Composition Operators on Spaces of Analytic Functions

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Functional analysis began a little more than 100 years ago

Questions had to do with interpreting differential operators
as linear transformations on vector spaces of functions

Sets of functions needed structure connected to the convergence
implicit in the limit processes of the operators

Concrete functional analysis developed with results on spaces
of integrable functions, with special classes of differential operators,
and sometimes used better behaved inverses of differential operators

The abstraction of these ideas led to:

Banach and Hilbert spaces

Bounded operators, unbounded closed operators, compact operators

Spectral theory as a generalization of Jordan form and diagonalizability

Multiplication operators as an extension of diagonalization of matrices

Concrete examples and development of theory interact:

Shift operators as an examples of asymmetric behavior possible

in operators on infinite dimensional spaces

Studying *composition operators* can be seen as extension of this process

The classical Banach spaces are spaces of functions on a set X: if φ is map of X onto itself, we can imagine a composition operator with symbol φ,

$$C_\varphi f = f \circ \varphi$$

for f in the Banach space.

This operator is formally linear:

$$(af + bg) \circ \varphi = af \circ \varphi + bg \circ \varphi$$

But other properties, like “Is $f \circ \varphi$ in the space?” clearly depend on the map φ and the Banach space of functions.

Several classical operators are composition operators. For example, we may regard $\ell^p(\mathbb{N})$ as the space of functions of $\mathbb{N}$ into $\mathbb{C}$ that are p^{th} power integrable with respect to counting measure by thinking x in ℓ^p as the function $x(k) = x_k$. If $\varphi : \mathbb{N} \to \mathbb{N}$ is given by $\varphi(k) = k + 1$, then $(C_\varphi x)(k) = x(\varphi(k)) = x(k+1) = x_{k+1}$, that is,

$$C_\varphi : (x_1, x_2, x_3, x_4, \cdots) \mapsto (x_2, x_3, x_4, x_5, \cdots)$$

so C_φ is the "backward shift".

In fact, backward shifts of all multiplicities can be represented as composition operators.

Moreover, composition operators often come up in studying other operators. For example, if we think of the operator of multiplication by z^2,

$$(M_{z^2} f)(z) = z^2 f(z)$$

it is easy to see that M_{z^2} commutes with multiplication by any bounded function. Also, C_{-z} commutes with M_{z^2}:

$$(M_{z^2} C_{-z} f)(z) = M_{z^2} f(-z) = z^2 f(-z)$$

and

$$(C_{-z} M_{z^2} f)(z) = C_{-z}(z^2 f(z)) = (-z)^2 f(-z) = z^2 f(-z)$$

In fact, in some contexts, the set of operators that commute with M_{z^2} is the algebra generated by the multiplication operators and the composition operator C_{-z}.

Also, Forelli showed that all isometries of $H^p(\mathbb{D})$, $1 < p < \infty$, $p \neq 2$, are weighted composition operators.

In these lectures, we will not consider absolutely arbitrary composition operators; a more interesting theory can be developed by restricting our attention to more specific cases.

Definition

Banach space of functions on set X is called a *functional Banach space* if

1. vector operations are the pointwise operations
2. $f(x) = g(x)$ for all x in X implies $f = g$ in the space
3. $f(x) = f(y)$ for all f in the space implies $x = y$ in X
4. $x \mapsto f(x)$ is a bounded linear functional for each x in X

We denote the linear functional in 4. by K_x, that is, for all f and x,

$$K_x(f) = f(x)$$

and if the space is a Hilbert space, K_x is the function in the space with

$$\langle f, K_x \rangle = f(x)$$

Examples

(1) $\ell^p(\mathbb{N})$ is a functional Banach space, as above

(2) $C([0,1])$ is a functional Banach space

(3) $L^p([0,1])$ is *not* a functional Banach space because

$$f \mapsto f(1/2)$$

is not a bounded linear functional on $L^p([0,1])$

We will consider functional Banach spaces whose functions are analytic on the underlying set X;

this what we mean by "Banach space of analytic functions"

Examples (cont'd) Some Hilbert spaces of analytic functions:

(4) Hardy Hilbert space: $X = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$$H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \ \text{ with } \ \|f\|_{H^2}^2 = \sum |a_n|^2 < \infty\}$$

where for f and g in $H^2(\mathbb{D})$, we have $\langle f, g\rangle = \sum a_n \overline{b_n}$

(5) Bergman Hilbert space: $X = \mathbb{D}$

$$A^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : \ \|f\|_{A^2}^2 = \int_{\mathbb{D}} |f(\zeta)|^2 \frac{dA(\zeta)}{\pi} < \infty\}$$

where for f and g in $A^2(\mathbb{D})$, we have $\langle f, g\rangle = \int f(\zeta)\overline{g(\zeta)}\, dA(\zeta)/\pi$

(6) Dirichlet space: $X = \mathbb{D}$

$$\mathcal{D} = \{f \text{ analytic in } \mathbb{D} : \ \|f\|_{\mathcal{D}}^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'(\zeta)|^2 \frac{dA(\zeta)}{\pi} < \infty\}$$

(7) generalizations where $X = \mathbf{B}_N$

If $\mathcal{H}$ is a Hilbert space of complex-valued analytic functions on the domain Ω in $\mathbb{C}$ or $\mathbb{C}^N$ and φ is an analytic map of Ω into itself, the *composition operator* C_φ on $\mathcal{H}$ is the operator given by

$$(C_\varphi f)(z) = f(\varphi(z)) \quad \text{for } f \text{ in } \mathcal{H}$$

At least formally, this defines C_φ as a linear transformation.

In this context, the study of composition operators was initiated about 40 years ago by Nordgren, Schwartz, Rosenthal, Caughran, Kamowitz, and others.

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Goal:

relate the properties of φ as a function with properties of C_φ as an operator.

Today, we'll mostly consider the Hardy Hilbert space, H^2

$$H^2 = \{f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

which is also described as

$$H^2 = \{f \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \int |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty\}$$

and for f in H^2

$$\|f\|^2 = \sup_{0<r<1} \int |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \int |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum |a_n|^2$$

For H^2, the Littlewood subordination theorem plus some easy calculations for changes of variables induced by automorphisms of the disk imply that

C_φ is bounded for all functions φ that are analytic and map $\mathbb{D}$ into itself

and the argument yields the following estimate of the norm for composition operators on H^2:

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\frac{1}{2}} \leq \|C_\varphi\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\frac{1}{2}}$$

This is the sort of result we seek, connecting the properties of the operator C_φ with the analytic and geometric properties of φ.

When an operator theorist studies an operator for the first time, questions are asked about the boundedness and compactness of the operator,

about norms,

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While the whole story is not known, much progress has been made $\cdots$

and we expect the answers to be given in terms of analytic and geometric properties of φ.

Very often, calculations with kernel functions give ways to connect the analytic and geometric properties of φ with the operator properties of C_φ.

For a point α in the disk $\mathbb{D}$, the kernel function K_α is the function in $H^2(\mathbb{D})$ such that for all f in $H^2(\mathbb{D})$, we have

$$\langle f, K_\alpha \rangle = f(\alpha)$$

f and K_α are in H^2, so $f(z) = \sum a_n z^n$ and $K_\alpha(z) = \sum b_n z^n$
for some coefficients. Thus, for each f in H^2,

$$\sum a_n \alpha^n = f(\alpha) = \langle f, K_\alpha \rangle = \sum a_n \overline{b_n}$$

The only way this can be true is for $b_n = \overline{\alpha^n} = \overline{\alpha}^n$ and

$$K_\alpha(z) = \sum \overline{\alpha}^n z^n = \frac{1}{1 - \overline{\alpha} z}$$

For a point α in the disk $\mathbb{D}$, because the kernel function K_α is a function in $H^2(\mathbb{D})$, we have

$$\|K_\alpha\|^2 = \langle K_\alpha, K_\alpha \rangle = K_\alpha(\alpha) = \frac{1}{1-\overline{\alpha}\alpha} = \frac{1}{1-|\alpha|^2}$$

These ideas show that $H^2(\mathbb{D})$ is functional Hilbert space and that $\|K_\alpha\| = (1-|\alpha|^2)^{-1/2}$

For each f in H^2 and α in the disk,

$$\langle f, C_\varphi^* K_\alpha \rangle = \langle C_\varphi f, K_\alpha \rangle = \langle f \circ \varphi, K_\alpha \rangle = f(\varphi(\alpha)) = \langle f, K_{\varphi(\alpha)} \rangle$$

Since this is true for every f, we see $C_\varphi^*(K_\alpha) = K_{\varphi(\alpha)}$

Further exploitation of this line of thought shows that C_φ is invertible if and only if φ is an automorphism of the disk and in this case, $C_\varphi^{-1} = C_{\varphi^{-1}}$

In addition to asking "When is C_φ bounded?" operator theorists would want to know "When is C_φ compact?"

Because

- analytic functions take their maxima at the boundary
- compact operators should take most vectors to much smaller vectors

expect C_φ compact implies $\varphi(\mathbb{D})$ is far from the boundary in some sense.

If $m(\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}) > 0$, then C_φ is not compact.

If $\|\varphi\|_\infty < 1$, then C_φ is compact.

In H^2 and similar spaces, as $|\alpha| \to 1$, then $\frac{1}{\|K_\alpha\|} K_\alpha \to 0$ weakly.

C_φ is compact if and only if C_φ^* is compact, and in this case, we must have

$$\left\| C_\varphi^* \left(\frac{1}{\|K_\alpha\|} K_\alpha \right) \right\| = \frac{\|K_{\varphi(\alpha)}\|}{\|K_\alpha\|} = \sqrt{\frac{1 - |\alpha|^2}{1 - |\varphi(\alpha)|^2}}$$

is going to zero.

Now if $\alpha \to \zeta$ non-tangentially with $|\zeta| = 1$ and the angular derivative $\varphi'(\zeta)$ exists, then the Julia-Caratheodory Theorem shows that $\frac{1-|\alpha|^2}{1-|\varphi(\alpha)|^2} \to \frac{1}{\varphi'(\zeta)}$

In particular, C_φ compact implies no angular derivative of φ is finite.

Theorem (1987, J.H. Shapiro)

Suppose φ is an analytic map of $\mathbb{D}$ into itself. For C_φ acting on $H^2(\mathbb{D})$,

$$\|C_\varphi\|_e^2 = \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{-\log|w|}$$

where N_φ is the Nevanlinna counting function.

Corollary

C_φ is compact on $H^2(\mathbb{D})$ if and only if $\displaystyle\limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{-\log|w|} = 0$

In some spaces larger than the Hardy Hilbert space, like the Bergman space,

C_φ is compact if and only if φ has no finite angular derivatives

Caughran and Schwartz (1975) showed that if C_φ is compact,

then φ has a fixed point in $\mathbb{D}$

and found spectrum of C_φ in terms of data at the fixed point.

This was the first of many results that show how the behavior of C_φ depends on the fixed points of φ. Digress to talk about fixed points.

If φ is a continuous map of $\overline{\mathbb{D}}$ into $\overline{\mathbb{D}}$, then φ must have a fixed point in $\overline{\mathbb{D}}$.

Only assume φ is analytic on $\mathbb{D}$, open disk!

Definition

Suppose φ is an analytic map of $\mathbb{D}$ into itself.

If $|b| < 1$, we say b *is a fixed point of* φ if $\varphi(b) = b$.

If $|b| = 1$, we say b *is a fixed point of* φ if $\lim_{r\to 1^-} \varphi(rb) = b$.

Julia-Caratheordory Theorem implies

If b is a fixed point of φ with $|b| = 1$, then $\lim_{r\to 1^-} \varphi'(rb)$ exists (call it $\varphi'(b)$) and $0 < \varphi'(b) \leq \infty$.

Denjoy-Wolff Theorem (1926)

If φ is an analytic map of $\mathbb{D}$ into itself, not the identity map, there is a unique fixed point, a, of φ in $\overline{\mathbb{D}}$ such that $|\varphi'(a)| \leq 1$.

For φ not an elliptic automorphism of $\mathbb{D}$, for each z in $\mathbb{D}$, the sequence

$$\varphi(z),\ \varphi_2(z) = \varphi(\varphi(z)),\ \varphi_3(z) = \varphi(\varphi_2(z)),\ \varphi_4(z) = \varphi(\varphi_3(z)),\ \cdots$$

converges to a and the convergence is uniform on compact subsets of $\mathbb{D}$.

This distinguished fixed point will be called the *Denjoy-Wolff point of* φ.

The Schwarz-Pick Lemma implies φ has at most one fixed point in $\mathbb{D}$ and if φ has a fixed point in $\mathbb{D}$, it must be the Denjoy-Wolff point.

Examples

(1) $\varphi(z) = (z + 1/2)/(1 + z/2)$ is an automorphism of $\mathbb{D}$ fixing 1 and -1.

The Denjoy-Wolff point is $a = 1$ because $\varphi'(1) = 1/3$ (and $\varphi'(-1) = 3$)

(2) $\varphi(z) = z/(2 - z^2)$ maps $\mathbb{D}$ into itself and fixes 0, 1, and -1.

The Denjoy-Wolff point is $a = 0$ because $\varphi'(0) = 1/2$ (and $\varphi'(\pm 1) = 3$)

(3) $\varphi(z) = (2z^3 + 1)/(2 + z^3)$ is an inner function fixing fixing 1 and -1

with Denjoy-Wolff point $a = 1$ because $\varphi'(1) = 1$ (and $\varphi'(-1) = 9$)

(4) Inner function $\varphi(z) = \exp(z + 1)/(z - 1)$ has a fixed point in $\mathbb{D}$,

Denjoy-Wolff point $a \approx .21365$, and infinitely many fixed points on $\partial\mathbb{D}$

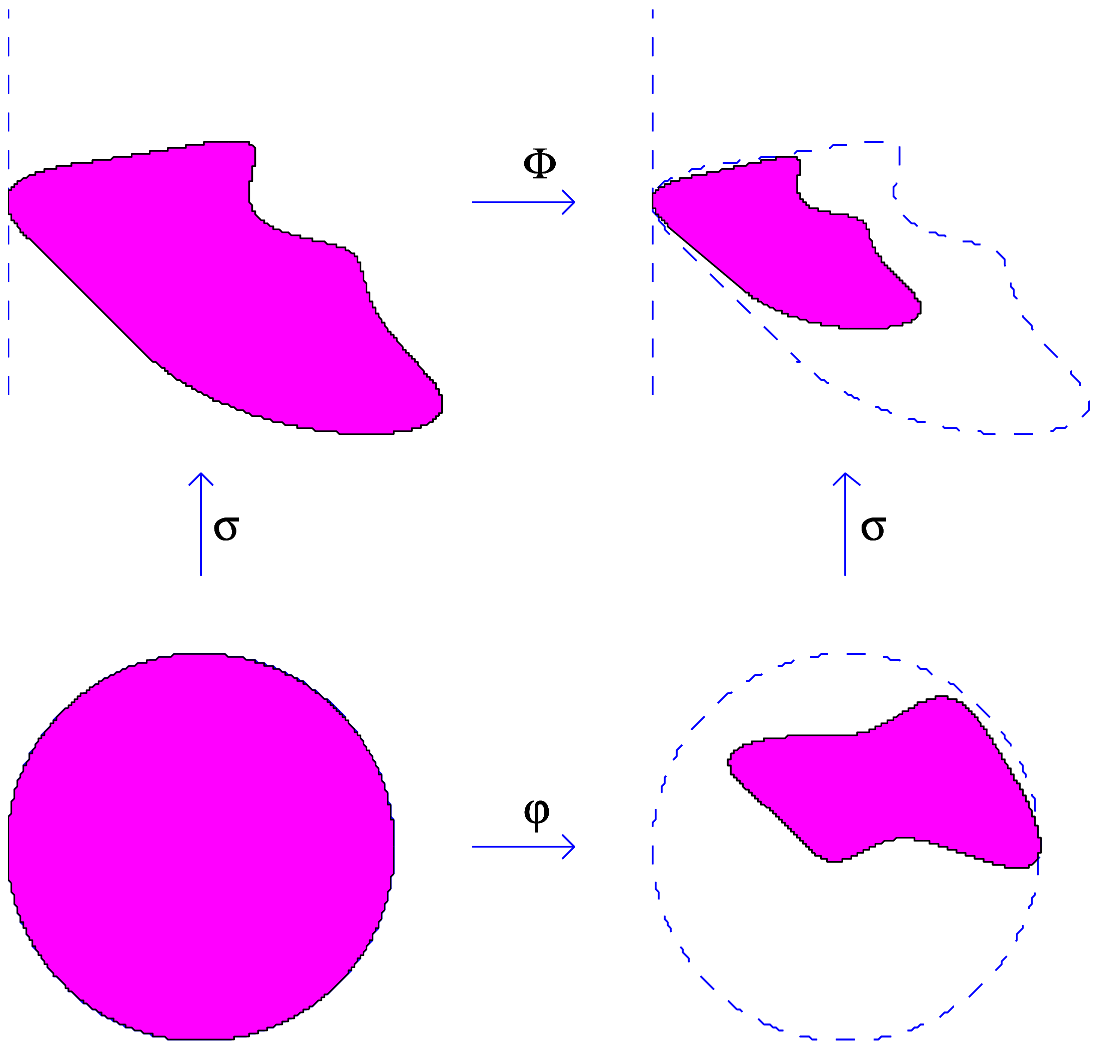
Denjoy-Wolff Thm suggests looking for a model for iteration of maps of $\mathbb{D}$

Five different types of maps of $\mathbb{D}$ into itself from the perspective of iteration,
classified by the behavior of the map near the Denjoy-Wolff point, a

In one of these types, $\varphi'(a) = 0$, (e.g., $\varphi(z) = (z^2 + z^3)/2$ with $a = 0$),
the model for iteration not yet useful for studying composition operators

In the other four types, when $\varphi'(a) \neq 0$, the map φ can be intertwined with
a linear fractional map and classified by the possible type of intertwining:
σ intertwines Φ and φ in the equality $\Phi \circ \sigma = \sigma \circ \varphi$

We want to do this with Φ linear fractional and σ univalent near a, so that σ is, locally, a change of variables. Using the notion of fundamental set, this linear fractional model becomes essentially unique [Cowen, 1981]



A linear fractional model in which φ maps $\mathbb{D}$ into itself with $a = 1$ and

$\varphi'(1) = \dfrac{1}{2}$, σ maps $\mathbb{D}$ into the right half plane, and $\Phi(w) = \dfrac{1}{2}\,w$

Linear Fractional Models:

- φ maps $\mathbb{D}$ into itself with $\varphi'(a) \neq 0$ (φ not an elliptic automorphism)
- Φ is a linear fractional automorphism of Ω onto itself
- σ is a map of $\mathbb{D}$ into Ω with $\quad \Phi \circ \sigma = \sigma \circ \varphi$

I. (plane dilation) $|a| < 1$, $\Omega = \mathbb{C}$, $\sigma(a) = 0$, $\Phi(w) = \varphi'(a)w$

II. (half-plane dilation) $|a| = 1$ with $\varphi'(a) < 1$, $\Omega = \{w : \operatorname{Re} w > 0\}$,

$\sigma(a) = 0$, $\Phi(w) = \varphi'(a)w$

III. (plane translation) $|a| = 1$ with $\varphi'(a) = 1$, $\Omega = \mathbb{C}$, $\Phi(w) = w + 1$

IV. (half-plane translation) $|a| = 1$ with $\varphi'(a) = 1$, $\Omega = \{w : \operatorname{Im} w > 0\}$,

(or $\Omega = \{w : \operatorname{Im} w < 0\}$), $\Phi(w) = w + 1$

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$\{\varphi_n(0)\}$ NOT an interpolating sequence

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$\{\varphi_n(0)\}$ IS an interpolating sequence

Theorem (Koenigs, 1884)

If φ is analytic map of $\mathbb{D}$ into itself, $\varphi(0) = 0$, and $0 < |\varphi'(0)| < 1$, then there is a unique map σ with $\sigma(0) = 0$, $\sigma'(0) = 1$, and $\sigma \circ \varphi = \varphi'(0)\sigma$

Moreover, if f is analytic (not the zero map) and λ is a number so that $f \circ \varphi = \lambda f$, then $\lambda = \varphi'(0)^n$ for some $n = 0, 1, 2, 3, \cdots$ and $f = c\,\sigma^n$ for some c

This is the plane dilation case (I.) with $a = 0$ and $\Phi(w) = \varphi'(0)w$

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Proof:

Starting with the second part, suppose f satisfies $f(\varphi(z)) = \lambda f(z)$ for some λ and all z in $\mathbb{D}$. Consider the Taylor series for f, $f(z) = \sum a_k z^k$ with first non-zero coefficient a_n, that is, $a_n \neq 0$ and $a_k = 0$ for k an integer, $k < n$. Since any non-zero multiple of f will work just as well, we suppose $a_n = 1$.

Compare the Taylor series for $f \circ \varphi$ and λf $\cdots$

get f unique and $\lambda = \varphi'(0)^n$

Theorem (Koenigs, 1884)

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Proof:

For the first part, define σ by $\sigma(z) = \lim\limits_{k\to\infty} \dfrac{\varphi_k(z)}{\varphi'(0)^k}$ where $\varphi_2(z) = \varphi(\varphi(z))$, $\varphi_3(z) = \varphi(\varphi_2(z))$, etc. Establish convergence; observe σ satisfies functional equation, $\sigma(0) = 0$, and $\sigma'(0) = 1$.

To finish, since the solution f from the earlier part was unique and σ^n also satisfies the conditions, $f = \sigma^n$.