# Compositional Semantics for Open Petri Nets based on Deterministic Processes 

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# Compositional Semantics for Open Petri Nets based on Deterministic Processes* 

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#### Abstract

In order to model the behaviour of open concurrent systems by means of Petri nets, we introduce open Petri nets, a generalization of the ordinary model where some places, designated as open, represent an interface of the system towards the environment. Besides generalizing the token game to reflect this extension, we define a truly concurrent semantics for open nets by extending the Goltz-Reisig process semantics of Petri nets.

We introduce a composition operation over open nets, characterized as a pushout in the corresponding category, suitable to model both interaction through open places and synchronization of transitions. The deterministic process semantics is shown to be compositional with respect to such composition operation. If a net $Z_{3}$ results as the composition of two nets $Z_{1}$ and $Z_{2}$, having a common subnet $Z_{0}$, then any two deterministic processes of $Z_{1}$ and $Z_{2}$ which "agree" on the common part, can be "amalgamated" to produce a deterministic process of $Z_{3}$. Vice versa, any deterministic process of $Z_{3}$ can be decomposed into processes of the component nets. The amalgamation and decomposition operations are shown to be inverse to each other, leading to a bijective correspondence between the deterministic processes of $Z_{3}$ and pair of deterministic processes of $Z_{1}$ and $Z_{2}$ which agree on the common subnet $Z_{0}$. Technically, our result is similar to the amalgamation theorem for data-types in the framework of algebraic specification. A possible application field of the proposed constructions and results is the modeling of interorganizational workflows, recently studied in the literature. This is illustrated by a running example.


## 1 Introduction

Among the various models of concurrent and distributed systems, Petri nets [Rei85] are certainly not the most expressive or the best-behaved. However, due to their intuitive graphical representation, Petri nets are widely used both in theoretical and applied research to specify and visualize the behaviour of systems. Especially when explaining the concurrent behaviour of a net to non-experts, one important feature of Petri nets is the possibility to describe their execution within the same visual notation, i.e., in terms of processes [GR83].

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Figure 1: Sample net modeling an interorganizational workflow.

However, when modeling reactive systems, i.e., concurrent systems with interacting subsystems, Petri nets force us to take a global perspective. In fact, ordinary Petri nets are not adequate to model open systems which can interact with their environment or, in a different view, which are only partially specified. This contradicts the common practice, e.g., in software engineering, where a large system is usually built out of smaller components.

Let us explain this problem in more detail by means of a typical application of Petri nets, the specification of workflows. A workflow describes a business process in terms of tasks and shared resources. Such descriptions are needed, for example, when interoperability of the workflows of different organizations is an issue, which is frequently the case, e.g., when applications of different enterprises shall be integrated over the Internet. A workflow net [vdA98] is a Petri net satisfying some structural constraints, like the existence of one initial and one final place, and a corresponding soundness condition: from each marking reachable from the initial one (one token on the initial place) we can reach the final marking (one token on the final place). An interorganizational workflow [vdA99] is modeled as a set of such workflow nets connected through additional places for asynchronous communication and synchronization requirements on transitions.

For instance, Fig. 1 shows an interorganizational workflow consisting of two local workflow nets Traveler and Agency related through communication places can, ack, bill, payment and ticket and a synchronization requirement between the two reserve transitions, modeled by a dashed line. The example describes the booking of a flight by a traveler in cooperation with a travel agency. After some initial negotiations (which is not modeled), both sides synchronize in the reservation of a flight. Then, the traveler may either acknowledge or cancel and re-enter the initial state. In both cases an asynchronous notification (e.g., a fax), modeled by the places ack and can, respectively, is sent to the travel agency. Next the local workflow of the traveler forks into two concurrent threads, the booking of a hotel and the payment of the bill. The trip can start when both tasks are completed and the ticket has been provided by the travel agency.

The overall net in Fig. 1 describes the system from a global perspective. Hence, the classical notion of behaviour (described, e.g., in terms of processes) is completely adequate. However, for a local subnet in isolation (like Traveler) which will only exhibit a meaningful behaviour when interacting with other subnets, this semantics is not appropriate because it does not take into account the possible interactions.

In order to overcome these limitations of ordinary Petri nets, we extend the basic model introducing open nets. An open net is a P/T Petri net with a distinguished set of places which are intended to represent the interface of the net towards the external world. Some similarities exist
with other approaches to net composition, like the Petri box calculus [BDH92, KEB94, KB99], the Petri nets with interface [NPS95, PW98] and the Petri net components [Kin97], which will be discussed in the conclusions. As a consequence of the (hidden, implicit) interaction between the net and the environment, some tokens can "freely" appear in or disappear from the open places. Besides generalizing the token game to reflect this changes, we provide a truly concurrent semantics by extending the ordinary (deterministic) process semantics [GR83] to open nets.

The embedding of an open net in a context is formally described by a morphism in a suitable category of open nets. Intuitively, in the target net new transitions can be attached to open places and, moreover, the interface towards the environment can be reduced by "closing" open places. Therefore, open net morphisms do not preserve but reflect the behaviour, i.e., any computation of the target (larger) net can be projected back to a computation of the source (smaller) net.

A composition operation is introduced over open nets. Two open nets $Z_{1}$ and $Z_{2}$ can be composed by specifying a common subnet $Z_{0}$ which embeds both in $Z_{1}$ and $Z_{2}$. Then the two nets can be glued along the common part. This is permitted only if the prescribed composition is consistent with the interfaces, i.e., only if the places of $Z_{1}$ and $Z_{2}$ which are used when connecting the two nets are actually open. The composition operation is characterized as a pushout in the category of open nets, where the conditions for the existence of the pushout nicely fit with the mentioned condition over interfaces.

Based on these concepts, the representation of the system of Fig. 1 in terms of two interacting open nets is given by the top part of Fig. 2, which comprises the two component nets Traveler and Agency, and the net Common which embeds into both components by means of open net morphisms. Places with incoming/outgoing dangling arcs are open. Observe that the common subnet Common of the components Traveler and Agency closely corresponds to the dashed items of Fig. 1, which represent the 'glue" between the two components. The net resulting from the composition of Traveler and Agency over the shared subnet Common is shown in the bottom part of Fig. 2.

Obviously, one would like to be able to establish a clear relationship between the behaviours of the component nets (in the example, the nets Traveler and Agency) and the behaviour of the composition (in the example, the net Global). We will show that indeed, the behaviour of the latter can be constructed "compositionally" out of the behaviours of the former, in the sense that two deterministic processes which "agree" on the shared part, can be synchronized to produce a deterministic process over the composed net. Vice versa, any deterministic process of the global net can be decomposed into deterministic processes of the component nets, which, in turn, can be synchronized to give the original process again. The top part of Fig. 3 shows two processes of the nets Traveler and Agency, the corresponding common projections over net Common and the process of Global arising from their synchronization.

The synchronization of processes, based on the composition of their underlying nets, resembles the amalgamation of data-types in the framework of algebraic specifications, and therefore we will speak of amalgamation of processes. In analogy with the amalgamation theorem for algebraic specifications [EM85], the main result of this paper shows that the amalgamation and decomposition constructions mentioned above are inverse to each other, establishing a bijective correspondence between the pairs of processes of two nets which agree on a common subnet and the processes of the net resulting from their composition.

The rest of the paper is organized as follows. Section 2 introduces the open Petri net model and the corresponding category. Section 3 extends the notion of process from ordinary to open nets and defines the operation of behaviour projection. Section 4 introduces the composition operation for open nets, based on a pushout in the category of open nets. Section 5 presents the compositionality result of the process semantics of open nets. Finally, Section 6 discusses some related work in the literature and outlines possible directions of future investigation. An extended abstract of this paper has been published as [BCEH01].


Figure 2: Interorganizational workflow as composition of open nets Traveler and Agency.


Figure 3: Amalgamation of processes for the nets Traveler an Agency.

## 2 Open nets

An open net is an ordinary P/T Petri net with a distinguished set of places which are intended to represent the interface of the net towards the external world (environment). As a consequence of the (hidden, implicit) interaction between the net and the environment, some tokens can freely appear in and disappear from the open places. Concretely, an open place can be either an input or an output place (or both), meaning that the environment can put or remove tokens from that place.

Given a set $X$ we will denote by $X^{\oplus}$ the free commutative monoid generated by $X$ and by $\mathbf{2}^{X}$ its powerset. Furthermore given a function $h: X \rightarrow Y$ we will denote by $h^{\oplus}: X^{\oplus} \rightarrow Y^{\oplus}$ its monoidal extension, while the same symbol $h: \mathbf{2}^{X} \rightarrow \mathbf{2}^{Y}$ denotes the extension of $h$ to sets.

Definition 1 (P/T Petrinet) A P/T Petri net is a tuple $N=(S, T, \sigma, \tau)$ where $S$ is the set of places, $T$ is the set of transitions (with $S \cap T=\emptyset$ ) and $\sigma, \tau: T \rightarrow S^{\oplus}$ are the functions assigning to each transition its pre- and post-set.

In the following we will denote by $\bullet(\cdot)$ and $(\cdot)^{\bullet}$ the monoidal extensions of the functions $\sigma$ and $\tau$ to functions from $T^{\oplus}$ to $S^{\oplus}$. Furthermore, given a place $s \in S$, the pre- and post-set of $s$ are defined by ${ }^{\bullet} s=\left\{t \in T \mid s \in t^{\bullet}\right\}$ and $s^{\bullet}=\left\{t \in T \mid s \in{ }^{\bullet} t\right\}$.

Definition 2 (Petri net category) Let $N_{0}$ and $N_{1}$ be Petri nets. A Petri net morphism $f: N_{0} \rightarrow N_{1}$ is a pair of total functions $f=\left\langle f_{T}, f_{S}\right\rangle$ with $f_{T}: T_{0} \rightarrow T_{1}$ and $f_{S}: S_{0} \rightarrow S_{1}$, such that for all $t_{0} \in T_{0},{ }^{\bullet} f_{T}\left(t_{0}\right)=f_{S}{ }^{\oplus}\left({ }^{\bullet} t_{0}\right)$ and $f_{T}\left(t_{0}\right)^{\bullet}=f_{S}{ }^{\oplus}\left(t_{0} \bullet\right)$ (see the diagram below).


The category of P/T Petri nets and Petri net morphisms is denoted by Net.
Petri net morphisms are closed under composition. This immediately follows by observing that given $f_{0}: N_{0} \rightarrow N_{1}$ and $f_{1}: N_{1} \rightarrow N_{2}$, we have $\left(f_{S_{1}} \circ f_{S_{0}}\right)^{\oplus}=f_{S_{1}} \oplus \circ f_{S_{0}} \oplus$.

Category Net is a lluf subcategory of the category Petri of [MM90]. The latter has the same objects, but more general morphisms which can map a place into a multiset of places.

We are now ready to introduce the notion of open net.
DEFINITION 3 (OPEN NET) An open net is a pair $Z=\left(N_{Z}, O_{Z}\right)$, where

- $N_{Z}=\left(S_{Z}, T_{Z}, \sigma_{Z}, \tau_{Z}\right)$ is an ordinary P/T Petri net and
- $O_{Z}=\left(O_{Z}^{+}, O_{Z}^{-}\right) \in \mathbf{2}^{S_{Z}} \times \mathbf{2}^{S_{Z}}$ are the input and output open places of the net.

Observe that the sets $O_{Z}^{+}$and $O_{Z}^{-}$are not necessarily disjoint, hence a place can be both an input and an output open place at the same time.

The notion of enabledness for a transition (or multiset of transitions) of an open net is the usual one, but, besides the changes produced to the state by the firing of the "internal" transitions of the net, one considers also the interaction with the environment, modelled by a kind of invisible actions producing/consuming tokens in the input/output places of the net. The actions of the environment which produce and consume tokens in an open place $s$ are denoted by $+_{s}$ and $-_{s}$, respectively.

DEFINITION 4 (FIRING) Let $Z$ be an open net. A sequential move can be (i) the firing of $a$ transition, i.e., $m \oplus{ }^{\bullet} t[t\rangle m \oplus t^{\bullet}$, with $m \in S_{Z}{ }^{\oplus}, t \in T_{Z}$; (ii) the creation of a token by the environment, i.e., $m[+s\rangle m \oplus s$, with $s \in O_{Z}^{+}, m \in S_{Z}{ }^{\oplus}$; (iii) the deletion of a token by the environment, i.e., $m \oplus s[-s\rangle m$, with $m \in S_{Z}{ }^{\oplus}$, $s \in O_{Z}^{-}$.

A parallel move is of the form

$$
m \oplus \bullet A \oplus m^{-}[A\rangle m \oplus A^{\bullet} \oplus m^{+}
$$

with $m \in S_{Z^{\oplus}}, A \in T_{Z}{ }^{\oplus}, m^{+} \in\left(O_{Z}^{+}\right)^{\oplus}, m^{-} \in\left(O_{Z}^{-}\right)^{\oplus}$.
Alternatively, the token game of an open net can be described as the behaviour of an ordinary net, called the closure of $Z$ and denoted by $\bar{Z}$. The net $\bar{Z}$ is obtained by adding transitions connected to open places which can freely produce/remove tokens from input/output places, i.e., $\bar{Z}=\left(T^{\prime}, S_{Z}, \sigma^{\prime}, \tau^{\prime}\right)$ where

- $T^{\prime}=T_{Z} \cup\left\{+{ }_{s} \mid s \in O_{Z}^{+}\right\} \cup\left\{-s \mid s \in O_{Z}^{-}\right\} ;$
- $\sigma^{\prime}\left(+_{s}\right)=0$ and $\tau^{\prime}\left(+_{s}\right)=s$ for any $s \in O_{Z}^{+}$;
- $\sigma^{\prime}(-s)=s$ and $\tau^{\prime}(-s)=0$ for any $s \in O_{Z}^{-}$;
and $\sigma^{\prime}, \tau^{\prime}$ coincide with $\sigma_{Z}, \tau_{Z}$ on the other transitions.
Example. The open nets for the local workflows Traveler and Agency of Fig. 1 are shown in the middle part of Fig. 2. Ingoing and outgoing arcs without source or target designate the input and output places, respectively. Observe that the synchronization transition reserve is common to both nets. Furthermore the communication places, like can, become open places.

DEFINITION 5 (OPEN NET MORPHISM) An open net morphism $f: Z_{1} \rightarrow Z_{2}$ is a Petri net morphism $f: N_{Z_{1}} \rightarrow N_{Z_{2}}$ such that, if we define

$$
\operatorname{in}(f)=\left\{s \in S_{1} \mid \cdot f_{S}(s)-f_{T}(\bullet s) \neq \emptyset\right\} \quad \text { and } \quad \operatorname{out}(f)=\left\{s \in S_{1} \mid f_{S}(s)^{\bullet}-f_{T}\left(s^{\bullet}\right) \neq \emptyset\right\}
$$

then

$$
\text { (i) } f_{S}^{-1}\left(O_{2}^{+}\right) \cup \operatorname{in}(f) \subseteq O_{1}^{+} \quad \text { and } \quad \text { (ii) } \quad f_{S}^{-1}\left(O_{2}^{-}\right) \cup \operatorname{out}(f) \subseteq O_{1}^{-} .
$$

The morphism $f$ is called an open net embedding if both components $f_{T}$ and $f_{S}$ are injective.
In the sequel, given an open net morphism $f=\left\langle f_{S}, f_{T}\right\rangle: Z_{1} \rightarrow Z_{2}$, to lighten the notation, we will omit the subscripts " $S$ " and " $T$ " in its place and transition components, writing $f(s)$ for $f_{S}(s)$ and $f(t)$ for $f_{T}(t)$.

A morphism $f: Z_{1} \rightarrow Z_{2}$ can be thought of as an "insertion" of net $Z_{1}$ into a larger net $Z_{2}$, which extends $Z_{1}$. In other words, $Z_{2}$ can be thought of as an instantiation of $Z_{1}$, where part of the unknown environment gets more specified. Conditions (i) and (ii) first require that open places are reflected and hence that places which are "internal" in $Z_{1}$ cannot be promoted to open places in $Z_{2}$. Furthermore, the context in which $Z_{1}$ is inserted can interact with $Z_{1}$ only through the open places. To understand how this is formalized, observe that for each place $s$ in $\operatorname{in}(f)$, its image $f(s)$ is in the post-set of a transition outside the image of $\bullet s$. Hence we can think that in $Z_{2}$ new transitions are attached to $s$ and can produce tokens in such place. This is the reason why condition (i) also asks any place in in $(f)$ to be an input open place of $Z_{1}$. Condition (ii) is analogous for output places.

The above intuition better fits with open net embeddings, and indeed most of the constructions in the paper will be defined for this subclass of open net morphisms. However, for technical reasons (e.g., to characterize the composition of open nets as a pushout) the more general notion of morphism is useful.
Example. As an example of open net morphism, consider the embedding of net Traveler into net Global of Fig. 4 (extracted from Fig. 2). Observe that the constraints characterizing open nets morphisms have an intuitive graphical interpretation:


Figure 4: The open net embedding of net Traveler into net Global.

- the connections of transitions to their pre-set and post-set have to be preserved. New connections cannot be added;
- in the larger net, a new arc may be attached to a place only if the corresponding place of the subnet has a dangling arc in the same direction. Dangling arcs may be removed, but cannot be added in the larger net. For instance, without the outgoing dangling arc from place can in net Traveler, i.e., if place can were not output open, the mapping in Fig. 4 would have not been a legal open net morphism.

Next we show that open net morphisms are closed under composition.

## PROPOSITION 6 Open net morphisms are closed under composition.

Proof. Let $f_{1}: Z_{1} \rightarrow Z_{2}$ and $f_{2}: Z_{2} \rightarrow Z_{3}$ be open net morphisms. Then $f_{2} \circ f_{1}$ is a morphism in Net. As for condition (i) of Definition 5, first observe that

$$
\begin{equation*}
\operatorname{in}\left(f_{2} \circ f_{1}\right) \subseteq \operatorname{in}\left(f_{1}\right) \cup f_{1}^{-1}\left(\operatorname{in}\left(f_{2}\right)\right) \tag{1}
\end{equation*}
$$

In fact,

$$
\begin{aligned}
\operatorname{in} & \left(f_{2} \circ f_{1}\right)= \\
& =\left\{s \in S_{1} \mid \bullet f_{2}\left(f_{1}(s)\right)-f_{2}\left(f_{1}(\bullet s)\right) \neq \emptyset\right\} \\
& =\left\{s \in S_{1} \mid \bullet f_{2}\left(f_{1}(s)\right)-f_{2}\left(\bullet f_{1}(s)\right) \neq \emptyset\right\} \cup\left\{s \in S_{1} \mid f_{2}\left(\bullet f_{1}(s)\right)-f_{2}\left(f_{1}(\bullet s)\right) \neq \emptyset\right\} \\
& \subseteq\left\{s \in S_{1} \mid f_{1}(s) \in \operatorname{in}\left(f_{2}\right)\right\} \cup\left\{s \in S_{1} \mid f_{2}\left(\bullet f_{1}(s)-f_{1}(\bullet s)\right) \neq \emptyset\right\} \\
& =f_{1}^{-1}\left(\operatorname{in}\left(f_{2}\right)\right) \cup\left\{s \in S_{1} \mid \bullet f_{1}(s)-f_{1}(\bullet s) \neq \emptyset\right\} \\
& =f_{1}^{-1}\left(\operatorname{in}\left(f_{2}\right)\right) \cup \operatorname{in}\left(f_{1}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \operatorname{in}\left(f_{2} \circ f_{1}\right) \\
& \\
& \left.\quad \subseteq \operatorname{in}\left(f_{1}\right) \cup f_{1}^{-1}\left(\operatorname{in}\left(f_{2}\right)\right) \quad \quad \quad \text { by using }(1)\right] \\
& \\
& \subseteq O_{1}^{+} \cup f_{1}^{-1}\left(O_{2}^{+}\right) \quad\left[\text { since, by def. of morphism, in }\left(f_{1}\right) \subseteq O_{1}^{+} \text {and in }\left(f_{2}\right) \subseteq O_{2}^{+}\right] \\
& \\
& \subseteq O_{1}^{+} \quad\left[\text { since, by def. of morphism, } f_{1}^{-1}\left(O_{2}^{+}\right) \subseteq O_{1}^{+}\right]
\end{aligned}
$$



Figure 5: Open net morphisms do not preserve the behaviour.

Furthermore, $\left(f_{2} \circ f_{1}\right)^{-1}\left(O_{3}^{+}\right)=f_{1}^{-1}\left(f_{2}^{-1}\left(O_{3}^{+}\right)\right) \subseteq f_{1}^{-1}\left(O_{2}^{+}\right) \subseteq O_{1}^{+}$, since $f_{1}$ and $f_{2}$ are morphisms. Thus, summing up,

$$
\left(f_{2} \circ f_{1}\right)^{-1}\left(O_{3}^{+}\right) \cup \operatorname{in}\left(f_{2} \circ f_{1}\right) \subseteq O_{1}^{+}
$$

Condition (ii), over output open places, can be proved in a totally analogous way.
By the previous proposition we can consider a category of open nets.
DEFINITION 7 (OPEN NETS CATEGORY) We will denote by ONet the category of open nets and open net morphisms.

We said that open net morphisms are designed to capture the idea of "insertion" of a net into a larger one. Hence it is natural to expect that they "reflect" the behaviour in the sense that given $f: Z_{0} \rightarrow Z_{1}$, the behaviour of $Z_{1}$ can be projected along the morphism to the behaviour of $Z_{0}$ (this fact will be formalized later, in Construction 13). Instead, differently from most of the morphisms considered over Petri nets, open net morphisms cannot be thought of as simulations since they do not preserve the behaviour. For instance, consider the open nets $Z_{0}$ and $Z_{1}$ in Fig. 5 and the obvious open net morphism between them. Then the firing sequence $0[+s\rangle s[t\rangle 0$ in $Z_{0}$ is not mapped to a firing sequence in $Z_{1}$.

There is an obvious forgetful functor from the category of open nets to the category of ordinary nets.

DEFINITION 8 We denote by $\mathcal{F}: \mathbf{O N e t} \rightarrow$ Net the forgetful functor defined by $\mathcal{F}(Z)=N_{Z}$ for any open net $Z$ and $\mathcal{F}\left(f: Z_{0} \rightarrow Z_{1}\right)=f: N_{Z_{0}} \rightarrow N_{Z_{1}}$ for any open net morphism $f$.

Since functor $\mathcal{F}$ acts on arrows as identity, with abuse of notation, given an open net morphism $f: Z_{0} \rightarrow Z_{1}$ we will often write $f: \mathcal{F}\left(Z_{1}\right) \rightarrow \mathcal{F}\left(Z_{2}\right)$ instead of $\mathcal{F}(f): \mathcal{F}\left(Z_{1}\right) \rightarrow \mathcal{F}\left(Z_{2}\right)$.

## 3 Deterministic processes of open nets

Similarly to what happens for ordinary nets, a process of an open net, providing a truly concurrent description of a (possibly nondeterministic) computation of the net, is an open net itself, satisfying suitable acyclicity and conflict freeness requirements, together with a mapping to the original net.

The open net underlying a process is an open occurrence net, namely an open net $K$ such that $N_{K}$ is an ordinary occurrence net and satisfying some additional conditions over open places. The open places in $K$ are intended to represent tokens which are produced/consumed by the environment in the considered computation. Consequently, every input open place is required to have an empty pre-set, i.e., to be minimal with respect to the causal order. In fact, an input open place in the post-set of some transition would correspond to a kind of generalized backward conflict: a token on this place could be generated in two different ways, i.e., by the firing of an "internal" transition or by the environment, and this would prevent one to interpret the place as a token occurrence.


Figure 6: A (nondeterministic) open occurrence net.

Observe that, instead, an output open place can be in the pre-set of a transition, as it happens for place $s$ in the open occurrence net of Fig. 6. The idea is that the token occurrence represented by place $s$ can be consumed either by the environment or by transition $t$.

Recall that for an ordinary net $N=(S, T, \sigma, \tau)$ the causal relation $<_{N}$ is defined as the least transitive relation over $S \cup T$ such that $x<_{K} y$ if $y \in x^{\bullet}$, for $x, y \in S \cup T$. The conflict relation $\#_{N}$ is defined as the least symmetric relation over $S \cup T$ such that $\mathbf{i}$ ) if ${ }^{\bullet} t \cap{ }^{\bullet} t^{\prime} \neq 0$ and $t \neq t^{\prime}$ then $t \#_{N} t^{\prime}$ (immediate conflict) and ii) if $x \# y<_{N} z$ then $x \#_{N} z$ (inheritance w.r.t. causality).

DEFINITION 9 (OPEN OCCURRENCE NET) An open occurrence net is an open net $K$ such that

1. $N_{K}$ is an ordinary occurrence net, namely in $N_{K}$ there are no backward conflicts, i.e., for any $t, t^{\prime} \in T_{K}$, if $t \neq t^{\prime}$ then $t^{\bullet} \cap t^{\bullet}=\emptyset$, the causal relation $<_{K}$ is a finitary strict partial order and the conflict relation $\#_{K}$ is irreflexive;
2. each input open place is minimal w.r.t. $<_{K}$, i.e., $\forall s \in O_{K}^{+} .{ }^{\bullet} s=\emptyset$.

We are now ready to introduce the notion of process for open nets.
DEFINITION 10 (OPEN NET PROCESS) A process of an open net $Z$ is a mapping $\pi: K \rightarrow Z$ where $K$ is an open occurrence net and $\pi: N_{K} \rightarrow N_{Z}$ is a Petri net morphism, such that

$$
\pi_{S}\left(O_{K}^{+}\right) \subseteq O_{Z}^{+} \quad \text { and } \quad \pi_{S}\left(O_{K}^{-}\right) \subseteq O_{Z}^{-}
$$

Observe that the mapping from the occurrence net $K$ to the the original net $Z$ is not, in general, an open net morphism. In fact, the process mapping, differently from open net morphisms, must be a simulation, i.e., it must preserve the behaviour. Furthermore, the image of an open place in $K$ must be an open place in $Z$, since tokens can be produced (consumed) by the environment only in input (output) open places of $Z$.

In the following, when the meaning is clear from the context, we will sometimes identify a process $\pi: K \rightarrow Z$ with the corresponding morphism $\pi: \mathcal{F}(K) \rightarrow \mathcal{F}(Z)$ in the category Net.

As usual, a process will be called deterministic if it represents a uniquely determined concurrent computation. First, an open occurrence net is deterministic if the underlying ordinary occurrence net is deterministic, i.e., each place is in the pre-set of at most one transition. Furthermore, the output open places must be maximal with respect to the causal order, i.e., an output open place cannot be in the pre-set of any transition. In fact, as already observed, an output open place $s$ which is in the pre-set of a transition $t$ represents a token occurrence which can be consumed either by the environment or by transition $t$. A process will be called deterministic if the underlying open occurrence net is deterministic.


Figure 7: An process of the open net Global and its projection to the subnet Traveler.

DEFINITION 11 (DETERMINISTIC OCCURRENCE NET AND PROCESS) An open occurrence net $K$ is called deterministic if

1. the underlying ordinary occurrence net $N_{K}$ is deterministic, i.e. $\forall s \in S_{K} .\left|s^{\bullet}\right| \leq 1$;
2. each output open place is maximal, i.e., $\forall s \in O_{K}^{-} \cdot s^{\bullet}=\emptyset$.

A process $\pi: K \rightarrow Z$ of an open net $Z$ is deterministic if $K$ is deterministic.
Example. A deterministic process for the open net Traveler is shown in Fig. 7 on the left. The morphism back to the original net Traveler is implicitly represented by the labeling. Observe that the requirement that each input place is minimal and each output place is maximal w.r.t. to the causal order of the process has a natural graphical interpretation: the absence of backward and forward conflicts extends to dangling arcs, i.e., in total, each place may have at most one ingoing and one outgoing arc.

Next we introduce a category of processes, where objects are processes and arrows are pairs of open net morphisms.

Definition 12 (CATEGORY of processes) We denote by Proc the category where objects are processes and, given two processes $\pi_{0}: K_{0} \rightarrow Z_{0}$ and $\pi_{1}: K_{1} \rightarrow Z_{1}$, an arrow $\psi: \pi_{0} \rightarrow \pi_{1}$ is a pair of open net morphisms $\psi=\left\langle\psi_{Z}: Z_{0} \rightarrow Z_{1}, \psi_{K}: K_{0} \rightarrow K_{1}\right\rangle$ such that the following diagram (indeed the underlying diagram in $\mathbf{N e t}$ ) commutes


### 3.1 Projecting processes along embeddings

Let $f: Z_{0} \rightarrow Z_{1}$ be an open net morphism. As mentioned before, it is natural to expect that each computation in $Z_{1}$ can be "projected" to $Z_{0}$, by considering only the part of the computation of the larger net which is visible in the smaller net. The above intuition is formalized, in the case of an open net embedding $f: Z_{0} \rightarrow Z_{1}$, by showing how a process of $Z_{1}$ can be projected along $f$ giving a process of $Z_{0}$.

CONSTRUCTION 13 (PROJECTION OF A PROCESS) Let $f: Z_{0} \rightarrow Z_{1}$ be an open net embedding and let $\pi_{1}: K_{1} \rightarrow Z_{1}$ be a process of $Z_{1}$. A projection of $\pi_{1}$ along $f$ is a pair $\left\langle\pi_{0}, \psi\right\rangle$ where $\pi_{0}: K_{0} \rightarrow Z_{0}$ is a process of $Z_{0}$ and $\psi: \pi_{0} \rightarrow \pi_{1}$ is an arrow in Proc, constructed as follows. Consider the pullback of $\pi_{1}$ and $f$ in Net, thus obtaining the net morphisms $\pi_{0}$ and $\psi_{K}$.


Then $K_{0}$ is obtained by taking $N_{K_{0}}$ with the smallest sets of open places which make $\psi_{K}: N_{K_{0}} \rightarrow N_{K_{1}}$ an open net morphism, namely

$$
O_{K_{0}}^{+}=\psi_{K}^{-1}\left(O_{K_{1}}^{+}\right) \cup \operatorname{in}\left(\psi_{K}\right) \quad \text { and } \quad O_{K_{0}}^{-}=\psi_{K}^{-1}\left(O_{K_{1}}^{-}\right) \cup \operatorname{out}\left(\psi_{K}\right)
$$

and $\psi=\left\langle\psi_{K}, f\right\rangle$.
The next proposition shows that the notion of projection is well-defined, and restricts to deterministic processes.

Proposition 14 The process $\pi_{0}: K_{0} \rightarrow Z_{0}$, as introduced in Construction 13, is well defined. Furthermore, the projection of a deterministic process is still a deterministic process.

Proof. First observe that $K_{0}$ is an open occurrence net. Since $f$ is injective, also $\psi_{K}$ is injective, and thus $N_{K_{0}}$ is isomorphic to the subnet of $N_{K_{1}}$ in the codomain of $\psi_{K}$, which is clearly an ordinary occurrence net. Furthermore, we must show that each open input place is minimal. Let $s \in O_{K_{0}}^{+}$. Then we have two possibilities:
i) $\psi_{K}(s) \in O_{K_{1}}^{+}$.

Observe that ${ }^{\bullet} s=\psi_{K}^{-1}\left({ }^{\bullet} \psi_{K}(s)\right)$. Since $K_{1}$ is an open occurrence net, ${ }^{\bullet} \psi_{K}(s)=\emptyset$ and thus ${ }^{-} s=0$.
ii) $s \in \operatorname{in}\left(\psi_{K}\right)$.

In this case ${ }^{\bullet} \psi_{K}(s)-\psi_{K}\left({ }^{\bullet} s\right) \neq \emptyset$. Recalling that $K_{1}$ is an occurrence net and thus $\left|{ }^{\bullet} \psi_{K}(s)\right| \leq 1$, we conclude that $\psi_{K}\left({ }^{\bullet} s\right)=\emptyset$. Hence, as desired, ${ }^{\bullet} s=\emptyset$.

Now, observe that $\pi_{0}$ is clearly a morphism in Net. Hence to conclude that $\pi_{0}$ is a well defined process it only remains to show that it also satisfies

$$
\pi_{0}\left(O_{K_{0}}^{+}\right) \subseteq O_{Z_{0}}^{+} \quad \text { and } \quad \pi_{0}\left(O_{K_{0}}^{-}\right) \subseteq O_{Z_{0}}^{-}
$$

Let us show, for instance, the first inclusion. Consider $s \in O_{K_{0}}^{+}$. Since, by construction, $O_{K_{0}}^{+}=$ $\psi_{K}^{-1}\left(O_{K_{1}}^{+}\right) \cup \operatorname{in}\left(\psi_{K}\right)$ we distinguish two possibilities:

1) $s \in \psi_{K}^{-1}\left(O_{K_{1}}^{+}\right)$

We have $f\left(\pi_{0}(s)\right)=\pi_{1}\left(\psi_{K}(s)\right) \in \pi_{1}\left(O_{K_{1}}^{+}\right)$and, by definition of process, $\pi_{1}\left(O_{K_{1}}^{+}\right) \subseteq O_{Z_{1}}^{+}$. Hence $\pi_{0}(s) \in f^{-1}\left(O_{Z_{1}}^{+}\right) \subseteq O_{Z_{0}}^{+}$, since $f$ is an open net morphism.
2) $s \in \operatorname{in}\left(\psi_{K}\right)$

In this case, ${ }^{\bullet} \psi_{K}(s)-\psi_{K}\left({ }^{\bullet} s\right) \neq \emptyset$. Since $K_{1}$ is an occurrence net, this means that there exists $t \in{ }^{\bullet} \psi_{K}(s)$ and $\psi_{K}\left({ }^{\bullet} s\right)=\emptyset$, i.e., ${ }^{\bullet} s=\emptyset$. Now observe that $\pi_{1}(t) \in{ }^{\bullet} \pi_{1}\left(\psi_{K}(s)\right)={ }^{\bullet} f\left(\pi_{0}(s)\right)$. Moreover, since the square in Construction 13 is a pullback, $\pi_{1}(t) \notin f\left(\bullet \pi_{0}(s)\right)$. In fact, if $\pi_{1}(t) \in$ $f\left({ }^{\bullet} \pi_{0}(s)\right)$ then there would be $t^{\prime}$ in $N_{K_{0}}$ such that $f\left(\pi_{0}\left(t^{\prime}\right)\right)=\pi_{1}(t)$, hence $t^{\prime} \in{ }^{\bullet} s$ and thus $\psi_{K}\left(t^{\prime}\right) \in{ }^{\bullet} \psi_{K}(s)$, which should be empty. Summing up $\pi_{1}(t)$ belongs to ${ }^{\bullet} f\left(\pi_{0}(s)\right)-f\left({ }^{\bullet} \pi_{0}(s)\right)$, which thereby is non-empty. Hence $\pi_{0}(s) \in \operatorname{in}(f)$.

Let us prove the second part, assume that $\pi_{1}: K_{1} \rightarrow Z_{1}$ is a deterministic process of $Z_{1}$. As in the general case, the net $N_{K_{0}}$ is isomorphic to the subnet of $N_{K_{1}}$ in the codomain of $\psi_{K}$, and thus it is an ordinary deterministic occurrence net. We already know that $\forall s \in O_{K_{0}}^{+} \cdot{ }^{\bullet} s=\emptyset$, and $\pi_{0}\left(O_{K_{0}}^{+}\right) \subseteq O_{Z_{0}}^{+}, \pi_{0}\left(O_{K_{0}}^{-}\right) \subseteq O_{Z_{0}}^{-}$. Thus we only need to show that $\forall s \in O_{K_{0}}^{-} . s=\emptyset$. Let $s \in O_{K_{0}}^{-}$. To prove that $s^{\bullet}=\emptyset$ just distinguish the case 1) $s \in \psi_{K}^{-1}\left(O_{K_{1}}^{-}\right)$and 2) $s \in \operatorname{out}\left(\psi_{K}\right)$. Then proceed exactly as in points (1) and (2) above, by substituting "-" and out $(\cdot)$ for + and in $(\cdot)$, respectively.

The process $\pi_{0}$ of $Z_{0}$ is uniquely determined up to isomorphism. Observe that fixing a representative in the isomorphism class of $\pi_{0}$ still we can have different choices for $\psi_{K}$ (obtained one from the other by composing with an automorphism over $N_{K_{0}}$ ).
Example. The embedding of Traveler into Global in Fig. 4 induces a projection of open net processes in the opposite direction. For instance, the right part of Fig. 7 shows a process of Global. Its projection along the embedding of Traveler into Global is shown on the left part of the same figure. Notice how transition acknowledged, which consumes a token in place ack, is replaced in the projection by a dangling output arc: an internal action in the larger net becomes an interaction with the environment in the smaller one.

REMARK 15 The construction of category Proc strictly resembles the construction of an arrow category. Denote by $\mathcal{N}:$ Proc $\rightarrow \mathbf{O N e t}$ the projection functor which maps each process $\pi: K \rightarrow Z$ to $Z$ and each process arrow $\left\langle\psi_{Z}, \psi_{K}\right\rangle$ to $\psi_{Z}$. Then, given an embedding $f: Z_{0} \rightarrow Z_{1}$ and a process $\pi_{1}: K_{1} \rightarrow Z_{1}$, a projection of $\pi_{1}$ along $f$, as defined above, is a cartesian arrow for $\pi_{1}$ and $f$.

If we restrict our attention to open net embeddings, thus obtaining the subcategories ONet * and Proc*, then the corresponding functor $\mathcal{N}^{*}$ is a fibration with total category Proc* and base category ONet ${ }^{*}$. Furthermore, the fibration $\mathcal{N}^{*}$ is split. In fact, the injectivity of the arrows in ONet* provides a choice of the pullbacks which are used for projections. Look at the diagram in Construction 13. When $f$ is injective, also $\psi_{K}$ is injective and thus we have a canonical choice $\left\langle K_{0}^{\prime}, \Psi_{K}^{\prime}, \pi_{0}^{\prime}\right\rangle$ for the construction, i.e.

- occurrence net $K_{0}^{\prime}$ :
$N_{K_{0}^{\prime}}$ is the subnet of $N_{K_{1}}$ identified as the image of $\psi_{K}$; the open places of $K_{0}^{\prime}$ are the open places in $K_{1}$ which belong to $K_{0}^{\prime}$ and the "interface places", namely the places in $K_{0}^{\prime}$ whose precondition is outside $K_{0}^{\prime}$, i.e.

$$
O_{K_{0}^{\prime}}^{+}=\left(O_{K_{1}}^{+} \cap S_{K_{0}^{\prime}}\right) \cup\left\{s \in S_{K_{0}^{\prime}}: \bullet s \cap\left(T_{K_{0}}-T_{K_{0}^{\prime}}\right) \neq \emptyset\right\}
$$

and $O_{K_{0}^{\prime}}^{-}$is defined in similar way.

- arrows $\psi_{K}^{\prime}$ and $\pi_{0}^{\prime}$ :
$\psi_{K}^{\prime}$ the inclusion of $K_{0}^{\prime}$ into $K_{1}$ and $\pi_{0}^{\prime}$ is uniquely determined by the requirement of commutativity.

The cleavage $c\left(f, \pi_{1}\right)=\left\langle\pi_{0}^{\prime},\left\langle f, \psi_{K}^{\prime}\right\rangle\right\rangle$ defined in this way is splitting.

## 4 Composing open nets

In this section we introduce a basic mechanism for composing open nets which will be characterized as a pushout construction in the category of open nets. Intuitively, two open nets $Z_{1}$ and $Z_{2}$ are composed by specifying a common subnet $Z_{0}$, and then by joining the two nets along $Z_{0}$. Consider, for instance, the open nets for the local workflows Traveler and Agency in the middle of Fig. 2. The two nets share the subnet Common depicted in the top of the same figure, which represents the "glue" between the two components. The net Global resulting from the


Figure 8: Category ONet does not have all pushouts.
composition of Traveler and Agency over the shared subnet Common is shown in the bottom part of Fig. 2. This composition is only defined if the embeddings of the components into the resulting net satisfy the constraints of open net morphisms. For example, if we remove the ingoing dangling arc of the place ticket in the net Traveler, the embedding of Common into Traveler would still represent a legal open net morphism. However, in this case the embedding of Traveler into Global would become illegal because of the new arc from issueTicket (see condition (i) of Definition 5).

Formally, given two nets $Z_{1}$ and $Z_{2}$ and a span $f_{1}: Z_{0} \rightarrow Z_{1}$ and $f_{2}: Z_{0} \rightarrow Z_{2}$, the composition operation constructs the corresponding pushout in ONet. Category ONet does not have all pushouts, while category Net does. We will see that this corresponds to the intuition that the composition operation can be performed in Net and then lifted to ONet, but only when it respects the interfaces specified by the various components, e.g., a new transition can be attached to a place only if such place is open. For instance it is possible to verify that there is no pushout for the arrows in Fig. 8, since intuitively the construction should merge all the places named $s$, attaching transition $t$ to a place in $Z_{1}$ which is not (output) open.

We start by recalling a characterization of pushouts in category Net.
Proposition 16 (PUSHOUT IN Net) Let $N_{1} \stackrel{f_{1}}{\leftarrow} N_{0} \xrightarrow{f_{2}} N_{2}$ be a span in Net. Then its pushout always exists, and can be defined as $N_{1} \xrightarrow{\alpha_{1}} N_{3} \stackrel{\alpha_{2}}{\leftarrow} N_{2}$, where the sets of places and transitions of $N_{3}$ are computed as the pushout in Set of the corresponding components:

$$
S_{3}=S_{1}+S_{0} S_{2} \quad \text { and } \quad T_{3}=T_{1}+T_{0} T_{2}
$$

with source and target functions defined by: for all $t \in T_{3}$, if $t=\alpha_{i}\left(t_{i}\right)$ with $t_{i} \in T_{i}$ and $i \in\{1,2\}$ then ${ }^{\bullet} t=\alpha_{i}{ }^{\oplus}\left({ }^{\bullet} t_{i}\right)$ and $t^{\bullet}=\alpha_{i}{ }^{\oplus}\left(t_{i} \bullet\right)$.

Next we formalize the condition which ensures the composability of a span in ONet.
DEFINITION 17 (COMPOSABLE SPAN) Let $Z_{1} \stackrel{f_{1}}{\leftarrow} Z_{0} \stackrel{f_{2}}{\leftrightarrows} Z_{2}$ be a span of open net morphisms. We say that $f_{1}$ and $f_{2}$ are composable if

1. $f_{2}\left(\operatorname{in}\left(f_{1}\right)\right) \subseteq O_{Z_{2}}^{+}$and $f_{2}\left(\operatorname{out}\left(f_{1}\right)\right) \subseteq O_{Z_{2}}^{-}$;
2. $f_{1}\left(\operatorname{in}\left(f_{2}\right)\right) \subseteq O_{Z_{1}}^{+}$and $f_{1}\left(\operatorname{out}\left(f_{2}\right)\right) \subseteq O_{Z_{1}}^{-}$.

In words, $f_{1}$ and $f_{2}$ are composable if the places which are used as interfaces by $f_{1}$, namely the places in $\left(f_{1}\right)$ and out $\left(f_{1}\right)$, are mapped by $f_{2}$ to input and output open places in $Z_{2}$, and also the symmetric condition holds. If, and only if, this condition is satisfied the pushout of $f_{1}$ and $f_{2}$ can be computed in Net and then lifted to ONet.


Figure 9: Pushout in ONet.

PROPOSITION 18 (PUSHOUTS IN ONet) Let $Z_{1} \stackrel{f_{1}}{\leftarrow} Z_{0} \xrightarrow{f_{2}} Z_{2}$ be a span in ONet (see the diagram in Fig. 9). Compute the pushout of the corresponding diagram in the category Net obtaining the net $N_{Z_{3}}$ and the morphisms $\alpha_{1}$ and $\alpha_{2}$, and then take as open places, for $x \in\{+,-\}$,

$$
O_{Z_{3}}^{x}=\left\{s_{3} \in S_{3} \mid \alpha_{1}^{-1}\left(s_{3}\right) \subseteq O_{Z_{1}}^{x} \wedge \alpha_{2}^{-1}\left(s_{3}\right) \subseteq O_{Z_{2}}^{x}\right\}
$$

Then $\left(\alpha_{1}, Z_{3}, \alpha_{2}\right)$ is the pushout in $\mathbf{O N e t}$ of $f_{1}$ and $f_{2}$ if and only if $f_{1}$ and $f_{2}$ are composable.
Proof. (if part) Let us show that, when $f_{1}$ and $f_{2}$ are composable, then $Z_{1} \xrightarrow{\alpha} Z_{3} \stackrel{\alpha_{2}}{\leftarrow} Z_{2}$ is a pushout in ONet.

We first prove that $\alpha_{1}$ and $\alpha_{2}$ are open net morphisms. The proof is carried out explicitly only for $\alpha_{1}$, since the case of $\alpha_{2}$ is completely analogous. First notice that

$$
\operatorname{in}\left(\alpha_{1}\right)=f_{1}\left(\operatorname{in}\left(f_{2}\right)\right)
$$

In fact, let $s_{1} \in \operatorname{in}\left(\alpha_{1}\right)$. Hence there exist a transition $t_{3} \in{ }^{\bullet} \alpha_{1}\left(s_{1}\right)-\alpha_{1}\left({ }^{\bullet} s_{1}\right)$. Since the square in Fig. 9 is a pushout in Net, there exists $s_{2} \in S_{2}$ such that $\alpha_{1}\left(s_{1}\right)=\alpha_{2}\left(s_{2}\right)$ and also $t_{2} \in{ }^{\bullet} s_{2}$ such that $\alpha_{2}\left(t_{2}\right)=t_{3}$ and $t_{2} \notin f_{2}\left(T_{0}\right)$. By using again the properties of pushouts, we deduce the existence of $s_{0} \in S_{0}$ such that $f_{1}\left(s_{0}\right)=s_{1}$ and $f_{2}\left(s_{0}\right)=s_{2}$. Now, $t_{2} \in{ }^{\bullet} f_{2}\left(s_{0}\right)-f_{2}\left(T_{0}\right) \subseteq$ $\cdot f_{2}\left(s_{0}\right)-f_{2}\left({ }^{\bullet} s_{0}\right)$. Hence $s_{0} \in \operatorname{in}\left(f_{2}\right)$ and thus $f_{1}\left(s_{0}\right)=s_{1} \in f_{1}\left(\operatorname{in}\left(f_{2}\right)\right)$. This proves that $\operatorname{in}\left(\alpha_{1}\right) \subseteq$ $f_{1}\left(\right.$ in $\left.\left(f_{2}\right)\right)$. The converse inclusion can be proved by reverting the proof steps.

Now, $\alpha_{1}$ is clearly a morphism in Net by construction. Furthermore, it satisfies also the condition $\alpha_{1}^{-1}\left(O_{Z_{3}}^{+}\right) \cup \operatorname{in}\left(\alpha_{1}\right) \subseteq O_{Z_{1}}^{+}$and $\alpha_{1}^{-1}\left(O_{Z_{3}}^{-}\right) \cup$ out $\left(\alpha_{1}\right) \subseteq O_{Z_{1}}^{-}$. For instance, the condition over input places is proved by noticing that $\alpha_{1}^{-1}\left(O_{Z_{3}}^{+}\right) \subseteq O_{Z_{1}}^{+}$by construction, and, in $\left(\alpha_{1}\right)=$ $f_{1}\left(\operatorname{in}\left(f_{2}\right)\right) \subseteq O_{Z_{1}}^{+}$by condition (2) of composability (Definition 17). Thus $\alpha_{1}$ is an open net morphism.

Moreover, for any pair of open net morphisms, $\beta_{1}: Z_{1} \rightarrow Z_{4}$ and $\beta_{2}: Z_{2} \rightarrow Z_{4}$, such that $\beta_{1} \circ f_{1}=\beta_{2} \circ f_{2}$, since $N_{Z_{1}} \xrightarrow{\alpha_{1}} N_{Z_{3}} \stackrel{\alpha_{2}}{\leftarrow} N_{Z_{2}}$ is a pushout in Net, there exists a unique arrow $h: Z_{3} \rightarrow Z_{4}$ in Net such that the diagram below commutes.


We only need to prove that $h$ is an open net morphism, by showing that it satisfies the condition over open places of Definition 5. Let us prove, for instance, that $h^{-1}\left(O_{4}^{+}\right) \cup \operatorname{in}(h) \subseteq O_{3}^{+}$. We divide the proof in two parts:
$-h^{-1}\left(O_{4}^{+}\right) \subseteq O_{3}^{+}$
Let $s_{3} \in h^{-1}\left(O_{4}^{+}\right)$, i.e., $s_{3} \in S_{3}$ and $h\left(s_{3}\right) \in O_{4}^{+}$. Let $s_{i} \in \alpha_{i}^{-1}\left(s_{3}\right)$ for some $i \in\{1,2\}$. By $h \circ \alpha_{i}=$
$\beta_{i}$ we have $\beta_{i}\left(s_{i}\right)=h\left(s_{3}\right) \in O_{4}^{+}$. Thus, since $\beta_{i}$ is an open net morphism, $s_{i} \in O_{i}^{+}$. In other words, $\alpha_{1}^{-1}\left(s_{3}\right) \subseteq O_{1}^{+}$and $\alpha_{2}^{-1}\left(s_{3}\right) \subseteq O_{2}^{+}$. Hence, by definition of $O_{3}^{+}, s_{3} \in O_{3}^{+}$.
$-\operatorname{in}(h) \subseteq O_{3}^{+}$
Let $s_{3} \in \operatorname{in}(h)$, namely ${ }^{\bullet} h\left(s_{3}\right)-h\left({ }^{\bullet} s_{3}\right) \neq \emptyset$. Observe that if $s_{3}=\alpha_{i}\left(s_{i}\right)$ for some $i \in\{1,2\}$, then we have that

$$
\begin{aligned}
\emptyset & \neq \bullet h\left(s_{3}\right)-h\left(\bullet s_{3}\right)= \\
& =\bullet h\left(\alpha_{i}\left(s_{i}\right)\right)-h\left(\bullet \alpha_{i}\left(s_{i}\right)\right) \\
& =\bullet \beta_{i}\left(s_{i}\right)-h\left(\bullet \alpha_{i}\left(s_{i}\right)\right) \\
& \subseteq \bullet \beta_{i}\left(s_{i}\right)-h\left(\alpha_{i}\left(\bullet s_{i}\right)\right) \quad\left[\text { since } \bullet \alpha_{i}\left(s_{i}\right) \supseteq \alpha_{i}\left(\bullet s_{i}\right)\right] \\
& =\bullet \beta_{i}\left(s_{i}\right)-\beta_{i}\left(\bullet s_{i}\right)
\end{aligned}
$$

Therefore $s_{i} \in \operatorname{in}\left(\beta_{i}\right)$, and thus, since $\beta_{i}$ is an open net morphism, $s_{i} \in O_{i}^{+}$. Hence $s_{3} \in \alpha_{i}^{-1}\left(O_{i}^{+}\right)$. Summing up, we deduce that $\alpha_{1}^{-1}\left(s_{3}\right) \subseteq O_{1}^{+}$and $\alpha_{2}^{-1}\left(s_{3}\right) \subseteq O_{2}^{+}$. Hence, by definition of $O_{3}^{+}$, $s_{3} \in O_{3}^{+}$.
(only if part) To prove composability of $f_{1}$ and $f_{2}$ is also necessary for ensuring that the pushout computed in Net is lifted to a pushout in ONet, suppose, for instance, that there exists $s_{2} \in$ $f_{2}\left(\operatorname{in}\left(f_{1}\right)\right)$ and $s_{2} \notin O_{2}^{+}$. Hence there is $s_{0} \in \operatorname{in}\left(f_{1}\right)$ such that $s_{2}=f_{2}\left(s_{0}\right)$.

Suppose, by contradiction, that the described construction gives a pushout $Z_{1} \xrightarrow{\alpha_{1}} Z_{3} \stackrel{\alpha_{2}}{\leftarrow} Z_{2}$ in ONet. Hence the places $s_{1}=f_{1}\left(s_{0}\right)$ and $s_{2}=f_{2}\left(s_{0}\right)$ have a common image $s_{3}=\alpha_{1}\left(s_{1}\right)=$ $\alpha_{2}\left(s_{2}\right)$. Since $s_{0} \in \operatorname{in}\left(f_{1}\right)$, there exists $t_{1} \in{ }^{\bullet} f_{1}\left(s_{0}\right)-f_{1}\left({ }^{\bullet} s_{0}\right)$. Thus $s_{3}=\alpha_{1}\left(s_{1}\right) \in \alpha_{1}\left(t_{1}\right)^{\bullet}$. Moreover, from the fact that $s_{2} \notin O_{2}^{+}$, by definition of open net morphism, we have $s_{2} \notin \mathrm{in}\left(\alpha_{2}\right)$. Hence there exists $t_{2} \in{ }^{\bullet} s_{2}$ such that $\alpha_{2}\left(t_{2}\right)=\alpha_{1}\left(t_{1}\right)$. Therefore there is $t_{0} \in T_{0}$ such that $f_{1}\left(t_{0}\right)=t_{1}$ and $f_{2}\left(t_{0}\right)=t_{2}$. But this contradicts the fact that $t_{1} \in{ }^{\bullet} f_{1}\left(s_{0}\right)-f_{1}\left({ }^{\bullet} s_{0}\right)$.

It is worth stressing that the pushout in ONet might exists also when $f_{1}$ and $f_{2}$ are not composable. This is the case for the diagram in Fig. 10.(a), which is a pushout in ONet, although the underlying diagram in Net is not a pushout. Indeed, $f_{1}$ and $f_{2}$ are not composable since, for instance, $f_{2}\left(\operatorname{out}\left(f_{1}\right)\right)=f_{2}\left(\left\{s_{0}\right\}\right)=\left\{s_{2}\right\} \nsubseteq O_{2}^{-}$. In this case the construction described in Proposition 18 does not work: it leads to the diagram in Fig. 10.(b), where the mappings $\alpha_{i}: Z_{i} \rightarrow Z_{3}$ are not open net morphisms, since, for instance $s_{1} \in$ out $\left(\alpha_{1}\right)$, but $s_{1} \notin O_{1}^{-}$.

One could be tempted to assume a different notion of composable span, i.e., to define $f_{1}$ and $f_{2}$ composable whenever their pushout exists in ONet. However, according to our intuition, morphisms $f_{1}$ and $f_{2}$ define a kind of "composition plan", which specifies that the images of $Z_{0}$ in $Z_{1}$ and $Z_{2}$ must be fused. The effect of the composition operation should be local, in the sense that nothing more than the images of $Z_{0}$ should be affected by the fusion. This fact is formalized by requiring that the pushout in ONet is obtained by lifting the pushout in Net. Observe that, instead, in the pushout depicted in Fig. 10.(a), also transitions $t_{1}$ and $t_{2}$, which are not in the common subnet $Z_{0}$, get fused.

## 5 Amalgamating deterministic processes

Let $f_{1}: Z_{0} \rightarrow Z_{1}$ and $f_{2}: Z_{0} \rightarrow Z_{2}$ be a composable span of open net embeddings and consider the corresponding composition, i.e., the pushout in ONet, as depicted in Fig. 9. We would like to establish a clear relationship among the behaviours of the involved nets. Roughly speaking, we would like that the behaviour of $Z_{3}$ could be constructed "compositionally" out of the behaviours of $Z_{1}$ and $Z_{2}$.

In this section we show how this can be done for deterministic processes. Given two deterministic processes $\pi_{1}$ of $Z_{1}$ and $\pi_{2}$ of $Z_{2}$ which "agree" on $Z_{0}$, we construct a deterministic process $\pi_{3}$ of $Z_{3}$ by "amalgamating" $\pi_{1}$ and $\pi_{2}$. Vice versa, each deterministic process $\pi_{3}$ of $Z_{3}$ can be projected over two deterministic processes $\pi_{1}$ and $\pi_{2}$ of $Z_{1}$ and $Z_{2}$, respectively, which


Figure 10: (a) A pushout in ONet of two non-composable arrows. (b) The pushout of the same arrows in Net.
can be amalgamated to produce $\pi_{3}$ again. Hence, all and only the deterministic processes of $Z_{3}$ can be obtained by amalgamating the deterministic processes of the components $Z_{1}$ and $Z_{2}$. This is formalized by showing that, working up to isomorphism, the amalgamation and decomposition operations are inverse to each other. This leads to a bijective correspondence between the processes of $Z_{3}$ and pair of processes of the components $Z_{1}$ and $Z_{2}$ which agree on the common subnet $Z_{0}$.

### 5.1 Pushout of deterministic occurrence open nets

As a first step towards the amalgamation of processes we identify a suitable condition which ensures that the pushout of deterministic occurrence open nets exists and produces a net in the same class. This condition will be used later to formalize the intuitive idea of processes of different nets which "agree" on a common part.

First, given a span $K_{1} \stackrel{f_{1}}{\leftarrow} K_{0} \xrightarrow{f_{2}} K_{2}$ we introduce the notion of causality relation induced by $K_{1}$ and $K_{2}$ over $K_{0}$. When the two nets are composed the corresponding causality relations get "fused". Hence, to avoid the creation of cyclic causal dependencies in the resulting net, the induced causality will be required to be a strict partial order.

DEFINITION 19 (INDUCED CAUSALITY AND CONSISTENT SPAN) Let $K_{1} \stackrel{f_{1}}{\leftarrow} K_{0} \xrightarrow{f_{2}} K_{2}$ be a span in ONet, where $K_{i}(i \in\{0,1,2\})$ are occurrence open nets. The relation of causality $<_{1,2}$ induced over $K_{0}$ by $K_{1}$ and $K_{2}$, through $f_{1}$ and $f_{2}$ is the least transitive relation such that for any $x_{0}, y_{0}$ in $K_{0}$, if $f_{1}\left(x_{0}\right)<1 f_{1}\left(y_{0}\right)$ or $f_{2}\left(x_{0}\right)<2 f_{2}\left(y_{0}\right)$ then $x_{0}<1,2 y_{0}$.

We say that the span is consistent, written $f_{1} \uparrow f_{2}$, if $f_{1}$ and $f_{2}$ are composable and the induced causality $<_{1,2}$ is a finitary strict partial order.

We next show that the composition operation in ONet, when applied to a consistent span of deterministic occurrence nets, produces a deterministic occurrence net. We first need a preliminary result.

Lemma 20 Let $K_{1} \stackrel{f_{1}}{\leftarrow} K_{0} \xrightarrow{f_{2}} K_{2}$ be a composable span in ONet, where $K_{i}(i \in\{0,1,2\})$ are deterministic occurrence open nets. Let $K_{1} \xrightarrow{\alpha_{1}} K_{3} \stackrel{\alpha_{2}}{\leftarrow} K_{2}$ be the pushout.


For any $x_{0}, y_{0}$ in $K_{0}$, if we let $x_{3}=\alpha_{1}\left(f_{1}\left(x_{0}\right)\right)=\alpha_{2}\left(f_{2}\left(x_{0}\right)\right)$ and $y_{3}=\alpha_{1}\left(f_{1}\left(y_{0}\right)\right)=\alpha_{2}\left(f_{2}\left(y_{0}\right)\right)$, then

$$
x_{0}<1,2 y_{0} \quad \text { iff } \quad x_{3}<3 y_{3}
$$

Proof. Below we will freely use the fact that open net morphisms, and thus, in particular $\alpha_{1}$ and $\alpha_{2}$, preserve the causality relation, in the sense that if $x_{i}<_{i} y_{i}$ in $K_{i}(i \in\{1,2\})$ then $\alpha_{i}\left(x_{i}\right)<3 \alpha_{i}\left(y_{i}\right)$.
$\Rightarrow$ Suppose that $x_{0}<1,2 y_{0}$. There are two possible cases:

- The causal dependence is directly induced by a causal dependence in $K_{1}$ or $K_{2}$, namely $f_{i}\left(x_{0}\right)<_{i} f_{i}\left(y_{0}\right)$ for some $i \in\{1,2\}$. Since $\alpha_{i}$ preserve causality, $\alpha_{i}\left(f_{i}\left(x_{0}\right)\right)<3 \alpha_{i}\left(f_{i}\left(y_{0}\right)\right)$, namely $x_{3}<3 y_{3}$.
- Otherwise, the causal dependence is generated by the transitive closure, namely there is $z_{0}$ such that $x_{0}<1,2 z_{0}<1,2 y_{0}$. Hence, an inductive reasoning allows us to conclude that $x_{3}<3$ $\alpha_{i}\left(f_{i}\left(z_{0}\right)\right)<3 y_{3}$ and thus $x_{3}<3 y_{3}$.
$(\Leftrightarrow)$ Let $\prec_{i}$ denote the immediate causality in $K_{i}$, i.e., $x \prec_{i} y$ if $x<_{i} y$ and there is no $z$ such that $x<_{i} z<_{i} y$. It is easy to see that for any $x_{3}, y_{3}$ in $K_{3}$,

$$
x_{3} \prec_{3} y_{3} \quad \text { iff } \quad \text { there are } i \in\{1,2\}, x_{i}, y_{i} \text { in } K_{i} \text { such that } x_{3}=\alpha_{i}\left(x_{i}\right), y_{3}=\alpha_{i}\left(y_{i}\right), x_{i} \prec_{i} y_{i}
$$

Assume that $x_{3}<_{3} y_{3}$. Then there is a $\prec_{3}$ chain $x_{3}=x_{3}^{1} \prec_{3} x_{3}^{2} \prec_{3} \ldots \prec_{3} x_{3}^{n}=y_{3}$. Let $C=$ $\left\{x_{3}^{1}, \ldots, x_{3}^{n}\right\}$. By the remark above, if $C$ is included in $\alpha_{i}\left(S_{i} \cup T_{i}\right)$ for some $i \in\{1,2\}$, then $f_{i}\left(x_{0}\right)<_{i} f_{i}\left(y_{0}\right)$, and thus $x_{0}<{ }_{1,2} y_{0}$. More generally, since $K_{3}$ is obtained as pushout of $K_{1}$ and $K_{2}$, the chain $C$ can be divided into $h+1$ segments $x_{3}, \ldots, x_{3}^{k_{1}}, \ldots, x_{3}^{k_{2}}, \ldots, x_{3}^{k_{h}}, \ldots y_{3}$, such that each segment is included in $\alpha_{i}\left(S_{i} \cup T_{i}\right)$ for some $i \in\{1,2\}$ and any "border" element $x_{3}^{k_{j}}$ is in $\alpha_{1}\left(S_{1} \cup T_{1}\right) \cap \alpha_{2}\left(S_{2} \cup T_{2}\right)$. By general properties of pushouts, for any $j$ we can find $x_{0}^{j} \in S_{0} \cup T_{0}$, such that $\alpha_{i}\left(f_{i}\left(x_{0}^{j}\right)\right)=x_{3}^{k_{j}}$ for $i \in\{1,2\}$.

Therefore, by the remark about immediate precedence in $K_{3}$, surely, for any $j$ there is some $i \in\{1,2\}$, such that

$$
\begin{equation*}
f_{i}\left(x_{0}^{j}\right)<_{i} f_{i}\left(x_{0}^{j+1}\right) \tag{2}
\end{equation*}
$$

and, similarly, $f_{i}\left(x_{0}\right)<i_{x} f_{i}\left(x_{0}^{1}\right)$ and $f_{i}\left(x_{0}^{h}\right)<i_{y} f_{i}\left(y_{0}\right)$ for suitable $i_{x}, i_{y} \in\{1,2\}$. But recalling the definition of induced causality, we deduce that $x_{0}<_{1,2} x_{0}^{1}<1,2 x_{0}^{2}<1,2 \ldots x_{0}^{k}<1,2 y_{0}$, and thus $x_{0}<1,2 y_{0}$.

PROPOSITION 21 Let $K_{1} \stackrel{f_{1}}{\leftarrow} K_{0} \xrightarrow{f_{2}} K_{2}$ be a composable span in ONet, where $K_{i}(i \in\{0,1,2\})$ are deterministic occurrence open nets, and let $K_{1} \xrightarrow{\alpha_{1}} K_{3} \stackrel{\alpha_{2}}{\leftarrow} K_{2}$ be the pushout in ONet.


Then $f_{1} \uparrow f_{2}$ if and only if the pushout object $K_{3}$ is a deterministic occurrence open net.

Proof. $(\Rightarrow)$ We know that $K_{3}$ is a well defined open net. To prove that $K_{3}$ is a deterministic open occurrence net we start showing that the underlying net $N_{K_{3}}$ is a deterministic occurrence net.
(1.a) causality $<_{3}$ is a strict partial order.

Assume, by contradiction, that $<_{3}$ is not irreflexive. Hence we can find a cycle of immediate causality in $K_{3}$, i.e., $x_{3}^{1} \prec_{3} x_{3}^{2} \prec_{3} \ldots \prec_{3} x_{3}^{n} \prec_{3} x_{3}^{1}$, an let $C=\left\{x_{3}^{1}, \ldots, x_{3}^{n}\right\}$. The cycle $C$ cannot be included in $\alpha_{i}\left(S_{i} \cup T_{i}\right)$ for some $i \in\{1,2\}$, otherwise $\prec_{i}$ would be cyclic in $K_{i}$. Hence there exists an item $x_{3} \in C \cap \alpha_{1}\left(S_{1} \cup T_{1}\right) \cap \alpha_{2}\left(S_{2} \cup T_{2}\right)$. Consider $x_{0}$ in $K_{0}$ such that $\alpha_{i}\left(f_{i}\left(x_{0}\right)\right)=x_{3}$. Since $x_{3}<3 x_{3}$, by Lemma 20, we have $x_{0}<1,2 x_{0}$ contradicting the hypothesis that the span is consistent.
(1.b) causality $<_{3}$ is finitary.

The proof can be done as in the point before, by exploiting the finitariness of causality in $K_{1}$ and $K_{2}$, and Lemma 20. Assuming the existence of a infinite descending chain of $<_{3}$ in $K_{3}$ we deduce that $<_{1,2}$ has an infinite descending chain in $K_{0}$, contradicting the assumption that the span is consistent and thus $<_{1,2}$ is finitary.

## (1.c) $K_{3}$ does not have forward conflicts

Suppose, by contradiction, that there exists a place $s_{3} \in S_{3}$ such that $\left|s_{3} \bullet\right|>1$. Let $t_{3}, t_{3}^{\prime} \in s_{3} \bullet$ such that $t_{3} \neq t_{3}^{\prime}$. Then surely $t_{3} \in \alpha_{1}\left(T_{1}\right)-\alpha_{2}\left(T_{2}\right)$ and $t_{3}^{\prime} \in \alpha_{2}\left(T_{2}\right)-\alpha_{1}\left(T_{1}\right)$, otherwise we would have a forward conflict in one of $K_{1}$ or $K_{2}$. Therefore $s_{3} \in \alpha_{1}\left(S_{1}\right) \cap \alpha_{2}\left(S_{2}\right)$. Let $s_{1} \in S_{1}$ such that $\alpha_{1}\left(s_{1}\right)=s_{3}$. Then $s_{1} \in \operatorname{out}\left(\alpha_{1}\right)$. But, since $s_{1}^{\bullet} \neq \emptyset$ this contradicts the assumption that $K_{1}$ is a deterministic open net.
(1.d) $K_{3}$ does not have backward conflicts

Assume, by contradiction, that there is a backward conflict in $K_{3}$, i.e. there are $t_{3}, t_{3}^{\prime} \in T_{3}$ with a common place in their post-set $s_{3} \in t_{3}{ }^{\bullet} \cap t_{3}^{\prime \bullet}$. Consider $s_{1} \in S_{1}$ such that $\alpha_{1}\left(s_{1}\right)=s_{3}$. Since $K_{2}$ and $K_{3}$ do not have backward conflicts, necessarily $t_{3} \in \alpha_{1}\left(T_{1}\right)-\alpha_{2}\left(T_{2}\right)$. Then $s_{1} \in \operatorname{in}\left(\alpha_{1}\right)$, and thus, since $\alpha_{1}$ is an open net morphism, $s_{1} \in O_{1}^{+}$. But this contradicts the fact that $K_{1}$ is an open occurrence net, since $s_{1} \in t_{1}{ }^{\bullet}$.

To conclude it remains to show the validity of the conditions over open places:
(2.a) $\forall s \in O_{3}^{-} . s^{\bullet}=\emptyset$

Same proof as point (1.c)
(2.b) $\forall s \in O_{3}^{+}$. $s=\emptyset$.

Same proof as point (1.d)
$(\Leftarrow)$ Let $K_{1} \stackrel{f_{1}}{\leftarrow} K_{0} \xrightarrow{f_{2}} K_{2}$ be a composable span in ONet, where $K_{i}(i \in\{0,1,2\})$ are deterministic occurrence open nets and assume that the pushout $K_{3}$ is an open deterministic net. We must show that induced causality $<_{1,2}$ is a finitary strict partial order. Let $f_{3}=\alpha_{1} \circ f_{1}=\alpha_{2} \circ f_{2}$. To conclude just recall that $<_{3}$ is a finitary strict partial order and then use the fact that, by Lemma 20, $x_{0}<1,2 y_{0}$ iff $f_{3}\left(x_{0}\right)<3 f_{3}\left(y_{0}\right)$.

### 5.2 Amalgamating deterministic processes

As mentioned before two deterministic processes $\pi_{1}$ of $Z_{1}$ and $\pi_{2}$ of $Z_{2}$ can be amalgamated only when they agree on the common subnet $Z_{0}$, an idea which is formalized by resorting to the notion of consistent span of deterministic occurrence open nets. In the rest of this section we will refer to a fixed pushout diagram in ONet, as represented in Fig. 9, where $f_{1}$ and $f_{2}$ are a composable span of open net embeddings.

(a)

(b)

Figure 11: Figures for Lemma 23.

DEFINITION 22 (AGREEMENT OF DETERMINISTIC PROCESSES) We say that two deterministic processes $\pi_{1}: K_{1} \rightarrow Z_{1}$ and $\pi_{2}: K_{2} \rightarrow Z_{2}$ agree on $Z_{0}$ if there are projections $\left\langle\pi_{0}, \psi^{i}\right\rangle$ along $f_{i}$ of $\pi_{i}$ for $i \in\{1,2\}$ such that $\psi_{K}^{1} \uparrow \psi_{K}^{2}$ (i.e., the span $K_{1} \stackrel{\psi_{K}^{1}}{\leftarrow} K_{0} \stackrel{\psi_{K}^{2}}{\rightarrow} K_{2}$ is consistent). In this case $\left\langle\pi_{0}, \psi^{1}\right\rangle$ and $\left\langle\pi_{0}, \psi^{2}\right\rangle$ are called agreement projections for $\pi_{1}$ and $\pi_{2}$.

Before introducing the notion of amalgamation we need to recall a simple technical result.
Lemma 23 1) Consider the diagram in Set depicted in Fig 11.(a). If the diagram is a pushout and $f$ is injective, then the diagram is also a pullback.
2) Consider a commuting diagram in a category $\mathbf{C}$, as depicted in Fig 11.(b). If the internal square, marked by PB , and the external one are pullbacks, then other internal square is a pullback as well.

DEFINITION 24 (AMALGAMATION OF PROCESSES) Let $\pi_{i}: K_{i} \rightarrow Z_{i}(i \in\{0,1,2,3\})$ be deterministic processes and let $\left\langle\pi_{0}, \psi^{1}\right\rangle$ and $\left\langle\pi_{0}, \psi^{2}\right\rangle$ be agreement projections of $\pi_{1}$ and $\pi_{2}$ along $f_{1}$ and $f_{2}$ (see Fig. 12.(a)). We say that $\pi_{3}$ is an amalgamation of $\pi_{1}$ and $\pi_{2}$, written $\pi_{3}=\pi_{1}+{ }_{\psi^{1}, \psi^{2}} \pi_{2}$, if there exist projections $\left\langle\pi_{1}, \phi^{1}\right\rangle$ and $\left\langle\pi_{2}, \phi^{2}\right\rangle$ of $\pi_{3}$ over $Z_{1}$ and $Z_{2}$, respectively, such that the upper square is a pushout in ONet.

We next give a more constructive characterization of process amalgamation, which also proves that the result is unique up to isomorphism.

Proposition 25 (AMALGAMATION CONSTRUCTION) Let $\pi_{1}: K_{1} \rightarrow Z_{1}$ and $\pi_{2}: K_{2} \rightarrow Z_{2}$ be deterministic processes that agree on $Z_{0}$, and let $\left\langle\pi_{0}, \psi^{1}\right\rangle$ and $\left\langle\pi_{0}, \psi^{2}\right\rangle$ be corresponding agreement projections. Then the amalgamation $\pi_{1}+{ }_{\psi^{1}, \psi^{2}} \pi_{2}$ is a process $\pi_{3}: K_{3} \rightarrow Z_{3}$, where the net $K_{3}$ is obtained as the pushout in ONet of $\psi_{K}^{1}: K_{0} \rightarrow K_{1}$ and $\psi_{K}^{2}: K_{0} \rightarrow K_{2}$ and the process mapping $\pi_{3}: K_{3} \rightarrow Z_{3}$ is uniquely determined by the universal property of the underlying pushout diagram in Net (see Fig. 12.(a)). Hence $\pi_{1}+{ }_{\psi^{1}, \psi^{2}} \pi_{2}$ is unique up to isomorphism.

Proof. We first show that $\pi_{3}$, defined as above, is a well-defined process of $Z_{3}$. Since by hypothesis $\psi_{K}^{1} \uparrow \psi_{K}^{2}$, we know by Proposition 21, that $K_{3}$ is a deterministic occurrence open net.

Furthermore, $\pi_{3}$ is an arrow in Net. To conclude that $\pi_{3}$ is a deterministic open net process we prove that $\pi_{3}\left(O_{K_{3}}^{+}\right) \subseteq O_{Z_{3}}^{+}$and $\pi_{3}\left(O_{K_{3}}^{-}\right) \subseteq O_{Z_{3}}^{-}$.

To this aim, we first observe that in the diagram of Fig. 12, the square with vertices $K_{1}, K_{3}$, $Z_{3}, Z_{1}$ is a pullback. Let us show, for instance, that the place component of the morphisms form a pullback. Actually, it suffices to show that given $s_{1} \in S_{Z_{1}}$ and $s_{3}^{\prime} \in S_{K_{3}}$ such that $\alpha_{1}\left(s_{1}\right)=\pi_{3}\left(s_{3}^{\prime}\right)$, there exists $s_{1}^{\prime} \in S_{K_{1}}$ such that $\phi_{K}^{1}\left(s_{1}^{\prime}\right)=s_{3}^{\prime}$. In fact, by commutativity of the diagram this implies that $\alpha_{1}\left(\pi_{1}\left(s_{1}^{\prime}\right)\right)=\alpha_{1}\left(s_{1}\right)$, and thus, by injectivity of $\alpha_{1}, \pi_{1}\left(s_{1}^{\prime}\right)=s_{1}$. Furthermore, uniqueness of $s_{0}^{\prime}$ follows from the injectivity of $\phi_{K}^{1}$. Hence, let us consider $s_{1} \in S_{Z_{1}}$ and $s_{3}^{\prime} \in S_{K_{3}}$ such that $\alpha_{1}\left(s_{1}\right)=\pi_{3}\left(s_{3}^{\prime}\right)=s_{3}$. If $s_{3}^{\prime}=\phi_{K}^{1}\left(s_{1}^{\prime}\right)$ for some $s_{1}^{\prime} \in S_{K_{1}}$, then we conclude. Otherwise, since


Figure 12: Amalgamation of open net processes.
the upper square is a pushout, necessarily $s_{3}^{\prime}=\phi_{K}^{2}\left(s_{2}^{\prime}\right)$ for some $s_{2}^{\prime} \in S_{K_{2}}$. Then $\alpha_{2}\left(\pi_{2}\left(s_{2}^{\prime}\right)\right)=$ $s_{3}=\alpha_{1}\left(s_{1}\right)$. Since the square $Z_{0}, Z_{1}, Z_{2}, Z_{3}$ is a pushout, this implies that there exists $s_{0}$ in $Z_{0}$ such that $f_{1}\left(s_{0}\right)=s_{1}$ and $f_{2}\left(s_{0}\right)=\pi_{2}\left(s_{2}^{\prime}\right)$. But, since the square $Z_{2}, K_{2}, K_{0}, Z_{0}$ is a pullback, there must be $s_{0}^{\prime} \in S_{K_{0}}$ such that $\psi_{K}^{2}\left(s_{0}^{\prime}\right)=s_{2}^{\prime}$. Hence, if we take $s_{1}^{\prime}=\psi_{K}^{1}\left(s_{0}^{\prime}\right)$, we have $\phi_{K}^{1}\left(x_{1}^{\prime}\right)=$ $\phi_{K}^{1}\left(\psi_{K}^{1}\left(s_{0}^{\prime}\right)\right)=\phi_{K}^{2}\left(\psi_{K}^{2}\left(s_{0}^{\prime}\right)\right)=\phi_{K}^{2}\left(s_{2}^{\prime}\right)=s_{3}^{\prime}$, as desired.

Now, take $s_{3}^{\prime} \in O_{K_{3}}^{+}$and consider $\pi_{3}\left(s_{3}^{\prime}\right)$. We distinguish the following (non exclusive) cases: - $\pi_{3}\left(s_{3}^{\prime}\right)=\alpha_{1}\left(s_{1}\right)$ for some $s_{1} \in S_{Z_{1}}$.

Since, as observed above, the square $K_{1}, K_{3}, Z_{3}, Z_{1}$ is a pullback, there is $s_{1}^{\prime} \in S_{K_{1}}$ such that $\phi_{K}^{1}\left(s_{1}^{\prime}\right)=s_{3}$ and $\pi_{1}\left(s_{1}^{\prime}\right)=s_{1}$. From the first equality, since $\phi_{K}^{1}$ is an open net morphism, we deduce that $s_{1}^{\prime} \in O_{K_{1}}^{+}$, and thus, by the second equality, since $\pi_{1}$ is a process, $s_{1} \in O_{Z_{1}}^{+}$.
$-\pi_{3}\left(s_{3}^{\prime}\right)=\alpha_{2}\left(s_{2}\right)$ for some $s_{2} \in S_{Z_{2}}$.
As above, we can conclude $s_{2} \in O_{Z_{2}}^{+}$.
Summing up the two cases, we have that $\alpha_{1}^{-1}\left(\pi_{3}\left(s_{3}\right)\right) \subseteq O_{Z_{1}}^{+}$and $\alpha_{2}^{-1}\left(\pi_{3}\left(s_{3}\right)\right) \subseteq O_{Z_{2}}^{+}$. Therefore, by construction of the pushout in ONet (see Proposition 18) $\pi_{3}\left(s_{3}\right) \in O_{Z_{3}}^{+}$. Thus $\pi_{3}\left(O_{K_{3}}^{+}\right) \subseteq O_{Z_{3}}^{+}$. The other inclusion, i.e., $\pi_{3}\left(O_{K_{3}}^{-}\right) \subseteq O_{Z_{3}}^{-}$, can be shown in a completely symmetric way.

The last thing to observe is that $\left\langle\pi_{i}, \phi^{i}\right\rangle$ is a projection of $\pi_{3}$ along $\alpha_{i}$, for $i \in\{1,2\}$. But this fact immediately follows from the above observations, since the squares $K_{i}, K_{3}, Z_{3}, Z_{i}$ are pullbacks in Net. Furthermore, $O_{i}^{+}=\phi_{K}^{i}{ }^{-1}\left(O_{3}^{+}\right) \cup$ in $\left(\phi_{K}^{i}\right)$. In fact, $\phi_{K}^{i}$ is an open net morphism and thus $\phi_{K}^{i-1}\left(O_{3}^{+}\right) \cup \operatorname{in}\left(\phi_{K}^{i}\right) \subseteq O_{i}^{+}$. To prove the other inclusion, for instance, when $i=1$, let $s_{1} \in O_{K_{1}}^{+}$. If $\phi_{K}^{1}\left(s_{1}\right) \in O_{K_{3}}^{+}$, we have that $s_{1} \in \phi_{K}^{1-1}\left(O_{K_{3}}^{+}\right)$. Otherwise, by recalling how the open places of the pushout object are defined (see Proposition 18), we deduce that there exists $s_{2} \in S_{K_{2}}$ such that $\phi_{K}^{2}\left(s_{2}\right)=\phi_{K}^{1}\left(s_{1}\right)$ and $s_{2} \notin O_{K_{2}}^{+}$. Since the upper square is a pushout, there must be $s_{0} \in S_{K_{0}}$ such that $\psi_{K}^{1}\left(s_{0}\right)=s_{1}$ and $\psi_{K}^{2}\left(s_{0}\right)=s_{2}$. Since $\psi_{K}^{1}$ is an open net morphism, this implies that $s_{0} \in O_{K_{0}}^{+}$. Since $s_{2} \notin O_{K_{2}}^{+}$and $\pi_{0}$ is a projection of $\pi_{2}$, we have that $s_{0} \in \operatorname{in}\left(\psi_{K}^{2}\right)$. Therefore, since the upper square is a pushout in Net, $s_{1} \in \operatorname{in}\left(\phi_{K}^{1}\right)$, as desired.

The amalgamation construction can be given a more elegant (although less constructive) characterization. In fact, process $\pi_{3}$ (and the corresponding process morphisms $\phi^{1}$ and $\phi^{2}$ ) can be obtained by taking the pushout in Proc of the arrows $\psi^{1}: \pi_{0} \rightarrow \pi_{1}$ and $\psi^{2}: \pi_{0} \rightarrow \pi_{2}$.

The next result shows how each deterministic process of a composed net can be constructed as the amalgamation of deterministic processes of the components.

PROPOSITION 26 (DECOMPOSITION OF PROCESSES) Let $\pi_{3}: K_{3} \rightarrow Z_{3}$ be a deterministic process of $Z_{3}$ and, for $i \in\{1,2\}$, let $\left\langle\pi_{i}, \phi^{i}\right\rangle$ be projections of $\pi_{3}$ along $\alpha_{i}$. Then process $\pi_{3}$ can be recovered as a suitable amalgamation of $\pi_{1}$ and $\pi_{2}$.

Proof. Let $\left\langle\pi_{i}, \phi^{i}\right\rangle$ be projections of $\pi_{3}$ along $\alpha_{i}$ for $i \in\{1,2\}$. Take any projection $\left\langle\pi_{0}, \psi^{1}\right\rangle$ of $\pi_{1}$ along $f_{1}$. The non-dotted part of the diagram below summarizes the situation:


Then projection $\left\langle\pi_{0}, \psi^{2}\right\rangle$ of $\pi_{2}$ along $f_{2}$ is obtained by defining $\psi_{K}^{2}$ as the arrow determined by the universal property of the pullback with vertices $K_{3}, Z_{3}, Z_{2}$ and $K_{2}$. To show that the projection is well-defined, first observe two facts

1) the square with vertices $K_{0}, Z_{0}, Z_{2}, K_{2}$ is indeed a pullback in Net.

In fact, by construction, the diagram commutes. Furthermore, in category Net the square with vertices $K_{0}, K_{3}, Z_{3}, Z_{0}$ is a pullback (since it can be viewed as the composition of two pullbacks $K_{0}, K_{1}, Z_{1}, Z_{0}$ and $\left.K_{1}, K_{3}, Z_{1}, Z_{3}\right)$. However the same square is composed out of $K_{0}, K_{2}, Z_{2}, Z_{0}$ and $K_{2}, K_{3}, Z_{3}, Z_{2}$. Hence, by Lemma 23 , also the square $K_{0}, Z_{0}, Z_{2}, K_{2}$ is a pullback in Net.
2) the upper square with vertices $K_{0}, K_{1}, K_{3}, K_{2}$ is a pushout in Net.

In fact, the vertical faces of the cube are pullbacks and the lower face is a pushout, hence, by the 3-cube lemma $\left[\mathrm{CEL}^{+} 96\right]$, we can conclude that the upper square is a pushout.

Let us prove that $\left\langle\pi_{0}, \psi^{2}\right\rangle$ is a well-defined projection of $\pi_{2}$ along $f_{2}$, by showing that

$$
O_{K_{0}}^{+}=\psi_{K}^{2-1}\left(O_{K_{2}}^{+}\right) \cup \operatorname{in}\left(\psi_{K}^{2}\right) \text { and } O_{K_{0}}^{-}=\psi_{K}^{2-1}\left(O_{K_{2}}^{-}\right) \cup \operatorname{out}\left(\psi_{K}^{2}\right)
$$

We restrict our attention to the first equality (the second one is proved by a symmetric reasoning), and we show the two inclusions separately.
$(\subseteq)$ Let $s_{0} \in O_{K_{0}}^{+}$. Since $\left\langle\pi_{0}, \psi^{1}\right\rangle$ is a projection of $\pi_{1}$, then $s_{0} \in \psi_{K}^{1-1}\left(O_{K_{1}}^{+}\right) \cup \operatorname{in}\left(\psi_{K}^{1}\right)$. We distinguish two cases

- Let $s_{0} \in \psi_{K}^{1-1}\left(O_{K_{1}}^{+}\right)$, i.e., $\psi_{K}^{1}\left(s_{0}\right) \in O_{K_{1}}^{+}$. Then, since $\left\langle\pi_{1}, \phi^{1}\right\rangle$ is a projection, again, $\psi_{K}^{1}\left(s_{0}\right) \in$ $\phi_{K}^{1-1}\left(O_{K_{3}}^{+}\right) \cup \operatorname{in}\left(\phi_{K}^{1}\right)$. If $\phi_{K}^{1}\left(\psi_{K}^{1}\left(s_{0}\right)\right) \in O_{K_{3}}^{+}$then, observing that $\phi_{K}^{2}\left(\psi_{K}^{2}\left(s_{0}\right)\right)=\phi_{K}^{1}\left(\psi_{K}^{1}\left(s_{0}\right)\right)$ and recalling that $\phi_{K}^{2}$ is an open net morphism, we conclude that $\psi_{K}^{2}\left(s_{0}\right) \in O_{K_{2}}^{+}$, and thus $s_{0} \in$ $\psi_{K}^{2-1}\left(O_{K_{2}}^{+}\right)$. If instead, $\psi_{K}^{1}\left(s_{0}\right) \in \operatorname{in}\left(\phi_{K}^{1}\right)$ then ${ }^{\bullet} \phi_{K}^{1}\left(\psi_{K}^{1}\left(s_{0}\right)\right)-\phi_{K}^{1}\left({ }^{\bullet} \psi_{K}^{1}\left(s_{0}\right)\right) \neq \emptyset$. Since $K_{1}$ is an occurrence open net and $\psi_{K}^{1}\left(s_{0}\right)$ is input open we have that ${ }^{\bullet} \psi_{K}^{1}\left(s_{0}\right)=\emptyset$. Thus, since the upper square is a pushout, ${ }^{\bullet} \psi_{K}^{2}\left(s_{0}\right)-\psi_{K}^{2}\left({ }^{\bullet} s_{0}\right) \neq \emptyset$. Hence $s_{0} \in \operatorname{in}\left(\psi_{K}^{2}\right)$.
- Let $s_{0} \in \operatorname{in}\left(\psi_{K}^{1}\right)$. Thus there exists $t_{1} \in{ }^{\bullet} \psi_{K}^{1}\left(s_{0}\right)-\psi_{K}^{1}\left({ }^{\bullet} s_{0}\right)$. Since the upper square is a pushout, $\phi_{K}^{1}\left(t_{1}\right) \in{ }^{\bullet} \phi_{K}^{2}\left(\Psi_{K}^{2}\left(s_{0}\right)\right)-\phi_{K}^{2}\left({ }^{\bullet} \psi_{K}^{2}\left(s_{0}\right)\right)$ hence $\psi_{K}^{2}\left(s_{0}\right) \in \operatorname{in}\left(\phi_{K}^{2}\right) \subseteq O_{K_{2}}^{+}$, since $\phi_{K}^{2}$ is an open net morphism. Hence $s_{0} \in \psi_{K}^{2-1}\left(O_{K_{2}}^{+}\right)$. Observe that, in particular, we have shown that $\psi_{K}^{2}\left(\operatorname{in}\left(\psi_{K}^{1}\right)\right) \subseteq O_{K_{2}}^{+}$.
$(\supseteq)$ Let $s_{0} \in \psi_{K}^{2-1}\left(O_{K_{2}}^{+}\right) \cup \operatorname{in}\left(\psi_{K}^{2}\right)$. We distinguish two cases
- Let $s_{0} \in \psi_{K}^{2-1}\left(O_{K_{2}}^{+}\right)$, i.e., $\psi_{K}^{2}\left(s_{0}\right) \in O_{K_{2}}^{+}$. Since $\left\langle\pi_{2}, \phi^{2}\right\rangle$ is a projection of $\pi_{3}$, we have that $\psi_{K}^{2}\left(s_{0}\right) \in \phi_{K}^{2-1}\left(O_{K_{3}}^{+}\right) \cup \operatorname{in}\left(\phi_{K}^{2}\right)$. If $\phi_{K}^{2}\left(\psi_{K}^{2}\left(s_{0}\right)\right) \in O_{K_{3}}^{+}$then, since $\phi_{K}^{1}$ is an open net morphism $\psi_{K}^{1}\left(s_{0}\right) \in O_{K_{1}}^{+}$and thus $s_{0} \in O_{K_{0}}^{+}$. If instead $\psi_{K}^{2}\left(s_{0}\right) \in \operatorname{in}\left(\phi_{K}^{2}\right)$ then ${ }^{\bullet} \phi_{K}^{2}\left(\psi_{K}^{2}\left(s_{0}\right)\right)-\phi_{K}^{2}\left({ }^{\bullet} \psi_{K}^{2}\left(s_{0}\right)\right) \neq$ $\emptyset$. Since the upper square is a pushout, this implies that ${ }^{\bullet} \psi_{K}^{1}\left(s_{0}\right)-\psi_{K}^{1}\left({ }^{\bullet} s_{0}\right) \neq \emptyset$ and thus $s_{0} \in \operatorname{in}\left(\psi_{K}^{1}\right)$.
- Let $s_{0} \in \operatorname{in}\left(\psi_{K}^{2}\right)$. Then ${ }^{\bullet} \psi_{K}^{2}\left(s_{0}\right)-\psi_{K}^{2}\left({ }^{\bullet} s_{0}\right) \neq \emptyset$. Since the upper square is a pushout, we have that ${ }^{\bullet} \phi_{K}^{1}\left(\psi_{K}^{1}\left(s_{0}\right)\right)-\phi_{K}^{1}\left({ }^{\bullet} \psi_{K}^{1}\left(s_{0}\right)\right) \neq \emptyset$. Since $\phi_{K}^{1}$ is an open net morphism, $\psi_{K}^{1}\left(s_{0}\right) \in O_{K_{1}}^{+}$and thus $s_{0} \in O_{K_{0}}^{+}$.

To conclude the proof, we need only to show that $\psi_{K}^{1} \uparrow \psi_{K}^{2}$. We observe that the upper square, which is known to be a pushout in Net, is also a pushout in ONet. To this aim we prove that, for $x \in\{+,-\}$,

$$
O_{K_{3}}^{x}=\left\{s_{3} \in S_{K_{3}} \mid \phi_{K}^{1-1}\left(s_{3}\right) \subseteq O_{3}^{x} \wedge \phi_{K}^{2-1}\left(s_{3}\right) \subseteq O_{K_{2}}^{x}\right\}
$$

Let us consider the condition on input places $(x=+)$. Let $s_{3} \in O_{K_{3}}^{+}$. Then, if $\phi_{i}^{-1}\left(s_{3}\right) \subseteq O_{K_{i}}^{+}$for $i \in\{1,2\}$, since $\phi_{i}$ is an open net morphism. For the converse inclusion, assume that

$$
\begin{equation*}
\phi_{K}^{1-1}\left(s_{3}\right) \subseteq O_{K_{1}}^{+} \quad \text { and } \quad \phi_{K}^{2-1}\left(s_{3}\right) \subseteq O_{K_{2}}^{+} \tag{3}
\end{equation*}
$$

Since the upper square is a pushout in Net, there is $s_{i} \in S_{i}$ (for some $i \in\{1,2\}$ ) such that $\phi_{i}\left(s_{i}\right)=$ $s_{3}$. Assume, without loss of generality, that there exists $s_{1} \in S_{1}$ such that $\phi_{K}^{1}\left(s_{1}\right)=s_{3}$. Hence, by (3), $s_{1} \in O_{K_{1}}^{+}$. Since $\pi_{1}$ is a projection of $\pi_{3}, O_{K_{1}}^{+}=\phi_{K}^{1-1}\left(O_{K_{3}}^{+}\right) \cup$ in $\left(\phi_{K}^{1}\right)$. If $s_{1} \in \phi_{K}^{1-1}\left(O_{K_{3}}^{+}\right)$ then we conclude. Otherwise, if $s_{1} \in \operatorname{in}\left(\phi_{K}^{1}\right)$ then there exists $t_{3} \in{ }^{\bullet} \phi_{K}^{1}\left(s_{1}\right)-\phi_{K}^{1}\left({ }^{\bullet} s_{1}\right)$. Since the upper square is a pushout in Net, there are $s_{2}$ in $K_{2}$ and $t_{2} \in{ }^{\bullet} s_{2}$ such that $\phi_{K}^{2}\left(s_{2}\right)=\phi_{K}^{1}\left(s_{1}\right)=s_{3}$ and $\phi_{K}^{2}\left(t_{2}\right)=t_{3}$. Since $s_{2} \in \phi_{K}^{2-1}\left(s_{3}\right)$, by (3) we have that $s_{2} \in O_{K_{2}}^{+}$, which contradicts the assumption that $K_{2}$ is an occurrence net since ${ }^{\bullet} s_{2} \neq \emptyset$.

The condition over output places $(x=-)$ is dealt with in a symmetric way, by exploiting the fact that the occurrence net $K_{3}$ is deterministic. This allows us to conclude that $\psi_{K}^{1} \uparrow \psi_{K}^{2}$ since this is a necessary condition to ensure that the pushout, computed in Net and lifted to ONet gives a deterministic occurrence open net (see Proposition 21).

The amalgamation and decomposition results for open net processes are summarized in a theorem which establishes a bijective correspondence between the processes of $Z_{1}$ and $Z_{2}$ which agree on $Z_{0}$ and the processes of $Z_{3}$. To formulate this result we need some preliminary observations.

Notice that an isomorphism $f: Z_{0} \rightarrow Z_{1}$ in ONet is an isomorphism $f: \mathcal{F}\left(Z_{1}\right) \rightarrow \mathcal{F}\left(Z_{2}\right)$ in Net such that $f\left(O_{0}^{+}\right)=O_{1}^{+}$and $f\left(O_{0}^{-}\right)=O_{1}^{-}$. Let $Z$ be an open net. Two deterministic processes of $Z, \pi: K \rightarrow Z$ and $\pi^{\prime}: K^{\prime} \rightarrow Z$, are called isomorphic written $\pi \simeq \pi^{\prime}$, if they are isomorphic in Proc, i.e., if there exists an isomorphism $\rho: K \rightarrow K^{\prime}$ in ONet such that (in Net) it holds $\pi \circ \rho=\pi^{\prime}$. In this case we will say that $\rho: \pi \rightarrow \pi^{\prime}$ is a process isomorphism (to mean that $\left\langle\rho, i d_{Z}\right\rangle$ is a process isomorphism). Let $\pi: K \rightarrow Z$ be a process. We denote by $[\pi]$ the set of processes of $Z$ isomorphic to $\pi$, i.e., $[\pi]=\left\{\pi^{\prime}: K^{\prime} \rightarrow Z \mid \pi^{\prime} \simeq \pi\right\}$. Then the set of (isomorphism classes of) processes of $Z$ is denoted by $\operatorname{DProc}(Z)$, i.e.,

$$
\operatorname{DProc}(Z)=\{[\pi] \mid \pi: K \rightarrow Z \text { is a deterministic process }\} .
$$

Given a span $Z_{1} \stackrel{f_{1}}{\leftarrow} Z_{0} \xrightarrow{f_{2}} Z_{2}$ in $\mathbf{O N e t}$, the isomorphism classes of deterministic processes of $Z_{1}$ and $Z_{2}$ which agree on $Z_{0}$, denoted by $\operatorname{DProc}\left(Z_{1} \stackrel{f_{1}}{\stackrel{ }{c}} Z_{0} \stackrel{f_{2}}{\rightarrow} Z_{2}\right)$, is the set

$$
\left\{\left[\pi_{1} \stackrel{\psi^{1}}{\leftarrow} \pi_{0} \xrightarrow{\psi^{2}} \pi_{2}\right] \mid \psi^{1}, \psi^{2} \text { agreement projections for } \pi_{1}, \pi_{2} \text { along } f_{1}, f_{2}\right\}
$$

where isomorphism of process spans is defined by $\left(\pi_{1} \stackrel{\psi^{1}}{\leftarrow} \pi_{0} \stackrel{\psi^{2}}{\rightarrow} \pi_{2}\right) \simeq\left(\pi_{1}^{\prime} \stackrel{\phi^{1}}{\leftarrow} \pi_{0}^{\prime} \xrightarrow{\phi^{2}} \pi_{2}^{\prime}\right)$ if there are process isomorphisms $\rho_{i}: \phi_{i} \rightarrow \phi_{i}^{\prime}$ such that the following diagram commutes


Observe that this implies that $\pi_{0}^{\prime} \in\left[\pi_{0}\right]$ and $\pi_{1}^{\prime}$ and that $\pi_{2}^{\prime}$ agree on $\pi_{0}^{\prime}$.
THEOREM 27 (AMALGAMATION THEOREM) Let $Z_{0}, Z_{1}, Z_{2}, Z_{3}$ be as in Fig. 9 and assume that the square is a pushout of two composable open net embeddings $f_{1}$ and $f_{2}$. Then there are composition and decomposition functions:

$$
\operatorname{Comp}: \operatorname{DProc}\left(Z_{1} \stackrel{f_{1}}{\leftarrow} Z_{0} \xrightarrow{f_{2}} Z_{2}\right) \rightarrow \operatorname{DProc}\left(Z_{3}\right)
$$

and

$$
\operatorname{Dec}: \operatorname{DProc}\left(Z_{3}\right) \rightarrow \operatorname{DProc}\left(Z_{1} \stackrel{f_{1}}{\leftarrow} Z_{0} \stackrel{f_{2}}{\rightarrow} Z_{2}\right)
$$

establishing a bijective correspondence between $\mathbf{D P r o c}\left(Z_{3}\right)$ and $\mathbf{D P r o c}\left(Z_{1} \stackrel{f_{1}}{\leftarrow} Z_{0} \stackrel{f_{2}}{\rightarrow} Z_{2}\right)$.
Proof Sketch. Let us define Comp : $\operatorname{DProc}\left(Z_{1} \stackrel{f_{1}}{\leftarrow} Z_{0} \stackrel{f_{2}}{\rightarrow} Z_{2}\right) \rightarrow \operatorname{DProc}\left(Z_{3}\right)$ by

$$
\operatorname{Comp}\left(\left[\pi_{1} \stackrel{\psi^{1}}{\leftarrow} \pi_{0} \stackrel{\psi^{2}}{\rightarrow} \pi_{2}\right]\right)=\left[\pi_{3}\right],
$$

where $\pi_{3}=\pi_{1}+{ }_{\psi^{1}, \psi^{2}} \pi_{2}$ is the amalgamation of $\pi_{1}$ and $\pi_{2}$ (see Definition 24). Furthermore $\operatorname{Dec}: \operatorname{DProc}\left(Z_{3}\right) \rightarrow \operatorname{DProc}\left(Z_{1} \stackrel{f_{1}}{\leftarrow} Z_{0} \stackrel{f_{2}}{\rightarrow} Z_{2}\right)$ is defined by

$$
\operatorname{Dec}\left(\left[\pi_{3}\right]\right)=\left[\pi_{1} \stackrel{\psi^{1}}{\leftarrow} \pi_{0} \xrightarrow{\psi^{2}} \pi_{2}\right],
$$

where $\pi_{1} \stackrel{\psi^{1}}{\leftarrow} \pi_{0} \xrightarrow{\psi^{2}} \pi_{2}$ is the decomposition of $\pi_{3}$ as defined in Proposition 26. Then it is possible to prove that Comp and Dec are well-defined and inverse to each other.

Example. The amalgamation theorem is exemplified in Fig. 3. Two processes for the component nets Traveler and Agency which agree on the shared subnet Common, i.e., such that their projections over Common coincide, can be amalgamated to produce a process for the composed net Global. Vice versa, each process of the net Global can be reconstructed as amalgamation of compatible processes of the component nets.

## 6 Conclusions and related work

In this paper we have introduced open nets, an extension of ordinary Petri nets which allows to specify open concurrent systems, interacting with an external environment. Open nets are endowed with a composition operation, suitable to model both interaction through open places and synchronization of transitions. The generalization to open nets of the Goltz-Reisig process semantics has been shown to be compositional with respect to the composition operation over open nets: if two nets $Z_{1}$ and $Z_{2}$ are composed, producing a net $Z_{3}$, then the processes of $Z_{3}$ can be obtained as amalgamations of processes of $Z_{1}$ and $Z_{2}$, and vice versa, any process of $Z_{3}$ can be decomposed into processes of the component nets. The amalgamation and decomposition
operations are shown to be inverse to each other, leading to a bijective correspondence between the processes of $Z_{3}$ and pair of processes of $Z_{1}$ and $Z_{2}$ which agree on the common subnet $Z_{0}$.

As mentioned in the introduction, the last result appears to be related to the amalgamation theorem for data-types in the framework of algebraic specifications [EM85]. There an amalgamation construction allows one to "combine" any two algebras $A_{1}$ and $A_{2}$ of algebraic specifications $S P E C_{1}$ and $S P E C_{2}$ having a common subspecification $S P E C_{0}$, if and only if the restrictions of $A_{1}$ and $A_{2}$ to $S P E C_{0}$ coincide. The amalgamation construction produces a unique algebra $A_{3}$ of specification $S P E C_{3}$, union of $S P E C_{1}$ and $S P E C_{2}$. The fact that the amalgamation of algebras is a pushout construction in the Grothendick's category of generalized algebras, suggests the possibility of having a similar characterization for process amalgamation using fibred categories (see also Remark 15).

Open nets have been partly inspired by the notion of open graph transformation system [Hec98], an extension of graph transformation for specifying reactive systems. In fact, P/T Petri nets can be seen as a special case of graph transformation systems [Cor96] and this correspondence extends to open nets and open graph transformation systems. However, a compositionality result corresponding to Theorem 27 is still lacking in this more general setting.

In the field of Petri nets, several other approaches to net composition have been proposed in the literature. Most of them can be classified as algebraic approaches. A first family, which dates back to the papers [NPW81, Win87a], considers a category of Petri nets where morphisms arise by viewing a Petri net as the signature of a multisorted algebra, the sorts being the places. Then an unfolding semantics is defined, which is characterized as a categorical right adjoint. This fact ensures its compositionality with respect to operations on nets defined in terms of categorical limits (e.g., net synchronization [Win87b]). The algebraic view is pushed forward in another seminal paper [MM90], where a Petri net is still seen as a signature, and its computational model (the category of deterministic processes in the sense of Best-Devillers [BD87]) is characterized as the free algebra (up to suitable axioms) over such a signature. Being obtained as a free construction, which in categorical terms provides a left adjoint, in this case the semantics is compositional with respect to operations defined in terms of colimits. However, in both cases, differently from what happens in our approach, there is no distinction between open and internal places. Basically, every place of a net $N$ can be implicitly seen as open because it can be used for connecting $N$ to other nets. On the other hand, the semantics (e.g., the notions of process in [GR83] or [MM90]) does not take into account explicitly the interaction with the environment.

A second, more recent class of approaches to Petri net composition aims at defining a "calculus of nets", where a set of process algebra-like operators allows to build complex nets starting from a suitable set of basic net components. For instance, in the Petri Box calculus [BDH92, KEB94, KB99] a special class of nets, called plain boxes (safe and clean nets), provides the basic components. Plain boxes are then combined by means of operations which can all be seen as an instance of refinement over suitable nets. More precisely, the authors identify a special family of nets, called operator boxes. Once a set of operator boxes is fixed, the composition is realized by refining such operator boxes with plain boxes, an operation which produces a net still identifiable with a plain box. The calculus is given a compositional semantics (both interleaving and concurrent). Although based on some common ideas, like the use of interface places, this approach is quite different from ours, since it mainly relies on refinement and it focuses on a special class of nets and on the possibility of defining a kind of process algebra over such nets, where plain boxes are constants and operator boxes are the operators of the algebra.

Another relevant approach in the second family, closer to ours, is presented in the papers [NPS95, PW98], which introduce an algebra of (labeled) Petri nets with interfaces. An interface consists of a set of public places and transitions, where a net can be extended and combined with other nets by means of composition operators. E.g., it is possible to add new transitions and places, to connect existing transitions and places by new arcs, to hide items in the net, etc. These operators can be used as basic constructors to build terms corresponding to nets with an interface. The representation of a Petri net via a term of the algebra of combinators
resembles the encoding of Petri nets into Milner action calculi [Mil96]. The pomset semantics of nets with interfaces, defined by using a notion of universal context for a net, is shown to be compositional with respect to the net combinators [PW98]. Despite some technical differences and the different focus, which in these papers is more on the syntactical aspects of the Petri net algebra, Petri nets with interface appear to have several analogies with open nets, and their relationship surely deserves a deeper investigation.

Finally we recall two approaches to Petri net components, i.e., Petri nets with distinguished interface places. Kindler [Kin97] introduces Petri net components with input and output places, which can be combined by means of an operation which connects the input places of a component to the output places of the other, and vice versa. A partial order semantics is introduced for components and it is proved to be compositional. Components can be viewed as particular open nets and, similarly, the composition operation for components can be seen as an instance of the composition operation for open nets. A very interesting idea in [Kin97], which we intend to explore also for open nets, is the introduction of a temporal logic, interpreted over processes, which can be used for reasoning in a modular way over distributed systems.

Basten [Bas98] considers components of Petri nets with interface places, called pins, of unspecified orientation, where nets can be fused together. A compositional operational semantics of Petri net components is described within a process algebra specifically designed for this purpose. This allows the verification of net components against requirements by means of equational reasoning. Moreover, the algebraic presentation of the operational semantics is used to formalize a notion of behavior inheritance between components.

The notions of projection and of amalgamation of processes can be extended to general (possibly nondeterministic) processes. We are currently working on the generalization of the amalgamation theorem to nondeterministic processes, which could represent a first step towards an unfolding semantics for open nets, in the style of Winskel [NPW81, Win87a], still compositional with respect to our composition operation.

It would be also interesting to extend the constructions and results in this paper to open high level nets, which have been already studied on a conceptual level in [PJHE98]. Part of the technical background is already available - for instance it has been shown in [PER95] how to construct pushouts of algebraic high level nets - but a suitable formalization of high level processes is still missing.

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