International Journal of Algebra, Vol. 13, 2019, no. 6, 259 - 295
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ija.2019.9724

'Compound Algebra' : Generalization of Complex Algebra

Ranjit Biswas

Department of Computer Science & Engineering
Jamia Hamdard University
Hamdard Nagar, New Delhi – 62, India

Abstract

In this paper a new type of numbers is discovered called by 'Compound Numbers' which is a generalized concept of the complex numbers. A new theory entitled "Theory of Objects" and the corresponding "Object Algebra" are first of all introduced where we can talk about various operations over the objects of a set, about a new concept of 'infinity', about zero object, and signed objects called by positive and negative objects; categorizations like prime objects, composite objects, neither prime nor composite objects, etc. The traditional notion of 'numbers' that we use in classical arithmetic is a particular case of 'object' of a 'region' algebraic structure. The introduction of imaginary number **i** in the classical 'Theory of Numbers' is very interesting but its history says that it took a long span of years to convince the mathematicians about the role and necessity of it in mathematics. Since then **i** has been playing extraordinary roles in various branches of Science, Engineering and many other broad fields. In our newly proposed "Theory of Objects", the notion of 'imaginary object' is introduced for a region where the existing concept of 'imaginary number' is a particular instance of the concept of 'imaginary object'. It is unearthed that the region C (set of complex numbers) does also have imaginary objects. The adjective 'imaginary' is completely a local issue with respect to the concerned region. Something may be an imaginary object for a region A, but may be a core internal member of another region B. For example, **i** is an imaginary object for a region R, but it is a core internal member of the region C. In the "Theory of Objects" developed in this paper, two imaginary objects **e** and **w** are unearthed for the region C (set of complex numbers). These imaginary objects **e** and **w** of C are called by 'compound numbers', and consequently a new number theory is introduced in the subject Number Theory entitled 'Theory of Compound Numbers'. In fact it is shown here that every complete region has at least one imaginary object. Finally a new type of Number Theory is introduced called by ‘Theory of A-numbers’. All the results here open a new type of algebra called by ‘Object Algebra’.

Mathematics Subject Classification: 08, 11

Keywords: region, object, positive object, negative object, onteger, bachelor, prime object, composite object, real object, imaginary object, complex object, compound number, im-number, rim, cim.

1. Introduction

Biswas in [7] introduced a new algebraic structure ‘region’ in Abstract Algebra. With several examples, it is shown in [7] in details with sufficient explanations that all the existing important algebraic structures viz. group, ring, module, field, linear space, algebra over a field, associative algebra over a field, Division Algebra are having a serious weakness in justifying many of the very fundamental operations of Elementary Algebra (and hence in justifying the fluent fundamental computations involved in the giant subject Mathematics, in different branches of Sciences and Engineering subjects). Region is the minimal algebraic structure which can validate the fundamental operations of Elementary Algebra, whereas the division algebra or any other existing standard algebraic structure can not. It is established in [7] that the subject Abstract Algebra has become now more complete and sound with the introduction and characterizations of this new algebraic structure ‘region’.

In this paper a new number called by ‘compound numbers’ is discovered as a generalized notion of the complex numbers, and a corresponding new kind of algebra called by ‘Compound Algebra’ is developed. The work is initiated by introducing “Object Algebra” in the “Theory of Objects”. And then another type of number theory is also discovered called by “Theory of A-numbers” corresponding to a complete region A. As a particular case, the existing theory of numbers is nothing but “Theory of R-numbers” where the complete region R is the set of real numbers.

History is regarded to be one of the greatest source of energy which encourages for new thoughts. Before going for detailed introduction of this work, let us make a quick visit to the history of numbers, which could be an interesting at this stage just for a quick perusal. History of Prime Numbers says that from around 1550 BC the Rhind Mathematical Papyrus made Egyptian fraction expansions of different forms for prime and composite numbers. However, the earliest surviving available records of the explicit study of prime numbers and composite come from Ancient Greek mathematics. Euclid's Elements (circa 300 BC) first time proved that there are infinite number of prime numbers. Even for constructing a list of prime numbers the Sieve of Eratosthenes was used. An Islamic mathematician Ibn al-Banna' al-Marrakushi observed that the sieve of Eratosthenes can be sped up by testing only the divisors up to the square root of the number to be tested for prime. Fibonacci migrated the innovations from Islamic mathematics back to Europe. His book Liber Abaci (1202) was the first book to describe trial division for testing the primality of a number using divisors which are less than or equal to the square root of the number. The theory of prime numbers has a lot of applications in computer science too, in particular in cryptography. In the Theory of Objects here a new notion of prime objects and composite objects are introduced, and it is observed that prime numbers are special cases of prime objects, composite numbers are special cases of composite objects.

In the existing mathematics **i** is the only imaginary number. But imaginary number **i** was not instantly created, was not instantly accepted by the mathematicians after its discovery. Its history says that it took several centuries to convince certain mathematicians to accept this new number **i**, and people now use **i** in everyday mathematics!. For example, it is used in studying various kinds of infinite series, to study that every polynomial equation has a solution if complex numbers are used, and there is a large list to mention the application domains of **i** in mathematics. Besides that, engineers use **i** to study stresses on beams and to study resonance, complex numbers are used to study the flow of fluid around objects like water flowing around a pipe, complex numbers are used in electric circuits, and help in transmitting radio waves, etc. Thus it was a revolutionary issue that **i** was created and accepted to be an essential element to enrich the subject mathematics. The surviving record of history of mathematics says that people were trying to use imaginary number even in 1st century. In 50 A.D., Heron of Alexandria studied the volume of an impossible section of a pyramid. But the problem then arose while facing to compute the value of $\sqrt{81-114}$, and consequently he gave up his attempt. After that for a very long time, none took interest to deal with imaginary number, although it wasn't for a lack of trying. In around 1500 A.D., the peculiar issue of computing square roots of negative numbers was reconsidered. Formulas for solving 3rd and 4th degree polynomial equations were discovered, and people realized that the necessity of work with square roots of negative numbers is genuine to proceed further for extension of the subject mathematics. In 1545 the first major work with imaginary numbers was the book entitled Ars Magna by Girolamo Cardan, in which he solved the equation x(10-x) = 40 and found the answer $5 \pm \sqrt{-15}$. Although he successfully found the answer, but he did not like imaginary numbers to be included in mathematics as an element for study. Rather his comment about such solution was "as subtle as it would be useless", and referred to working with the imaginary numbers as a kind of "mental torture." Most of the mathematicians supported Cardan for nearly one century years of time!. It is Rene Descartes who introduced the standard form $a \pm b\sqrt{-1}$ for complex numbers in 1637 A.D. But, it is again a fact that Descartes too didn't like complex numbers in mathematics. However, Euler in 1777 used the symbol **i** to stand for √-1, writing the complex number $a \pm b\sqrt{-1}$ in the form a+ib which seems to be now easier to the mathematicians. Argand in 1806 proposed how to plot complex numbers in a plane, and thereafter this plane is being called by 'Argand Plane'. In 1833, Hamilton proposed to express complex numbers as pairs of real numbers, like as the complex number a+bi to be expressed as (a,b). Although this idea was simple but it made the topic more popular and useful in a easy manner. It took several genius mathematicians such as Weierstrass, Schwarz, Dedekind, Holder, Poincare, Eduard, Burnet, Cauchy, Niels Henrik Able, Mobius, to name a few only out of many, to work several years to convince the world to accept complex numbers.

In the Theory of Objects developed in this paper, a new concept of 'imaginary object' is introduced. It is observed that the classical concept of 'imaginary numbers' (or, complex numbers) are just one particular instance of the notion of 'imaginary objects'. Although the birth of the 'imaginary numbers' took place long before, but interestingly it is observed now that the same happened out of a very particular 'region', not out of a set or out of a Division Algebra!. This fact is unearthed and explained in details in this work. It is explained that by virtue of their respective definitions and independently owned properties, neither Division Algebra nor any of the existing algebraic structures alone can produce a sound theory on the prime numbers, composite numbers, imaginary numbers and complex numbers. The basic philosophy introduced in the Theory of Objects is that the status 'imaginary' is purely <u>local</u> with respect to the region concerned. According to this philosophy, one object could be imaginary with respect to one set, and may not be imaginary with respect to another set. This could be realized by an example of our social life. See that a person may be a 'guest' for a family but could be well a 'core member' of another family in our society!. The status of 'guest' is thus local to the concerned family, and similarly the status of 'core family member' is also a local issue with respect to another family. For another example, a person may be a 'foreigner' in one country, but he is very well a true citizen of another country! The status of 'foreigner' is thus local to the concerned country, and similarly the status of 'bonafide citizen' is also a local issue with respect to another country. Similarly, for instance, in the 'Theory of Objects' it is shown that 2i is an imaginary object for the set R but not so for the set C. But C itself has its own imaginary objects according to the Theory of Objects. Two imaginary objects **e** and **w** are unearthed for C in this paper and then 'Theory of Compound Numbers' is developed. In fact it is shown here that every complete region has at least one imaginary object. This result opens a new kind of elementary algebra which may be called as 'Object Algebra'.

After that, another concept in the Theory of Objects is introduced. It is shown that every complete region A has its own 'Theory of Numbers' called by 'Theory of A-numbers'. The classical 'Theory of Numbers' of the existing volume of mathematics is just one instance of it to be called by 'Theory of RR-numbers' corresponding to the particular complete region RR. It is claimed that the "Theory of Objects" will play a huge role to the Number Theorists in a new direction. In due time, the 'Number Theorists' may alternatively be re-designated with the broader title 'Object Theorists' as they may need to cultivate the broad area 'Theory of Objects' in pursuance of cultivating the 'Theory of Numbers' in a much better style and in a generalized fashion. In fact, one of the major contributions in this work is that several new type of numbers are discovered and it is established that every complete region has its own Theory of Numbers. Consequently the existing 'Theory of Numbers' needs to be updated, extended and viewed in a new style.

2. Preliminaries of the 'Region'

The important algebraic structures viz. groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, Division Algebras have made the subject 'Algebra' very rich and well equipped to deal with various algebraic computations at elementary level to higher level. Nevertheless it is unearthed in [7] that these algebraic structure and huge volume of literature on them are not sufficient for Mathematics. There are many serious problems and issues in mathematics which do not fall under the jurisdictions of these algebraic structures to deal appropriately. There is a genuine vacuum unearthed in the family of all existing standard algebraic structures. Consequently, in [7] it is sufficiently justified that this family needs inclusion of an appropriate member who can take the responsibility unlike the existing standard algebraic structures. An algebraist can introduce a number of algebraic structures if he desires. But the question arises about their role in Algebra, in Mathematics. A new algebraic structure is not supposed to be a redundant one to the subject 'Algebra' to unnecessarily cater to the existing huge volume of literature of Algebra. It must have some unique as well as advance kind of roles which the existing algebraic structures do not have by their respective definitions and independently owned properties. It must have unique capabilities to enrich Mathematics which none of the existing algebraic structures can claim. And this objective a new and very important algebraic structure called by "Region" is developed in Abstract Algebra. Few definitions which are used in this paper in our subsequent work are presented below from Region Algebra.

2.1 Region

Consider a non-null set A with three binary operators $\oplus$, $*$ and $\bullet$ defined over it such that for a given field $(F,+,.)$, the following three conditions are satisfied:-

(i) $(A,\oplus,*)$ forms a field,

(ii) $(A,\oplus,\bullet)$ forms a linear space over the field $(F,+,.)$, and

(iii) A satisfies the property of "Compatibility with the scalars of the field F", i.e. $\forall$ a, b $\in$ F and $\forall$ x,y $\in$ A,

$$(a\bullet x) * (b\bullet y) = (a.b)\bullet(x*y).$$

Then the algebraic structure $(A, \oplus, *, \bullet)$ is called a **Region** over the field (F, +, **.**). If there is no confusion, we may simply use the notation A to represent the region $(A, \oplus, *, \bullet)$, for brevity.

2.1 Partitioned Region

Consider a real region $A = (A, \oplus, *, \bullet)$. Suppose that A forms a chain with respect to a total order relation (say, denoted by the notation '$\leq$'). Then the real region A is called a chain region with respect to the total order relation '$\leq$'. A real region $A = (A, \oplus, *, \bullet)$ is called a Partitioned Region if the following conditions are satisfied :

(i) A is an infinite region,
(ii) A is a chain region with respect to a total order relation '$\leq$', and
(iii) the characteristic of A is zero.

Here A is called a 'partitioned region' because of the fact that it induces a partition P_A of A into three mutually disjoint non-null sets denoted by A^+, A^- and $\{0_A\}$ such that

(i) $A^+ = \{ a : a \in A \text{ and } 0_A < a\}$
(ii) $A^- = \{ a : a \in A \text{ and } a < 0_A\}$.

Clearly, $\forall\, a \in A^+$, $\sim a \in A^-$ and $\forall\, b \in A^-$, $\sim b \in A^+$.

(Note : It may be recalled from the properties of the chain that : $a < b$ iff $a \leq b$ and $a \neq b$, where "$\leq$" is the total order relation of the chain A, and similarly $a > b$ iff $b \leq a$ and $b \neq a$).

This partition P_A, once made, is regarded as an absolute partition of the region A corresponding to its total order relation '$\leq$' in the sense that this partition generates the sign of every object of the complete region A, positive or negative, which will remain absolute throughout the complete literature henceforth. However for a different type of total order relation defined over the region A we will get a different partition of A. But the set $\{0_A\}$ is common to all such possible partitions.

2.2 Extended Region

Consider an infinite region $A = (A, \oplus, *, \bullet)$. The extended region of A is the region itself with all its infinity objects, if any. The infinity objects are not basically the core member of the region A, but to be included into it. At this point of time we do not consider any method about 'how to find out all the infinity objects of an infinite region'. However for a portioned region the method is rather easier.

Consider a partitioned region $A = (A, \oplus, *, \bullet)$. If we now include two more objects $+\propto_A$ and $-\propto_A$ in A as two permanent guests, then the set $A^E = A \cup \{+\propto_A, -\propto_A\}$ is the 'extended region' of the region A.

The two guest objects $+\propto_A$ and $-\propto_A$ are called infinities, and are defined as below:

(i) $+\propto_A = \frac{x_A}{0_A}$ where x_A ($\neq 0_A$) is any positive object of the region A, and

(ii) $-\propto_A = \frac{z_A}{0_A}$ where z_A ($\neq 0_A$) is any negative object of the region A.

The extended region of the partitioned region A is denoted by the notation A^E. However, if there is no confusion then we may use the notation A itself to denote the extended region of A. Note that an extended region is not a region. For a partitioned region, it is just a superset of the set A containing two more objects.

But whenever we say that 'A is an extended region', it will simply mean that A is a region with all its infinities as permanent guests. At this stage we do not explore to study whether there are more infinities other than the two guest objects $+\propto_A$ and $-\propto_A$ for a partitioned region.

An extended region A^E may also be called as 'extended real region' if the corresponding region A is a real region by virtue of the definition of a partitioned region.

For the region C, there are many infinities to be included into it to call it an extended region. For example $\forall$ a, b $\in$ R, the object a+ib is an infinity object for C if either a or b or both are the infinity object of the region R. The extended region of C is denoted by the notation C^E. Further future study on the topic of extended region will make the literature richer.

3. Theory of Objects

'Region' is the most practiced algebra in school/college education, research, scientific and engineering calculations, etc. It is the minimum mandatory algebra to study science, mathematics, engineering, and other areas. Its objects (elements) with the support of the axioms play various roles to expose themselves for induction in various branches of mathematics, and they exercise among themselves too with various characteristic properties. This phenomenon develops a new direction in mathematics called by **"Theory of Objects".**

This work provides the beginning of the "Theory of Objects". Presently the theory is at its baby stage, and is initiated in this work with the following three topics:-

1. "Prime Objects" and "Composite Objects" in a Region
2. "Imaginary Objects" and "Compound Objects" in a Region, and
3. "Theory of Numbers" : Every Complete Region has its own.

The subsection 3.1 introduces the topic "Prime Objects" and "Composite Objects" in a Region, and then the work introduces the topics "Imaginary Objects" and "Compound Objects" in a Region, and also then introduces the topic "Theory of Numbers" : Every Complete Region has its own.

3.1 'Prime Objects' and 'Composite Objects'

In our school mathematics, we speak about 'prime numbers' and 'composite numbers'. They are members of the set R of real numbers. A prime number (or a prime) is a natural number greater than 1 that has no positive divisors other than 1 and itself. A natural number greater than 1 that is not a prime number is called a composite number. In the Theory of Objects, they are basically objects of the region R. In this subsection we introduce the notion of 'prime object' and 'composite object' in any arbitrary region A = $(A, \oplus, *, \bullet)$. We consider here simple regions only, not necessarily the complete regions. First of all we introduce the notion of 'bachelor set' in a given region, and then we use the notion of bachelor set to define the concept of 'exact division' in a bachelor set.

3.1.1 'Bachelor Set' in a Region

Let A be a region. A subset B of the region A is called a 'bachelor set' in A if
(i) $1_A \in B$, $0_A \notin B$ and
(ii) $\forall\ x\ (\neq 1_A) \in B$, $x^{-1} \notin B$.

A bachelor set can never be a null set because the smallest bachelor set in a region A is the singleton $\{1_A\}$. Also, it is obvious from the above definition that the self-inverse objects (like an element x, where $x^2 = 1_A$) other than 1_A of the region A are not the members of any bachelor set of A. Clearly A itself can not be a bachelor set in A.

Any subset S of a bachelor set B in the region A is also a bachelor set in A if $1_A \in S$.

It can be verified that if B is a bachelor set in a region A, then the set

$$\tilde{B} = \{y : y = x^{-1} \text{ where } x \in B\}$$

is also a bachelor set in A. This set $\tilde{B}$ is called the 'conjugate bachelor' of the bachelor set B in the region A.

Clearly, conjugate of the conjugate of B is B itself. The union of two bachelors in A need not be a bachelor in A, but the intersection of two bachelors will be a bachelor in A.
For every bachelor set B in A, $B \cap \tilde{B} = \{1_A\}$.
If B and C are two bachelors in the region A, then the conjugate of $(B \cap C) = \tilde{B} \cap \tilde{C}$. If $B = \tilde{B}$, then the only case is that $B = \tilde{B} = \{1_A\}$.

Example 3.1.1
Consider the region RR. Clearly the following are true :

(i) the set N of natural numbers is a bachelor set in the region RR.

(ii) The set M = {1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, …} = { m : m = 1/n, n $\in$ N, where N is the set of natural numbers} is a bachelor set in the region RR.
(iii) The set L = {1, 78.261, 9287, 83.5} is also a bachelor set in the region RR.

Example 3.1.2
The set R^+ of all positive real numbers is not a bachelor set in the region RR.

Proposition 3.1.1
If the set B of cardinality n is a bachelor set in the region A, then B has 2^{n-1} number of distinct sub-bachelors.

Proof:
For n = 1, the result is true because the only possibility is that B = {1_A}.
Now consider the case n > 1. The two trivial sub-bachelors are {1_A} and B. The cardinality of the set B–{1_A} is (n-1) which is having 2^{n-1} number of subsets including the null set and the set B–{1_A} itself. Adding the common element 1_A to each of these 2^{n-1} subsets will create 2^{n-1} number of bachelor sets of A, being all the sub-bachelors of B. Hence proved.

There are four types of division operations defined in a region. We introduce here the operation of 'Exact Division' in a bachelor set in the region A, which is a kind of division of an element of a bachelor set B by another element of the same bachelor set B.

3.1.2 'Exact Division' in a Bachelor Set

Let B be a bachelor set in the region A. Consider two objects x, y $\in$ B. We say that the object x <u>exactly divides</u> the object y in B, denoted by the notation "x $|_B$ y", if $\exists$ z $\in$ B such that $\frac{y}{x} = z$ holds good in the region A.
In another words, we say that the object x <u>exactly divides</u> the object y in B, denoted by the notation "x $|_B$ y", if $y*x^{-1} \in$ B.
The notation "$|_B$" signifies the operation of 'exact division' in the bachelor set B of the region A = (A, $\oplus$,*, $\bullet$), and the notation "$\nmid_B$" signifies the operation "can not exactly divide" in B.

The following results are straightforward.

Proposition 3.1.2
(i) x $|_B$ x $\forall$ x $\in$ B.
(ii) 1_A $|_B$ x $\forall$ x $\in$ B.
(ii) for x $\neq$ y, if x $|_B$ y then y $\nmid_B$ x, where x, y $\in$ B.

Proof:
(i) Since $1_A \in$ B, we have $x*x^{-1} \in$ B. Hence proved.

(ii) Obvious.

(iii) if $x \,|_B\, y$ then we have $y*x^{-1} \in B$.
Therefore $(y*x^{-1})^{-1} \notin B$, which means $x*y^{-1} \notin B$. Hence Proved.

Proposition 3.1.3

It may happen that for a given pair of objects x, y in a bachelor B in a region A, neither $x \,|_B\, y$ nor $y \,|_B\, x$.

Proof:

Consider a bachelor C in the region A where x, y are in C and $x \,|_C\, y$ (such that $\frac{y}{x} = z$). Now consider the set $B = C - \{z\}$.

Clearly B is a bachelor in the region A, where both x and y are in a bachelor B but neither $x \,|_B\, y$ nor $y \,|_B\, x$. Hence proved.

3.1.3 'Composite Objects' and 'Prime Objects'

We introduce now the notion of 'Composite Objects' and 'Prime Objects' in a region with respect to a bachelor set B of it.

'Composite Object'

Let B be a bachelor set of a region A. An object $x \in B$ is called a 'Composite Object' in B, if $\exists$ p, q $\in B - \{1_A\}$ such that $x = p*q$ in A.

'Prime Object'

An object $x \in B - \{1_A\}$ is called a 'Prime Object' in B if x is not a composite object in B.

It may be noted that any composite or prime object in B must be a member of B. By virtue of the construction here, there is no reason to check whether the element 0_A and the self-inverse elements (other than 1_A) of the region A are 'prime' or 'composite' or 'neither prime nor composite' in any bachelor set in the region A, as they can not be members of any bachelor set in A.
However, 1_A is the only element in any bachelor B which is neither a prime object nor a composite object. For every other object x (i.e. if $x \neq 1_A$) in B, x is by default either a prime object or a composite object. Thus the following proposition is straightforward.

Proposition 3.1.4

There can not be any object x in the bachelor B in the region A which is both prime and composite.

If may be noted here that an object x may be prime in a bachelor B of a region A, but may not be so in another bachelor C of the same region A, even if $x \in B, C$ both.

Thus, for a given region, the property of prime, composite and 'neither prime nor composite' is dependent upon the concerned bachelor set, and they must be members of the concerned bachelor set. For a given bachelor set, checking an object of a region whether prime or composite or 'neither prime nor composite' with respect to this bachelor set is an invalid issue if the object itself be not a member of the bachelor set.

3.1.4 Partition of a Bachelor Set

For a bachelor set V in a region A, an important partition of the set V can be made into three subsets **:** the set of Prime objects in V, the set of Composite objects in V, and the set of neither Prime nor Composite objects in V, as shown in Figure 3.1 below. This is a partition of the bachelor set V because there can not be any object in the set V which is both a prime object and a composite object simultaneously in V.

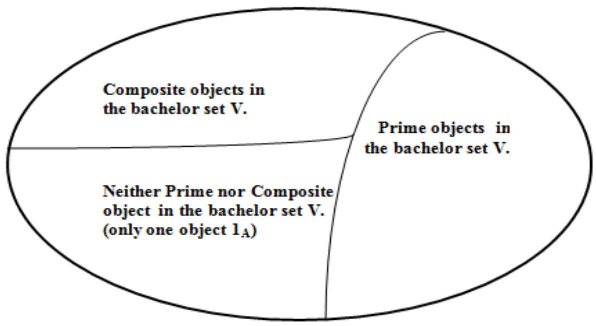

Figure 3.1. Prime, Composite and 'neither prime nor composite' objects in a bachelor set V in the region A : partitioned into three subsets

The following proposition is now straightforward.

Proposition 3.1.5
If x is a prime (composite) object in a bachelor B of a region R then x^{-1} is a prime (composite) object in the conjugate bachelor $\tilde{B}$, and conversely.

We present below examples of the notion of prime objects and composite objects in a bachelor set in a region.

Example 3.1.3
Consider the region RR. Consider the bachelor set N of the region RR where N = {1, 2, 3, 4, 5, 6, 7, 8, …} = the set of natural numbers.

Clearly, the members 4, 6, 8, 9, 10, 12, 14, ….. are composite objects of the bachelor N here in the region RR; and the members 2, 3, 5, 7, 11, 13, …. are prime objects of the bachelor N in RR. Actually these are popularly known as 'composite numbers' and 'prime numbers' respectively in the existing literature of the classical 'Theory of Numbers'. There can not be any object in the bachelor N which is both prime and composite. It is known to us that a prime number (or a prime) is a natural number greater than 1 that has no positive divisors other than 1 and itself. A natural number greater than 1 that is not a prime number is called a composite number.

And 1 is the only object in the bachelor N which is neither a prime object nor a composite object (see Figure 3.2). There is no object in the bachelor N which is both prime and composite object. In fact this is a very much known result in the existing classical 'Theory of Numbers' that the integer 1 is neither a prime number nor a composite number.

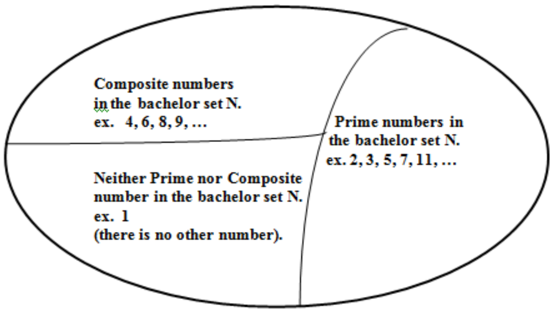

Figure 3.2. Partitioned into : Prime, Composite and 'neither prime nor composite' numbers in the bachelor set N (of natural numbers) in the region RR

Another example of prime and composite objects is given below.

Example 3.1.4

Consider the region RR. Consider the bachelor set M of the region RR where M = {1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, …} = { m **:** m = 1/n, n $\in$N, where N is the set of natural numbers}. Clearly, the members 1/4, 1/6, 1/8, 1/9, 1/10, 1/12, ….. are composite objects of the bachelor M here in the region RR; and the members 1/2, 1/3, 1/5, 1/7, 1/11, 1/13, ….. are prime objects of M in RR (see Figure 3.3). And 1 is the only object in the bachelor M which is neither a prime object nor a composite object. There is no object in the bachelor M which is both prime and composite.

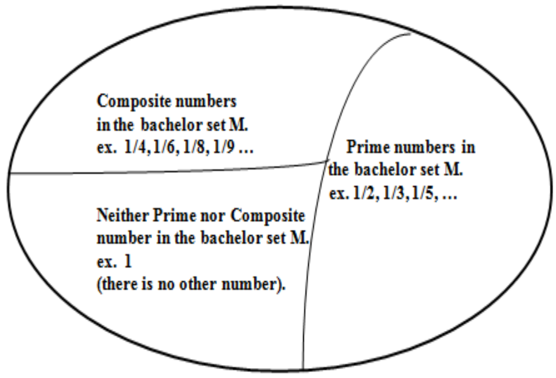

Figure 3.3. Partitioned into : Prime, Composite and 'neither prime nor composite' numbers in the bachelor set M in the region RR

Example 3.1.5

Consider the bachelor L = {1, 78.261, 9287, 83.5} of the region RR. Clearly, the members 78.261, 9287, 83.5 are prime objects in the bachelor L; there does not exist any composite object in L. And 1 is the only object in the bachelor L which is neither prime object nor composite object. There can not be any object in any bachelor which is both prime and composite (which may be verified to be true for the bachelor L here).

The above mentioned examples show that the classical prime numbers (in the existing classical 'Theory of Numbers') are particular case of prime objects in the region RR with respect to its bachelor set N. It may be noted that the notion of prime objects and composite objects are defined over any region, need not necessarily be in a complete region.

4. Compound Algebra

In this section we introduce the notion of 'imaginary object' and 'complex object' of a region. However, we will also see here that a region A may or may not have imaginary object. A region even may have more than one imaginary objects too. Imaginary objects of a region A are not members of A and so they are called 'imaginary' with respect to the concerned region A only (i.e. it is purely a local characteristics property with respect to the region concerned). An imaginary object of a region A could be core member of other regions (other than A). Consequently, a core member of a region A could be imaginary object of some other region(s). Just imagine an analogous concept that a person Mr. P may be a stranger to a family, but he is a core member of another family. And similarly a person Mr. P may be the core family member of a family but he may be a stranger to another family. Thus an object of a region A could be an imaginary object of another region B, but can not be an imaginary object of the same region A itself. Every region has its own set of imaginary objects (if exist). This set could be null set too for a region. Two sets of imaginary objects corresponding to two distinct regions may be disjoint or overlapping.

4.1 'Existence' of Imaginary Objects

Consider an extended region A^E of the region A = $(A, \oplus, *, \bullet)$ as defined in the previous section. For the region A, any member of the set A is called a "real object" of the region A.
If something is not a member of the extended region A^E, we can not call it a real object of the region A. But quite naturally curiosity arises about: When do we say that a region A has an imaginary object ?

We present here an important set of two conditions called by **'Qualification Conditions'** which are to be fulfilled in order to have guarantee that there exists at least one Imaginary Object of a Region A.

A region may or may not have an imaginary object. First of all we define the terms 'valid expression' and 'constant expression' over a region A.

Valid Expression in a Region

Let $E_1(x)$ and $E_2(x)$ be two single variable expressions valid in the region A. In region mathematics we say that an expression is regarded to be a valid expression in a region A if it can be computed in A with the valid operations of A, assuming the inclusion of two infinities in A. Thus for a valid expression E(x), every computed value of E(x) for any x of A must be in the extended region of A.

Constant Expression in a Region

Let E(x) be a single variable expression valid in the region A. Then an expression like $E(x) \equiv a$, where a is a given real object of A, is called a constant expression in A.

Imaginary Object of a Region A: 'Qualification Conditions'

We define here two necessary conditions to be fulfilled for existence of an Imaginary Object of a region A.

Consider two distinct valid expressions $E_1(x)$ and $E_2(x)$ in the extended region of A. Then we say that the region A has an 'imaginary object' if the following two conditions are fulfilled:
(i) at least one of the two expressions $E_1(x)$ and $E_2(x)$ is not a constant expression in the region A, and
(ii) the equality (not identity) $E_1(x) = E_2(x)$ is not satisfied by any element of the extended region of A.
These two conditions are called **'Qualification Conditions'** corresponding to the pair of distinct expressions $E_1(x)$ and $E_2(x)$ which are necessarily to be fulfilled for possible existence of an imaginary object of the region A.

We may choose infinite number of pairs of expressions for $E_1(x)$ and $E_2(x)$ over a given region A. But there is no method described here for choosing two appropriate expressions for $E_1(x)$ and $E_2(x)$ which can offer us or guarantee us the existence of at least one imaginary object of the region A. At present let us do the exercise by trial and failure to explore the possibility of the existence of at least one imaginary object of the region A.

For example, if we choose two distinct expressions $f(x) = 2x+3$ and $g(x) = 7$ then the pair of functions f and g satisfies the condition(i) of the 'qualification conditions' in the region R, but does not satisfy the condition(ii).

If we choose two distinct expressions $f(x) = x+3$ and $g(x) = x+7$ then the pair of functions f and g satisfies the condition(i) of the 'qualification conditions' in the region R, but does not satisfy the condition(ii) because the infinity objects of the extended region of R satisfy the equation $f(x) = g(x)$.

However, if we choose two distinct expressions $f(x) = x^2+4$ and $g(x) = 0$ then the pair of expressions f and g satisfies both the conditions of the 'qualification conditions' in the region R.

Consider the region C. If we choose $f(z) = 5z+3$ and $g(z) = 9+3i$ then the pair of distinct functions f and g satisfies the condition(i) of the 'qualification conditions' in the region C, but does not satisfy the condition(ii). But if we choose $f(z) = 5+3i$ and $g(z) = 9+3i$ then the pair of these distinct functions f and g does not satisfy the condition(i) of the 'qualification conditions' in the region C.

If we choose two distinct expressions $f(z) = z+3i$ and $g(z) = z+7i$ then the pair of functions f and g satisfies the condition(i) of the 'qualification conditions' in the region C, but does not satisfy the condition(ii) because many of the infinity objects of the extended region of C satisfy the equation $f(z) = g(z)$.

Now let us suppose that **i** is an imaginary object of the region A coming out of the equation $E_1(x) = E_2(x)$ where $E_1(x)$ and $E_2(x)$ are two valid distinct expressions in

the extended region of A satisfying the above two conditions. Then we accept that both $E_1(\mathbf{i})$ and $E_2(\mathbf{i})$ can be computed satisfying the equality $E_1(\mathbf{i}) = E_2(\mathbf{i})$.
Suppose that $E_1(\mathbf{i}) = E_2(\mathbf{i}) = a$ (say). Here a is obviously a member of the region A, but **i** is not a member of the region A.

Let us designate this imaginary object **i** of A to be an atomic imaginary object. Then any expression $E(\mathbf{i}, x_1, x_2, x_3, \ldots, x_n)$ with respect to the operations $\oplus, *, \bullet$ of the region A over its outer field F is called a "complex object" of the region A if the value of $E(\mathbf{i}, x_1, x_2, x_3, \ldots, x_n)$ is not a member of the extended region of A, where the variables $x_1, x_2, x_3, \ldots, x_n$ assume objects from A. There may exist nil or one or more number of atomic imaginary object in a region A, and corresponding to every imaginary object (if exists) there exists a set of complex objects of the region A.

It may be noted here that by definition (as stated and explained above) we can only realize about the <u>existence</u> of an imaginary object of a region A if exists, but we can not trace its <u>identity</u> immediately. Because an imaginary object of a region A is not a member of A, and consequently we do not know where we can search it from, where it has come from. It is fact that on this issue we officially know nothing beyond the boundary of the set A at this stage. It is an open problem to us at this moment for further study and research on this issue.

Example 4.1.1
Consider the region RR. If we take $E_1(x) \equiv x^2 + 1$ and $E_2(x) \equiv 2x - 1$, then $E_1(x)$ and $E_2(x)$ are distinct satisfying both the conditions of the 'Qualification Conditions'. Therefore, there exists at least one imaginary object of RR.
If we take $E_1(x) \equiv x^2 + 1$ and $E_2(x) \equiv 0$, then in this case too we observe that both the 'Qualification Conditions' are fulfilled. This guarantees that the RR region does have at least one imaginary object.

But, by the above examples, we are not sure here whether there exist only finite number of imaginary objects or infinite number of imaginary objects for the region RR.

Example 4.1.2
Consider the region RR. Let us take two distinct expressions $E_1(x) \equiv 3/(x-1)$ and $E_2(x) \equiv 3/(x-4)$. Then we can not conclude from this example whether there exists at least one imaginary object of the region RR, although both $E_1(x)$ and $E_2(x)$ are valid expressions $E_1(x)$ and $E_2(x)$ in the extended region of RR. It is in fact because of the reason that the condition(ii) of the 'Qualification Conditions' is not fulfilled here.

Example 4.1.3
Consider the region RR. Let us take $E_1(x) \equiv 3/(x-1)$ and $E_2(x) \equiv 7/(x-4)$. Then we can not conclude from this example whether there exists at least one imaginary object of the region RR.

Example 4.1.4
Consider the simple trivial region ($Z_2, \oplus, \mathbf{.}, \mathbf{.}$) where $Z_2 = \{0, 1\}$, $\oplus$ is the "addition modulus 2" operator and '**.**' is the 'multiplication modulus 2' operator of real numbers. We see that if we consider the two expressions $E_1(x) \equiv 2x+1$ and $E_2(x) \equiv 0$ in the region ($Z_2, \oplus, \mathbf{.}, \mathbf{.}$), then we observe that the 'Qualification Conditions' are fulfilled here. Therefore, there exists at least one imaginary object of this region Z_2.

However, if we take $E_1(x) \equiv x^2 +1$ and $E_2(x) \equiv 0$ then it does not help us to know the existence of any imaginary object of Z_2. It is in fact because of the reason that the condition(ii) of the 'Qualification Conditions' is not fulfilled here for Z_2.

It is justified in in region algebra that mathematically there exist infinite number of distinct 1-D complete regions.

Proposition 4.1.1
Every complete region has at least one imaginary object.
Proof. (This proposition is established in the next section)

4.2 Im-numbers and Imaginary Numbers : rim and cim

Instead of any region A = (A, $\oplus$,*, $\bullet$), let us consider now a particular region RR. Since RR is a complete region, its characteristic is zero. Therefore according to the Proposition 5.2 (established in the next section) it has at least one imaginary object. In the Theory of Objects, let us call these imaginary objects of the region RR by a special name 'imaginary numbers'.
But now there arises a conflict (of title) because of the fact that the existing 'Theory of Numbers' has also a notion of 'imaginary numbers'. To avoid confusion between the existing concept of 'imaginary numbers' and our notion of 'imaginary numbers' for the particular region RR, we will henceforth call our notion of 'imaginary numbers' by the abbreviated term 'im-numbers'. It is obvious that all the imaginary numbers are im-numbers, but at this moment we can not answer whether the converse is true or not.

We call the im-numbers for the set of real numbers R by the term R-im or **rim** (in short). The existing 'Theory of Numbers' says that **i** (= $\sqrt{-1}$) is a rim.

Similarly, if there exist 'imaginary objects' of the region C of complex numbers then we will call each of them by the term C-im or **cim** (in short).

4.3 "Square Root" of an object

For a given object z of a region A, if $\exists x \in A$ such that $x^2 = z$ then we say that x is a real square root object (or, simply may be called 'square root') of the object z, denoted by $\sqrt{z} = x$.

An object of a region A may have nil or more number of real square roots. Clearly 0_A and 1_A are the only objects for which the object itself is the square root of it respectively. However 1_A may have more than one square roots which are 1_A and $\sim 1_A$.

Example 4.3.1
Consider the region RR. Clearly the object 9 of RR has a square root and the object -9 does not have any square root. Hence -9 has at least one imaginary square root. It implies that the region RR does have at least one imaginary object.

4.4 "nth Root" of an object
For a given object z of a region A, if $\exists x \in A$ such that $x^n = z$ then we say that x is a real nth root object (or, simply may be called 'nth root') of the object z denoted by $\sqrt[n]{z}$, where n is a positive integer.
An object may have nil or more number of real nth roots. In case, for a given z the equation $x^n = z$ is not satisfied by any $x \in A$, then we say that z has at least one 'imaginary nth root'; and at the same time we understand the existence of at least one 'imaginary object' of the region A.

4.5 Classical set of Complex Numbers : a particular instance

For an arbitrary region A, knowing about the 'possible existence' of some imaginary objects of it is not a straightforward task. Consequently, knowing the 'identities' of the imaginary objects of it (if exist) is also not a straightforward task, unlike knowing the imaginary objects of the region RR which is a particular case. Nevertheless, according to our Theory of Objects there is no guarantee at this stage that **:** "the set of all imaginary objects of the region RR is exactly equal to the set of complex numbers". It is an open problem now for us. However it is now guaranteed that the classical set C of complex numbers is a subset of the set of all imaginary objects of the region RR.

4.6 Logarithm of Objects

Consider a region A. For two objects x and y of the region A, the logarithm of an object x to the base y is denoted by the notation $\log_y(x)$ which is the real number b such that $y^b = x$. We will discuss the issue for $x = 0_A$ or for $y = 0_A$ later on. We could see that if A and B are two distinct complete regions, then the real numbers like $\log_{2_A} 4_A$ (i.e. logarithm of the object 4_A to the base 2_A) and $\log_{2_B} 4_B$ are not equal. The objects like 4_A, 4_B etc. are introduced later here.

4.7 Compound Numbers

In this section we discover a new concept termed as "Compound Numbers": which is another new direction unearthed in the classical 'Theory of Numbers'.

Consider the two distinct expressions $f(x) \equiv x^2 + 1$ and $g(x) \equiv 0$. Clearly f and g satisfy the qualification conditions mentioned in subsection 4.1. There is no x in the region RR (set R) which satisfies the equation f(x) = g(x). It indicates that there is at least one rim in R.
It is in fact well known to everybody that R has one rim which is **i** (= $\sqrt{-1}$) . At this moment we will not debate on the issue "How many distinct atomic rims R does have of kind **i**", unless we do further work on it. As in the existing literatures on the classical Theory of Numbers, there is one and only one atomic rim which is **i**, of course along with infinite number of other rims of kind (a+ib).

Now let us consider the following analysis very carefully:

The analysis is done with the help of examples.
Consider the region C. Consider the function $f : C \to C$ given by

$$f(z) = (|z|^2 + 2) + 3i.$$

Consider another function $g : C \to C$ given by

$$g(z) = 1 + 3i.$$

Both f and g are functions of complex variable. It is obvious that f(z) is not a constant expression. It outputs different results for different values of z in general. Now, it may be observed that there is no object z of the region C which satisfies the equation f(z) = g(z). Thus the pair of expressions f(z) and g(z) satisfies the **'Qualification Conditions'** which are necessarily to be fulfilled for possible existence of an imaginary object of the region C (as mentioned in subsection-4.1). Consequently, it indicates that there is at least one imaginary object (cim) in C. Say **e** is one atomic cim in C generated from the above equation f(z) = g(z). It means that **e** is an imaginary object of C for which the equality f(**e**) = g(**e**) = z_0 holds good, where $z_0 \in C$. And hence we define this imaginary object **e** for the region C to be such that |**e**| = **i**.

Clearly **e** does not belong to C (analogous to the statement that: **i** does not belong to R). Therefore **e** can not be written in the form of **e** = a+ib where a, b are real numbers. The notion of rim **i** has provided the mathematicians a unique scope to solve any real equation of type f(x) = g(x) where both f(x) and g(x) are simultaneously not constant functions with unequal values. Similarly the notion of cim **e** has provided the scope to solve a complex equation of above type f(z) =

g(z) where both f(z) and g(z) are simultaneously not constant functions with unequal values.
Both **i** and **e** are philosophically discovered in a common way. See that on executing an operation over **i** the result happens to be in R, and similarly on executing an operation over **e** the result happens to be in C. Because square of **i** is in R and modulus of **e** is in C.

Let us now solve the following problem to show an application of **e** in Mathematics.

Problem
Solve the complex equation

$$|f(z)|^2 + z = g(z),$$

where $f(z) = (4+3i)z+(3-5i)$ and $g(z) = z-1$.

If we solve this complex equation, we get one of its roots given by $z = z_1 + \mathbf{e}\, z_2$, where $z_1 = \frac{4-3i}{25}$ and $z_2 = \frac{3+29i}{25}$.
It is to be noted that **i** is an im-member of the region R, not of the region C. And similarly **e** is an imaginary member of the region C, not of the region R. Thus for the real objects z_1 and non-zero z_2 of C, if **e** is one cim of C then the object $d = (z_1 + \mathbf{e}\, z_2)$ is not a member in C, i.e. d is not a real object of C (this situation is analogous to the case where for x_1 and non-zero x_2 of R, the object $d = (x_1 + \mathbf{i}\, x_2)$ is not a member in R). Such an object $d = (z_1 + \mathbf{e}\, z_2)$ is a complex object of the region C and is called by the term **compound number** in C.

A cim or a compound number of C is not a core member of C. It has taken birth by virtue of the definition of 'imaginary object of a region' as defined earlier in subsection 4.1.

Next let us consider another example as below.
Consider the region C. Consider the function $f : C \to C$ given by

$$f(z) = z + \arg(z).$$

Consider another function $g : C \to C$ given by

$$g(z) = 2z.$$

Both f and g are functions of complex variable. It is obvious that f(z) is not a constant expression. It outputs different results for different values of z in general. Similarly g(z) is also not a constant expression as it outputs different results for different values of z. Now, it may be observed that there is no object z of the region C which satisfies the equation f(z) = g(z). Thus the pair of expressions f(z) and g(z) satisfies the **'Qualification Conditions'** which are necessarily to be fulfilled for possible existence of an imaginary object of the region C (as mentioned in subsection-4.1). Consequently, it indicates that there is at least one more imaginary object (cim) in C. Say **w** is the cim in C generated from the above equation f(z) = g(z). It means that **w** is an imaginary object of C for which the equality f(**w**) = g(**w**) = z_0 holds good, where $z_0 \in$ C. Clearly **w** does not belong to C (analogous to the statement that: **e** does not belong to C, **i** does not belong to R), but z_0 is obviously a member of the region C. Therefore **w** can not be written in the form of **w** = a+ib where a, b are real numbers. Thus the notion of cim **e** has provided the scope to solve here the complex equation of type f(z) = g(z).

It is to be noted that **i** is an im-member of the region R, not of the region C. And similarly **w** is an imaginary member of the region C, not of the region R. Thus for the real objects z_1 and non-zero z_2 of C, if **w** is one cim of C then the object d = (z_1 + **w** z_2) is not a member in C, i.e. d is not a real object of C (this situation is analogous to the case where for x_1 and non-zero x_2 of R, the object d = (x_1 + **i** x_2) is not a member in R). Such an object d = (z_1 + **w** z_2) is a complex object of the region C and is another example of **compound number** in C.

The cim **w** or the compound number of C is not a core member of C. It has taken birth by virtue of the definition of 'imaginary object of a region' as defined earlier.

4.7.1 Two parts of a compound number

Let us consider the imaginary object **e** of the region C. Consider a compound number d = (z_1 + **e** z_2) of the region C. Here the complex number z_1 is a real object of C and is called the 'complex part' of the compound number d; and the complex number z_2 is also a real object of C and is called the 'imaginary part' of the compound number d.

If d = (3+4i) + **e** (7-2i) is a compound number of the region C then the complex number (3+4i) is the 'complex part' of d and the complex number (7-2i) is the 'imaginary part' of d. Both the 'complex part' and 'imaginary part' of a compound number are complex numbers. As a trivial case the 'complex part' of the cim **e** is (0+i0) and the 'imaginary part' is (1+i0). Corresponding to every atomic cim, there exist infinite number of compound numbers.

Similarly let us consider the imaginary object **w** of the region C and a corresponding compound number d = (z_1 + **w** z_2) of the region C. Here the complex number z_1 is a real object of C and is called the 'complex part' of the compound number d; and the complex number z_2 is also a real object of C and is called the 'imaginary part' of the compound number d. If d = (9+2i) + **w**(3-8i) is a compound number of the region C then the complex number (9+2i) is the 'complex part' of d and the complex number (3-8i) is the 'imaginary part' of d. Both the 'complex part' and 'imaginary part' of a compound number are complex numbers. As a trivial case the 'complex part' of the cim **w** is (0+i0) and the 'imaginary part' is (1+i0). Corresponding to every atomic cim, there exist infinite number of compound numbers.

In general, suppose that $R_1, R_2, R_3, \ldots, R_n$ are n number of regions. A region may or may not have imaginary object. Even if a region R_i has an imaginary object, we need to explore how many more imaginary objects does R_i have. If e_i is an imaginary object of the region R_i and if a, b are real objects of R_i then $(a + b\, e_i)$ is a complex object of the region R_i. However, for the particular region C, its complex objects are called by compound numbers.

4.7.2 No confusion about the 'existence' of cim

If x is in R then the equation $x^2 + 1 = 0$ is not satisfied by any x of R and thus there may exist one or more solutions of this equation in the form of $x = x_1 + \mathbf{i}\, x_2$ which are 'imaginary objects' of the region R in the Theory of Objects (which we call as complex numbers in our classical Number Theory).

The equation $f(z) = g(z)$ where $f(z) = (|z|^2 + 2) + 3i$ and $g(z) = 1 + 3i$ can not be solved for z in C. This situation leads to the existence of at least one cim. Consequently, it is to be very carefully noted that searching for x from R for satisfying the equation $x^2 + 1 = 0$ and searching for z satisfying the equation $f(z) = g(z)$ where $f(z) = (|z|^2 + 2) + 3i$ and $g(z) = 1 + 3i$ are basically same type of problems in our new mathematics on Theory of Objects. Only difference is that these two searching problems are to be executed on two different platforms (two different regions). In the first case we do search for real numbers x and y from the jurisdiction R only, whereas in the second case we do search for a complex number z from the jurisdiction C only. We must be careful about our boundary of the concerned region while searching for solutions of valid equations in that region. Thus, there is no confusion in the existence of at least one atomic cim of C, but its further characterization are to be done in future research work.

History says that after the discovery of the rim **i** for the set R of numbers, a new number system took birth which is the set C of complex numbers. The giant C came into existence by the birth of one object which is **i** for R. It is to be philosophically viewed that the existing notion of 'complex numbers' is with respect to its base-root which is 'real numbers'. Also for an example, see that '5i' is an imaginary number to the set R, not to the set C!. To the set C, the number '5i' is a core family-member having 100% degree of belongingness in C. It is to be clearly understood that the issue of 'imaginary' or 'complex' is an relative issue, but local to the concerned region. One object may be a core family (not an imaginary object) to a region A, but it could be an imaginary object to another region B (not a core family member)!.

The set R of real numbers is conceptualized first, and later by the discovery of **i** the mathematicians discovered the birth of the classical set C of complex numbers. In an analogous way we claim that picking-up the region C and by the discovery of the cim **e** (and other atomic cims, if exist of C) has led to the discovery of a new set of numbers. Let us call this new set by the set of "Compound Numbers" denoted by **E** which is corresponding to the imaginary object **e**. Our immediate need is to discover the fundamental operations on **E** (like additions, multiplications, etc.) and then to study **E** as a possible algebra, and more**.**

It is obvious that **E** forms a group with respect to the binary operator '**+**' defined as below :

for the compound numbers $d_1 = z_{11} + \mathbf{e}\, z_{12}$ and $d_2 = z_{21} + \mathbf{e}\, z_{22}$ of **E**, define $(d_1 + d_2)$ by :

$$d_1 + d_2 = (z_{11} + z_{21}) + \mathbf{e}\,(z_{12} + z_{22}),$$

which is obviously a compound number in **E**.

It may be observed that the philosophy behind the birth of **i** and **e** is almost analogous. The following table will show a comparative information about the birth and growth of **i** and **e**.

Unearthing the cim **w** has led to the discovery of another new set of numbers. Let us call this new set by the set of "Compound Numbers" denoted by **W** which is corresponding to the imaginary object **w**. Our immediate need is to discover the fundamental operations on **W** (like additions, multiplications, etc.) and then to study **W** as a possible algebra, and more**.** It is obvious that **W** forms a group with respect to the binary operator '**+**' defined as below :
for the compound numbers $d_1 = z_{11} + \mathbf{w}\, z_{12}$ and $d_2 = z_{21} + \mathbf{w}\, z_{22}$ of **W**, define $(d_1 + d_2)$ by :

$$d_1 + d_2 = (z_{11} + z_{21}) + \mathbf{w}\,(z_{12} + z_{22}),$$

which is obviously a compound number in **W**.

In the "Theory of Objects", the two sets **E** and **W** of Compound Numbers introduced here are just at their own infant stage, but undoubtedly they are two new sets of numbers discovered here. With a rigorous amount of research work on the two sets **E** and **W** of numbers, it will surely take its own shape in future to update the existing classical "Theory of Numbers". Without giving further justifications, we claim that there are possibly many more sets of numbers (besides the two sets **E** and **W** of numbers) yet to be unearthed.

Table 1. A comparative datatable about **i** and **e**

Sr. No.	about **i**	about **e**
1	It is an 'imaginary object' of the region R.	It is an 'imaginary object' of the region C.
2	In the existing Theory of Numbers it is called 'imaginary number'.	In the newly developed Theory of Objects it is called 'compound number'.
3	It is created with some issues arose while working with the set R of real numbers (however in our new mathematics we say in a different way like: It is created with some issues arose while working with the region R).	It is created with some issues arose while working with the region C.
4	Its definition by birth says that : On executing an operation over **i** the result happens to be in R. The operation is 'square'.	Its definition by birth says that : On executing an operation over **e** the result happens to be in C. The operation is 'modulus'.
5	The complex number a+**i**b has two parts. Both the parts are real numbers.	The compound number g+**e**z has two parts. Both the parts are complex numbers.
6	A complex number a+**i**b can be considered in a 2-D geometry.	A compound number z_1+**e**z_2 can be considered in a 4-D geometry.
7	Set of complex numbers is denoted by C. It plays a huge role in Mathematics, Science and many other giant domains. Complex Algebra is a rich algebra in Mathematics.	Set of compound numbers corresponding to the imaginary object **e** is denoted by E. This set E forms an abelian group with respect to the binary operation '+' defined above. Compound Algebra is yet to be developed further in the context of our proposed new mathematics.

5. Theory of A-Numbers

In earlier sections, while introducing the notion of prime and composite objects and then the notion of imaginary and compound objects, we have considered a general region which need not be a complete region. There are regions which are complete and there are regions which are not. In this section we introduce the "Theory of Objects" corresponding to every complete region A titled as "Theory of A-Numbers". It has been shown that corresponding to a region A, there may exist infinite number of distinct complete regions (i.e. 1-D complete regions).

In this subsection we develop a new theory called by "Theory of A-numbers" corresponding to a complete region A. If a region K is not a complete region, the "Theory of K-numbers" does not exist for it; however the topics of prime and composite objects, imaginary and compound objects, can be well studied in any region K, be it a complete region or not. Suppose that A is a complete region. We first of all define the concept of 'Object Linear Continuum Line' in the complete region A.

5.1 Object Linear Continuum Line

The notion of 'Object Linear Continuum Line' in a complete region A is explained as below.
A line can be drawn on plain paper on which one point may be fixed to be the location for the zero object 0_A, with all positive objects of A having their respective locations to the right and all negative objects of A having their respective locations to the left of the zero object 0_A, as explained earlier. Thus the 'positive direction' of the line can be called to be X_A-axis and the 'negative direction' of the line can be called to be $X_A{}^1$-axis. And the line which the objects of the complete region A is considered to lie upon is called the Object Linear Continuum Line for the complete region A (see Figure 5.4).

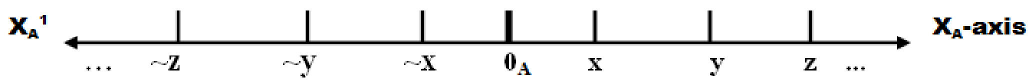

Figure 5.4. Object Linear Continuum Line of the complete region A, a general view

By the distance between two objects x and y of the complete region A, we mean the corresponding metric distance ρ(x,y) of the normed complete metric space A. The distance of a positive object x_A from the origin is $\|x_A\|$ = ρ(x_A,0_A) = x_a, and the distance of a negative object ~ x_A from the origin is = -x_a (imposing minus sign). For example, see a collection of consecutive equi-spaced points on the object line as shown in the Figure 5.5 below.

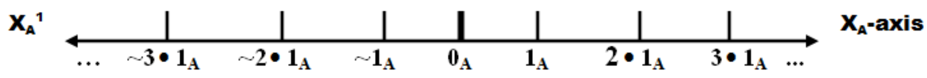

Figure 5.5. Object Linear Continuum Line of the complete region A with a collection of consecutive equi-spaced object points

The term 'equi-spaced' in the caption of Figure 5.5 is well understood in the sense of the corresponding metric (or norm) of the complete region A,
i.e. for any real number r, $\rho(r \bullet 1_A, (r+1) \bullet 1_A)$ = positive constant (independent of the real number r), in the complete region A.

Since A = $(A, \oplus, *, \bullet)$ is complete (normed complete metric space), there are no "points missing" from it (inside or at the boundary). Since A is a chain, every object of A has a unique address on this Object Linear Continuum Line $X_A{}^1X_A$; and conversely i.e. corresponding to every address (point) on this Object Linear Continuum Line $X_A{}^1X_A$ there is a unique object of the region A.

5.2 Unit Length

We define now the concept of 'Unit Length' in a complete region A. Consider a complete region A = $(A, \oplus, *, \bullet)$. If x_A is a positive object on the object linear continuum line, then the distance of the point x_A from the point O (the location of the object 0_A on the object linear continuum line $X_A{}^1X_A$) is a positive real number denoted by the notation x_a. We use the classical practiced convention to say that $\sim x_A$ is at a distance of $-x_a$ (imposing minus sign) from the point O, although as per definition of metric a distance can not be a negative quantity.

For $x_A \in A$, we have

$$\|x_A\| = \begin{cases} x_A & \textit{if } x_A \textit{ is a positive object} \\ -x_A & \textit{if } x_A \textit{ is a negative object} \end{cases}$$

because $\rho(0_A, x_A) = \rho(0_A, \sim x_A) = |x_a|$.

Corresponding to the unit element 1_A of the complete region A, the positive real number 1_a (where $1_a = \| 1_A \| = \rho(0_A, 1_A)$) is called the 'unit length' in the Theory of A-numbers. Thus the unit length is defined by the the distance of the object 1_A from 0_A. Clearly 0_a being the $\| 0_A \|$ is equal to the real number 0.

Proposition 5.1
For every complete region A, the unit length is equal to 1.

Proof. Straightforward from the axiom of the normed linear space.
From the identity $\|1_a \bullet 1_A\| = \|1_A\|$, we have $|1_a|.\|1_A\| = \|1_A\|$.
Therefore, $|1_a| = 1$, i.e. $1_a = 1$. Hence proved.

An Important Note

We have $x_a = x.1_a = x$, where $x_A = x_a \bullet 1_A$ and $\|x_A\| = |x_a| = \rho(0_A, x_A)$.
In the Theory of Objects, we will use the notation x_a instead of x (although $x_a = x$) in most part of its literature because of the following fact:
(i) the small letter 'a' in x_a tells that the name of the concerned region is 'A'.
(ii) the real number x_a signifies that corresponding to it there is a unique object x_A of the region A situated on the Object Linear Continuum Line of A at a distance x_a from the origin 0_A.
(iii) if x_a is a positive real number then the corresponding object x_A is a positive object in A, if if x_a is a negative real number then the corresponding object x_A is a negative object in A, and if if x_a is equal to zero(0) then the corresponding object x_A is 0_A.

5.3 Ontegers in a complete region

In this subsection we introduce the notion of 'Onteger' in a complete region A = $(A, \oplus, *, \bullet)$. The word 'onteger' is not a valid word in English dictionary. It is an abbreviated word for "Object Integer". The concept of 'ontegers' will be the basic element in developing the new number theory entitled 'Theory of A-numbers'.

Consider an object x_A in the complete region A.
Therefore $x_A = x_a \bullet 1_A$, $\|x_A\| = x_a$ where x_A is a positive object. And $\sim x_A = -x_a \bullet 1_A$, $\|\sim x_A\| = x_a$ where $\sim x_A$ is a negative object;.

Onteger
If m is any real integer, then the object m_A of the complete region A is called an 'object integer' or 'onteger' in the 'Theory of A-numbers'.

Thus the ontegers in the 'Theory of A-numbers' are 0_A, $\oplus 1_A$, $\sim 1_A$, $\oplus 2_A$, $\sim 2_A$, $\oplus 3_A$, $\sim 3_A$, ….. etc. The ontegers $\oplus 1_A$, $\oplus 2_A$, $\oplus 3_A$, $\oplus 4_A$, …….. etc. are 'positive ontegers' and the ontegers $\sim 1_A$, $\sim 2_A$, $\sim 3_A$, $\sim 4_A$, …….. etc. are 'negative ontegers' in the 'Theory of A-numbers'. The onteger 0_A is neither a positive onteger nor a negative onteger. Obviously, the set of all ontegers of the complete region A is a countable set. However, it may be true that norm of some of the ontegers of the complete region A are integers in R. In a complete region A, the ontegers 0_A, $\oplus 2_A$, $\sim 2_A$, $\oplus 4_A$, $\sim 4_A$, ….. are even ontegers and the ontegers $\oplus 1_A$, $\sim 1_A$, $\oplus 3_A$, $\sim 3_A$ $\oplus 5_A$, $\sim 5_A$ …….. are odd ontegers.

For a given complete region A the distance between two consecutive ontegers on the object linear continuum line will be always a constant real number.

Thus, we have for the complete region A,
…… $= \rho(\sim 3_A, \sim 2_A) = \rho(\sim 2_A, \sim 1_A) = \rho(\sim 1_A, 0_A) = 1 = \rho(0_A, 1_A) = \rho(1_A, 2_A) = \rho(2_A, 3_A) = \rho(3_A, 4_A) =$ ……… ,
and similarly for the complete region B,
…… $= \rho_1(\sim 3_B, \sim 2_B) = \rho_1(\sim 2_B, \sim 1_B) = \rho_1(\sim 1_B, 0_B) = 1 = \rho_1(0_B, 1_B) = \rho_1(1_B, 2_B) = \rho_1(2_B, 3_B) = \rho_1(3_B, 4_B) =$ ………. .

For any real number r, $\rho(r \bullet 1_A, (r+1) \bullet 1_A)$ = positive constant 1, which is independent of the real number r. For any two real numbers r and k, we have $\rho(r \bullet 1_A, (r+k) \bullet 1_A) = |k|.1$.

5.4. ‘R_A value’ of a real number x

Let A be a complete region. Corresponding to the complete region A, consider the 1-to-1 mapping $R_A : R \to R$ defined by

$$R_A(x) = x.1_a = x_a \quad \forall x \in R.$$

Then the real number x_a is called the ‘R_A value’ of the real number x denoted by $R_A(x) = x_a$ corresponding to the complete region A.

Clearly, in that case $R_A(-x) = -x_a$. Also $R_A(0) = 0_a = 0$, and $R_A(1) = 1_a = 1$.

For $x_A \in A$, we have

$$\|x_A\| = |R_A(x)| = \begin{cases} x_a & \text{if } x_A \text{ is a positive object} \\ -x_a & \text{if } x_A \text{ is a negative object} \end{cases}$$

because $\rho(0_A, x_A) = \rho(0_A, \sim x_A) = |x_a|$.

Consider the above defined 1-to-1 mapping $R_{RR} : R \to R$ for the complete region RR. It is obvious that $R_{RR} : R \to R$ is an identity mapping.

5.5 ‘Natural A-ontegers’

In the Theory of A-numbers, the positive ontegers $\oplus 1_A$, $\oplus 2_A$, $\oplus 3_A$, $\oplus 4_A$, … are called the Natural A-ontegers.

Proposition 5.2
Every complete region has at least one imaginary object.

Proof. Consider any complete region A = (A, $\oplus$,*, $\bullet$). By definition of complete region, its characteristic is zero. In our literature, by complete region we mean 1-D region calculus. We take help of an example here. Consider the equation $x_A^2 \oplus 1_A = 0_A$ in the region A.

We will show that the equation $x_A^2 \oplus 1_A = 0_A$ is not satisfied by any object of A, where both LHS and RHS of this equation are valid expressions in A satisfying the necessary 'qualification conditions' as mentioned in subsection 4.1.

Let us prove it by contradiction.
i.e. if possible, suppose that for an object x_A of A we have

$$x_A^2 \oplus 1_A = 0_A$$

Or, $(x_a \bullet 1_A)^2 \oplus 1_A = 0_A$

Or, $x_a^2 \bullet 1_A^2 \oplus 1_A = 0_A$

Or, $x_a^2 \bullet 1_A \oplus 1_a \bullet 1_A = 0_A$

Or, $(x_a^2 + 1_a) \bullet 1_A = 0_A$

Or, $(x^2 + 1) \bullet 1_A = 0_A$

We must have either $(x^2 + 1) = 0$ or $1_A = 0_A$, which is a contradiction because the equality $(x^2 + 1) = 0$ is not true for any real number.
Therefore, there is no real object x_A of the region A which can satisfy the equation

$$x_A^2 \oplus 1_A = 0_A$$

Consequently, it produces one imaginary object of the complete region A which is ∍ (say). Hence the result.

(**Note:** It may be noted that the equation $x_A^2 \oplus 1_A = 0_A$ produces different imaginary objects for different complete region A. It may also be noted that although C does not form an 1-D region calculus (i.e. 1-D complete region), but it does not mean that C will not have any imaginary object.)

Proposition 5.3
If A is a complete region then for any real numbers x and y the following results are true:

(i) $x_a \pm y_a = (x \pm y)_a$
(ii) $(x_a)^n = (x^n)_a$ where n is an integer.
(iii) $x_A \oplus y_A = (x+y)_A$
(iv) $x_A \sim y_A = (x-y)_A$
(v) $(x_A)^n = (x^n)_A$ where n is an integer.

Proof.
(i) $x_a \pm y_a = x.1_a \pm y.1_a = (x \pm y).1_a = (x \pm y)_a$. Hence Proved.
(ii) $(x_a)^n = (x.1_a)^n = x^n.(1_a)^n = x^n.1_a = (x^n)_a$. Hence Proved.
(iii) $x_A \oplus y_A = (x_a \bullet 1_A \oplus y_a \bullet 1_A) = (x_a \oplus y_a) \bullet 1_A = (x+y)_a \bullet 1_A = (x+y)_A$. Hence Proved.
(iv) proof is similar to (iii).
(v) $(x_A)^n = (x_a \bullet 1_A)^n = (x_a)^n \bullet (1_A)^n = (x_a)^n \bullet 1_A =$
$(x^n)_a \bullet 1_A = (x^n)_A$. Hence Proved.

Proposition 5.4
If A is a complete region, then $\exists$ infinite set of trio x, y, z $\in$ A such that the relation $x^n \oplus y^n = z^n$ is satisfied for n = 2.
Proof. Take the case for $x = 3_A$, $y = 4_A$ and $z = 5_A$.

Now, $3_A^2 \oplus 4_A^2 = (3_a \bullet 1_A)^2 \oplus (4_a \bullet 1_A)^2$
$= (3_a)^2 \bullet (1_A)^2 \oplus (4_a)^2 \bullet (1_A)^2$
$= 9_a \bullet 1_A^2 \oplus 16_a \bullet 1_A^2$
$= 25_a \bullet 1_A^2$
$= (5_a \bullet 1_A)^2$
$= 5_A^2$

This particular result can be used to generate infinite number of similar but distinct results. Hence proved.

5.6 $\mathcal{E}_A$-Complex Objects

Consider any complete region A = (A, $\oplus$,*, $\bullet$). Since it is a complete region, by definition its characteristic is zero. As per Proposition 5.2 every such region has at least one imaginary object; this is an important result because existence of any new such imaginary object leads to a new object algebra. Consider any imaginary object of A, which is $\mathcal{E}_A$ (say).
Then $\forall x_A, y_A \in A$ the object $(x_A \oplus \varepsilon_A y_A)$ is called an "$\mathcal{E}_A$-Complex Objects" corresponding to the region A. In that case the object x_A is called the 'real part' and the object y_A is called the 'imaginary part' of the $\mathcal{E}_A$-Complex object. Obviously both real part and imaginary part of an $\mathcal{E}_A$-Complex object are real objects of the region A.

5.7 $\mathcal{E}$-Complex Object

The $\mathcal{E}$-Complex Object is in fact a particular case of $\mathcal{E}_A$-Complex objects. Consider the infinite region A = (A, $\oplus$,*, $\bullet$) whose characteristic is zero. It is shown that the equation $x_A^2 \oplus 1_A = 0_A$ is not satisfied by any object of A. Suppose

that the corresponding particular imaginary object $\mathcal{E}_A$ is denoted by another notation $\mathcal{E}$.
Thus we have the result $\varepsilon^2 \oplus 1_A = 0_A$ i.e. $\mathcal{E}^2 = \sim 1_A$.
Then $\forall x_A, y_A \in A$, the object $(x_A \oplus \varepsilon_A y_A)$ is called an $\mathcal{E}$-Complex Object corresponding to the region A.

The set $C_A = \{ z_A = (x_A \oplus \varepsilon_A y_A) : x_A, y_A \in A \}$ is called the set of all $\mathcal{E}$-Complex Objects corresponding to the region A.

(There should not be any confusion between the two notations C_A and c_A. In the Theory of A-numbers, c_A is an object of the region A where C_A is the set of all $\mathcal{E}$-Complex Objects corresponding to the region A).

5.8 Algebra of $\mathcal{E}$-Complex Object

In this section a new type of algebra is developed called by 'Algebra of $\mathcal{E}$-Complex Object'. Consider the set C_A of all $\mathcal{E}$-Complex Objects corresponding to the region A. Denote the $\mathcal{E}$-Complex Object $(0_A \oplus \varepsilon_A 0_A)$ by the notation $\mathcal{E}_0$ and the $\mathcal{E}$-Complex Object $(1_A \oplus \varepsilon_A 0_A)$ by the notation $\mathcal{E}_1$.
If $z_A = (x_A \oplus \varepsilon_A y_A)$ be an $\mathcal{E}$-complex object, then we define its conjugate $\mathcal{E}$-complex object given by

$$\overline{z_A} = (x_A \sim \varepsilon y_A).$$

Define the following operations over the set C_A corresponding to the region A = (A, $\oplus$, *, $\bullet$). If there is no confusion, let us use the same notations $\oplus$ and * of A for the case of the set C_A too (although their definitions are different in A and C_A).

(1) Addition & Subtraction
If $z_{A1} = (x_{A1} \oplus \varepsilon y_{A1})$ and $z_{A2} = (x_{A2} \oplus \varepsilon y_{A2})$ be two $\mathcal{E}$-complex objects, then define addition of them using the identical notation $\oplus$ as below:

$$z_{A1} \oplus z_{A2} = (x_{A1} \oplus \varepsilon y_{A1}) \oplus (x_{A2} \oplus \varepsilon y_{A2}) = (x_{A1} \oplus x_{A2}) \oplus \varepsilon (y_{A1} \oplus y_{A2})$$

which clearly belongs to C_A;

and define subtraction as below:

$$z_{A1} \sim z_{A2} = (x_{A1} \oplus \varepsilon y_{A1}) \sim (x_{A2} \oplus \varepsilon y_{A2}) = (x_{A1} \sim x_{A2}) \oplus \varepsilon (y_{A1} \sim y_{A2})$$

which clearly belongs to C_A.

(2) Multiplication
If $z_{A1} = (x_{A1} \oplus \varepsilon y_{A1})$ and $z_{A2} = (x_{A2} \oplus \varepsilon y_{A2})$ be two $\mathcal{E}$-complex objects, then define multiplication of them using the identical notation $*$ as below

$$\begin{aligned} z_{A1} * z_{A2} &= (x_{A1} \oplus \varepsilon y_{A1}) * (x_{A2} \oplus \varepsilon y_{A2}) \\ &= (x_{A1} * x_{A2} \sim y_{A1} * y_{A2}) \oplus \varepsilon (x_{A1} * y_{A2} \oplus y_{A1} * x_{A2}) \end{aligned}$$

which clearly belongs to C_A.

(3) Scalar Multiplication

For $k \in R$ and for $z_A = (x_A \oplus \varepsilon y_A) \in C_A$, define the scalar multiplication as below :

$k \bullet z_A = (k \bullet x_A \oplus \varepsilon\ k \bullet y_A)$, which clearly belongs to C_A.

Proposition 5.5

If $z_{A1} = (x_{A1} \oplus \varepsilon y_{A1})$ and $z_{A2} = (x_{A2} \oplus \varepsilon y_{A2})$ be two ε-complex objects in C_A, then $z_{A1} * z_{A2} = \varepsilon_0$ iff at least one of z_{A1} and z_{A2} is ε_0 (i.e. there is no zero divisor).

Proof :

Suppose that $z_{A1} * z_{A2} = \varepsilon_0$.

$\Rightarrow (x_{A1} * x_{A2} \sim y_{A1} * y_{A2}) \oplus \varepsilon(x_{A1} * y_{A2} \oplus y_{A1} * x_{A2}) = \varepsilon_0$.

$\Rightarrow (x_{A1} * x_{A2} \sim y_{A1} * y_{A2}) = 0_A$ and $(x_{A1} * y_{A2} \oplus y_{A1} * x_{A2}) = 0_A$.

$\Rightarrow x_{A1} * x_{A2} = y_{A1} * y_{A2}$ and $x_{A1} * y_{A2} = \sim y_{A1} * x_{A2}$

$\Rightarrow$ either $x_{A1}^2 + y_{A1}^2 = 0_A$ or $x_{A2}^2 + y_{A2}^2 = 0_A$ or both.

Hence proved.

It may be observed that the set C_A forms a group with respect to the binary operation $\oplus$, and the set $C_A - \{\varepsilon_0\}$ forms a group with respect to the binary operation *.

Several other algebraic properties of C_A can be studied in future work.

6. Conclusion

In this paper a new theory called by "Theory of Objects" along with its algebra called by 'Object Algebra' is introduced at the outset. Although this theory is at its baby stage, but it is initiated here with four topics as follows:

1. "Prime Objects" and "Composite Objects" in a Region
2. "Imaginary Objects" and "Compound Objects" in a Region,
3. Compound Numbers : a generalized concept of the complex numbers, and
3. a new type of "Theory of Numbers" : Every Complete Region has its own.

The existing notion of 'prime numbers' is a special case of 'prime objects', and the existing notion of 'composite numbers' is a special case of 'composite objects'. We define imaginary objects (if exist) of a region. As a particular case of imaginary object we study the existing notion of imaginary number i of the set R of real numbers, which is called by 'rim' in the Theory of Objects.

Another major breakthrough in Object Algebra we unearth is that the region C (set of complex numbers) has at least two imaginary objects. Any atomic imaginary object of C is called by the notation 'cim' of C. Two distinct cims of C we have unearthed here which we name by e and w. If x and y are in R, then corresponding to the rim i of R the object (x+iy) is a complex number. The object (x+iy) is a complex object in the jurisdiction of R, but in the jurisdiction of some other region it may not be a complex object. The basic unique property of an imaginary object is that it is a local property in some region, but may be a core member of some other region. Analogously, if z_1 and z_2 are in C then corresponding to the cim **e** of C the object (z_1 + **e** z_2) is a compound number for the set C. The rim **i** is imaginary for R, not for C; and thus the rim **i** is a core member of C, not of R. Therefore rim **i** is a real object of C as per definition of real object of a region. The cim **e** is imaginary for C, not for any other region in general. Being the imaginary object in C, the cim **e** is not a member of C, i.e. not a real object of C. Thus we have happened to see now the birth of a new type of numbers called by 'compound numbers' of the form (z_1 + **e** z_2) where z_1 and z_2 are in C. All the compound numbers of the form (z_1 + **e** z_2) are compound numbers with respect to the cim **e.** But there is another cim **w** of C is also discoveres. The set of all compound numbers corresponding to the cim **e** is denoted by **E** and the set of all compound numbers corresponding to the cim **w** is denoted by **W**. We need to analyze the two sets **E** and **W** more precisely, by identifying precisely all its members, characteristic properties, results, etc. which will be our future course of research work. In the "Theory of Objects", the two sets **E** and **W** of Compound Numbers introduced here is just at its infant stage, but undoubtedly it is a new set of numbers discovered here. With a rigorous amount of research work on the sets **E** and **W** of numbers, it will surely take its own shape in future to update the existing classical "Theory of Numbers". For studying prime and composite objects, imaginary objects, etc. we have considered simple regions, not complete regions. But, mathematically there are infinite number of distinct complete regions exist in our mathematics of Object Algebra. We then introduce "Theory of A-numbers" which is developed if A is a complete region, otherwise not valid. We have identified 'What are the minimum properties which need to be satisfied by a set A so that a new Geometry can be developed over the platform A?'. In every complete region A, there are ontegers ……, $\sim 3_A$, $\sim 2_A$, $\sim 1_A$, 0_A, 1_A, 2_A, 3_A, 4_A, 5_A, ….., and a particular instance of ontegers are the integers ….., -3, -2, -1, 0, 1, 2, 3, 4, ….. which are the ontegers in the complete region RR.

Consequently, upon the discovery of the Theory of Objects and new types of numbers, we need to revisit many of the existing famous results in our future work, viz:

(i) R, C, H, O are the only normed division algebras.

(ii) the associative real division algebras are real numbers, complex numbers, and quaternions.

(iii) The Cayley algebra is the only non-associative division algebra.

(iv) The algebras of real numbers, complex numbers, quaternions, and Cayley numbers are the only ones where multiplication by unit "vectors" is distance-preserving.

With the notion of "Theory of Objects" introduced here, it is sure that in due time the 'Number Theorists' can be re-designated with a new title 'Object Theorists' as the areas of cultivation will not be limited to just numbers but to the objects.

References

[1] Althoen, S. C. and Kugler, L. D., When is R^2 a Division Algebra?, *American Math. Monthly*, **90** (1983), 625-635. https://doi.org/10.1080/00029890.1983.11971297

[2] Acheson, David, *The Calculus Story: A Mathematical Adventure,* First Edition, Oxford University Press, UK, 2017.

[3] Adamson, I. T., *Introduction to Field Theory*, Dover Publications, 2007. https://doi.org/10.1017/cbo9780511600593

[4] Apostol, Tom M., *Introduction to Analytic Number Theory*, Springer-Verlag, Heidelberg Berlin, 1976. https://doi.org/10.1017/cbo9780511600593

[5] Artin, Michael, *Algebra*, Prentice Hall, New York, 1991. https://doi.org/10.1007/978-3-0348-9124-0

[6] Baron, Margaret E., *The Origins of the Infinitesimal Calculus,* Dover Publications, Inc., Mineola, New York, 2003.

[7] Biswas, Ranjit, Region : the unique algebraic structure in Abstract Algebra, *International Journal of Algebra*, **13** (2019), no. 5, 185 – 238. https://doi.org/10.12988/ija.2019.9621

[8] Biswas, Ranjit, Birth of Compound Numbers, *Turkish Journal of Analysis and Number Theory*, **2** (6) (2016), 208-219. https://doi.org/10.12691/tjant-2-6-4

[9] Biswas, Ranjit, Introducing Soft Statistical Measures, *The Journal of Fuzzy Mathematics*, **22** (4) (2014), 819-851.

[10] Biswas, Ranjit, "Atrain Distributed System" (ADS): An Infinitely Scalable Architecture for Processing Big Data of Any 4Vs, in 'Computational Intelligence for Big Data Analysis Frontier Advances and Applications': edited by D. P. Acharjya, Satchidananda Dehuri and Sugata Sanyal, Springer International Publishing Switzerland, Part-1 2015,1-53. https://doi.org/10.1007/978-3-319-16598-1_1

[11] Biswas, Ranjit, *Is 'Fuzzy Theory' An Appropriate Tool For Large Size Problems?*, in the book-series of SpringerBriefs in Computational Intelligence, Springer, Heidelberg, 2016. https://doi.org/10.1007/978-3-319-26718-0

[12] Biswas, Ranjit, *Is 'Fuzzy Theory' An Appropriate Tool For Large Size Decision Problems?*, Chapter-8 in Imprecision and Uncertainty in Information Representation and Processing, in the series of STUDFUZZ, Springer, Heidelberg, 2016.

[13] Beachy, J. A., and Blair, W.D., *Abstract Algebra*, 2nd Ed., Waveland Press, Prospect Heights, III, 1996.

[14] Bourbaki, Nicolas, *Elements of the History of Mathematics*, Springer-Verlag, Berlin, 1994.

[15] Bourbaki, Nicolas, *Elements of Mathematics : Algebra I*, New York, Springer-Verlag, 1998.

[16] Boyer, Carl B. and Merzbach, Uta C., *A History of Mathematics*, 3rd Edition. John Wiley & Sons, Hoboken, New Jersey, 2011.

[17] Boyer, Carl B., *History of Analytic Geometry*, Dover Publications, Inc., Mineola, New York, 2004.

[18] Boyer, Carl B., *The History of the Calculus and Its Conceptual Development*, Dover Publications, Inc., Mineola, New York, 1959.

[19] Cajori, Florian, *A History of Elementary Mathematics – With Hints on Methods of Teaching*, Read Books. UK, 2010.

[20] Copson, E.T., *Metric Spaces*, Cambridge University Press, 1968. https://doi.org/10.1017/cbo9780511566141

[21] Davenport, Harold; Montgomery, Hugh L., *Multiplicative Number Theory*, Graduate texts in mathematics, Springer-Verlag New York Inc, 2000. https://doi.org/10.1007/978-1-4757-5927-3

[22] Dixon, G.M., Division Algebras: *Octonions Quaternions Complex Numbers and the Algebraic Design of Physics*, Kluwer Academic Publishers, Dordrecht, 2010.

[23] Edwards, Harold M., *Fermat's Last Theorem: a Genetic Introduction to Algebraic Number Theory*, Graduate Texts in Mathematics. Springer-Verlag. New York, 2000. https://doi.org/10.1007/978-1-4684-9307-8

[24] Ellis, G., *Rings and Fields*, Oxford University Press, 1993.

[25] Fraenkel, A., Bar-Hillel, Y., and Lev, A., *Foundations of Set Theory*, North Holland, Amsterdam.

[26] Franklin, Joel L., *Matrix Theory*, Mineola, N.Y., 2000.

[27] Hardy, G.H., Wright, Edward M., Roger Heath-Brown, Silverman, Joseph. and Wile, Andrew, *Introduction to the Theory of Numbers*, (6th Edition) Oxford University Press, UK, 2008.

[28] Herstein, I. N., *Topics in Algebra*, Wiley Eastern Limited, New Delhi, 2001.

[29] Hungerford, T., *Algebra*, Graduate Texts in Mathematics, Vol. 73, Springer-Verlag, New York, 1974.

[30] Jacobson, N., *Basic Algebra I*, 2nd Ed., W. H. Freeman & Company Publishers, San Francisco, 1985.

[31] Jacobson, N., *Basic Algebra II*, 2nd Ed., W. H. Freeman & Company Publishers, San Francisco, 1989.

[32] Jacobson, N., *The Theory of Rings*, American Mathematical Society Mathematical Surveys. Vol. I, American Mathematical Society, New York, 1943. https://doi.org/10.1090/surv/002

[33] Lam, T. Y., *Exercises in Classical Ring Theory*, Problem Books in Mathematics, Springer-Verlag, New York, 1995. https://doi.org/10.1007/978-1-4757-3987-9

[34] Lang, Serge, *Undergraduate Algebra*, (3rd ed.). Berlin, New York, Springer-Verlag, 2005.

[35] Loney, S. L., *The Elements of Coordinate Geometry*, Part-I, Macmillan Student Edition, Macmillan India Limited, Madras, 1975.

[36] Lorenz, Falko, *Algebra,* Volume II: Fields with Structures, Algebras and Advanced Topics, Springer, 2008.

[37] Matsumura, Hideyuki, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986.

[38] Nahin, Paul J., *An Imaginary Tale: The Story of* $\sqrt{-1}$, Princeton University Press, 2007.

[39] Pierce, Richard S., *Associative Algebras*, Graduate Texts in Mathematics, Studies in the History of Modern Science, Springer-Verlag, Berlin, 1982. https://doi.org/10.1007/978-1-4757-0163-0

[40] Reyes, Mitchell, The Rhetoric in Mathematics: Newton, Leibniz, the Calculus, and the Rhetorical Force of the Infinitesimal, *Quarterly Journal of Speech*, **90** (2004), 159-184.

[41] Lidl, Rudolf and Niederreiter, Harald, *Finite Fields* (2nd ed.), Cambridge University Press, 2008. https://doi.org/10.1017/cbo9780511525926

[42] Robbins, Herbert and Courant, Richard, *What is Mathematics?: An Elementary Approach to Ideas and Methods*, 2nd edition, Oxford University Press, New York, 1996.

[43] Rosen, Kenneth H., *Elementary Number Theory* (6th ed.), Pearson Education, USA, 2010.

[44] Rudin, Walter, *Real and Complex Analysis*, McGraw Hills Education, India, 2006.

[45] Saltman, D. D., *Lectures on Division Algebras,* Providence, RI, Amer. Math. Society, 1999. https://doi.org/10.1090/cbms/094

[46] Sardar, Ziauddin, Ravetz, Jerry, Loon, Borin Van Loon, *Introducing Mathematics*, Icon Books Ltd, London, Reprint edition, 2015.

[47] Shult, Ernest E., *Points and Lines*, Universitext, Springer, 2011. https://doi.org/10.1007/978-3-642-15627-4

[48] Simmons, G. F., *Introduction to Topology and Modern Analysis*, McGraw Hill, New York, 1963.

[49] Van der Waerden and Bartel Leendert, *Algebra*, Springer-Verlag, New York, 1991.

[50] Vinogradov, I.M., *Elements of Number Theory* (reprint of the 1954 ed.). Mineola, NY, Dover Publications. Mineola, New York, 2003.

Received: August 1, 2019; Published: August 17, 2019