\mathcal{F}_T)$ and $A \in \mathcal{F}_t$.

Indeed, by (M) and (T), we have

$$\begin{aligned} & \phi_t(1_A X) - 1_{A^c} \|X\|_\infty = \phi_t(1_A X - 1_{A^c} \|X\|_\infty) \le \phi_t(X) \\ \le \quad & \phi_t(1_A X + 1_{A^c} \|X\|_\infty) = \phi_t(1_A X) + 1_{A^c} \|X\|_\infty \,, \end{aligned}$$

and (LP) follows by multiplying through with 1_A. Under (N), (LP) is equivalent to

$$1_A \phi_t(X) = \phi_t(1_A X) \quad \text{for all } X \in L^\infty(\mathcal{F}_T) \text{ and } A \in \mathcal{F}_t \,.$$

Definition 3.2 *We call a dynamic monetary utility function $(\phi_t)_{t=0}^T$ time-consistent if*

$$\phi_{t+1}(X) \ge \phi_{t+1}(Y) \quad \textit{implies} \quad \phi_t(X) \ge \phi_t(Y) \tag{3.1}$$

for all $X, Y \in L^\infty(\mathcal{F}_T)$ and $t = 0, \dots, T-1$.

Due to the properties (N), (M) and (T), time-consistency of dynamic monetary utility functions on $L^\infty(\mathcal{F}_T)$ is equivalent to the dynamic programming principle

$$\phi_t(X) = \phi_t(\phi_{t+1}(X)) \quad \text{for all} \quad X \in L^\infty(\mathcal{F}_T) \text{ and } t = 0, \dots, T-1 \,. \tag{3.2}$$

Concepts equivalent or similar to (3.1) or (3.2) have been studied in different contexts, see for instance, [24, 25, 15, 13, 32, 14, 11, 3, 29, 28, 4, 5, 30, 33, 12, 9, 31, 23, 26, 16, 27].

3.1 Generators

For a dynamic monetary utility function $(\phi_t)_{t=0}^T$, we denote by F_t the restriction of ϕ_t to $L^\infty(\mathcal{F}_{t+1})$ and call $(F_t)_{t=0}^{T-1}$ the generators of $(\phi_t)_{t=0}^T$. It follows from (3.2) that a time-consistent dynamic monetary utility function is uniquely given by its generators. One can also start with an arbitrary family

$$F_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t) \,, \quad t = 0, \dots, T-1 \,,$$

of monetary utility functions and define the time-consistent monetary utility function $(\phi_t)_{t=0}^T$ by backwards induction:

$$\begin{aligned} \phi_T(X) &:= X \\ \phi_t(X) &:= F_t(\phi_{t+1}(X)) \,, \quad t \le T-1 \,. \end{aligned}$$

It is clear that every ϕ_t is $\mathcal{F}_t$-concave if and only if each F_t is so.

3.2 Duality

In this section we provide duality results for time-consistent dynamic concave monetary utility functions on $L^\infty(\mathcal{F}_T)$ in terms of one-step penalty functions. In order to define them, we first have to introduce for $t = 1, \dots, T$, the set of one-step transition densities

$$\mathcal{D}_t := \left\{ \xi \in L^1_+(\mathcal{F}_t) \mid \mathrm{E}_\mathbb{P}\left[\xi \mid \mathcal{F}_{t-1}\right] = 1 \right\} .$$

Every sequence $(\xi_{t+1}, \dots, \xi_T) \in \mathcal{D}_{t+1} \times \dots \times \mathcal{D}_T$ induces a $\mathbb{P}$-martingale $(M^\xi_r)_{r=0}^T$ by

$$M^\xi_r := \begin{cases} 1 & \text{for } r \le t \\ \xi_{t+1} \cdot \dots \cdot \xi_r & \text{for } r = t+1, \dots, T \end{cases}$$

and a probability measure $\mathbb{Q}^\xi$ in $\mathcal{P}$ with density

$$\frac{d\mathbb{Q}^\xi}{d\mathbb{P}} = M^\xi_T \, .$$

On the other hand, every probability measure $\mathbb{Q}$ in $\mathcal{P}$ leads to a non-negative martingale

$$M^\mathbb{Q}_t := \mathrm{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t\right] , \quad t = 0, \dots, T \, .$$

Then,

$$\left\{ M^\mathbb{Q}_{t-1} = 0 \right\} \subset \left\{ M^\mathbb{Q}_t = 0 \right\} \quad \text{for all } 1 \le t \le T \, ,$$

and the sequence

$$\xi^\mathbb{Q}_t := \begin{cases} \frac{M^\mathbb{Q}_t}{M^\mathbb{Q}_{t-1}} & \text{on } \left\{ M^\mathbb{Q}_{t-1} > 0 \right\} \\ 1 & \text{on } \left\{ M^\mathbb{Q}_{t-1} = 0 \right\} \end{cases} \quad \text{for } t = 1, \dots, T \, ,$$

is an element in $\mathcal{D}_1 \times \dots \times \mathcal{D}_T$ that induces the measure $\mathbb{Q}$.

We will work with the convention

$$\mathrm{E}_\mathbb{Q}\left[X \mid \mathcal{F}_t\right] := \mathrm{E}_\mathbb{P}\left[\xi^\mathbb{Q}_{t+1} \cdot \dots \cdot \xi^\mathbb{Q}_T X \mid \mathcal{F}_t\right] , \quad X \in L(\mathcal{F}_T) \, , \quad t = 0, \dots T-1 \, .$$

$\mathrm{E}_\mathbb{P}\left[\xi^\mathbb{Q}_{t+1} \cdot \dots \cdot \xi^\mathbb{Q}_T X \mid \mathcal{F}_t\right]$ is a version of $\mathrm{E}_\mathbb{Q}\left[X \mid \mathcal{F}_t\right]$ that is defined up to $\mathbb{P}$-almost sure equality, whereas $\mathrm{E}_\mathbb{Q}\left[X \mid \mathcal{F}_t\right]$ is only defined up to $\mathbb{Q}$-almost sure equality.

By $\bar{L}_+(\mathcal{F}_t)$ we denote all $\mathcal{F}_t$-measurable functions from Ω to $[0, \infty]$. The conditional expectation of a random variable $X \in \bar{L}_+(\mathcal{F}_t)$ is, as usual, understood as

$$\mathrm{E}_\mathbb{P}\left[X \mid \mathcal{F}_t\right] := \lim_{n \to \infty} \mathrm{E}_\mathbb{P}\left[X \wedge n \mid \mathcal{F}_t\right] .$$

Definition 3.3 *For $t \in \{0, \dots, T-1\}$, we call a mapping*

$$\varphi_t : \mathcal{D}_{t+1} \to \bar{L}_+(\mathcal{F}_t)$$

a one-step penalty function if it satisfies the following two conditions:

(i) $\operatorname{ess\,inf}_{\xi \in \mathcal{D}_{t+1}} \varphi_t(\xi) = 0$

(ii) $\varphi_t(1_A \xi + 1_{A^c} \xi') = 1_A \varphi_t(\xi) + 1_{A^c} \varphi_t(\xi')$ *for all $\xi, \xi' \in \mathcal{D}_{t+1}$ and $A \in \mathcal{F}_t$.*

For $\mathbb{Q} \in \mathcal{P}$, we set

$$\varphi_t(\mathbb{Q}) := \varphi_t(\xi^\mathbb{Q}_{t+1}) \, .$$

A dynamic penalty function on $\mathcal{D}$ consists of a sequence $(\varphi_t)_{t=0}^{T-1}$ of one-step penalty functions.

The following theorem shows that every dynamic penalty function on $\mathcal{D}$ induces a time-consistent dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$. Note that (3.7) below is a dual representation of a whole family of risk measures $(\phi_t)_{t=0}^T$ in terms of a single penalty function. It is an extension of the dual representation of a time-consistent dynamic coherent risk measure in terms of one m-stable (or rectangular) set of probability measures (see [3, 11, 29, 14, 9]). Formula (3.5) can be seen as a discrete version of the continuous-time representation (37) in [5].

Theorem 3.4 *Let $(\varphi_t)_{t=0}^{T-1}$ be a dynamic penalty function on $\mathcal{D}$. Then*

$$F_t(X) := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \left\{ \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t(\mathbb{Q}) \right\}, \quad t = 0, \ldots, T-1, \tag{3.3}$$

defines generators of a time-consistent concave monetary utility function $(\phi_t)_{t=0}^T$ with the following representations:

$$\phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=t+1}^{s} \varphi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right] \tag{3.4}$$

$$= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=t+1}^{T} \varphi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right] \tag{3.5}$$

$$= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=1}^{s} \varphi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right] \tag{3.6}$$

$$= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=1}^{T} \varphi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right] \tag{3.7}$$

for $0 \le t < s \le T$ and $X \in L^\infty(\mathcal{F}_s)$.

Proof. It can easily be checked that for all $t = 0, \ldots, T-1$,

$$F_t(X) := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \left\{ \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t(\mathbb{Q}) \right\}$$

defines a concave monetary utility function from $L^\infty(\mathcal{F}_{t+1})$ to $L^\infty(\mathcal{F}_t)$. Therefore, the family $(F_t)_{t=0}^{T-1}$ induces a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$. It remains to show (3.4)–(3.7). To do this we define the mappings $\tilde{\phi}_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ by

$$\tilde{\phi}_t(X) := \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=1}^{T} \varphi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right], \quad X \in L^\infty(\mathcal{F}_T).$$

If $X \in L^\infty(\mathcal{F}_s)$ for $0 \le t < s \le T$, then

$$
\begin{aligned}
\tilde{\phi}_t(X) &= \operatorname*{ess\,inf}_{(\xi_1,\dots,\xi_T)\in\mathcal{D}_1\times\dots\times\mathcal{D}_T} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_T\left(X+\sum_{j=1}^{T}\varphi_{j-1}(\xi_j)\right) \mid \mathcal{F}_t\right] \\
&= \sum_{j=1}^{t} \operatorname*{ess\,inf}_{\xi_j\in\mathcal{D}_j}\varphi_{j-1}(\xi_j) \qquad (3.8) \\
&+ \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{D}_{t+1}\times\dots\times\mathcal{D}_s} \Bigg\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_s\left(X+\sum_{j=t+1}^{s}\varphi_{j-1}(\xi_j)\right) \mid \mathcal{F}_t\right] \\
&\qquad + \operatorname*{ess\,inf}_{(\xi_{s+1},\dots,\xi_T)\in\mathcal{D}_{s+1}\times\dots\times\mathcal{D}_T} \sum_{j=s+1}^{T} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_{j-1}\;\varphi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] \Bigg\} \qquad (3.9)
\end{aligned}
$$

The terms (3.8) and (3.9) are both equal to 0. For (3.8) this follows directly from condition (i) of Definition 3.3. For (3.9) we prove it by induction over T: Fix $(\xi_{t+1},\dots,\xi_s) \in \mathcal{D}_{t+1}\times\dots\times\mathcal{D}_s$. If $T = s+1$, then (3.9) equals

$$\operatorname*{ess\,inf}_{\xi_{s+1}\in\mathcal{D}_{s+1}} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_s\;\varphi_s(\xi_{s+1}) \mid \mathcal{F}_t\right]. \tag{3.10}$$

By condition (ii) of Definition 3.3, the family $\{\varphi_s(\xi_{s+1}) \mid \xi_{s+1} \in \mathcal{D}_{s+1}\}$ is directed downwards. Therefore, it follows by Beppo Levi's dominated convergence theorem that $\operatorname{ess\,inf}$ in (3.10) can be taken inside the conditional expectation. By condition (i) of Definition (3.3), this shows that (3.10) is equal to zero. Now, assume $T \ge s+2$ and

$$\operatorname*{ess\,inf}_{(\xi_{s+1},\dots,\xi_{T-1})\in\mathcal{D}_{s+1}\times\dots\times\mathcal{D}_{T-1}} \sum_{j=s+1}^{T-1} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_{j-1}\;\varphi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] = 0\,.$$

Then, to prove that (3.9) is equal to zero, it is enough to show that for fixed $(\xi_{t+1},\dots,\xi_{T-1}) \in \mathcal{D}_{t+1}\times\dots\times\mathcal{D}_{T-1}$, the term

$$\operatorname*{ess\,inf}_{\xi_T\in\mathcal{D}_T} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\cdot\dots\cdot\xi_{T-1}\;\varphi_{T-1}(\xi_T) \mid \mathcal{F}_t\right]$$

is zero. As above, this follows because φ_{T-1} satisfies condition (ii) of Definition 3.3 and therefore, the $\operatorname{ess\,inf}$ can be taken inside the conditional expectation.

Since (3.8) and (3.9) are both equal to zero, we have

$$
\begin{aligned}
\tilde{\phi}_t(X) &= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X+\sum_{j=1}^{s}\varphi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right] \\
&= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X+\sum_{j=t+1}^{T}\varphi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right] \\
&= \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X+\sum_{j=t+1}^{s}\varphi_{j-1}(\mathbb{Q}) \mid \mathcal{F}_t\right]. \qquad (3.11)
\end{aligned}
$$

Next, we show $\phi_t = \tilde{\phi}_t$ by induction over s. If $s = t+1$, then we obtain from (3.11) that

$$\phi_t(X) = F_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \varphi_t(\mathbb{Q}) \mid \mathcal{F}_t\right] = \tilde{\phi}_t(X) \quad \text{for all } X \in L^\infty(\mathcal{F}_s)\,.$$

Now, assume $s \geq t+2$ and $\phi_t(Y) = \tilde{\phi}_t(Y)$ for all $Y \in L^\infty(\mathcal{F}_{s-1})$. If $X \in L^\infty(\mathcal{F}_s)$, then $F_{s-1}(X) \in L^\infty(\mathcal{F}_{s-1})$, and we get

$$\begin{aligned}
&\phi_t(X) = \phi_t(F_{s-1}(X)) = \tilde{\phi}_t(F_{s-1}(X)) \\
= &\operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_{s-1})\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_{s-1}} \mathrm{E}_{\mathbb{P}}\left[(\xi_{t+1}\cdot\dots\cdot\xi_{s-1})\left(F_{s-1}(X) + \sum_{j=t+1}^{s-1}\varphi_{j-1}(\xi_j)\right) \mid \mathcal{F}_t\right] \\
= &\operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_{s-1})\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_{s-1}} \mathrm{E}_{\mathbb{P}}\Bigg[(\xi_{t+1}\cdot\dots\cdot\xi_{s-1}) \\
&\qquad \left(\operatorname*{ess\,inf}_{\xi_s\in\mathcal{D}_s} \mathrm{E}_{\mathbb{P}}\left[\xi_s X \mid \mathcal{F}_{s-1}\right] + \sum_{j=t+1}^{s}\varphi_{j-1}(\xi_j)\right) \mid \mathcal{F}_t\Bigg]\,.
\end{aligned} \tag{3.12}$$

By condition (ii) of Definition 3.3, the family

$$\mathrm{E}_{\mathbb{P}}\left[\xi_s X \mid \mathcal{F}_{s-1}\right] + \varphi_{s-1}(\xi_s)\,, \quad \xi_s \in \mathcal{D}_s$$

is directed downwards. Therefore, we can take the ess inf in (3.12) outside the conditional expectation and arrive at

$$\phi_t(X) = \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{D}_{t+1}\times\cdots\times\mathcal{D}_s} \mathrm{E}_{\mathbb{P}}\left[(\xi_{t+1}\cdot\dots\cdot\xi_s)\left(X + \sum_{j=t+1}^{s}\varphi_{j-1}(\xi_j)\right) \mid \mathcal{F}_t\right] = \tilde{\phi}_t(X)\,,$$

which concludes the proof. □

It can easily be checked that generators of the form (3.3) and the corresponding dynamic monetary utility functions (3.4)–(3.7) have the following continuity property:

Definition 3.5 *For $0 \leq t \leq s \leq T$, we call a mapping $I : L^\infty(\mathcal{F}_s) \to L^\infty(\mathcal{F}_t)$ continuous from above if*

$$I(X^n) \to I(X) \quad \mathbb{P}\text{-almost surely}$$

for every sequence $(X^n)_{n\geq 1}$ in $L^\infty(\mathcal{F}_s)$ that decreases $\mathbb{P}$-almost surely to $X \in L^\infty(\mathcal{F}_s)$.

We call a dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $L^\infty(\mathcal{F}_T)$ continuous from above if all ϕ_t are continuous from above.

On the other hand, every time-consistent dynamic concave monetary utility function with generators that are continuous from above has a representation of the form (3.4). This will be shown in Corollary 3.8 below. But first we need the following definition and lemma.

Definition 3.6 *For a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^{T-1}$ with generators $(F_t)_{t=0}^{T-1}$ that are continuous from above we define for all $t = 0,\dots,T-1$ and $\xi_{t+1} \in \mathcal{D}_{t+1}$,*

$$\varphi_t^{\min}(\xi_{t+1}) := \operatorname*{ess\,sup}_{X\in L^\infty(\mathcal{F}_{t+1})} \left\{F_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right]\right\}$$

and call $(\varphi_t^{\min})_{t=0}^{T-1}$ the minimal dynamic penalty function of $(\phi_t)_{t=0}^T$.

In the subsequent lemma we provide a conditional dual representation result for concave generators F_t in terms of $\varphi_t^{\min}$. Similar results are proved in [29, 3, 12, 9, 6, 31, 26]. Since our setup is slightly different, we provide a proof that is adapted to it.

Lemma 3.7 *Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$ with generators $(F_t)_{t=0}^{T-1}$ that are continuous from above. Then $(\varphi_t^{\min})_{t=0}^{T-1}$ is the smallest dynamic penalty function such that*

$$F_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{P}} \left\{ \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\min}(\mathbb{Q}) \right\} \tag{3.13}$$

for all $t = 0, \dots, T-1$ and $X \in L^\infty(\mathcal{F}_{t+1})$.

Proof. We fix $t \in \{0, \dots, T-1\}$ and introduce the sets

$$\mathcal{B}_t := \{X \in L^\infty(\mathcal{F}_{t+1}) \mid F_t(X) \ge 0\}$$

and

$$\mathcal{C}_t := \left\{ X \in L^\infty(\mathcal{F}_{t+1}) \mid \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\min}(\mathbb{Q}) \ge 0 \text{ for all } \mathbb{Q} \in \mathcal{P} \right\} .$$

It follows directly from the definition of $\varphi_t^{\min}$ that

$$\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\min}(\mathbb{Q}) \ge F_t(X)$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$ and $\mathbb{Q} \in \mathcal{P}$. This shows that $\mathcal{B}_t$ is contained in $\mathcal{C}_t$. In the following we are going to show $\mathcal{C}_t \subset \mathcal{B}_t$. Assume this is not the case. Then there exists $X^* \in \mathcal{C}_t \setminus \mathcal{B}_t$. Hence, the set $A := \{F_t(X^*) < 0\}$ has positive $\mathbb{P}$-measure and $1_A X^*$ is still in $\mathcal{C}_t \setminus \mathcal{B}_t$. The mapping

$$X \mapsto I(X) := \mathrm{E}\left[F_t(X)\right]$$

is a concave monetary utility function from $L^\infty(\mathcal{F}_{t+1})$ to $\mathbb{R}$ that is continuous from above. Hence, it can be deduced from the Krein–Šmulian theorem that

$$\mathcal{B} := \{X \in L^\infty(\mathcal{F}_{t+1}) \mid I(X) \ge 0\}$$

is $\sigma(L^\infty(\mathcal{F}_{t+1}), L^1(\mathcal{F}_{t+1}))$-closed, see for instance, the proof of Theorem 3.2 in Delbaen [10] or Remark 4.3 in Cheridito et al. [9]. Since it does not contain $1_A X^*$, it follows from the separating hyperplane theorem that there exists a $\mathbb{Q} \in \mathcal{P}$ such that

$$\mathrm{E}_{\mathbb{Q}}\left[1_A X^*\right] < \inf_{X\in\mathcal{B}} \mathrm{E}_{\mathbb{Q}}\left[X\right] \le \inf_{X\in\mathcal{B}_t} \mathrm{E}_{\mathbb{Q}}\left[X\right] . \tag{3.14}$$

Since the family $\{\mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] \mid X \in \mathcal{B}_t\}$ is directed downwards, we obtain from Beppo Levi's monotone convergence theorem that

$$\inf_{X\in\mathcal{B}_t} \mathrm{E}_{\mathbb{Q}}\left[X\right] = \mathrm{E}_{\mathbb{Q}}\left[\operatorname*{ess\,inf}_{X\in\mathcal{B}_t} \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] \right] .$$

Moreover, by the translation property of F_t, $\varphi_t^{\min}$ can be written as

$$\varphi_t^{\min}(\mathbb{Q}) = \operatorname*{ess\,sup}_{X\in\mathcal{B}_t} \mathrm{E}_{\mathbb{Q}}\left[-X \mid \mathcal{F}_t\right] .$$

Therefore, it follows from (3.14) that

$$\mathrm{E}_{\mathbb{Q}}\left[1_A X^*\right] < \mathrm{E}_{\mathbb{Q}}\left[-\varphi_t^{\min}(\mathbb{Q})\right] ,$$

and hence,

$$\mathrm{E}_{\mathbb{Q}}\left[\mathrm{E}_{\mathbb{Q}}\left[1_A X^* \mid \mathcal{F}_t\right] + \varphi_t^{\min}(\mathbb{Q})\right] < 0\,.$$

But this contradicts $1_A X^* \in \mathcal{C}_t$. Thus, we must have $\mathcal{B}_t = \mathcal{C}_t$. Note that for all $X \in L^\infty(\mathcal{F}_{t+1})$,

$$F_t(X) = \operatorname{ess\,sup}\left\{m \in L^\infty(\mathcal{F}_t) \mid X - m \in \mathcal{B}_t\right\}\,,$$

and therefore,

$$\begin{aligned}
& F_t(X) = \operatorname{ess\,sup}\left\{m \in L^\infty(\mathcal{F}_t) \mid \mathrm{E}_{\mathbb{Q}}\left[X - m \mid \mathcal{F}_t\right] + \varphi_t^{\min}(\mathbb{Q}) \geq 0 \text{ for all } \mathbb{Q} \in \mathcal{P}\right\} \\
= \quad & \operatorname{ess\,sup}\left\{m \in L^\infty(\mathcal{F}_t) \mid \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\min}(\mathbb{Q}) \geq m \text{ for all } \mathbb{Q} \in \mathcal{P}\right\} \\
= \quad & \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] + \varphi_t^{\min}(\mathbb{Q})\,.
\end{aligned}$$

Finally note that for every function φ_t from $\mathcal{D}_{t+1}$ to $\bar{L}_+(\mathcal{F}_t)$ that satisfies

$$F_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{D}_{t+1}} \left\{\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] + \varphi_t(\xi_{t+1})\right\}$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$, we have

$$F_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] \leq \varphi_t(\xi_{t+1})$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$ and $\xi_{t+1} \in \mathcal{D}_{t+1}$, and therefore, $\varphi_t^{\min} \leq \varphi_t$. □

The following corollary is an immediate consequence of Theorem 3.4 and Lemma 3.7.

Corollary 3.8 *Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function that is continuous from above. Then*

$$\begin{aligned}
\phi_t(X) \quad := \quad & \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=t+1}^{s} \varphi_{j-1}^{\min}(\mathbb{Q}) \mid \mathcal{F}_t\right] \\
= \quad & \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=t+1}^{T} \varphi_{j-1}^{\min}(\mathbb{Q}) \mid \mathcal{F}_t\right] \\
= \quad & \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=1}^{s} \varphi_{j-1}^{\min}(\mathbb{Q}) \mid \mathcal{F}_t\right] \\
= \quad & \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}} \mathrm{E}_{\mathbb{Q}}\left[X + \sum_{j=1}^{T} \varphi_{j-1}^{\min}(\mathbb{Q}) \mid \mathcal{F}_t\right]
\end{aligned}$$

for all $0 \leq t < s \leq T$ and $X \in L^\infty(\mathcal{F}_s)$.

The following theorem is related to Corollary 4.8 in [9], which shows that a relevant time-consistent monetary utility function $(\phi_t)_{t=0}^T$ is completely determined by ϕ_0.

Theorem 3.9 *Let $(\varphi_t)_{t=0}^{T-1}$ be a dynamic penalty function on $\mathcal{D}$ with corresponding time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $L^\infty(\mathcal{F}_T)$. Assume there exists an $X \in L^\infty(\mathcal{F}_T)$ and a probability measure $\mathbb{Q}_X$ equivalent to $\mathbb{P}$ such that*

$$\phi_0(X) = \mathrm{E}_{\mathbb{Q}_X}\left[X + \sum_{j=1}^{T} \varphi_{j-1}(\mathbb{Q}_X)\right]\,.$$

Then

$$\phi_t(X) = \mathrm{E}_{\mathbb{Q}_X}\left[X + \sum_{j=t+1}^{T} \varphi_{j-1}(\mathbb{Q}_X) \mid \mathcal{F}_t\right]$$

for all $t = 1, \dots, T-1$.

Proof. By Theorem 3.4, we have

$$\phi_t(X) \le \mathrm{E}_{\mathbb{Q}_X}\left[X + \sum_{j=t+1}^{T} \varphi_{j-1}(\mathbb{Q}_X) \mid \mathcal{F}_t\right]$$

for all $t = 0, \dots, T-1$. Now, assume that there exists a $t \in \{1, \dots, T-1\}$ such that

$$\phi_t(X) < \mathrm{E}_{\mathbb{Q}_X}\left[X + \sum_{j=t+1}^{T} \varphi_{j-1}(\mathbb{Q}_X) \mid \mathcal{F}_t\right]$$

on a set with positive $\mathbb{P}$-measure. Then,

$$\begin{aligned}
\phi_0(X) &= \mathrm{E}_{\mathbb{Q}_X}\left[X + \sum_{j=1}^{T} \varphi_{j-1}(\mathbb{Q}_X)\right] \\
&= \mathrm{E}_{\mathbb{Q}_X}\left[\mathrm{E}_{\mathbb{Q}_X}\left[X + \sum_{j=t+1}^{T} \varphi_{j-1}(\mathbb{Q}_X) \mid \mathcal{F}_t\right] + \sum_{j=1}^{t} \varphi_{j-1}(\mathbb{Q}_X)\right] \\
&> \mathrm{E}_{\mathbb{Q}_X}\left[\phi_t(X) + \sum_{j=1}^{t} \varphi_{j-1}(\mathbb{Q}_X)\right] \\
&\ge \operatorname*{ess\,inf}_{(\xi_1,\dots,\xi_T)\in\mathcal{D}_1\times\dots\times\mathcal{D}_T} \mathrm{E}_{\mathbb{P}}\left[\xi_1 \cdot \dots \cdot \xi_T \left(X + \sum_{j=1}^{T} \varphi_{j-1}(\xi_j)\right)\right] \\
&= \phi_0(X)\,,
\end{aligned}$$

which is absurd. □

3.3 Examples

3.3.1 Time-consistent dynamic Average-Value-at-Risk

For every $t = 0, \dots, T-1$, let α_t be an element of $L^\infty(\mathcal{F}_t)$ such that $0 < \alpha_t \le 1$. Consider the generators

$$F_t(X) := \operatorname*{ess\,inf}_{\xi_{t+1}\in\mathcal{D}_{t+1}\,,\,\xi_{t+1}\le\alpha_t^{-1}} \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\, X \mid \mathcal{F}_t\right]\,, \quad X \in L^\infty(\mathcal{F}_{t+1}).$$

Then, $-F_t$ is a conditional Average-Value-at-Risk on $L^\infty(\mathcal{F}_{t+1})$ at the level α_t; see [18] for the definition of the unconditional Average-Value-at-Risk. The minimal dynamic penalty function $(\varphi_t^{\min})_{t=0}^{T-1}$ of the induced time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^{T}$ is given by

$$\varphi_t^{\min}(\xi_{t+1}) = \operatorname{ess\,sup}_{X\in L^\infty(\mathcal{F}_{t+1})}\left\{F_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right]\right\} = \begin{cases} 0 & \text{if } \xi_{t+1} \le \alpha_t^{-1} \\ \infty & \text{else} \end{cases}.$$

Hence,

$$\varphi^{\min}(\mathbb{Q}) := \sum_{j=1}^{T} \varphi_{j-1}^{\min}(\mathbb{Q}) = \begin{cases} 0 & \text{if } \xi_j^{\mathbb{Q}} \leq \alpha_{j-1}^{-1} \text{ for all } j = 1, \dots, T \\ \infty & \text{else} \end{cases}$$

and

$$\phi_t(X) = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}} \mathrm{E}_{\mathbb{Q}} \left[X + \varphi^{\min}(\mathbb{Q}) \mid \mathcal{F}_t \right] = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{Q}} \mathrm{E}_{\mathbb{Q}} \left[X \mid \mathcal{F}_t \right] ,$$

where

$$\mathcal{Q} := \left\{ \mathbb{Q} \in \mathcal{P} \mid \xi_j^{\mathbb{Q}} \leq \alpha_{j-1}^{-1} \text{ for all } j = 1, \dots, T \right\} .$$

$(\rho_t)_{t=0}^T = (-\phi_t)_{t=0}^T$ is a time-consistent dynamic Average-Value-at-Risk at the dynamic level $(\alpha_0, \dots, \alpha_{T-1})$.

3.3.2 Time-consistent dynamic entropic risk measure

For all $t = 0, \dots, T-1$, let $\alpha_t \in L^\infty(\mathcal{F}_t)$ with $\alpha_t > 0$ and define F_t by

$$F_t(X) = -\alpha_t^{-1} \log \mathrm{E}_{\mathbb{P}} \left[\exp(-\alpha_t X) \mid \mathcal{F}_t \right] , \quad X \in L^\infty(\mathcal{F}_{t+1}) .$$

Then, $-F_t$ is a conditional entropic risk measure on $L(\mathcal{F}_{t+1})$ with risk aversion parameter α_t; see [17, 18, 4, 5, 12, 9]. It is well known that the minimal dynamic penalty function $(\varphi_t^{\min})_{t=0}^{T-1}$ of the induced time-consistent concave monetary utility function $(\phi_t)_{t=0}^T$ is given by

$$\varphi_t^{\min}(\xi_{t+1}) = \alpha_t^{-1} \mathrm{E}_{\mathbb{P}} \left[\xi_{t+1} \, \log(\xi_{t+1}) \mid \mathcal{F}_t \right]$$

Hence,

$$\varphi^{\min}(\mathbb{Q}) := \sum_{j=1}^{T} \varphi_{j-1}^{\min}(\mathbb{Q}) = \sum_{j=1}^{T} \mathrm{E}_{\mathbb{Q}} \left[\log \left(\left(\xi_{t+1}^{\mathbb{Q}} \right)^{\alpha_t^{-1}} \right) \mid \mathcal{F}_t \right]$$

and

$$\phi_t(X) \quad = \quad \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{P}} \mathrm{E}_{\mathbb{Q}} \left[X + \varphi^{\min}(\mathbb{Q}) \mid \mathcal{F}_t \right] , \quad X \in L^\infty(\mathcal{F}_T) , \quad t = 0, \dots, T .$$

3.3.3 Dynamic variational preferences

Recall that $\mathcal{R}^\infty$ denotes the space of all essentially bounded adapted processes $(X_t)_{t=0}^T$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. We here understand X_t as a cashflow at time t. Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$ and $\beta > 0$ a depreciation factor. The transform

$$W_t(.) := \beta^{-t} \phi_t(\beta^t .) , \quad t = 0, \dots, T ,$$

is still a dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$. It satisfies the β-modified time-consistency condition

$$W_{t+1}(X) \geq W_{t+1}(Y) \quad \text{implies} \quad W_t(\beta X) \geq W_t(\beta Y) ,$$

or equivalently, the β-modified dynamic programming principle

$$W_t(X) = W_t(\beta W_{t+1}(\beta^{-1} X)) \quad \text{for all } X \in L^\infty(\mathcal{F}_T) \text{ and } t = 0, \dots, T-1 .$$

Now, let u be an increasing continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then, the functionals

$$V_t(X) := W_t\left(\sum_{j=t}^{T} \beta^{j-t} u(X_j)\right), \quad X \in \mathcal{R}^\infty\,,\; t = 0, \dots, T\,,$$

satisfy the recursive relation

$$V_t(X) = u(X_t) + W_t(\beta\, V_{t+1}(X))\,, \quad t = 0, \dots, T-1\,.$$

This class of dynamic preferences is axiomatized in [27], where they are called dynamic variational preferences.

4 Dynamic monetary risk measures for stochastic processes

In this section the risky objects are stochastic processes $X \in \mathcal{R}^\infty$ modelling discounted value processes or discounted cumulated cash flows; for instance, the discounted market value of a portfolio, the discounted equity value of a firm or the discounted surplus of an insurance company. This interpretation of $X \in \mathcal{R}^\infty$ is the same as in [3, 7, 8, 9, 23] but different from the one in Subsection 3.3.3 above or the one in [21], where X is understood as a sequence of cash flows. As before, we are interested in monetary risk measures ρ_t but find it more convenient to work with the corresponding monetary utility functions $\phi_t = -\rho_t$. In the following, we generalize the definitions of Section 3 to this more general setup. For $0 \le t \le s \le T$, we define the projection $\pi_{t,s} : \mathcal{R}^\infty \to \mathcal{R}^\infty$ by

$$\pi_{t,s}(X)_r := 1_{\{t \le r\}} X_{r \wedge s}\,, \quad r = 0, \dots, T\,.$$

and denote

$$\mathcal{R}^\infty_{t,s} := \pi_{t,s}(\mathcal{R}^\infty)\,.$$

Definition 4.1 *Let $t \in \{0, \dots, T\}$. A monetary utility function on $\mathcal{R}^\infty_{t,T}$ is a mapping $\phi_t : \mathcal{R}^\infty_{t,T} \to L^\infty(\mathcal{F}_t)$ with the following properties:*

(N) **Normalization:** $\phi_t(0) = 0$

(M) **Monotonicity:** $\phi_t(X) \ge \phi_t(Y)$ *for all* $X, Y \in \mathcal{R}^\infty_{t,T}$ *such that* $X \ge Y$

(T) **Translation property:** $\phi_t(X + m1_{[t,T]}) = \phi_t(X) + m$ *for all* $X \in \mathcal{R}^\infty_{t,T}$ *and* $m \in L^\infty(\mathcal{F}_t)$

We call ϕ_t $\mathcal{F}_t$-concave if it satisfies

(C) **$\mathcal{F}_t$-concavity:** $\phi_t(\lambda X + (1-\lambda)Y) \ge \lambda\phi_t(X) + (1-\lambda)\phi_t(Y)$ *for all* $X, Y \in \mathcal{R}^\infty_{t,T}$ *and* $\lambda \in L^\infty(\mathcal{F}_t)$ *such that* $0 \le \lambda \le 1$

For $X \in \mathcal{R}^\infty$ we set

$$\phi_t(X) := \phi_t \circ \pi_{t,T}(X)\,.$$

A dynamic monetary utility function on $\mathcal{R}^\infty$ is a family $(\phi_t)_{t=0}^T$ such that each ϕ_t is a monetary utility function on $\mathcal{R}^\infty_{t,T}$. If all ϕ_t satisfy (C)*, then we call $(\phi_t)_{t=0}^T$ a dynamic concave monetary utility function on $\mathcal{R}^\infty$.*

As in the case of risk measures for random variables, it can be deduced from (M) and (T) that ϕ_t satisfies the

(LP) **Local property:** $1_A\phi_t(X) = 1_A\phi_t(1_A X)$ for all $X \in \mathcal{R}^\infty$ and $A \in \mathcal{F}_t$

Definition 4.2 *We call a dynamic monetary utility function* $(\phi_t)_{t=0}^T$ *on* $\mathcal{R}^\infty$ *time-consistent if for all* $X, Y \in \mathcal{R}^\infty$ *and* $t = 0, \dots, T-1$,

$$X_t = Y_t \quad \textit{and} \quad \phi_{t+1}(X) \geq \phi_{t+1}(Y)$$

implies

$$\phi_t(X) \geq \phi_t(Y) .$$

It can easily be deduced from (N), (M) and (T) that time-consistency of a dynamic monetary risk measure on $\mathcal{R}^\infty$ is equivalent to the following dynamic programming principle:

$$\phi_t(X) = \phi_t(X_t 1_{\{t\}} + \phi_{t+1}(X) 1_{[t+1,T]}) \quad \text{for all } X \in \mathcal{R}^\infty \text{ and } t = 0, \dots, T-1 . \tag{4.15}$$

4.1 Aggregators and generators

For a time time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$, we define the aggregators

$$G_t : L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t) , \quad t = 0, \dots, T-1$$

by

$$G_t(X_t, X_{t+1}) := \phi_t(X) ,$$

where X is the process in $\mathcal{R}^\infty_{t,t+1}$ given by

$$X_r := \begin{cases} 0 & \text{for } r < t \\ X_t & \text{for } r = t \\ X_{t+1} & \text{for } r \geq t+1 \end{cases} .$$

Clearly, G_t has the following three properties:

(G1) $G_t(0,0) = 0$

(G2) $G_t(X_t, X_{t+1}) \geq G_t(Y_t, Y_{t+1})$ if $X_t \geq Y_t$ and $X_{t+1} \geq Y_{t+1}$

(G3) $G_t(X_t + m, X_{t+1} + m) = G_t(X_t, X_{t+1}) + m$ for all $m \in L(\mathcal{F}_t)$,

and it can be seen from (4.15) that the whole dynamic functional $(\phi_t)_{t=0}^T$ is uniquely determined by the aggregators $(G_t)_{t=0}^{T-1}$. In fact, every sequence of functionals $(G_t)_{t=0}^{T-1}$ satisfying (G1)–(G3) defines a time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ by

$$\begin{aligned} \phi_T(X) &= X \\ \phi_t(X) &= G_t(X_t, \phi_{t+1}(X)) , \quad t \leq T-1 . \end{aligned}$$

(Notice the formal similarities to the aggregators in the theory of recursive utilities for consumption streams, see for instance, [24] and [15].)

It is clear that $(\phi_t)_{t=0}^T$ is concave if and only if all G_t satisfy

(G4) $G_t(\lambda X_t + (1-\lambda) Y_t, \lambda X_{t+1} + (1-\lambda) Y_{t+1}) \geq \lambda G_t(X_t, X_{t+1}) + (1-\lambda) G_t(Y_t, Y_{t+1})$
for all $X_t, Y_t \in L^\infty(\mathcal{F}_t)$, $X_{t+1}, Y_{t+1} \in L^\infty(\mathcal{F}_{t+1})$ and $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$.

By (G3), we can write G_t as

$$G_t(X_t, X_{t+1}) = X_t + G_t(0, X_{t+1} - X_t) = X_t + H_t(X_{t+1} - X_t) , \tag{4.16}$$

for the mapping $H_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ given by

$$H_t(X) := G_t(0, X) .$$

It follows from (G1)–(G3) that H_t has the following three properties:

(H1) $H_t(0) = 0$
(H2) $H_t(X) \geq H_t(Y)$ for $X, Y \in L^\infty(\mathcal{F}_{t+1})$ with $X \geq Y$
(H3) $H_t(X + m) \leq H_t(X) + m$ for all $X \in L^\infty(\mathcal{F}_{t+1})$ and $m \in L^\infty_+(\mathcal{F}_t)$

(H1) and (H2) are clear, and (H3) holds because for $X \in L^\infty(\mathcal{F}_{t+1})$ and $m \in L^\infty_+(\mathcal{F}_t)$,

$$H_t(X + m) = G_t(0, X + m) = m + G_t(-m, X) \leq m + G_t(0, X) = m + H_t(X)\,.$$

On the other hand, every sequence $(H_t)_{t=0}^{T-1}$ of mappings satisfying (H1)–(H3) induces aggregators $(G_t)_{t=0}^{T-1}$ of a time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$. Indeed, if H_t satisfies (H1)–(H3) and G_t is given by

$$G_t(X_t, X_{t+1}) = X_t + H_t(X_{t+1} - X_t)\,,$$

then it clearly satisfies (G1) and (G3). (G2) follows from (H2) and (H3) because for $X_t, Y_t \in L^\infty(\mathcal{F}_t)$ such that $X_t \geq Y_t$ and $X_{t+1}, Y_{t+1} \in L^\infty(\mathcal{F}_{t+1})$ such that $X_{t+1} \geq Y_{t+1}$, we get from (H2) and (H3) that

$$\begin{aligned} & G_t(X_t, X_{t+1}) = X_t + H_t(X_{t+1} - X_t) \geq X_t + H_t(Y_{t+1} - X_t) \\ \geq \quad & X_t + H_t(Y_{t+1} - Y_t) - (X_t - Y_t) = Y_t + H_t(Y_{t+1} - Y_t)\,. \end{aligned}$$

We call $(H_t)_{t=0}^{T-1}$ the generators of $(\phi_t)_{t=0}^T$. It is clear that G_t satisfies (G4) if and only if H_t fulfils

(H4) $H_t(\lambda X + (1-\lambda)Y) \geq \lambda H_t(X) + (1-\lambda) H_t(Y)$ for all $\lambda \in L^\infty(\mathcal{F}_t)$ such that $0 \leq \lambda \leq 1$.

4.2 Duality

For $t = 1, \ldots, T$, we define the set

$$\mathcal{E}_t := \left\{ \xi \in L^1_+(\mathcal{F}_t) \mid \mathrm{E}_{\mathbb{P}}\left[\xi \mid \mathcal{F}_{t-1}\right] \leq 1 \right\}\,.$$

Every sequence $(\xi_{t+1}, \ldots, \xi_T) \in \mathcal{E}_{t+1} \times \cdots \times \mathcal{E}_T$ induces a $\mathbb{P}$-supermartingale $(M^\xi_r)_{r=0}^T$ by

$$M^\xi_r := \begin{cases} 1 & \text{for } r \leq t \\ \xi_{t+1} \cdot \cdots \cdot \xi_r & \text{for } r = t+1, \ldots, T \end{cases}$$

Definition 4.3 *A one-step penalty function on $\mathcal{E}_{t+1}$ is a mapping*

$$\psi_t : \mathcal{E}_{t+1} \to \bar{L}_+(\mathcal{F}_t)$$

that satisfies the two properties:

(i) $\operatorname{ess\,inf}_{\xi \in \mathcal{E}_{t+1}} \psi_t(\xi) = 0$

(ii) $\psi_t(1_A \xi + 1_{A^c} \xi') = 1_A \psi_t(\xi) + 1_{A^c} \psi_t(\xi')$ *for all $\xi, \xi' \in \mathcal{E}_{t+1}$ and $A \in \mathcal{F}_t$*

A dynamic penalty function on $\mathcal{E}$ is a sequence $(\psi_t)_{t=0}^{T-1}$ of one-step penalty functions.

Theorem 4.4 *Let $(\psi_t)_{t=0}^{T-1}$ be a dynamic penalty function on $\mathcal{E}$. Then*

$$H_t(X) := \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] + \psi_t(\xi_{t+1}) \right\}\,, \quad t = 0, \ldots, T-1\,, \tag{4.17}$$

defines generators of a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ that has the following representations:

$$\phi_t(X) = X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{E}_{t+1}\times\dots\times\mathcal{E}_s} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{s} M_j^{\xi}\Delta X_j + M_{j-1}^{\xi}\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] \tag{4.18}$$

$$= X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_T)\in\mathcal{E}_{t+1}\times\dots\times\mathcal{E}_T} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{T} M_j^{\xi}\Delta X_j + M_{j-1}^{\xi}\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] \tag{4.19}$$

for all $0 \le t < s \le T$ and $X \in \mathcal{R}^\infty_{t,s}$, where we used the notation $\Delta X_j := X_j - X_{j-1}$.

Proof. It can easily be checked that for every $t = 0, \dots, T-1$,

$$H_t(X) := \operatorname*{ess\,inf}_{\xi_{t+1}\in\mathcal{E}_{t+1}} \left\{\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right] + \psi_t(\xi_{t+1})\right\}$$

defines an $\mathcal{F}_t$-concave mapping from $L^\infty(\mathcal{F}_{t+1})$ to $L^\infty(\mathcal{F}_t)$ that satisfies (H1)–(H3). Therefore, the family $(H_t)_{t=0}^{T-1}$ induces a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$. To show (4.18) and (4.19), we define the mappings $\tilde{\phi}_t : \mathcal{R}^\infty_{t,T} \to L^\infty(\mathcal{F}_t)$ by

$$\tilde{\phi}_t(X) := X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_T)\in\mathcal{E}_{t+1}\times\dots\times\mathcal{E}_T} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{T} M_j^{\xi}\,\Delta X_j + M_{j-1}^{\xi}\,\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right]. \tag{4.20}$$

Exactly as in the proof of Theorem 3.4, it can be deduced from conditions (i) and (ii) of Definition 4.3 that

$$\tilde{\phi}_t(X) = X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{E}_{t+1}\times\dots\times\mathcal{E}_s} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{s} M_j^{\xi}\,\Delta X_j + M_{j-1}^{\xi}\,\psi_{j-1}(\xi_j) \mid \mathcal{F}_t\right] \tag{4.21}$$

for all $X \in \mathcal{R}^\infty_{t,s}$. Now, we can show $\phi_t = \tilde{\phi}_t$ by induction over s. First, assume that $X \in \mathcal{R}^\infty_{t,t+1}$. Then, by (4.21),

$$\begin{aligned} \phi_t(X) = G_t(X_t, \phi_{t+1}(X)) = G_t(X_t, X_{t+1}) = X_t + H_t(\Delta X_{t+1}) \\ = \quad X_t + \operatorname*{ess\,inf}_{\xi_{t+1}\in\mathcal{E}_{t+1}} \left\{\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\Delta X_{t+1} \mid \mathcal{F}_t\right] + \psi_t(\xi_{t+1})\right\} = \tilde{\phi}_t(X)\,. \end{aligned}$$

Now, assume $X \in \mathcal{R}^\infty_{t,s}$ for $s \ge t+2$ and $\phi_t(Y) = \tilde{\phi}_t(Y)$ for all $Y \in \mathcal{R}^\infty_{t,s-1}$. By (4.15), we have $\phi_t(X) = \phi_t(Y)$ for

$$Y = 1_{[t,s-1)}X + 1_{[s-1,T]}\phi_{s-1}(X) \in \mathcal{R}^\infty_{t,s-1}\,,$$

and therefore,

$$
\begin{aligned}
& \phi_t(X) = \phi_t(Y) = \tilde{\phi}_t(Y) \\
= \; & \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_{s-1}) \in \mathcal{E}_{t+1} \times \dots \times \mathcal{E}_{s-1}} \mathrm{E}_{\mathbb{P}} \left[\sum_{j=t+1}^{s-1} M_j^{\xi} \Delta Y_j + M_{j-1}^{\xi} \psi_{j-1}(\xi_j) \mid \mathcal{F}_t \right] \\
= \; & \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_{s-1}) \in \mathcal{E}_{t+1} \times \dots \times \mathcal{E}_{s-1}} \mathrm{E}_{\mathbb{P}} \left[\sum_{j=t+1}^{s-2} M_j^{\xi} \Delta X_j + M_{s-1}^{\xi} [\phi_{s-1}(X) - X_{s-2}] \right. \\
& \left. \qquad + \sum_{j=t+1}^{s-1} M_{j-1}^{\xi} \psi_{j-1}(\xi_j) \mid \mathcal{F}_t \right] \\
= \; & \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_{s-1}) \in \mathcal{E}_{t+1} \times \dots \times \mathcal{E}_{s-1}} \mathrm{E}_{\mathbb{P}} \left[\sum_{j=t+1}^{s-2} M_j^{\xi} \Delta X_j \right. \\
& \qquad + M_{s-1}^{\xi} [\Delta X_{s-1} + \operatorname*{ess\,inf}_{\xi_s \in \mathcal{E}_s} \{ \mathrm{E}_{\mathbb{P}} [\xi_s \Delta X_s \mid \mathcal{F}_{s-1}] + \psi_{s-1}(\xi_s) \}] \qquad (4.22) \\
& \left. \qquad + \sum_{j=t+1}^{s-1} M_{j-1}^{\xi} \psi_{j-1}(\xi_j) \mid \mathcal{F}_t \right] .
\end{aligned}
$$

where for $s = t+2$, the term $\sum_{j=t+1}^{s-2} M_j^{\xi} \Delta X_j$ is understood as 0. By condition (ii) of Definition 4.3, the family

$$\mathrm{E}_{\mathbb{P}} [\xi_s \Delta X_s \mid \mathcal{F}_{s-1}] + \psi_{s-1}(\xi_s) \, , \quad \xi_s \in \mathcal{E}_s$$

is directed downwards. Therefore, we can take the ess inf in (4.22) outside of the conditional expectation $\mathrm{E}_{\mathbb{P}} [. \mid \mathcal{F}_t]$ and arrive at

$$\phi_t(X) = \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s) \in \mathcal{E}_{t+1} \times \dots \times \mathcal{E}_s} \mathrm{E}_{\mathbb{P}} \left[\sum_{j=t+1}^{s} M_j^{\xi} \Delta X_j + M_{j-1}^{\xi} \psi_{j-1}(\xi_j) \right] = \tilde{\phi}_t(X) \, .$$

□

It can easily be checked that aggregators of the form (4.17) are continuous from above in the sense of Definition 3.5 and utility functions of the form (4.18)–(4.19) are continuous from above in the following more general sense:

Definition 4.5 *Let $0 \le t \le s \le T$. We call a mapping $I : \mathcal{R}^{\infty}_{t,s} \to L^{\infty}(\mathcal{F}_t)$ continuous from above if $I(X^n) \to I(X)$ $\mathbb{P}$-almost surely for all $(X^n)_{n \ge 1}$ and X in $\mathcal{R}^{\infty}_{t,T}$ such that X^n_r decreases to X_r $\mathbb{P}$-almost surely for all $r = t, \dots, s$.*

We call a dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^{\infty}$ continuous from above if every ϕ_t is continuous from above.

Lemma 4.6 *A time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ is continuous from above if and only if all of the corresponding aggregators $(G_t)_{t=0}^{T-1}$ are continuous from above, which is the case if and only if all of the associated generators $(H_t)_{t=0}^{T-1}$ are continuous from above.*

Proof. It is obvious that $(\phi_t)_{t=0}^T$ is continuous from above if and only if all aggregators $(G_t)_{t=0}^{T-1}$ are continuous from above. Now, fix t and assume that G_t is continuous from above. If $(X^n)_{n\geq 1}$ and X are in $L^\infty(\mathcal{F}_{t+1})$ such that $(X^n)_{n\geq 1}$ decreases to X $\mathbb{P}$-almost surely, then we have

$$H_t(X^n) = G_t(0, X^n) \searrow G_t(0, X) = H_t(X) \quad \mathbb{P}-\text{almost surely.}$$

Hence, H_t is continuous from above. On the other hand, if we assume that H_t is continuous from above, $(X_t^n)_{n\geq 1}$ and X are in $L^\infty(\mathcal{F}_t)$ such that $(X^n)_{n\geq 1}$ decreases to X $\mathbb{P}$-almost surely and $(X_{t+1}^n)_{n\geq 1}$ and X_{t+1} are in $L^\infty(\mathcal{F}_{t+1})$ such that $(X^n)_{n\geq 1}$ decreases to X $\mathbb{P}$-almost surely, then

$$G_t(X_t^n, X_{t+1}^n) = X_t^n + H_t(X_{t+1}^n - X_t^n) \leq X_t^n + H_t(X_{t+1}^n - X_t) \searrow X_t + H_t(X_{t+1} - X_t) = G_t(X_t, X_{t+1})$$

$\mathbb{P}$-almost surely. This shows that G_t is continuous from above. □

Definition 4.7 *For a time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ with generators $(H_t)_{t=0}^{T-1}$ that are continuous from above, we define for $\xi_{t+1} \in \mathcal{E}_{t+1}$ and $t = 0, \dots, T-1$,*

$$\psi_t^{\min}(\xi_{t+1}) := \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F}_{t+1})} \left\{ H_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] \right\}$$

and call $(\psi_t^{\min})_{t=0}^{T-1}$ the minimal dynamic penalty function of $(\phi_t)_{t=0}^T$.

Lemma 4.8 *Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function on $\mathcal{R}^\infty$ with generators $(H_t)_{t=0}^{T-1}$ that are continuous from above. Then $(\psi_t^{\min})_{t=0}^{T-1}$ is the smallest dynamic penalty function on $\mathcal{E}$ such that*

$$H_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} X \mid \mathcal{F}_t\right] + \psi_t^{\min}(\xi_{t+1}) \right\} \tag{4.23}$$

for all $t = 0, \dots, T-1$ and $X \in L^\infty(\mathcal{F}_{t+1})$.

Proof. Fix $t \in \{0, \dots, T-1\}$ and consider $\hat{\Omega} := \{t, t+1\} \times \Omega$ with the σ-algebra $\hat{\mathcal{F}}_{t+1}$ generated by all sets of the form $\{j\} \times A_j$ for $j = t, t+1$ and $A_j \in \mathcal{F}_j$. Let $\hat{\mathbb{P}}$ be the probability measure on $(\hat{\Omega}, \hat{\mathcal{F}}_{t+1})$ given by $\hat{\mathbb{P}}[\{j\} \times A_j] := \frac{1}{2}\mathbb{P}[A_j]$ for $j = t, t+1$ and $A_j \in \mathcal{F}_j$. By $\hat{\mathcal{F}}_t$ we denote the σ-algebra on $\hat{\Omega}$ generated by all sets of the form $\{t, t+1\} \times A_t$ for $A_t \in \mathcal{F}_t$. Then we have

$$L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) = L^\infty(\hat{\Omega}, \hat{\mathcal{F}}_{t+1}, \hat{\mathbb{P}}),$$

and the aggregator G_t can be viewed as a concave monetary utility function from $L^\infty(\hat{\Omega}, \hat{\mathcal{F}}_{t+1}, \hat{\mathbb{P}})$ to $L^\infty(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$. Clearly, it is continuous from above. Therefore, it follows from Lemma 3.7 that

$$G_t(X_t, X_{t+1}) = \operatorname*{ess\,inf}_{(a, \xi_{t+1}) \in L^\infty_{0,1}(\mathcal{F}_t) \times \mathcal{D}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[(1-a)X_t + a\xi_{t+1} X_{t+1} \mid \mathcal{F}_t\right] + \zeta_t(a, \xi_{t+1}) \right\}, \tag{4.24}$$

where

$$L^\infty_{0,1}(\mathcal{F}_t) := \{a \in L^\infty(\mathcal{F}_t) \mid 0 \leq a \leq 1\}$$

and

$$\begin{aligned}
\zeta_t(a, \xi_{t+1}) &:= \operatorname*{ess\,sup}_{(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1})} \left\{ G_t(X_t, X_{t+1}) - \mathrm{E}_{\mathbb{P}}\left[(1-a)X_t + a\,\xi_{t+1} X_{t+1} \mid \mathcal{F}_t\right] \right\} \\
&= \operatorname*{ess\,sup}_{(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1})} \left\{ G_t(0, \Delta X_{t+1}) - \mathrm{E}_{\mathbb{P}}\left[a\,\xi_{t+1} \Delta X_{t+1} \mid \mathcal{F}_t\right] \right\} \\
&= \operatorname*{ess\,sup}_{X \in L^\infty(\mathcal{F}_{t+1})} \left\{ H_t(X) - \mathrm{E}_{\mathbb{P}}\left[a\,\xi_{t+1} X \mid \mathcal{F}_t\right] \right\}.
\end{aligned}$$

This shows that

$$H_t(X) \quad = \quad G_t(0,X) = \operatorname*{ess\,inf}_{\xi_{t+1}\in\mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right] + \psi_t^{\min}(\xi_{t+1}) \right\}$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$. The minimality of $\psi_t^{\min}$ follows because every function $\psi_t : \mathcal{E}_{t+1} \to \bar{L}_+(\mathcal{F}_t)$ that fulfills

$$H_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1}\in\mathcal{E}_{t+1}} \left\{ \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right] + \psi_t(\xi_{t+1}) \right\} \quad \text{for all } X \in L^\infty(\mathcal{F}_{t+1})$$

must also satisfy

$$H_t(X) - \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}X \mid \mathcal{F}_t\right] \le \psi_t(\xi_{t+1}) \quad \text{for all } X \in L^\infty(\mathcal{F}_{t+1}) \text{ and } \xi_{t+1} \in \mathcal{E}_{t+1}\,,$$

and therefore, $\psi_t^{\min} \le \psi_t$. □

The following corollary is an immediate consequence of Theorem 4.4 and Lemma 4.8.

Corollary 4.9 *Let $(\phi_t)_{t=0}^T$ be a time-consistent dynamic concave monetary utility function that is continuous from above. Then*

$$\begin{aligned}
\phi_t(X) \quad &= \quad X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_s)\in\mathcal{E}_{t+1}\times\cdots\times\mathcal{E}_s} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{s} M_j^\xi \Delta X_j + M_{j-1}^\xi \psi_{j-1}^{\min}(\xi_j) \mid \mathcal{F}_t\right] \\
&= \quad X_t + \operatorname*{ess\,inf}_{(\xi_{t+1},\dots,\xi_T)\in\mathcal{E}_{t+1}\times\cdots\times\mathcal{E}_T} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{T} M_j^\xi \Delta X_j + M_{j-1}^\xi \psi_{j-1}^{\min}(\xi_j) \mid \mathcal{F}_t\right]
\end{aligned}$$

for all $0 \le t < s \le T$ and $X \in R_{t,s}^\infty$.

Our next result is the extension of Theorem 3.9 for risk measures that depend on stochastic processes.

Theorem 4.10 *Let $(\psi_t)_{t=0}^{T-1}$ be a dynamic penalty function on $\mathcal{E}$ with corresponding time-consistent dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$. Assume there exists an $X \in \mathcal{R}^\infty$ and an element $(\xi_1^X,\dots,\xi_T^X)$ in $\mathcal{E}_1\times\cdots\times\mathcal{E}_T$ with $M_T^{\xi^X} > 0$ such that*

$$\phi_0(X) = X_0 + \mathrm{E}_{\mathbb{P}}\left[\sum_{j=1}^{T} M_j^{\xi^X} \Delta X_j + M_{j-1}^{\xi^X} \psi_{j-1}(\xi_j^X)\right].$$

Then

$$\phi_t(X) = X_t + \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{T} M_j^{\xi^X} \Delta X_j + M_{j-1}^{\xi^X} \psi_{j-1}(\xi_j^X) \mid \mathcal{F}_t\right]$$

for all $t = 1,\dots,T-1$.

Proof. By Theorem 4.4, we have

$$\phi_t(X) \le X_t + \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{T} M_j^{\xi^X} \Delta X_j + M_{j-1}^{\xi^X} \psi_{j-1}(\xi_j^X) \mid \mathcal{F}_t\right]$$

for all $t = 0, \dots, T-1$. If there exists a $t \in \{1, \dots, T-1\}$ such that

$$\phi_t(X) < X_t + \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{T} M_j^{\xi^X} \Delta X_j + M_{j-1}^{\xi^X} \psi_{j-1}(\xi_j^X) \mid \mathcal{F}_t\right]$$

on a set with positive $\mathbb{P}$-measure. Then,

$$\begin{aligned}
&\phi_0(X) = X_0 + \mathrm{E}_{\mathbb{P}}\left[\sum_{j=1}^{T} M_j^{\xi^X} \Delta X_j + M_{j-1}^{\xi^X} \psi_{j-1}(\xi_j^X)\right] \\
= \quad & X_0 + \mathrm{E}_{\mathbb{P}}\left[\sum_{j=1}^{t} M_j^{\xi^X} \Delta X_j + M_{j-1}^{\xi^X} \psi_{j-1}(\xi_j^X) + \mathrm{E}_{\mathbb{P}}\left[\sum_{j=t+1}^{T} M_j^{\xi^X} \Delta X_j + M_{j-1}^{\xi^X} \psi_{j-1}(\xi_j^X) \mid \mathcal{F}_t\right]\right] \\
> \quad & X_0 + \mathrm{E}_{\mathbb{P}}\left[\sum_{j=1}^{t} M_j^{\xi^X} \Delta X_j + M_{j-1}^{\xi^X} \psi_{j-1}(\xi_j^X) + \phi_t(X) - X_t\right] \\
\geq \quad & X_0 + \operatorname*{ess\,inf}_{(\xi_1,\dots,\xi_T) \in \mathcal{E}_1 \times \cdots \times \mathcal{E}_T} \mathrm{E}_{\mathbb{P}}\left[\sum_{j=1}^{T} M_j^{\xi} \Delta X_j + M_{j-1}^{\xi} \psi_{j-1}(\xi_j)\right] \\
= \quad & \phi_0(X)\,,
\end{aligned}$$

which is a contradiction. □

4.3 Composed generators

In this section we study generators of the composed form

$$H_t(X) = h_t(F_t(X))\,, \tag{4.25}$$

where $F_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ is a monetary utility function from $L^\infty(\mathcal{F}_{t+1})$ to $L^\infty(\mathcal{F}_t)$ and h_t a function from $\mathbb{R}$ to $\mathbb{R}$ satisfying

(h1) $h_t(0) = 0$
(h2) h_t is increasing
(h3) $|h_t(x) - h_t(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Then $H_t = h_t \circ F_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ satisfies the properties (H1), (H2) and (H3). Hence, $(h_t, F_t)_{t=0}^{T-1}$ induces a time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$. If in addition to (h1)–(h3), h_t is concave and F_t $\mathcal{F}_t$-concave, then H_t satisfies (H4), and $(\phi_t)_{t=0}^T$ is a time-consistent dynamic concave monetary utility function on $\mathcal{R}^\infty$.

By standard convex duality, every concave function $h : \mathbb{R} \to \mathbb{R}$ can be represented as

$$h(x) = \min_{y \in \mathbb{R}} \{xy + h^*(y)\}\ ,$$

where h^* is the concave conjugate given by

$$h^*(y) := \sup_{x \in \mathbb{R}} \{xy - h(x)\}\ .$$

As a consequence, we have the following

Proposition 4.11 *If $F_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ is given by*

$$F_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{D}_{t+1}} \{\mathrm{E}_{\mathbb{P}}[\xi_{t+1} X \mid \mathcal{F}_t] + \varphi_t(\xi_{t+1})\}$$

for a one-step penalty function φ_t on $\mathcal{D}_{t+1}$ and h_t is a concave function from $\mathbb{R}$ to $\mathbb{R}$ satisfying (h1)–(h3)*. Then $H_t = h_t \circ F_t$ can be represented as*

$$H_t(X) = \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \{\mathrm{E}_{\mathbb{P}}[\xi_{t+1} X \mid \mathcal{F}_t] + \psi_t(\xi_{t+1})\}, \quad X \in L^\infty(\mathcal{F}_{t+1}),$$

for the one-step penalty function ψ_t on $\mathcal{E}_{t+1}$ given by

$$\psi_t(\xi_{t+1}) = \mathrm{E}_{\mathbb{P}}[\xi_{t+1} \mid \mathcal{F}_t]\, \varphi_t\left(\frac{\xi_{t+1}}{\mathrm{E}_{\mathbb{P}}[\xi_{t+1} \mid \mathcal{F}_t]}\right) + h_t^*(\mathrm{E}_{\mathbb{P}}[\xi_{t+1} \mid \mathcal{F}_t]).$$

Proof. Since h_t satisfies (h2) and (h3), we have $h_t^*(y) = \infty$ for all $y \notin [0,1]$, and therefore

$$\begin{aligned} & H_t(X) = h_t(F_t(X)) \\ = \; & h_t\left(\operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{D}_{t+1}} \{E_{\mathbb{P}}[\xi_{t+1} X \mid \mathcal{F}_t] + \varphi_t(\xi_{t+1})\}\right) \\ = \; & \inf_{0 \le y \le 1} \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{D}_{t+1}} \left\{ y \{E_{\mathbb{P}}[\xi_{t+1} X \mid \mathcal{F}_t] + \varphi_t(\xi_{t+1})\} + h_t^*(y) \right\} \\ = \; & \operatorname*{ess\,inf}_{\xi_{t+1} \in \mathcal{E}_{t+1}} \left\{ E_{\mathbb{P}}[\xi_{t+1} X \mid \mathcal{F}_t] + \lambda \varphi_t\left(\frac{\xi_{t+1}}{\lambda}\right) + h_t^*(\lambda) \right\} \end{aligned}$$

for $\lambda := E_{\mathbb{P}}[\xi_{t+1} \mid \mathcal{F}_t]$. □

In the following we are going to discuss different specifications of the function h_t. This leads to extensions of some of the examples of Section 5 of Cheridito et al. [9].

4.3.1 Risk measures that depend only on the final value

If $h_t(x) = x$, then the aggregators reduce to

$$G_t(X_t, X_{t+1}) = X_t + h_t(F_t(X_{t+1} - X_t)) = F_t(X_{t+1})$$

and the corresponding $(\phi_t)_{t=0}^T$ is a time-consistent dynamic utility function on $L^\infty(\mathcal{F}_T)$ as in Section 3.

4.3.2 Risk measures that depend on a weighted average over time

If $h_t(x) = \gamma_t x$ for a constant $\gamma_t \in (0,1)$, then

$$G_t(X_t, X_{t+1}) = X_t + h_t(F_t(X_{t+1} - X_t)) = (1 - \gamma_t) X_t + \gamma_t F_t(X_{t+1}).$$

Consider the transformed generators $\tilde{F}_t(.) := \gamma_t F_t(\gamma_t^{-1}.)$ and the corresponding time-consistent dynamic monetary utility function on $L^\infty(\mathcal{F}_T)$ given by

$$\tilde{\phi}_t = \tilde{F}_t \circ \cdots \circ \tilde{F}_{T-1}.$$

Then it can easily be checked that the time-consistent dynamic monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ induced by the aggregators $(G_t)_{t=0}^T$ is given by

$$\phi_t(X) = \tilde{\phi}_t\left(\sum_{j=t}^T \delta_j^t X_j\right), \quad t = 0, \ldots T-1,$$

where

$$\delta_j^t = \begin{cases} 1-\gamma_t & \text{for } j=t \\ \gamma_t \cdot \dots \cdot \gamma_{j-1}(1-\gamma_j) & \text{for } t<j<T \\ \gamma_t \cdot \dots \cdot \gamma_{T-1} & \text{for } j=T \end{cases} .$$

In particular, for $\gamma_t = \frac{T-t}{T-t+1}$, $t=0,\dots,T-1$, we get

$$\phi_t(X) = \tilde{\phi}_t\left(\frac{1}{T-t+1}\sum_{j=t}^{T} X_j\right).$$

4.3.3 Risk measures defined by worst stopping

For $h_t(x) = x \wedge 0$, the aggregators become

$$G_t(X_t, X_{t+1}) = X_t + h_t(F_t(X_{t+1} - X_t)) = X_t \wedge F_t(X_{t+1}) \tag{4.26}$$

and the ϕ_t are of the form

$$\phi_t(X) = \operatorname*{ess\,inf}_{\tau\in\Theta_t} \tilde{\phi}_t(X_\tau), \quad X \in \mathcal{R}^\infty,$$

where Θ_t is the set of all $\{t,\dots,T\}$-valued stopping times and $(\tilde{\phi}_t)_{t=0}^T$ is the time-consistent dynamic concave monetary utility function on $L^\infty(\mathcal{F}_T)$ given by

$$\tilde{\phi}_t(X) = F_t \circ \dots \circ F_{T-1}(X), \quad X \in L^\infty(\mathcal{F}_T).$$

This can directly be seen by checking that

$$\operatorname*{ess\,inf}_{\tau\in\Theta_t} \tilde{\phi}_t(X_\tau), \quad t=0,\dots,T$$

is a time-consistent dynamic monetary utility function on $\mathcal{R}^\infty$ whose aggregators are given by (4.26).

4.3.4 Trade-off functions

Instead of specifying the function h_t directly, one can start with a continuous decreasing function $g_t : \mathbb{R} \to \mathbb{R}$ such that $g_t(0) = 0$ and define the corresponding aggregator G_t by

$$G_t(X_t, X_{t+1}) := \operatorname{ess\,sup}\{m \in L^\infty(\mathcal{F}_t) \mid (X_t - m, X_{t+1} - m) \in \mathcal{B}_t\},$$

where the one-step acceptance set $\mathcal{B}_t$ is given by

$$\mathcal{B}_t = \{(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) \mid g_t(X_t) \le F_t(X_{t+1})\}.$$

The function g_t specifies the trade-off between risk at time t and $t+1$ of an acceptable process $X \in \mathcal{R}^\infty_{t,t+1}$. It can easily be checked that the inverse h_t of the strictly increasing function $g_t(-x)+x$ satisfies (h1)–(h3), and $\mathcal{B}_t$ can be written as

$$\mathcal{B}_t = \{(X_t, X_{t+1}) \in L^\infty(\mathcal{F}_t) \times L^\infty(\mathcal{F}_{t+1}) \mid X_t + h_t \circ F_t(X_{t+1} - X_t) \ge 0\}.$$

Hence, the generator H_t is given by $h_t \circ F_t$. Note h_t is concave if and only if g_t is convex. For $g_t(x) = 0$ we get $h_t(x) = x$, and we are back in the case of Subsection 4.3.1. The case $g_t(x) = (1 - 1/\gamma_t)x$ for $0 < \gamma_t < 1$ corresponds to $h_t(x) = \gamma_t x$ of Subsection 4.3.2. The function $h_t(x) = x \wedge 0$ of Subsection 4.3.3 is not bijective. Therefore, it cannot be obtained from a trade-off function g_t.

Example 4.12 Our last example is built on trade-off functions of the form $g_t(x) := \exp(-\gamma_t x) - 1$ for $\gamma_t > 0$. In this case there exists no closed form expression for h_t. But we have the following relation between the concave conjugates of $\tilde{g}_t = -g_t$ and h_t:

$$h_t^*(y) = \begin{cases} y\tilde{g}_t^*(1/y-1) & \text{for } y \in (0,1] \\ \infty & \text{for } y \notin (0,1] \end{cases} .$$

Indeed, $h_t^*(y) = \infty$ for $y \notin (0,1]$ is an immediate consequence of the fact that h_t is bijective and satisfies (h2) and (h3). For $y \in (0,1]$, we can write

$$\begin{aligned} & h_t^*(y) = \sup_{x\in\mathbb{R}} \{h_t(x) - xy\} = \sup_{x\in\mathbb{R}} \left\{h_t(h_t^{-1}(-x)) - h_t^{-1}(-x)y\right\} \\ = \;& y \cdot \sup_{x\in\mathbb{R}} \left\{-x/y - h_t^{-1}(-x)\right\} = y \cdot \sup_{x\in\mathbb{R}} \left\{(1-1/y)\,x - (h_t^{-1}(-x) + x)\right\} \\ = \;& y \cdot \sup_{x\in\mathbb{R}} \{\tilde{g}_t(x) - (1/y-1)\,x\} = y\tilde{g}_t^*(1/y-1) \; . \end{aligned}$$

For $y > 0$, the concave conjugate of $\tilde{g}_t(x) = -g_t(x) = 1 - \exp(-\gamma_t x)$ is given by

$$\tilde{g}_t^*(y) = 1 + \frac{y}{\gamma_t}\log\left(\frac{y}{\gamma_t}\right) - \frac{y}{\gamma_t} \, .$$

Hence,

$$h_t^*(y) = y + \frac{1-y}{\gamma_t}\log\left(\frac{1-y}{\gamma_t y}\right) - \frac{1-y}{\gamma_t} \, .$$

We now combine h_t with the entropic generator

$$F_t(X) = -\alpha_t^{-1}\log \mathrm{E}_{\mathbb{P}}\left[\exp(-\alpha_t X) \mid \mathcal{F}_t\right] , \quad X \in L^\infty(\mathcal{F}_{t+1})$$

with minimal penalty function

$$\varphi_t^{\min}(\xi_{t+1}) = \alpha_t^{-1}\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\log(\xi_{t+1}) \mid \mathcal{F}_t\right] ,$$

Then it follows from from Proposition 4.11 that the minimal dynamic penalty function of the dynamic concave monetary utility function $(\phi_t)_{t=0}^T$ on $\mathcal{R}^\infty$ induced by $(h_t, F_t)_{t=0}^{T-1}$ is given by

$$\psi_t(\xi_{t+1}) = \alpha_t^{-1}\mathrm{E}_{\mathbb{P}}\left[\xi_{t+1}\log(\xi_{t+1}) \mid \mathcal{F}_t\right] - \alpha_t^{-1}\lambda\log(\lambda) + \lambda + \frac{1-\lambda}{\gamma_t}\log\left(\frac{1-\lambda}{\gamma_t\lambda}\right) - \frac{1-\lambda}{\gamma_t} ,$$

for $\xi_{t+1} \in \mathcal{E}_{t+1}$ and $\lambda := \mathrm{E}_{\mathbb{P}}\left[\xi_{t+1} \mid \mathcal{F}_t\right]$.

References

[1] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D., (1997). Thinking coherently. *RISK*, **10**, November, 68–71.

[2] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D., (1999). Coherent Risk Measures. *Mathematical Finance*, **9**(3), 203–228.

[3] Artzner, Ph., Delbaen, F., Eber, J.M., Heath, D., Ku, H., (2003). Coherent multiperiod risk adjusted values and Bellman's principle. Preprint.

[4] Barrieu, P., El Karoui, N. (2004). Optimal derivative design under dynamic risk measures. in Mathematics of Finance, Contemporary Mathematics.

[5] Barrieu, P., El Karoui, N. (2005). Pricing, Hedging and Optimally Designing Derivatives via Minimization of Risk Measures, Preprint.

[6] Bion-Nadal, J. (2004). Conditional risk measures and robust represenation of convex risk measures. Ecole Polytechnique, CMAP, preprint no. 557.

[7] Cheridito, P., Delbaen F., Kupper M. (2004). Coherent and convex risk measures for bounded càdlàg processes, *Stoch. Proc. Appl.* **112**(1), 1–22.

[8] Cheridito, P., Delbaen F., Kupper M. (2005). Coherent and convex risk measures for unbounded càdlàg processes. *Finance and Stochastics*, **9**, 369–387.

[9] Cheridito, P., Delbaen F., Kupper M. (2006). Dynamic monetary risk measures for bounded discrete-time processes. *Electronic Journal of Probability* **11**, 57–106.

[10] Delbaen, F., (2002). Coherent risk measures on general probability spaces. *Advances in Finance and Stochastics*, 1–37, Springer-Verlag, Berlin.

[11] Delbaen, F., (2003). The structure of m-stable sets and in particular of the set of risk neutral measures. Preprint.

[12] Detlefsen, K and G. Scandolo (2005). Conditional and dynamic convex risk measures. *Finance and Stochastics*, **9**(4), 539–561.

[13] Duffie, D., Epstein, L.G. (1992). Stochastic differential utility. *Econometrica*, **60**(2), 353–394.

[14] Epstein, L.G., Schneider, M. (2003). Recursive multiple-priors. *J. Econom. Theory*, **113**(1), 1–31.

[15] Epstein, L.G., Zin, S.E. (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theoretical framework. *Econometrica* **57**(4), 937–969.

[16] Föllmer, H., Penner, I., (2006). Convex risk measures and the dynamics of their penalty functions. Preprint, Humboldt-University Berlin.

[17] Föllmer, H., Schied, A., (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, **6**(4), 429–447.

[18] Föllmer, H., Schied, A., (2004). Stochastic Finance, An Introduction in Discrete Time. Second Edition. de Gruyter Studies in Mathematics 27.

[19] Frittelli, M., Gianin, E.R. (2002). Putting order in risk measures. *Journal of Banking and Finance* **26**(7), 1473-1486

[20] Frittelli, M., Rosazza Gianin, E. (2004). Dynamic convex risk measures. Risk Measures for the 21st Century, Chapter 12, Wiley Finance.

[21] Frittelli, M., Scandolo, G. (2005). Risk measures and capital requirements for processes. forthcoming in Math. Finance.

[22] Gilboa, I., Schmeidler, D. (1989). Maxmin expected utility with non-unique priop. J.Math. Econom. **18**, 141–153.

[23] Jobert, A., Rogers, L.C.G. (2005). *Valuations and dynamic convex risk measures.* Preprint.

[24] Koopmans, T.C. (1960). Stationary ordinal utility and impatience. *Econometrica*, **28**(2), 287–309.

[25] Kreps, M.K., Porteus, E.L. (1978). Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, **46**(1), 185–200.

[26] Klöppel, S., Schweizer, M. (2005). Dynamic Indifference Valuation via Convex Risk Measures. Preprint, ETH Zurich.

[27] Maccheroni, F., Marinacci, M., Rustichini, A. (2005). *Dynamic variational preferences.* Preprint.

[28] Peng, S. (2004). *Non-linear expectations, non-linear evaluations and risk measures.* Lecture Notes in Mathematics 1856. Springer.

[29] Riedel, F. (2004). Dynamic coherent risk measures. *Stoch. Proc. Appl.*, **112**(2), 185–200.

[30] Roorda B., Schumacher J.M., Engwerda J. (2005). *Coherent acceptability measures in multiperiod models.* Math. Finance **15**(4), 589–612.

[31] Ruszczyński, A., Shapiro, A. (2005). *Conditional risk mappings.* Preprint, Rutgers University.

[32] Wang, T. (2003). Conditional preferences and updating. *J. Econom Theory*, **108**(2), 286–321.

[33] Weber, S. (2006). Distribution-invariant risk measures, information, and dynamic consistency. *Math. Finance*, **16**(2), 419–441.