COMPOSITIONS AND CONVEX COMBINATIONS OF ASYMPTOTICALLY REGULAR FIRMLY NONEXPANSIVE MAPPINGS ARE ALSO ASYMPTOTICALLY REGULAR

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Abstract

Because of Minty's classical correspondence between firmly nonexpansive mappings and maximally monotone operators, the notion of a firmly nonexpansive mapping has proven to be of basic importance in fixed point theory, monotone operator theory, and convex optimization. In this note, we show that if finitely many firmly nonexpansive mappings defined on a real Hilbert space are given and each of these mappings is asymptotically regular, which is equivalent to saying that they have or "almost have" fixed points, then the same is true for their composition. This significantly generalizes the result by Bauschke from 2003 for the case of projectors (nearest point mappings). The proof resides in a Hilbert product space and it relies upon the Brezis-Haraux range approximation result. By working in a suitably scaled Hilbert product space, we also establish the asymptotic regularity of convex combinations.

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1 Introduction and Standing Assumptions

Throughout this paper,

(1)	X is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$

and induced norm $\|\cdot\|$. We assume that

(2)	$m \in \{2, 3, 4, \ldots\}$ and $I := \{1, 2, \ldots, m\}$.

Recall that an operator $T\colon X \to X$ is *firmly nonexpansive* (see, e.g., [2], [10], and [11] for further information) if $(\forall x \in X)(\forall y \in X)\ \|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$ and that a set-valued operator $A\colon X \rightrightarrows X$ is *maximally monotone* if it is *monotone*, i.e., for all (x, x^*) and (y, y^*) in the graph of A, we have $\langle x - y, x^* - y^* \rangle \geq 0$ and if the graph of A cannot be properly enlarged without destroying monotonicity. (We shall write $\operatorname{dom} A = \{x \in X \mid Ax \neq \varnothing\}$ for the *domain* of A, $\operatorname{ran} A = A(X) = \bigcup_{x \in X} Ax$ for the *range* of A, and $\operatorname{gr} A$ for the *graph* of A.) These notions are equivalent (see [13] and [9]) in the sense that if A is maximally monotone, then its *resolvent* $J_A := (\operatorname{Id} + A)^{-1}$ is firmly nonexpansive, and if T is firmly nonexpansive, then $T^{-1} - \operatorname{Id}$ is maximally monotone. (Here and elsewhere, Id denotes the identity operator on X.) The Minty parametrization (see [13] and also [2, Remark 23.22(ii)]) states that if A is maximally monotone, then

$$\operatorname{gr} A = \{(J_A x, x - J_A x) \mid x \in X\}. \tag{3}$$

In optimization, one main problem is to find zeros of maximally monotone operators — these zeros may correspond to critical points or solutions to optimization problems. In terms of resolvents, the corresponding problem is that of finding fixed points. For background material in fixed point theory and monotone operator theory, we refer the reader to [2], [4], [5], [8], [10], [11], [17], [18], [19], [20], [21], [22], [23], and [24].

The aim of this note is to provide approximate fixed point results for compositions and convex combinations of finitely many firmly nonexpansive operators.

The first main result (Theorem 4.6) substantially extends a result by Bauschke [1] on the compositions of projectors to the composition of firmly nonexpansive mappings. The second main result (Theorem 5.5) extends a result by Bauschke, Moffat and Wang [3] on the convex combination of firmly nonexpansive operators from Euclidean to Hilbert space.

The remainder of this section provides the standing assumptions used throughout the paper.

Even though the main results are formulated in the given Hilbert space X, it will turn out that the key space to work in is the product space

$$X^m := \big\{\mathbf{x} = (x_i)_{i\in I} \;\big|\; (\forall i \in I)\ x_i \in X\big\}. \tag{4}$$

This product space contains an embedding of the original space X via the *diagonal* subspace

$$\mathbf{\Delta} := \big\{\mathbf{x} = (x)_{i\in I} \;\big|\; x \in X\big\}. \tag{5}$$

We also assume that we are given m firmly nonexpansive operators $T_1, \ldots, T_m$; equivalently, m resolvents of maximally monotone operators $A_1, \ldots, A_m$:

$$(\forall i \in I) \quad T_i = J_{A_i} = (\mathrm{Id} + A_i)^{-1} \ \text{ is firmly nonexpansive.} \tag{6}$$

We now define various pertinent operators acting on X^m. We start with the Cartesian product operators

$$\mathbf{T}\colon X^m \to X^m \colon (x_i)_{i\in I} \mapsto (T_i x_i)_{i\in I} \tag{7}$$

and

$$\mathbf{A}\colon X^m \rightrightarrows X^m \colon (x_i)_{i\in I} \mapsto (A_i x_i)_{i\in I}. \tag{8}$$

Denoting the identity on X^m by $\mathbf{Id}$, we observe that

$$J_{\mathbf{A}} = (\mathbf{Id} + \mathbf{A})^{-1} = T_1 \times \cdots \times T_m = \mathbf{T}. \tag{9}$$

Of central importance will be the *cyclic right-shift operator*

$$\mathbf{R}\colon X^m \to X^m \colon (x_1, x_2, \ldots, x_m) \mapsto (x_m, x_1, \ldots, x_{m-1}) \tag{10}$$

and for convenience we set

$$\mathbf{M} = \mathbf{Id} - \mathbf{R}. \tag{11}$$

We also fix strictly positive *convex coefficients* (or weights) $(\lambda_i)_{i\in I}$, i.e.

$$(\forall i \in I) \quad \lambda_i \in \,]0,1[\;\text{ and } \sum_{i\in I} \lambda_i = 1. \tag{12}$$

Let us make X^m into the Hilbert product space

$$\mathbf{X} := X^m, \quad \text{with } \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i\in I} \langle x_i, y_i \rangle . \tag{13}$$

The orthogonal complement of $\boldsymbol{\Delta}$ with respect to this standard inner product is known (see, e.g., [2, Proposition 25.4(i)]) to be

$$\boldsymbol{\Delta}^\perp = \big\{ \mathbf{x} = (x_i)_{i\in I} \;\big|\; \sum_{i\in I} x_i = 0 \big\}. \tag{14}$$

Finally, given a nonempty closed convex subset C of X, the *projector* (nearest point mapping) onto C is denoted by P_C. It is well known to be firmly nonexpansive.

2 Properties of the Operator M

In this section, we collect several useful properties of the operator $\mathbf{M}$, including its Moore-Penrose inverse (see [12] and e.g. [2, Section 3.2] for further information.). To that end, the following result—which is probably part of the folklore—will turn out to be useful.

Proposition 2.1 *Let Y be a real Hilbert space and let B be a continuous linear operator from X to Y with adjoint B^* and such that* $\operatorname{ran} B$ *is closed. Then the Moore-Penrose inverse of B satisfies*

$$B^\dagger = P_{\operatorname{ran} B^*} \circ B^{-1} \circ P_{\operatorname{ran} B}. \tag{15}$$

Proof. Take $y \in Y$. Define the corresponding set of least squares solutions (see, e.g., [2, Proposition 3.25]) by $C := B^{-1}(P_{\operatorname{ran} B} y)$. Since $\operatorname{ran} B$ is closed, so is $\operatorname{ran} B^*$ (see, e.g., [2, Corollary 15.34]); hence[1], $U := (\ker B)^\perp = \overline{\operatorname{ran} B^*} = \operatorname{ran} B^*$. Thus, $C = B^\dagger y + \ker B = B^\dagger y + U^\perp$. Therefore, since $\operatorname{ran} B^\dagger = \operatorname{ran} B^*$ (see, e.g., [2, Proposition 3.28(v)]), $P_U(C) = P_U B^\dagger y = B^\dagger y$, as claimed. ■

Before we present various useful properties of $\mathbf{M}$, let us recall the notion of a *rectangular* (which is also known as star or 3* monotone, see [6]) operator. A monotone operator $B\colon X \rightrightarrows X$ is *rectangular* if $(\forall (x, y^*) \in \operatorname{dom} B \times \operatorname{ran} B)\ \sup_{(z,z^*)\in \operatorname{gr} B} \langle x - z, z^* - y^* \rangle < +\infty$.

[1] $\ker B = B^{-1}0 = \big\{ x \in X \;\big|\; Bx = 0 \big\}$ denotes the *kernel* (or nullspace) of B.

Theorem 2.2 *Define*[2]

$$\mathbf{L}\colon \boldsymbol{\Delta}^\perp \to \mathbf{X}\colon \mathbf{y} \mapsto \sum_{i=1}^{m-1} \frac{m-i}{m}\mathbf{R}^{i-1}\mathbf{y}. \tag{16}$$

Then the following hold.

(i) $\mathbf{M}$ *is continuous, linear, and maximally monotone with* $\operatorname{dom}\mathbf{M} = \mathbf{X}$.

(ii) $\mathbf{M}$ *is rectangular.*

(iii) $\ker\mathbf{M} = \ker\mathbf{M}^* = \boldsymbol{\Delta}$.

(iv) $\operatorname{ran}\mathbf{M} = \operatorname{ran}\mathbf{M}^* = \boldsymbol{\Delta}^\perp$ *is closed.*

(v) $\operatorname{ran}\mathbf{L} = \boldsymbol{\Delta}^\perp$.

(vi) $\mathbf{M}\circ\mathbf{L} = \mathbf{Id}|_{\boldsymbol{\Delta}^\perp}$.

(vii) $\mathbf{M}^{-1}\colon \mathbf{X} \rightrightarrows \mathbf{X}\colon \mathbf{y} \mapsto \begin{cases} \mathbf{L}\mathbf{y} + \boldsymbol{\Delta}, & \textit{if } \mathbf{y} \in \boldsymbol{\Delta}^\perp; \\ \varnothing, & \textit{otherwise.} \end{cases}$

(viii) $\mathbf{M}^\dagger = P_{\boldsymbol{\Delta}^\perp}\circ\mathbf{L}\circ P_{\boldsymbol{\Delta}^\perp} = \mathbf{L}\circ P_{\boldsymbol{\Delta}^\perp}$.

(ix) $\mathbf{M}^\dagger = \sum_{k=1}^{m} \frac{m-(2k-1)}{2m}\mathbf{R}^{k-1}$.

Proof. (i): Clearly, $\operatorname{dom}\mathbf{M} = \mathbf{X}$ and $(\forall \mathbf{x} \in \mathbf{X})\ \|\mathbf{R}\mathbf{x}\| = \|\mathbf{x}\|$. Thus, $\mathbf{R}$ is nonexpansive and therefore $\mathbf{M} = \mathbf{Id} - \mathbf{R}$ is maximally monotone (see, e.g., [2, Example 20.27]).

(ii): See [2, Example 24.14] and [1, Step 3 in the proof of Theorem 3.1] for two different proofs of the rectangularity of $\mathbf{M}$.

(iii): The definitions of $\mathbf{M}$ and $\mathbf{R}$ and the fact that $\mathbf{R}^*$ is the cyclic left shift operator readily imply that $\ker\mathbf{M} = \ker\mathbf{M}^* = \boldsymbol{\Delta}$.

(iv), (vi), and (vii): Let $\mathbf{y} = (y_1,\ldots,y_m) \in \mathbf{X}$. Assume first that $\mathbf{y} \in \operatorname{ran}\mathbf{M}$. Then there exists $\mathbf{x} = (x_1,\ldots,x_m)$ such that $y_1 = x_1 - x_m$, $y_2 = x_2 - x_1$, ..., and $y_m = x_m - x_{m-1}$. It follows that $\sum_{i\in I} y_i = 0$, i.e., $\mathbf{y} \in \Delta^\perp$ by [2, Proposition 25.4(i)]. Thus,

$$\operatorname{ran}\mathbf{M} \subseteq \boldsymbol{\Delta}^\perp. \tag{17}$$

Conversely, assume now that $\mathbf{y} \in \boldsymbol{\Delta}^\perp$. Now set

$$\mathbf{x} := \mathbf{L}\mathbf{y} = \sum_{i=1}^{m-1} \frac{m-i}{m}\mathbf{R}^{i-1}\mathbf{y}. \tag{18}$$

[2] Here and elsewhere we write S^n for the n-fold composition of an operator S.

It will be notationally convenient to wrap indices around i.e., $y_{m+1} = y_1$, $y_0 = y_m$ and likewise. We then get

$$(\forall i \in I) \quad x_i = \frac{m-1}{m} y_i + \frac{m-2}{m} y_{i-1} + \cdots + \frac{1}{m} y_{i+2}. \tag{19}$$

Therefore,

$$\sum_{i\in I} x_i = \frac{m-1}{m} \sum_{i\in I} y_i + \frac{m-2}{m} \sum_{i\in I} y_i + \frac{1}{m} \sum_{i\in I} y_i = \frac{m-1}{2} \sum_{i\in I} y_i = 0. \tag{20}$$

Thus $\mathbf{x} \in \mathbf{\Delta}^{\perp}$ and

$$\operatorname{ran} \mathbf{L} \subseteq \Delta^{\perp}. \tag{21}$$

Furthermore,

$$(\forall i \in I) \quad x_i - x_{i-1} = \frac{m-1}{m} y_i - \frac{1}{m} y_{i-1} - \frac{1}{m} y_{i-2} - \cdots - \frac{1}{m} y_{i+1} \tag{22a}$$

$$= y_i - \frac{1}{m} \sum_{j\in I} y_j = y_i. \tag{22b}$$

Hence $\mathbf{M}\mathbf{x} = \mathbf{x} - \mathbf{R}\mathbf{x} = \mathbf{y}$ and thus $\mathbf{y} \in \operatorname{ran} \mathbf{M}$. Moreover, in view of (iii),

$$\mathbf{M}^{-1}\mathbf{y} = \mathbf{x} + \ker \mathbf{M} = \mathbf{x} + \mathbf{\Delta}. \tag{23}$$

We thus have shown

$$\mathbf{\Delta}^{\perp} \subseteq \operatorname{ran} \mathbf{M}. \tag{24}$$

Combining (17) and (24), we obtain $\operatorname{ran} \mathbf{M} = \mathbf{\Delta}^{\perp}$. We thus have verified (vi), and (vii). Since $\operatorname{ran} \mathbf{M}$ is closed, so is $\operatorname{ran} \mathbf{M}^*$ (by, e.g., [2, Corollary 15.34]). Thus (iv) holds.

(viii)&(v): We have seen in Proposition 2.1 that

$$\mathbf{M}^{\dagger} = P_{\operatorname{ran} \mathbf{M}^*} \circ \mathbf{M}^{-1} \circ P_{\operatorname{ran} \mathbf{M}}. \tag{25}$$

Now let $\mathbf{z} \in \mathbf{X}$. Then, by (iv), $\mathbf{y} := P_{\operatorname{ran} \mathbf{M}} \mathbf{z} = P_{\mathbf{\Delta}^{\perp}} \mathbf{z} \in \mathbf{\Delta}^{\perp}$. By (vii), $\mathbf{M}^{-1}\mathbf{y} = \mathbf{L}\mathbf{y} + \mathbf{\Delta}$. So $\mathbf{M}^{\dagger}\mathbf{z} = P_{\operatorname{ran} \mathbf{M}^*} \mathbf{M}^{-1} P_{\operatorname{ran} \mathbf{M}} \mathbf{z} = P_{\operatorname{ran} \mathbf{M}^*} \mathbf{M}^{-1}\mathbf{y} = P_{\mathbf{\Delta}^{\perp}}(\mathbf{L}\mathbf{y} + \mathbf{\Delta}) = P_{\mathbf{\Delta}^{\perp}} \mathbf{L}\mathbf{y} = \mathbf{L}\mathbf{y} = (\mathbf{L} \circ P_{\mathbf{\Delta}^{\perp}})\mathbf{z}$ because $\operatorname{ran} \mathbf{L} \subseteq \mathbf{\Delta}^{\perp}$ by (21). Hence (viii) holds. Furthermore, by (iv) and e.g. [2, Proposition 3.28(v)], $\operatorname{ran} \mathbf{L} = \operatorname{ran} \mathbf{L} \circ P_{\mathbf{\Delta}^{\perp}} = \operatorname{ran} \mathbf{M}^{\dagger} = \operatorname{ran} \mathbf{M}^* = \mathbf{\Delta}^{\perp}$ and so (v) holds.

(ix): Note that $P_{\mathbf{\Delta}^{\perp}} = \mathbf{Id} - P_{\mathbf{\Delta}}$ and that $P_{\mathbf{\Delta}} = m^{-1} \sum_{j\in I} \mathbf{R}^j$. Hence

$$P_{\mathbf{\Delta}^{\perp}} = \mathbf{Id} - \frac{1}{m} \sum_{j\in I} \mathbf{R}^j. \tag{26}$$

Thus, by (viii) and (16),

$$\mathbf{M}^{\dagger} = \mathbf{L} \circ P_{\mathbf{\Delta}^{\perp}} = \frac{1}{m} \sum_{i=1}^{m-1} (m-i)\mathbf{R}^{i-1} \circ \Big(\mathbf{Id} - \frac{1}{m} \sum_{j\in I} \mathbf{R}^j \Big) \tag{27}$$

$$= \frac{1}{m}\sum_{i=1}^{m-1}(m-i)\mathbf{R}^{i-1} - \frac{1}{m^2}\sum_{i=1}^{m-1}(m-i)\sum_{j\in I}\mathbf{R}^{i+j-1}. \tag{28}$$

Re-arranging this expression in terms of powers of $\mathbf{R}$ and simplifying leads to

$$\mathbf{M}^\dagger = (\mathbf{Id} - \mathbf{R})^\dagger = \sum_{k=1}^{m} \frac{m-(2k-1)}{2m}\mathbf{R}^{k-1}. \tag{29}$$

■

Remark 2.3 Suppose that $\widetilde{\mathbf{L}}\colon \mathbf{\Delta}^\perp \to \mathbf{X}$ satisfies $\mathbf{M}\circ\widetilde{\mathbf{L}} = \mathbf{Id}\,|_{\mathbf{\Delta}^\perp}$. Then

$$\mathbf{M}^{-1}\colon \mathbf{X} \rightrightarrows \mathbf{X}\colon \mathbf{y} \mapsto \begin{cases} \widetilde{\mathbf{L}}\mathbf{y} + \mathbf{\Delta}, & \text{if } \mathbf{y} \in \mathbf{\Delta}^\perp; \\ \varnothing, & \text{otherwise.} \end{cases} \tag{30}$$

One may show that $\mathbf{M}^\dagger = P_{\mathbf{\Delta}^\perp}\circ\widetilde{\mathbf{L}}\circ P_{\mathbf{\Delta}^\perp}$ and that $P_{\mathbf{\Delta}^\perp}\circ\widetilde{\mathbf{L}} = \mathbf{L}$ (see (16)). Concrete choices for $\widetilde{\mathbf{L}}$ and $\mathbf{L}$ are

$$\mathbf{\Delta}^\perp \to \mathbf{X}\colon (y_1, y_2, \ldots, y_m) \mapsto (y_1, y_1+y_2, \ldots, y_1+y_2+y_3+\cdots+y_m); \tag{31}$$

however, the range of the latter operator is not equal $\mathbf{\Delta}^\perp$ whenever $X \neq \{0\}$.

Remark 2.4 Denoting the *symmetric part* of $\mathbf{M}$ by $\mathbf{M}_+ = \frac{1}{2}\mathbf{M} + \frac{1}{2}\mathbf{M}^*$ and defining the *quadratic form* associated with $\mathbf{M}$ by $q_{\mathbf{M}}\colon \mathbf{x} \to \frac{1}{2}\langle \mathbf{x}, \mathbf{M}\mathbf{x}\rangle$, we note that [1, Proposition 2.3] implies that[3] $\operatorname{ran}\mathbf{M}_+ = \operatorname{dom} q^*_{\mathbf{M}} = \mathbf{\Delta}^\perp$.

Fact 2.5 (Brezis-Haraux) (See [6] and also, e.g.,[2, Theorem 24.20].) *Suppose A and B are monotone operators on X such that $A+B$ is maximally monotone,* $\operatorname{dom} A \subseteq \operatorname{dom} B$*, and B is rectangular. Then* $\operatorname{int}\operatorname{ran}(A+B) = \operatorname{int}(\operatorname{ran} A + \operatorname{ran} B)$ *and* $\overline{\operatorname{ran}(A+B)} = \overline{\operatorname{ran} A + \operatorname{ran} B}$.

Applying the Brezis-Haraux result to our given operators $\mathbf{A}$ and $\mathbf{M}$, we obtain the following.

Corollary 2.6 *The operator $\mathbf{A}+\mathbf{M}$ is maximally monotone and* $\overline{\operatorname{ran}(\mathbf{A}+\mathbf{M})} = \overline{\mathbf{\Delta}^\perp + \operatorname{ran}\mathbf{A}}$.

Proof. Since each A_i is maximally monotone and recalling Theorem 2.2(i), we see that $\mathbf{A}$ and $\mathbf{M}$ are maximally monotone. On the other hand, $\operatorname{dom}\mathbf{M} = \mathbf{X}$. Thus, by the well known sum theorem for maximally monotone operators (see, e.g, [2, Corollary 24.4(i)]), $\mathbf{A}+\mathbf{M}$ is maximally monotone. Furthermore, by Theorem 2.2(ii)&(iv), $\mathbf{M}$ is rectangular and $\operatorname{ran}\mathbf{M} = \mathbf{\Delta}^\perp$. The result therefore follows from Fact 2.5. ■

[3] Recall that the *Fenchel conjugate* of a function f defined on X is given by $f^*\colon x^* \mapsto \sup_{x\in X}\big(\langle x, x^*\rangle - f(x)\big)$.

3 Composition

We now use Corollary 2.6 to study the composition. When $m = 2$, then Theorem 3.1(v) also follows from [14, page 124].

Theorem 3.1 *Suppose that* $(\forall i \in I)\ 0 \in \overline{\operatorname{ran}(\operatorname{Id} - T_i)}$. *Then the following hold.*

(i) $\mathbf{0} \in \overline{\operatorname{ran}(\mathbf{A} + \mathbf{M})}$.

(ii) $(\forall \varepsilon > 0)\ (\exists (\mathbf{b}, \mathbf{x}) \in \mathbf{X} \times \mathbf{X})\ \|\mathbf{b}\| \leq \varepsilon$ *and* $\mathbf{x} = \mathbf{T}(\mathbf{b} + \mathbf{R}\mathbf{x})$.

(iii) $(\forall \varepsilon > 0)\ (\exists (\mathbf{c}, \mathbf{x}) \in \mathbf{X} \times \mathbf{X})\ \|\mathbf{c}\| \leq \varepsilon$ *and* $\mathbf{x} = \mathbf{c} + \mathbf{T}(\mathbf{R}\mathbf{x})$.

(iv) $(\forall \varepsilon > 0)\ (\exists \mathbf{x} \in \mathbf{X})\ (\forall i \in I)\ \|T_{i-1} \cdots T_1 x_m - T_i T_{i-1} \cdots T_1 x_m - x_{i-1} + x_i\| \leq (2i - 1)\varepsilon$, *where* $x_0 = x_m$.

(v) $(\forall \varepsilon > 0)\ (\exists x \in X)\ \|x - T_m T_{m-1} \cdots T_1 x\| \leq m^2 \varepsilon$.

Proof. (i): The assumptions and (3) imply that $(\forall i \in I)\ 0 \in \overline{\operatorname{ran} A_i}$. Hence, $\mathbf{0} \in \overline{\operatorname{ran} \mathbf{A}}$. Obviously, $\mathbf{0} \in \mathbf{\Delta}^{\perp}$. It follows that $\mathbf{0} \in \overline{\mathbf{\Delta}^{\perp} + \operatorname{ran} \mathbf{A}}$. Thus, by Corollary 2.6, $\mathbf{0} \in \overline{\operatorname{ran}(\mathbf{A} + \mathbf{M})}$.

(ii): Fix $\varepsilon > 0$. In view of (i), there exists $\mathbf{x} \in \mathbf{X}$ and $\mathbf{b} \in \mathbf{X}$ such that $\|\mathbf{b}\| \leq \varepsilon$ and $\mathbf{b} \in \mathbf{A}\mathbf{x} + \mathbf{M}\mathbf{x}$. Hence $\mathbf{b} + \mathbf{R}\mathbf{x} \in (\mathbf{Id} + \mathbf{A})\mathbf{x}$ and thus $\mathbf{x} = J_{\mathbf{A}}(\mathbf{b} + \mathbf{R}\mathbf{x}) = \mathbf{T}(\mathbf{b} + \mathbf{R}\mathbf{x})$.

(iii): Let $\varepsilon > 0$. By (ii), there exists $(\mathbf{b}, \mathbf{x}) \in \mathbf{X} \times \mathbf{X})$ such that $\|\mathbf{b}\| \leq \varepsilon$ and $\mathbf{x} = \mathbf{T}(\mathbf{b} + \mathbf{R}\mathbf{x})$. Set $\mathbf{c} = \mathbf{x} - \mathbf{T}(\mathbf{R}\mathbf{x}) = \mathbf{T}(\mathbf{b} + \mathbf{R}\mathbf{x}) - \mathbf{T}(\mathbf{R}\mathbf{x})$ Then, since $\mathbf{T}$ is nonexpansive, $\|\mathbf{c}\| = \|\mathbf{T}(\mathbf{b} + \mathbf{R}\mathbf{x}) - \mathbf{T}(\mathbf{R}\mathbf{x})\| \leq \|\mathbf{b}\| \leq \varepsilon$.

(iv): Take $\varepsilon > 0$. Then, by (iii), there exists $\mathbf{x} \in \mathbf{X}$ and $\mathbf{c} \in \mathbf{X}$ such that $\|\mathbf{c}\| \leq \varepsilon$ and $\mathbf{x} = \mathbf{c} + \mathbf{T}(\mathbf{R}\mathbf{x})$. Let $i \in I$. Then $x_i = c_i + T_i x_{i-1}$. Since $\|c_i\| \leq \|\mathbf{c}\| \leq \varepsilon$ and T_i is nonexpansive, we have

$$\begin{aligned} \|T_i T_{i-1} \cdots T_1 x_0 - x_i\| &\leq \|T_i T_{i-1} \cdots T_1 x_0 - T_i x_{i-1}\| + \|T_i x_{i-1} - x_i\| && \text{(32a)} \\ &\leq \|T_i T_{i-1} \cdots T_1 x_0 - T_i x_{i-1}\| + \varepsilon. && \text{(32b)} \end{aligned}$$

We thus obtain inductively

$$\|T_i T_{i-1} \cdots T_1 x_0 - x_i\| \leq i\varepsilon. \tag{33}$$

Hence,

$$\|T_{i-1} \cdots T_1 x_0 - x_{i-1}\| \leq (i - 1)\varepsilon. \tag{34}$$

The conclusion now follows from adding (33) and (34), and recalling the triangle inequality.

(v): Let $\varepsilon > 0$. In view of (iv), there exists $\mathbf{x} \in \mathbf{X}$ such that

$$(\forall i \in I) \quad \|T_{i-1} \cdots T_1 x_m - T_i T_{i-1} \cdots T_1 x_m - x_{i-1} + x_i\| \leq (2i - 1)\varepsilon \tag{35}$$

where $x_0 = x_m$. Now set $(\forall i \in I)$ $e_i = T_{i-1} \cdots T_1 x_m - T_i T_{i-1} \cdots T_1 x_m - x_{i-1} + x_i$. Then $(\forall i \in I)$ $\|e_i\| \le (2i-1)\varepsilon$. Set $x = x_m$. Then

$$\sum_{i=1}^{m} e_i = \sum_{i=1}^{m} T_{i-1} \cdots T_1 x_m - T_i T_{i-1} \cdots T_1 x_m - x_{i-1} + x_i \tag{36}$$

$$= x - T_m T_{m-1} \cdots T_1 x. \tag{37}$$

This, (35), and the triangle inequality imply that

$$\|x - T_m T_{m-1} \cdots T_1 x\| \le \sum_{i=1}^{m} \|e_i\| \le \sum_{i=1}^{m} (2i-1)\varepsilon = m^2 \varepsilon. \tag{38}$$

This completes the proof. ■

Corollary 3.2 *Suppose that* $(\forall i \in I)$ $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T_i)}$. *Then* $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T_m T_{m-1} \cdots T_1)}$.

Proof. This follows from Theorem 3.1(v). ■

Remark 3.3 The converse implication in Corollary 3.2 fails in general: indeed, consider the case when $X \neq \{0\}$, $m = 2$, and $v \in X \smallsetminus \{0\}$. Now set $T_1 \colon X \to X \colon x \mapsto x + v$ and set $T_2 \colon X \to X \colon x \mapsto x - v$. Then $0 \notin \overline{\operatorname{ran}(\operatorname{Id} - T_1)} = \{-v\}$ and $0 \notin \overline{\operatorname{ran}(\operatorname{Id} - T_2)} = \{v\}$; however, $T_2 T_1 = \operatorname{Id}$ and $\overline{\operatorname{ran}(\operatorname{Id} - T_2 T_1)} = \{0\}$.

Remark 3.4 Corollary 3.2 is optimal in the sense that even if $(\forall i \in I)$ we have $0 \in \operatorname{ran}(\operatorname{Id} - T_i)$, we cannot deduce that $0 \in \operatorname{ran}(\operatorname{Id} - T_m T_{m-1} \cdots T_1)$: indeed, suppose that $X = \mathbb{R}^2$ and $m = 2$. Set $C_1 := \operatorname{epi} \exp$ and $C_2 := \mathbb{R} \times \{0\}$. Suppose further that $T_1 = P_{C_1}$ and $T_2 = P_{C_2}$. Then $(\forall i \in I)$ $0 \in \operatorname{ran}(\operatorname{Id} - T_i)$; however, $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T_2 T_1)} \smallsetminus \operatorname{ran}(\operatorname{Id} - T_2 T_1)$.

4 Asymptotic Regularity

The following notions (taken from Bruck and Reich's seminal paper [7]) will be very useful to obtain stronger results.

Definition 4.1 ((strong) nonexpansiveness and asymptotic regularity) *Let* $S \colon X \to X$. *Then:*

(i) *S is* nonexpansive *if* $(\forall x \in X)(\forall y \in X)$ $\|Sx - Sy\| \le \|x - y\|$.

(ii) *S is* strongly nonexpansive *if S is nonexpansive and whenever* $(x_n)_{n \in \mathbb{N}}$ *and* $(y_n)_{n \in \mathbb{N}}$ *are sequences in X such that* $(x_n - y_n)_{n \in \mathbb{N}}$ *is bounded and* $\|x_n - y_n\| - \|Sx_n - Sy_n\| \to 0$, *it follows that* $(x_n - y_n) - (Sx_n - Sy_n) \to 0$.

(iii) *S is asymptotically regular if* $(\forall x \in X)$ $S^n x - S^{n+1} x \to 0$.

The next result illustrates that strongly nonexpansive mappings generalize the notion of a firmly nonexpansive mapping. In addition, the class of strongly nonexpansive mappings is closed under compositions. (In contrast, the composition of two (necessarily firmly nonexpansive) projectors may fail to be firmly nonexpansive.)

Fact 4.2 (Bruck and Reich) *The following hold.*

(i) *Every firmly nonexpansive mapping is strongly nonexpansive.*

(ii) *The composition of finitely many strongly nonexpansive mappings is also strongly nonexpansive.*

Proof. (i): See [7, Proposition 2.1]. (ii): See [7, Proposition 1.1]. ■

The sequences of iterates and of differences of iterates have striking convergence properties as we shall see now. In passing, we note that Fact 4.3(i) also appears in [14, Theorem 3.7.(b)] even in certain Banach spaces.

Fact 4.3 (Bruck and Reich) *Let $S\colon X \to X$ be strongly nonexpansive and let $x \in X$. Then the following hold.*

(i) *The sequence $(S^n x - S^{n+1} x)_{n\in\mathbb{N}}$ converges strongly to the unique element of least norm in $\overline{\operatorname{ran}(\operatorname{Id} - S)}$.*

(ii) *If $\operatorname{Fix} S = \varnothing$, then $\|S^n x\| \to +\infty$.*

(iii) *If $\operatorname{Fix} S \neq \varnothing$, then $(S^n x)_{n\in\mathbb{N}}$ converges weakly to a fixed point of S.*

Proof. (i): See [7, Corollary 1.5]. (ii): See [7, Corollary 1.4]. (iii): See [7, Corollary 1.3]. ■

Suppose $S\colon X \to X$ is asymptotically regular. Then, for every $x \in X$, $0 \leftarrow S^n x - S^{n+1} x = (\operatorname{Id} - S)S^n x \in \operatorname{ran}(\operatorname{Id} - S)$ and hence $0 \in \overline{\operatorname{ran}(\operatorname{Id} - S)}$. The opposite implication fails in general (consider $S = -\operatorname{Id}$), but it is true for strongly nonexpansive mappings. Under the assumption that S is firmly nonexpansive, the following result also follows from [16, Corollary 2].

Corollary 4.4 *Let $S\colon X \to X$ be strongly nonexpansive. Then S is asymptotically regular if and only if $0 \in \overline{\operatorname{ran}(\operatorname{Id} - S)}$.*

Proof. "⇒": Clear. "⇐": Fact 4.3(i). ■

Corollary 4.5 *Set $S := T_m T_{m-1} \cdots T_1$. Then S is asymptotically regular if and only if $0 \in \overline{\operatorname{ran}(\operatorname{Id} - S)}$.*

Proof. Since each T_i is firmly nonexpansive, it is also strongly nonexpansive by Fact 4.2(i). By Fact 4.2(ii), S is strongly nonexpansive. Now apply Corollary 4.4. Alternatively, $0 \in \overline{\operatorname{ran}(\operatorname{Id} - S)}$ by Corollary 3.2 and again Corollary 4.4 applies. ■

We are now ready for our first main result. When $m = 2$, then the conclusion also follows from [14, page 124].

Theorem 4.6 *Suppose that each T_i is asymptotically regular. Then $T_m T_{m-1} \cdots T_1$ is asymptotically regular as well.*

Proof. Theorem 3.1(v) implies that $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T_m T_{m-1} \cdots T_1)}$. The conclusion thus follows from Corollary 4.5. ■

As an application of Theorem 4.6, we obtain the main result of [1].

Example 4.7 Let $C_1, \ldots, C_m$ be nonempty closed convex subsets of X. Then the composition of the corresponding projectors, $P_{C_m} P_{C_{m-1}} \cdots P_{C_1}$ is asymptotically regular.

Proof. For every $i \in I$, the projector P_{C_i} is firmly nonexpansive, hence strongly nonexpansive, and $\operatorname{Fix} P_{C_i} = C_i \neq \varnothing$. Suppose that $(\forall i \in I)\ T_i = P_{C_i}$, which is thus asymptotically regular by Corollary 4.4. Now apply Theorem 4.6. ■

5 Convex Combination

In this section, we use our fixed weights $(\lambda_i)_{i \in I}$ (see (12)) to turn X^m into a Hilbert product space *different from* $\mathbf{X}$ considered in the previous sections. Specifically, we set

$$\mathbf{Y} := X^m \quad \text{with} \quad \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i \in I} \lambda_i \langle x_i, y_i \rangle \tag{39}$$

so that $\|\mathbf{x}\|^2 = \sum_{i \in I} \lambda_i \|x_i\|^2$. We also set

$$\mathbf{Q} \colon X^m \to X^m \colon \mathbf{x} \mapsto (\bar{x})_{i \in I}, \quad \text{where } \bar{x} := \sum_{i \in I} \lambda_i x_i. \tag{40}$$

Fact 5.1 (See [2, Proposition 28.13].) *In the Hilbert product space $\mathbf{Y}$, we have $P_{\mathbf{\Delta}} = \mathbf{Q}$.*

Corollary 5.2 *In the Hilbert product space $\mathbf{Y}$, the operator $\mathbf{Q}$ is firmly nonexpansive and strongly nonexpansive. Furthermore,* $\operatorname{Fix} \mathbf{Q} = \mathbf{\Delta} \neq \varnothing$, $\mathbf{0} \in \operatorname{ran}(\mathbf{Id} - \mathbf{Q})$, *and $\mathbf{Q}$ is asymptotically regular.*

Proof. By Fact 5.1, the operator $\mathbf{Q}$ is equal to the projector $P_{\mathbf{\Delta}}$ and hence firmly nonexpansive. Now apply Fact 4.2(i) to deduce that $\mathbf{Q}$ is strongly nonexpansive. It is clear that $\operatorname{Fix} \mathbf{Q} = \mathbf{\Delta}$ and that $\mathbf{0} \in \operatorname{ran}(\mathbf{Id} - \mathbf{Q})$. Finally, recall Corollary 4.4 to see that $\mathbf{Q}$ is asymptotically regular. ■

Proposition 5.3 *In the Hilbert product space* $\mathbf{Y}$*, the operator* $\mathbf{T}$ *is firmly nonexpansive.*

Proof. Since each T_i is firmly nonexpansive, we have $(\forall \mathbf{x} = (x_i)_{i\in I} \in \mathbf{Y})(\forall \mathbf{y} = (y_i)_{i\in I} \in \mathbf{Y})$ $\|T_i x_i - T_i y_i\|^2 \leq \langle x_i - y_i, T_i x_i - T_i y_i\rangle \Rightarrow \|\mathbf{Tx} - \mathbf{Ty}\|^2 = \sum_{i\in I} \lambda_i \|T_i x_i - T_i y_i\|^2 \leq \sum_{i\in I} \lambda_i \langle x_i - y_i, T_i x_i - T_i y_i\rangle = \langle \mathbf{x} - \mathbf{y}, \mathbf{Tx} - \mathbf{Ty}\rangle$. ■

Theorem 5.4 *Suppose that* $(\forall i \in I)$ $0 \in \overline{\operatorname{ran}(\operatorname{Id} - T_i)}$*. Then the following hold in the Hilbert product space* $\mathbf{Y}$*.*

(i) $\mathbf{0} \in \overline{\operatorname{ran}(\mathbf{Id} - \mathbf{T})}$

(ii) $\mathbf{T}$ *is asymptotically regular.*

(iii) $\mathbf{Q} \circ \mathbf{T}$ *is asymptotically regular.*

Proof. (i): This follows because $(\forall \mathbf{x} = (x_i)_{i\in I})$ $\|\mathbf{x} - \mathbf{Tx}\|^2 = \sum_{i\in I} \lambda_i \|x_i - T_i x_i\|^2$.

(ii): Combine Fact 4.2(i) with Corollary 4.4.

(iii): On the one hand, $\mathbf{Q}$ is firmly nonexpansive and asymptotically regular by Corollary 5.2. On the other hand, $\mathbf{T}$ is firmly nonexpansive and asymptotically regular by Proposition 5.3 and (ii). Altogether, the result follows from Theorem 4.6. ■

We are now ready for our second main result, which concerns convex combinations of firmly nonexpansive mappings. For further results in this direction—namely convex combinations of strongly nonexpansive mappings in Banach spaces—we refer the reader also to [15].

Theorem 5.5 *Suppose that each* T_i *is asymptotically regular. Then* $\sum_{i\in I} \lambda_i T_i$ *is asymptotically regular as well.*

Proof. Set $S := \sum_{i\in I} \lambda_i T_i$. Fix $x_0 \in X$ and set $(\forall n \in \mathbb{N})$ $x_{n+1} = Sx_n$. Set $\mathbf{x}_0 = (x_0)_{i\in I} \in X^m$ and $(\forall n \in \mathbb{N})$ $\mathbf{x}_{n+1} = (\mathbf{Q} \circ \mathbf{T})\mathbf{x}_n$. Then $(\forall n \in \mathbb{N})$ $\mathbf{x}_n = (x_n)_{i\in I}$. Now $\mathbf{Q} \circ \mathbf{T}$ is asymptotically regular by Theorem 5.4(iii); hence, $\mathbf{x}_n - \mathbf{x}_{n+1} = (x_n - x_{n+1})_{i\in I} \to \mathbf{0}$. Thus, $x_n - x_{n+1} \to 0$ and therefore S is asymptotically regular. ■

Remark 5.6 Theorem 5.5 extends [3, Theorem 4.11] from Euclidean to Hilbert space. One may also prove Theorem 5.5 along the lines of the paper [3]; however, that route takes longer.

Remark 5.7 Similarly to Remark 3.4, one cannot deduce that if each T_i has fixed points, then $\sum_{i\in I} \lambda_i T_i$ has fixed points as well: indeed, consider the setting described in Remark 3.4 for an example.

We conclude this paper by showing that we truly had to work in $\mathbf{Y}$ and not in $\mathbf{X}$; indeed, viewed in $\mathbf{X}$, the operator $\mathbf{Q}$ is generally not even nonexpansive.

Theorem 5.8 *Suppose that $X \neq \{0\}$. Then the following are equivalent in the Hilbert product space* $\mathbf{X}$.

(i) $(\forall i \in I)\ \lambda_i = 1/m$.

(ii) $\mathbf{Q}$ *coincides with the projector* $P_{\boldsymbol{\Delta}}$.

(iii) $\mathbf{Q}$ *is firmly nonexpansive.*

(iv) $\mathbf{Q}$ *is nonexpansive.*

Proof. "(i)⇒(ii)": [2, Proposition 25.4(iii)]. "(ii)⇒(iii)": Clear. "(iii)⇒(iv)": Clear. "(iv)⇒(i)": Take $e \in X$ such that $\|e\| = 1$. Set $\mathbf{x} := (\lambda_i e)_{i\in I}$ and $y := \sum_{i\in I} \lambda_i^2 e$. Then $\mathbf{Qx} = (y)_{i\in I}$. We compute $\|\mathbf{Qx}\|^2 = m\|y\|^2 = m\big(\sum_{i\in I}\lambda_i^2\big)^2$ and $\|\mathbf{x}\|^2 = \sum_{i\in I}\lambda_i^2$. Since $\mathbf{Q}$ is nonexpansive, we must have that $\|\mathbf{Qx}\|^2 \leq \|\mathbf{x}\|^2$, which is equivalent to

$$m\Big(\sum_{i\in I}\lambda_i^2\Big)^2 \leq \sum_{i\in I}\lambda_i^2 \tag{41}$$

and to

$$m\sum_{i\in I}\lambda_i^2 \leq 1. \tag{42}$$

On the other hand, applying the Cauchy-Schwarz inequality to the vectors $(\lambda_i)_{i\in I}$ and $(1)_{i\in I}$ in $\mathbb{R}^m$ yields

$$1 = 1^2 = \Big(\sum_{i\in I}\lambda_i \cdot 1\Big)^2 \leq \big\|(\lambda_i)_{i\in I}\big\|^2\big\|(1)_{i\in I}\big\|^2 = m\sum_{i\in I}\lambda_i^2. \tag{43}$$

In view of (42), the Cauchy-Schwarz inequality (43) is actually an equality which implies that $(\lambda_i)_{i\in I}$ is a multiple of $(1)_{i\in I}$. We deduce that $(\forall i \in I)\ \lambda_i = 1/m$. ■

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