COMPOUND CHANNELS, TRANSITION EXPECTATIONS AND LIFTINGS

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Abstract.

In Section 1, we introduce the notion of lifting as a generalization of the notion of compound state introduced in [21], [22] and we show that this notion allows an unified approach to the problems of quantum measurement and of signal transmission through quantum channels. The dual of a linear lifting is a transition expectation in the sense of [3] and we characterize those transition expectations which arise from compound states in the sense of [22].

In Section 2, we characterize those liftings whose range is contained in the closed convex hull of product states and we prove that the corresponding quantum Markov chains [2] are uniquely determined by a classical generalization of both the quantum random walks of [4] and the locally diagonalizable states considered in [3].

In Section 4, as a first application of the above results, we prove that the attenuation (beam splitting) process for optical communication treated in [21] can be described in a simpler and more general way in terms of liftings and of transition expectations. The error probabilty of information transmission in the attenuation process is rederived from our new description. We also obtain some new results concerning the explicit computation of error probabilities in the squeezing case.

Key Words.

Compound State, Transition Expectation, Lifting, Channel, Quantum Probability, Quantum Markov Chain, Beam Splitting, Optical Communication

AMS Classification.

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Introduction

The following situation is very common both in classical and quantum physics: one considers two systems, denoted respectively 1, 2 and their algebras of observables, $\mathcal{A}_1, \mathcal{A}_2$. One usually assumes that the interaction between the two systems is switched on at a sharp time t_o before which the two systems are considered to be independent. During the interaction the two systems merge into a larger system denoted $(1, 2)$ whose algebra of observables $\mathcal{A}$ contains both $\mathcal{A}_1$ and $\mathcal{A}_2$, in the sense that there are embeddings

$$j_1 : \mathcal{A}_1 \to \mathcal{A} \ ; \ j_2 : \mathcal{A}_2 \to \mathcal{A} \tag{0.1}$$

and that any physical information on the state of system 1 or of system 2 after the interaction can be obtained by choosing a state φ on $\mathcal{A}$, i.e. a state of the composite system $(1, 2)$, and considering its restriction on the algebra $j_1(\mathcal{A}_1)$ (resp. $j_2(\mathcal{A}_2)$). In most applications one chooses

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \ ; \ \varphi = \varphi_1 \otimes \varphi_2 \tag{0.2}$$

$$j_1(a_1) = a_1 \otimes 1_2 \ ; \ j_2(a_2) = 1_1 \otimes a_2, \qquad a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2 \tag{0.3}$$

(i.e.,) the algebra of the compound system is described by a tensor product. In the present paper we shall confine our analysis to such a situation.

The choice of the state φ depends on the initial states of the two systems and on the interaction between them. In connection with this situation one studies several problems depending on the interpretation of the systems 1 and 2. For example:

i) the state φ_2 of system 2, **after** the interaction, is known (e.g., an output signal, a pointer in a measurement apparatus) and one wants to know the state φ_1 of system 1 **before** the interaction (e.g., an input signal, the state of a microsystem which has interacted with the apparatus).

ii) as in (i), exchanging the roles of 1 and 2. ¿From the mathematical point of view, this exchange is trivial, but we want to underline that our approach avoids the separation of a *macroworld*, described by classical physics, from a *microworld*, described by quantum physics.

iii) the initial state of the composite system $(1, 2)$ is known and one wants to know the state of system 1 (system 2).

iv) the state φ_1 of system 1, **before** the interaction (e.g., the preparation of a microsystem) and the form of the interaction, are known and one wants to know the state of system 1 **after** the interaction.

In all these cases the goal is to construct a map from the state space of a system to the state space of another system. In the literature on quantum information and communication systems, such a map is called a **channel** [20]. An important class of channels are those from the state space of an algebra $\mathcal{A}_1$ into the state space of the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$. This channels are called **liftings**; more generally, a lifting should be through as a channel from a sub-system to a compound system. An important example of liftings are the duals of transition expectations.

1. %par

Recall (cf. Definition (1.3) below) that if $\mathcal{A}_1, \mathcal{A}_2$ are C*-algebras, a **transition expectation** from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to $\mathcal{A}_1$ is a completely positive linear map $\mathcal{E} : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_1$ satisfying (1.6).

Transition expectations play a crucial role in the construction of quantum Markov chains and they arise naturally within the framework of measurement theory in the following way: the composite system $(1,2)$ undergoes an evolution $u_t : \mathcal{A} \to \mathcal{A}$ $(t \in R)$, which is a one-parameter group of $*-$automorphisms of $\mathcal{A}$. This means that the state φ of $(1,2)$ evolves according to the law

$$\varphi_t := \varphi \circ u_t \tag{0.4}$$

and the state φ_1 of the system 1 evolves according to the **reduced evolution**:

$$\varphi_{1,t}(a_1) := \varphi_1(E_2 \circ u_t \circ j_1(a_1)) \ ; \ a_1 \in \mathcal{A}_\infty \tag{0.5}$$

where j_1 is given by (0.3) and $E_2 : \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_1$ is the Umegaki conditional expectation characterized by

$$E_2(a_1 \otimes a_2) = a_1 \varphi_2(a_2) \ ; \ a_1 \in \mathcal{A}_1 \quad ; \ a_2 \in \mathcal{A}_2 \tag{0.6}$$

Let us fix a time T representing the moment when the experiment ends (ideally $T = +\infty$) and consider the linear map $\mathcal{E}_T : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_1$ characterized by

$$\mathcal{E}_T(a_1 \otimes a_2) = E_2(u_T(a_1 \otimes a_2)) \ ; \ a_1 \in \mathcal{A}_1 \ ; \ a_2 \in \mathcal{A}_2 \tag{0.7}$$

Then $\mathcal{E}_T$ is a transition expectation. If one is ready to accept that the evolution u_T does not take place inside the algbra $\mathcal{A}_1 \otimes \mathcal{A}_2$ but is a representation of $\mathcal{A}_1 \otimes \mathcal{A}_2$ into another algebra (usually much larger), then in some cases and in a certain technical sense, (0.7) represents the most general class of transition expectations (cf. Theorem (1.4) below).

An *instrument* in the sense of the operational approach to quantum measurment is obtained by taking the restriction of a transition expectation $\mathcal{E}$ to a subalgebra $\mathcal{C}_1 \otimes \mathcal{A}_2$ of $\mathcal{A}_1 \otimes \mathcal{A}_2$ where $\mathcal{C}_1$ is a σ-finite abelian von Neumann sup-algebra of $\mathcal{A}_1$. In this case it is known that, if $\mathcal{C}_1$ is σ-finite, then there exists a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathcal{C}_1$ is isomorphic to $L^\infty(\Omega, \mathcal{F}, P)$ and the points $\omega \in \Omega$ are interpreted as macroscopic parameters of the apparatus. If $\mathcal{A}_1 = \mathcal{C}_1$, i.e. if $\mathcal{A}_1$ is an abelian von Neumann algebra, the isomorphism (cf. [28])

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 = L^\infty(\Omega, \mathcal{F}, P) \otimes \mathcal{A}_2 \cong L^\infty(\Omega, \mathcal{F}, P; \mathcal{A}_2) \tag{0.8}$$

implies that the elements of $\mathcal{A}$, i.e., the observables of the composite system $(1,2)$ can be interpreted as functions $(\Omega, \mathcal{F}, P) \to \mathcal{A}_2$, i.e., as **operator valued** random variables. Thus interpreting $(\Omega, \mathcal{F}, P)$ as the sample space of a classical stochastic process, the operational scheme becomes equivalent to the theory of **operator valued classical processes** (cf. [7]).

¿From this point of view an instrument in the operational sense is an object which is only *half-quantum*. The physical motivations for this choice go back to some ideas of Ludwig according to which the measurement apparatus is usually a macroscopic body so that classical probability should be sufficient for its description. Several authors have introduced variations and modifications of Ludwig ideas, however, since all the examples of physical interest of instruments in operational sense, produced up to now, are the restrictions of liftings, we feel that the latter notion plays a more natural and fundamental role.

In conclusion of the present introduction, we show the theory of lifting includes the so calld *operational approach.*

1. CHANNELS, LIFTINGS AND TRANSITION EXPECTATIONS

For a C*-algebra $\mathcal{A}$, we denote $\mathcal{S}(\mathcal{A})$ the convex set of its states. In this paper all C*- and W*-algebras are realized on some separable Hilbert spaces and, unless explicitly stated, the tensor products are those induced by the tensor products of the corresponding Hilbert spaces. If $\mathcal{A}$ is a von Neumann algebra, $\mathcal{S}(\mathcal{A})$ denotes the set of its normal states and $\mathcal{S}(\mathcal{A})_{extr}$ the set of extremal states. Both $\mathcal{S}(\mathcal{A})$ and $\mathcal{S}(\mathcal{A})_{extr}$ are measurable spaces with their Borel structure and the set of probability measures on $\mathcal{S}(\mathcal{A})$ ($\mathcal{S}(\mathcal{A})_{extr}$) is denoted $Prob\mathcal{S}(\mathcal{A})$ ($Prob\mathcal{S}(\mathcal{A})_{extr}$). If $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, a **channel** from $\mathcal{A}$ to $\mathcal{B}$ is a map $\Lambda^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{B})$. If Λ^* is affine we speak of **a linear channel** . If Λ^* is $w*$-continuous and linear, then it can be extended by linearity to a linear map (still denoted Λ^*) from $\mathcal{A}^*$ to $\mathcal{B}^*$. Its dual $\Lambda : \mathcal{B} \rightarrow \mathcal{A}$ is a positive map. If it is completely positive, we call it a **Markovian operator**. Such channels have been studied with some applications by several authors (c.f., [20], [25] and references quoted theirs). Certain quantum channels are naturally associated to classical Markovian kernels on the measurable space $\mathcal{S}(\mathcal{B}) \times \mathcal{S}(\mathcal{A})$. In fact, given such a Markovian kernel, i.e., a measurable map

$$p : \omega \in \mathcal{S}(\mathcal{A}) \longrightarrow p(\,\cdot\,|\omega) \in Prob(\mathcal{S}(\mathcal{B}))$$

one can define a channel in the following way: for any state $\varphi \in \mathcal{S}(\mathcal{A})$ one fixes a convex decomposition

$$\varphi = \int_{\mathcal{S}(\mathcal{A})} \omega d\mu_\varphi(\omega)$$

and defines the channel $\Lambda^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{B})$ through the identity

$$\Lambda^*\varphi := \int_{\mathcal{S}(\mathcal{A})} d\mu_\varphi(\omega) \int_{\mathcal{S}(\mathcal{B})} \omega' p(d\omega'|\omega)$$

The channel Λ^* is usually nonlinear since the map $\varphi \rightarrow \mu_\varphi$ is affine if and only if $\mathcal{S}(\mathcal{A})$ is a simplex ([8] Theorem (4.1.15) and Corollary (4.1.17)) and this is the case if and only if $\mathcal{A}$ is Abelian ([8] Example (4.1.6)). On the other hand, given a linear channel Λ^* one might try to associate to it a Markovian kernel $p(\,\cdot\,|\omega)$ on the measurable space $\mathcal{S}(\mathcal{B}) \times \mathcal{S}(\mathcal{A})$, by fixing, for each $\omega \in \mathcal{S}(\mathcal{A})$, a convex decomposition

$$\Lambda^*\omega := \int_{\mathcal{S}(\mathcal{B})} \omega' p(d\omega'|\omega)$$

The possibility of choosing such a decomposition so to assure the measurability of the map

$$p : \omega \in \mathcal{S}(\mathcal{A}) \longrightarrow p(\,\cdot\,|\omega) \in Prob(\mathcal{S}(\mathcal{B}))$$

as well as the study of the support of these measures give rise to some subtle measure theoretic problems which will be discussed elsewhere. In many examples however, these Markovian kernels can be explicitly constructed and, at least on a subset of the states and have good support and measurability properties.

Definition 1.1: Let $\mathcal{A}_1, \mathcal{A}_2$ be C*-algebras and let $\mathcal{A}_1 \otimes \mathcal{A}_2$ be a fixed C*-tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. A **lifting** from $\mathcal{A}_1$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a $w*$-continuous map

$$\mathcal{E}^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2) \tag{1.1}$$

If $\mathcal{E}^*$ is affine and its dual is a completely positive map, we call it a linear lifting; if it maps pure states into pure states, we call it **pure**.

Remark: Also in the nonlinear case some kinds of *complete positivity requirement* should be included in the definition of lifting. However, the theory of nonlinear completely positive maps is still in its infancy and the some is true for a satisfactory *dualization* of it. Therefore we leave the general question open for further developments and we limit ourselves to present some examples of nonlinear liftings which are of some interest for the applications.

To every lifting from $\mathcal{A}_1$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$ we can associate two channels: one from $\mathcal{A}_1$ to $\mathcal{A}_1$, defined by

$$\Lambda_1^* \rho_1(a_1) := (\mathcal{E}^* \rho_1)(a_1 \otimes 1) \quad ; \quad \forall a_1 \in \mathcal{A}_1 \tag{1.2}$$

another from $\mathcal{A}_1$ to $\mathcal{A}_2$, defined by

$$\Lambda_2^* \rho_1(a_2) := (\mathcal{E}^* \rho_1)(1 \otimes a_2) \quad ; \quad \forall a_2 \in \mathcal{A}_2 \tag{1.3}$$

In general, a state $\varphi \in \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ such that

$$\varphi \mid_{\mathcal{A}_1 \otimes 1} = \rho_1 \quad ; \quad \varphi \mid_{1 \otimes \mathcal{A}_2} = \rho_2 \tag{1.4}$$

has been called [21], [23] a **compound state** of the states $\rho_1 \in \mathcal{S}(\mathcal{A}_1)$ and $\rho_2 \in \mathcal{S}(\mathcal{A}_2)$. In classical probability theory, also the term **coupling** between ρ_1 and ρ_2 is used [13].

The following problem is important in several applications: Given a state $\rho_1 \in \mathcal{S}(\mathcal{A}_1)$ and a channel $\Lambda^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_2)$, find a standard lifting $\mathcal{E}^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ such that $\mathcal{E}^* \rho_1$ is a compound state of ρ_1 and $\Lambda^* \rho_1$. Several particular solutions of this problem have been proposed in [9], [10], [11], [21], [22], [23], however an explicit description of all the possible solutions to this problem is still missing.

Definition 1.2: A lifting from $\mathcal{A}_1$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$ is called **nondemolition** for a state $\rho_1 \in \mathcal{S}(\mathcal{A}_1)$ if ρ_1 is invariant for Λ_1^* i.e., if for all $a_1 \in \mathcal{A}_1$

$$(\mathcal{E}^* \rho_1)(a_1 \otimes 1) = \rho_1(a_1) \tag{1.5}$$

The idea of this definition being that the interaction with system 2 does not alter the state of system 1.

Definition 1.3: Let $\mathcal{A}_1, \mathcal{A}_2$ be C*-algebras and let $\mathcal{A}_1 \otimes \mathcal{A}_2$ be a fixed C*-tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. A **transition expectation** from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to $\mathcal{A}_1$ is a completely positive linear map $\mathcal{E} : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_1$ satisfying

$$\mathcal{E}(1_{\mathcal{A}_1} \otimes 1_{\mathcal{A}_2}) = 1_{\mathcal{A}_1}. \tag{1.6}$$

Remark: The notion of *nondemolition lifting*, discussed here is essentially (i.e., up to minor technicalities) included in the more abstract notion of **state extension** introduced by Cecchini and Petz [10], [11] (cf. also Cecchini and Kümmerer [11]).

The two interpretations of the notion of standard lifting, which shall be used in the present paper, are the following:

(1) The measurement process

$\mathcal{A}_1$ (resp. $\mathcal{A}_2$) is interpreted as the algebra of observables of a system (resp. a measurement apparatus) and $\mathcal{E}^*$ describes the interaction between system and apparatus as well as the preparation of the apparatus. If $\rho_1 \in \mathcal{S}(\mathcal{A}_1)$ is the preparation of the system, i.e., its state before the interaction with the apparatus, then $\Lambda_1^*\rho_1 \in \mathcal{S}(\mathcal{A}_1)$ (resp. $\Lambda_2^*\rho_1 \in \mathcal{S}(\mathcal{A}_2)$) is the state of the system (resp. of the apparatus) after the measurement.

(2) The signal transmission process

An input signal is transmitted and received by an apparatus which produces an output signal. Here $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) is interpreted as the algebra of observables of the input (resp. output) signal and $\mathcal{E}^*$ describes the interaction between the input signal and the receiver as well as the preparation of the receiver. If $\rho_1 \in \mathcal{S}(\mathcal{A}_1)$ is the input signal, then the state $\Lambda_2^*\rho_1 \in \mathcal{S}(\mathcal{A}_2)$, defined by (1.3) is the state of the (observed) output signal.

An important lifting related to this signal transmission is one due to a quantum communication process discussed below (Example 1a and 4).

In several important applications the state ρ_1 of the system before the interaction (preparation, input signal) is not known and one would like to know this state knowing only $\Lambda_2^*\rho_1 \in \mathcal{S}(\mathcal{A}_2)$, i.e., the state of the apparatus after the interaction (output signal). From a mathematical point of view this problem is not well posed, since usually the map Λ_2^* is not invertible. The best one can do in such cases is to acquire a control on the description of those input states which have the same image under Λ_2^* and then choose among them according to some statistical criterion. Another widely applied procedure is to postulate, on the basis of some experimental information, that the input state belongs to an a priori given restricted class of states and to choose among these ones by some statistical criterion.

In the following we describe several examples of liftings which appear frequently in the applications.

Example 1: Isometric liftings.
Let $V : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ be an isometry

$$V^*V = 1_{\mathcal{H}_1}. \tag{1.7}$$

Then the map

$$\mathcal{E} : x \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) \to V^*xV \in \mathcal{B}(\mathcal{H}_1) \tag{1.8}$$

is a transition expectation, and the associated lifting maps a density matrix $w_1 \in T(\mathcal{H}_1)$ into $\mathcal{E}^*w_1 = Vw_1V^*$. Liftings of this type are called isometric. Every isometric lifting is a pure lifting. Isometric liftings have turned out to play a relevant role in some mathematical models for superconductivity [14].

Example 1a: The attenuation (or beam splitting) lifting.
It is the particular isometric lifting characterized by the properties.

$$\mathcal{H}_1 = \mathcal{H}_2 =: \Gamma(\mathbf{C}) = \text{the Fock space over } \mathbf{C}$$

$$V : \Gamma(\mathbf{C}) \to \Gamma(\mathbf{C}) \otimes \Gamma(\mathbf{C}) \tag{1.9}$$

is characterized by the expression

$$V|\theta >= |\alpha\theta > \otimes|\beta\theta > \tag{1.10}$$

where $|\theta >$ is the normalized coherent vector parametrized by $\theta \in \mathbf{C}$ and $\alpha, \beta \in \mathbf{C}$ are such that

$$|\alpha|^2 + |\beta|^2 = 1 \tag{1.11}$$

Notice that this liftings maps coherent states into products of coherent states. So it maps the simplex of the so called **classical states** (i.e., the convex combinations of coherent vectors) into itself. Restricted to these states it is of convex product type in the sense of Definition 2.1 below, but it is not of convex product type on the set of all states.

Denoting, for $\theta \in \mathbf{C}$, ω_θ the state on $\mathcal{B}(\Gamma(\mathbf{C}))$ defined by

$$\omega_\theta(b) =< \theta, b\theta > \quad ; \ b \in \mathcal{B}(\Gamma(\mathbf{C})) \tag{1.12}$$

we see that, for any $b \in \mathcal{B}(\Gamma(\mathbf{C}))$

$$(\mathcal{E}^*\omega_\theta)(b \otimes 1) = \omega_{\alpha\theta}(b) \tag{1.13}$$

hence this lifting is not nondemolition.

Interpretation: $\Gamma(\mathbf{C})$ is the space of a 1-mode EM field (signal). V represents the interaction, of the signal with an apparatus (e.g., a receiver or a semi–transparent mirror). In $\Gamma(\mathbf{C}) \otimes \Gamma(\mathbf{C})$ the second factor is the space of the apparatus.

Equation (1.10) means that, by the effect of the interaction, a coherent signal (beam) $|\theta>$ splits into 2 signals (beams) still coherent, but of lower intensity. Because of (1.11), the total intensity (energy) is preserved by the transformation.

Example 1b: Superposition beam splitting.

The only difference with Example (1a.) is the form of V, which in this case is

$$V|\theta>= \frac{1}{\sqrt{2}}(|\alpha\theta> \otimes|\beta\theta> -i|\beta\theta> \otimes|\alpha\theta>) \tag{1.14}$$

One easily checks that V extends linearly to an isometry of the form (1.9).

This isometric lifting is not of convex product type in the sense of Definition 2.1 of the next section, neither it is a nondemolition lifting.

Example 2: The compound lifting.

Let $\Lambda^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_2)$ be a channel. For any $\rho_1 \in \mathcal{S}(\mathcal{A}_2)$ in the closed convex hull of the external states, fix a decomposition of ρ_1 as a convex combination of extremal states in $\mathcal{S}(\mathcal{A}_1)$

$$\rho_1 = \int_{\mathcal{S}(\mathcal{A}_1)} \omega_1 p(d\omega_1 \mid \rho_1) \tag{1.15}$$

where $p(\ \cdot \mid \rho_1)$ is a Borel measure on $\mathcal{S}(\mathcal{A}_1)$ with support in the extremal states, and define

$$\mathcal{E}^*\rho_1 := \int_{\mathcal{S}(\mathcal{A}_1)} \omega_1 \otimes \Lambda^*\omega_1 p(d\omega_1 \mid \rho_1) \tag{1.16}$$

Then $\mathcal{E}^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_2 \otimes \mathcal{A}_2)$ is a lifting, nonlinear even if Λ^* is linear, and nondemolition for ρ_1. In Section 2, we shall see that the most general lifting, mapping $\mathcal{S}(\mathcal{A}_1)$ into the closed convex hull of the external product states on $\mathcal{A}_1 \otimes \mathcal{A}_2$ is essentially of this type. Here "essentially" means that, in order to recover the most general case, we shall weaken, from the original definition of compound state in [22], the condition that $p(d\omega_1 \mid \rho_1)$ is concentrated on the extremal states.

Therefore once a channel is given, a lifting of convex product type can be constructed by (1.16), and the converse is also true due to (1.3):

$$\textbf{channel} \longleftrightarrow \textbf{lifting}.$$

For example, the von Neumann quantum measurement process is written, in our terminology, as follows: Having measured an observable $A = \sum_n a_n P_n$ (spectral decomposition with discrete spectrum) in a state ρ, the state after this measurement will be

$$\Lambda^*\rho = \sum_n P_n \rho P_n$$

and a lifting $\mathcal{E}^*$, of convex product type, associated to this channel Λ^* and to a fixed decomposition of ρ as $\rho = \sum_n \lambda_n \rho_n$ $(\rho_n \in \mathcal{S}(\mathcal{A}_1))$ is given by :

$$\mathcal{E}^*\rho = \sum_n \lambda_n \rho_n \otimes \Lambda^* \rho_n \tag{1.17}$$

A more sophisticated example of lifting of this type is a reduction of a state associated with an open system dynamics. Namely, if a system Σ_1 described by a Hilbert space $\mathcal{H}$ interacts with an external system Σ_2 described by another Hilbert space $\mathcal{K}$ and the initial states of Σ_1 and Σ_2 are ρ and σ, respectively, then the combined state θ_t of Σ_1 and Σ_2 at time t after the interaction between the two systems is given by

$$\theta_t = U_t^*(\rho \otimes \sigma)U_t$$

where $U_t = \exp(itH)$ with the total Hamiltonian H of Σ_1 and Σ_2. A channel is obtained by taking the partial trace with respect to $\mathcal{K}$ i.e.,

$$\rho \to \Lambda_t^* \rho = tr_{\mathcal{K}} \theta_t .$$

A lifting associated to this channel is given by (1.17) with Λ_t^* above.

Example 3: Canonical form of a lifting

Let $\mathcal{A}_1 = \mathcal{B}(\mathcal{H}_1)$, $\mathcal{A}_2 = \mathcal{B}(\mathcal{H}_2)$. The most general linear lifting $\mathcal{E}^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ has the form

$$w_1 \in T(\mathcal{H}_1) \to \sum K_i(w_1 \otimes 1)K_i^* \in T(\mathcal{H}_1 \otimes \mathcal{H}_2) \tag{1.18}$$

for some $K_i \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that

$$\sum K_i^* K_i = 1 \tag{1.19}$$

This is a simple consequence of Krein's Lemma.

Example 4: The lifting for quantum communication channel.

Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{K}_1$, $\mathcal{K}_2$ be Hilbert spaces. Denote α the amplification

$$\alpha : b_2 \in \mathcal{B}(\mathcal{H}_2) \to \alpha(b_2) = b_2 \otimes 1_{\mathcal{K}_2} \in \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{K}_2) \tag{1.20}$$

Let

$$\gamma : \mathcal{B}(\mathcal{H}_2 \otimes \mathcal{K}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{K}_1)$$

be a completely positive identity preserving map and, for $\sigma_1 \in \mathcal{S}(\mathcal{B}(\mathcal{K}_1))$ denote $\bar{\sigma}_1^{(2)}$ the conditional expectation

$$\bar{\sigma}_1^{(2)} : a_1 \otimes b_1 \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{K}_1) \to a_1 \sigma_1(b_1) \in \mathcal{B}(\mathcal{H}_1) \cong \mathcal{B}(\mathcal{H}_1) \otimes 1_{\mathcal{K}_1} \tag{1.21}$$

Then the lifting and the channel describing quantum communication processes are defined by

$$\mathcal{E}^* = \gamma^* \circ \bar{\sigma}_1^{(2)*} \tag{1.22}$$

$$\Lambda^* \rho = \alpha^* \circ \mathcal{E}^*(\rho) = tr_{\mathcal{K}_2} \gamma^*(\rho \otimes \sigma_1); \;\; \rho \in \mathcal{S}(\mathcal{B}(\mathcal{H}_1)) \tag{1.23}$$

where ρ and σ_1 correspond to an input state and a noise state, respectively (c.f. [21]).

Moreover the following remark, extending an unpublished result of A.Frigerio, shows that the above model of the quantum communication process is universal among the transition expectations, provided one chooses the space of the representation large enough.

Theorem 1.4: Let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ for a separable, infinite dimensional, Hilbert space $\mathcal{H}$ and let $\mathcal{E} : \mathcal{B} \otimes \mathcal{B} \longrightarrow \mathcal{B}$ be a normal transition expectation. Then there exist a normal state φ on the W^*-algebra $(\otimes\mathcal{B})^3 \otimes \mathbf{M}_2 =: \mathcal{C}$ ($\mathbf{M}_2$ is the algebra of 2×2 complex matrices), and a unitary element U of

$$\mathcal{A} := \mathcal{C} \otimes \mathcal{B} \cong (\otimes\mathcal{B})^4 \otimes \mathbf{M}_2 \tag{1.24}$$

such that , denoting $\pi : \mathcal{B} \otimes \mathcal{B} \to \mathcal{A}$ the normal representation (amplification)

$$\pi(x) = (\, 1_{\mathcal{B}^2} \otimes x \,)\, 0$$

Proof: From Kraus' Lemma [18] we know that, since $\mathcal{E}$ is normal, there exist a coutable (since $\mathcal{H}$ is separable family) $a_i \in \mathcal{B} \otimes \mathcal{B}$ $(i \in N)$ such that , identifying $\mathcal{B}$ with $\mathcal{B} \otimes 1$ one has

$$\mathcal{E}(x) = \sum_{i \in N} a_i^* x a_i \qquad x \in \mathcal{B} \otimes \mathcal{B}$$

If $\mathcal{H}$ is infinite dimensional in $\mathcal{B} \otimes \mathcal{B}$ there exist isometries u_i $(i \in N)$ such that for each i, j $u_i^* u_j = \delta_{ij}$. Thus , defining

$$v := \sum_{i=1}^{d} u_i \otimes a_i \in (\otimes\mathcal{B})^4 \tag{1.28}$$

one finds

$$v^*(1_{\mathcal{B}^2} \otimes x)v = 1_{\mathcal{B}^2} \otimes \mathcal{E}(x) \qquad x \in \mathcal{B} \otimes \mathcal{B} \tag{1.29}$$

In particular , since $v^*v = 1_{\mathcal{B}^4}$, v is a partial isometry with initial projection the identity. Denote $e = vv^*$ its final projection and define the unitary operator

$$U = (\, v \,)\, 1 - e$$

2. CONVEX COMBINATIONS OF PRODUCT STATES

One of the main differences between classical and quantum probability is that, while all the measures on a product space are in the closed convex hull (for the weak topology) of product measures, it is not true that all the states on the tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ of two general C^*–algebras are limits (in some topology) of convex combinations of product states.

In particular, the image under a general lifting $\mathcal{E}^*$ of a state φ will usually not be a convex combination of product states.

However the class of liftings with this porperty is particulary interesting because we expect that in this class some features of quantum probability will mix with some features of classical probability. This class is defined as follows:

Definition 2.1: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be W^*–algebras. A lifting $\mathcal{E}^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ will be called **of convex product type**, or shortly a **convex product lifting** , if any state $\omega \in \mathcal{S}(\mathcal{A}_1)$ is mapped by $\mathcal{E}^*$ into a convex combination of product states on $\mathcal{A}_1 \otimes \mathcal{A}_2$. If this property holds only for any state ω in a subset $\mathcal{F} \subseteq \mathcal{S}(\mathcal{A}_1)$ then $\mathcal{E}^*$ is called a convex product lifting with respect to the family $\mathcal{F}$.

For any von Neumann algebra $\mathcal{A}$, the set $\mathcal{S}(\mathcal{A})$ of all its states has a natural structure of measurable space with its Borel σ–algebra. In the following any probability measure on $\mathcal{S}(\mathcal{A})$ will be meant with respect to this σ–algebra.

Definition 2.2: A **convex decomposition** of $\varphi \in \mathcal{S}(\mathcal{A})$ is a probability measure μ on $\mathcal{S}(\mathcal{A})$ satisfying

$$\varphi = \int_{\mathcal{S}(\mathcal{A})} \omega d\mu(\omega) \tag{2.1}$$

If μ is pseudosupported, in the sense of [8], in the set of extremal states of $\mathcal{S}(\mathcal{A})$, we speak of an **extremal convex decomposition** of φ.

Proposition 2.1: To every lifting of convex product type $\mathcal{E}^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$, one can associate a pair

$$\{p_\rho(d\omega_1),\ p_\rho(d\omega_2|\omega_1)\} \tag{2.2}$$

with the following properties:

(i) $p_\rho(d\omega_1)$ is a probability measure on $\mathcal{S}(\mathcal{A}_1)$

(ii) $p_\rho(d\omega_2|\omega_1)$ is a Markovian kernel from $\mathcal{S}(\mathcal{A}_1)$ to $\mathcal{S}(\mathcal{A}_2)$. Conversely every pair (2.2) satisfying (i) and (ii) above determines, via (2.4) and (2.5), a unique convex product lifting.

Proof: For $\mathcal{E}^*$ as in Definition 2.1, let us fix a state $\rho_1 \in \mathcal{S}(\mathcal{A}_1)$ and also a decomposition of $\mathcal{E}^*\rho_1$ as a convex combination of product states

$$\mathcal{E}^*\rho_1 = \int_{\mathcal{S}(\mathcal{A}_1)\times\mathcal{S}(\mathcal{A}_2)} \omega_1 \otimes \omega_2 dp(\omega_1, \omega_2|\rho_1) \tag{2.3}$$

Denoting $p_{\rho_1}(d\omega_2|\omega_1)$ the conditional probability of $p(\,\cdot\,|\rho_1)$ on the σ–algebra of the first factor and $dp_{\rho_1}(\omega_1)$ the marginal of $p(\,\cdot\,|\rho_1)$ on the first factor, we obtain

$$\mathcal{E}^*\rho_1 = \int_{\mathcal{S}(\mathcal{A}_1)} \int_{\mathcal{S}(\mathcal{A}_2)} \omega_1 \otimes \omega_2 dp_{\rho_1}(\omega_1) p_{\rho_1}(d\omega_2|\omega_1) = \int_{\mathcal{S}(\mathcal{A}_1)} \omega_1 \otimes \Lambda^*_\rho \omega_1 dp_{\rho_1}(\omega_1) \tag{2.4}$$

$$\Lambda^*_{\rho_1}\omega_1 := \int_{\mathcal{S}(\mathcal{A}_2)} \omega_2 p_{\rho_1}(d\omega_2|\omega_1) \tag{2.5}$$

Thus any lifting $\mathcal{E}^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$, of convex product type, has the form (2.4), where p_{ρ_1} is a probability measure on $\mathcal{S}(\mathcal{A}_1)$ and the map $\Lambda^*_{\rho_1} : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_2)$, is given by (2.5). Notice that $\Lambda^*_{\rho_1}$ is a channel in the sense of Section 1 and it is usually nonlinear both in ω_1 and ρ_1.

Conversely, given p_{ρ_1} and $\Lambda^*_{\rho_1}$ as above, if we define $\mathcal{E}^*$ by (2.3), then clearly $\mathcal{E}^*$ is a convex product lifting from $\mathcal{S}(\mathcal{A}_1)$ to $\mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$. Finally it is clear that the map

$$(\omega_1, S_2) \in \mathcal{S}(\mathcal{A}_1) \times \text{Borel}(\mathcal{S}(\mathcal{A}_2)) \to p_{\rho_1}(\mathcal{S}_2|\omega_1) \in [0,1]$$

is a classical Markovian kernel on the Borel space $\mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)$.

Remark: If in (2.3) one conditions on the σ-algebra of the second factor rather than on the first one, the resulting lifting is

$$\mathcal{E}^*\rho_1 = \int_{\mathcal{S}(\mathcal{A}_1)} \int_{\mathcal{S}(\mathcal{A}_2)} \omega_1 \otimes \omega_2 dq_{\rho_1}(\omega_2) dq_{\rho_1}(d\omega_1|\omega_2)$$

where $dq_{\rho_1}(\omega_2)$ is a probability measure on $\mathcal{S}(\mathcal{A}_2)$ and $dq_{\rho_1}(d\omega_1|\omega_2)$ a Markovian kernel from $\mathcal{S}(\mathcal{A}_2)$ to $\mathcal{S}(\mathcal{A}_1)$.

Let us now consider **the relation between the liftings of convex product type and Markov chains**.

The dual of a linear lifting is a transition expectation, therefore to any linear lifting one can associate a quantum Markov chain [2] in a standard way.

If the lifting is of convex product type, then we can take advantage of this special structure to extend the construction of quantum Markov chains to the case of a not necessarily linear lifting $\mathcal{E}^*$. In what follows we describe this procedure.

If $\mathcal{E}^* : \mathcal{S}(\mathcal{A}_2) \to \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ is a lifting of convex product type, then it has the form:

$$\mathcal{E}^*\rho_2 = \int_{\mathcal{S}(\mathcal{A}_1)} \int_{\mathcal{S}(\mathcal{A}_2)} \omega^1 \otimes \omega^2 p(d\omega^1, d\omega^2|\rho_2) \tag{2.6}$$

Notice that $p(d\omega^1, d\omega^2|\rho_2)$ can be considered as a Markovian Kernel on the space

$$\mathcal{S}_{12} := \mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)$$

which is constant on the first conditioning, i.e.,

$$p(d\omega^1, d\omega^2|\rho_1, \rho_2) = p(d\omega^1, d\omega^2|\rho_2) \quad ; \quad \omega_1, \rho_1, \in \mathcal{S}(\mathcal{A}_1) \ , \ \omega_2, \rho_2 \in \mathcal{S}(\mathcal{A}_2) \tag{2.7}$$

Clearly (2.6) is a state on $\mathcal{A}_1 \otimes \mathcal{A}_2$. If we apply $\mathcal{E}^*$ to ω^2 in (2.6), we obtain the following state on $(\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes \mathcal{A}_2$:

$$\int_{\mathcal{S}_{12}} p(d\omega_1^1, d\omega_1^2, |\rho_2) \omega_1^1 \otimes \mathcal{E}^*\omega_1^2 =$$

$$= \int_{\mathcal{S}_{12}} \int_{\mathcal{S}_{12}} p(d\omega_1^1, d\omega_1^2 | \rho_2) p(d\omega_2^1, d\omega_2^2 | \omega_1^2) \omega_1^1 \otimes \omega_2^1 \otimes \omega_2^2$$

where

$$\omega_i^1 \in \mathcal{S}(\mathcal{A}_1) \; and \; \omega_i^2 \in \mathcal{S}(\mathcal{A}_2).$$

Applying again $\mathcal{E}^*$ to ω_2^2 we find

$$\int_{\mathcal{S}_{12}} \int_{\mathcal{S}_{12}} \int_{\mathcal{S}_{12}} p(d\omega_1^1, d\omega_1^2 | \rho_2) p(d\omega_2^1, d\omega_2^2 | \omega_1^2) p(d\omega_3^1, d\omega_3^2 | \omega_2^2) \omega_1^1 \otimes \omega_2^1 \otimes \omega_3^1 \otimes \omega_3^2$$

At the n–th iteration we obtain the state $\mathcal{E}^*_{n]} \rho_2$ on $(\otimes \mathcal{A}_1)^n \otimes \mathcal{A}_2$, defined by:

$$\mathcal{E}^*_{n]} \rho_2 := \int_{\mathcal{S}_{12}^n} \left(\otimes_{i=1}^n \omega_i^1 \right) \otimes \omega_n^2 \cdot \Pi_{i=2}^n p(d\omega_i^1, d\omega_i^2 | \omega_{i-1}^2) p(d\omega_1^1, d\omega_1^2, | \rho_2) \tag{2.8}$$

This suggests to introduce the classical Markov process

$$\xi_n := (\xi_n^1, \xi_n^2) : (\Omega, \mathcal{F}, P) \to \mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2) = S_{12} \tag{2.9}$$

with the transition function given by (2.7) and initial distribution $p(\cdot | \rho_2)$. This transition probability has a nice interpretation in terms of signal + noise : if system1 represents the noise and system 2 the signal, then condition (2.7) means that the joint distribution at time $(n+1)$ depends on the signal at time n, but not on the noise at time n : a natural assumption if we think of the noises at defferent times, as generated by independent causes. Now let $\mathcal{A} := \otimes_{\mathbf{N}} \mathcal{A}_1$. The identification

$$a_1 \otimes \ldots \otimes a_n \cong a_1 \otimes a_2 \otimes \ldots \otimes a_n \otimes 1 \otimes 1 \otimes \cdots$$

induces a natural identification of $(\otimes \mathcal{A}_1)^n$ with a sub-algebra $\mathcal{A}_{[2,n]}$ of $\mathcal{A} = \otimes_{\mathbf{N}} \mathcal{A}_1$ (the product of the first n-factors).

In particular, if $\rho_2 \in \mathcal{S}(\mathcal{A}_2)$ is a state on $\mathcal{A}_2$, the restriction of $\mathcal{E}^*_{n]} \rho_2$ on $(\otimes \mathcal{A}_1)^n$ is a state on $(\otimes^n \mathcal{A}_1)$ and, with the above identification, we can consider it a state $\rho_{[1,n]}$ on $\mathcal{A}$. Following from all above, in particular (2.8), we obtain

Proposition 2.2: For any $\rho_2 \in \mathcal{S}(\mathcal{A}_2)$ the limit

$$\lim_{n \to \infty} \rho_{[1,n]} =: \varphi \tag{2.10}$$

exists pointwise weakly on $\mathcal{A}$. Moreover, if E_ξ denotes the mean with respect to the process $\{\xi_n\}$, defined by (2.9), then one has

$$\varphi = E_\xi \left(\otimes_{n \in \mathbf{N}} \xi_n^1 \right). \tag{2.11}$$

3. CENTRALIZER LIFTINGS

In this Section we introduce an interesting class of nonlinear liftings generalizing the construction of [22]. It is shown that the Cecchini-Petz notion of **state extension** [11], introduced after [22] and for totally independent reasons, is a generalization of our construction hence, a fortiori, of the one in [22].

Recall that a linear map E from a C*-algebra $\mathcal{A}$ to a C*-algebra $\mathcal{B}$ is called **anti-completely positive** if the map $\bar{E} : \mathcal{A} \to \mathcal{B}$, defined by

$$\bar{E}(a) := E(a^*) \quad ; \quad a \in \mathcal{A} \tag{3.1}$$

is completely positive antilinear, i.e., for any natural integer n, any $a_1, \dots, a_n \in \mathcal{A}$ and any $b_1, \dots, b_n \in \mathcal{B}$, one has

$$\sum_{jk} b_j^* E(a_k^* a_j) b_k = \sum_{jk} b_j^* \bar{E}(a_j^* a_k) b_k \geq 0$$

Proposition 3.1: Let $\mathcal{A}_1, \mathcal{A}_2$ be W*-algebras, let $\mathcal{A}_1 \otimes^{(o)} \mathcal{A}_2$ denote their algebraic tensor product. For $\rho \in \mathcal{S}(\mathcal{A}_1)$ let $\mathcal{A}_1^\rho$ denote the centralizer of ρ and let $E : \mathcal{A}_2 \to \mathcal{A}_1^\rho$ be any anticompletely positive identity preserving linear map. Then there exists a unique state φ_ρ on $\mathcal{A}_1 \otimes^{(o)} \mathcal{A}_2$ such that

$$\varphi_\rho(a_1 \otimes a_2) := \rho(a_1 E(a_2)) \quad ; \quad a_1 \in \mathcal{A}_1 \ , \ a_2 \in \mathcal{A}_2$$

Proof: Let n be a natural integer and let $b_1, \dots, b_n \in \mathcal{A}_1$ and $a_1, \dots, a_n \in \mathcal{A}_2$. By assumption the $\mathcal{A}_1^\rho$-valued $n \times n$ matrix $B = (B_{kj})$ defined by

$$B_{kj} := E(a_j^* a_k) = \bar{E}(a_k^* a_j)$$

is of positive type, hence it has the form $B = M^* M$ for some $\mathcal{A}_1^\rho$-valued $n \times n$ matrix $M = (M_{kj})$. One has therefore

$$\varphi_\rho(|\sum_j a_j \otimes b_j|^2) = \sum_{jk} \rho\Big(b_j^* b_k E(a_j^* a_k)\Big) =$$

$$= \sum_{jkh} \rho\Big(b_j^* b_k M_{hk}^* M_{hj}\Big) =$$

$$= \sum_{jkh} \rho\Big(M_{hj} b_j^* b_k M_{hk}^*\Big) =$$

$$= \sum_h \rho\Big([\sum_j b_j M_{hj}^*]^* \cdot [\sum_k b_k M_{hk}^*]\Big) \geq 0$$

Remark: Clearly

$$| \varphi_\rho(a_1 \otimes a_2) | \leq \| a_1 \| \cdot \| a_2 \| \quad ; \quad a_1 \in \mathcal{A}_1 \ , \ a_2 \in \mathcal{A}_2$$

hence φ_ρ is continuous for the greatest cross norm on $\mathcal{A}_1 \otimes^{(o)} \mathcal{A}_2$. Cecchini and Petz [12] have proved that it is also continuous for the smallest C*-norm [28]. (This is clear if the centralizer of ρ, i.e., $\mathcal{A}_1^\rho$, is abelian because in that case all the C*-norms on $\mathcal{A}_1 \otimes^{(o)} \mathcal{A}_2$ coincide with the minimal C*-norm ([28], Proposition 1.22.5)). Moreover it is easy to check that the operator E, defined by (3.1) is an example of Cecchini' s λ-operator [9]. In this case in fact the Tomita involution J_1 acts as the identity on the cyclic space of $\mathcal{A}_1^\rho$, the centralizer of $\mathcal{A}_1$, therefore the identity (3.1) is precisely the defining relation of the λ-operator.

If $\mathcal{A}_1^\rho$ is a discrete abelian algebra generated by a partition (e_j) of the identity, then any positive map E, from $\mathcal{A}_2$ to $\mathcal{A}_1^\rho$ is also completely and anti-completely positive and it has the form

$$E(a_2) = \sum_j \varphi_j(a_2) e_j \quad ; \quad a_2 \in \mathcal{A}_2 \tag{3.2}$$

with $\varphi_j \in \mathcal{S}(\mathcal{A}_2)$. In this case it is immediate to verify that

$$\varphi_\rho = \sum_j \rho_j \otimes \varphi_j \tag{3.3}$$

where φ_j is given by (3.2) and

$$\rho_j := \rho(e_j(\ \cdot\ )e_j)$$

In general, whenever the state φ_ρ, defined by (3.3), is continuous, the map $\rho \mapsto \varphi_\rho$ defines a lifting $\mathcal{E}^*$ in the sense of Definition 1.1. This lifting is in general nonlinear since the map E in (3.1) may depend on ρ.

For example, if $\mathcal{A}_1$ is the algebra of all operators on some Hilbert space and ρ has the form $\rho = tr(w\ \cdot\ )$ for some density matrix w with spectral decomposition given by

$$w = \sum_j p_j e_j \tag{3.4}$$

then the centralizer $\mathcal{A}_1^\rho$ is the closed linear span of the (e_j) and if the φ_j are chosen to be of the form

$$\varphi_j = \Lambda^*\left(\frac{\rho_j}{\rho_j(1)}\right) \tag{3.5}$$

for some channel $\Lambda^* : \mathcal{S}(\mathcal{A}_1) \to \mathcal{S}(\mathcal{A}_2)$, then equation (3.3) becomes of the same form as (1.17) giving an example of nonlinear compound lifting.

4. ERROR PROBABILITY FOR OPTICAL COMMUNICATION

An optical communication process studied by several authors (cf.[21] for a mathematical analysis), the so-called **attenuation process**, can be described by the isometric lifting described in Example 1.a. This description is simpler than the previous ones and allows quicker computations. This statement is illustrated with the computation of several error probabilities related to this model.

Before introducing these computations, we briefly review some basic facts about the notions of quantum coding and of error probability in quantum control communication processes along the lines of [24].

Suppose that, by some procedure, we encode an information representing it by a sequence of letters $c^{(1)}, \ldots, c^{(n)}, \ldots$, where $c^{(k)}$ is an element in a set $\mathcal{C}$ of symbols called the alphabet.

A **quantum code** is a map which associates to each symbol (or sequence of symbols) in $\mathcal{C}$ a quantum state, representing an optical signal. Sometimes one uses a state as two codes: one for input and one for output.

In the sequel we shall only consider a two symbols alphabet:

$$\mathcal{C} = \{0, 1\} \tag{4.1}$$

One example of quantum code $\Xi = \{\xi_0, \xi_1\}$ where ξ_i is the quantum state corresponding to the symbol $c_i \in \mathcal{C}$, is obtained by choosing ξ_0 as the vacuum state and ξ_1 another state such as a coherent or a squeezed state of a one-mode field.

Two states (quantum codes) $\xi_0^{(1)}$ and $\xi_1^{(1)}$ in the input system are transmitted to the output system through a channel Λ^*. We here assume a Z-type signal transmission, namely that the input signal "0", represented by the state $\xi_0^{(1)}$, is error free in the sense that it goes always to the output signal "0" represented by $\xi_0^{(2)}$, while the input signal "1", represented by the state $\xi_1^{(1)}$, is not error free in the sense that its output can give rise to both states $\xi_0^{(2)}$ or $\xi_1^{(2)}$ with different probabilities.

The **error probability** q_e is then the probability that the input signal "1" is recognized as the output signal "0", so that it is given by

$$q_e = tr\Lambda^*(\xi_1^{(1)})\xi_0^{(2)} \tag{4.2}$$

In the case of quantum attenuation process, this error probability is written by using the attenuation operator V given in Example (1.a) with the construction (Example (4)) of quantum lifting:

$$q_e = tr_{\mathcal{H}_2} tr_{\mathcal{K}_2}(V\xi_1^{(1)}V^*)\xi_0^{(2)} \tag{4.3}$$

There are two main ways, called **pulse modulation**, to code the symbols of the alphabet $\mathcal{C}$. We briefly explain them for completeness. A pulse is an optical signal, represented by a non vacuum state of the EM field; its energy is here called the **height** of the pulse. To a single symbol of the alphabet $\mathcal{C}$ one associates one or more pulses. Time is discretized and each time interval between t_k and t_{k+1} has length τ. Each time interval corresponds to a single symbol of the alphabet.

(1) PCM (Pulse Code Modulation) : To the k-th symbol a_k of the input sequence, one associates N pulses starting at a time t_k. The ordered set of these pulses is denoted x_k. For instance, for the alphabet $\{a_0, a_1\}$, for $N = 5$ and choosing the elementary pulses to be the vacuum (i.e., no pulse) denoted 0, and another fixed pulse, e.g., a coherent state, denoted 1, the code x_k corresponding to a_k is determined by $x_0 = (1, 0, 1, 0, 0), x_1 = (0, 1, 1, 0, 1)$ and so on. For this modulation, we need N slots (sites) in one time interval (e.g., between t_k and t_{k+1}) to fully represent all M signals; $2^{N-1} < M \leq 2^N$.

(2) PPM (Pulse Position Modulation) : In this case there is only one non-vacuum pulse in each time interval of length τ. The code x_k expressing a signal a_k is determined by the position of the non vacuum pulse, so that we need M slots (sites) in each time interval in order to express M signals; For instance in the same notations as above, $x_0 = (0, 0, 0, 1, 0), x_2 = (0, 1, 0, 0, 0)$.

Given (4.2), the error probability of PCM with the t_0-tuple error correcting the following (4.3) and (4.4), respectively:

$$P_e^{PCM} = \sum_{j=t_0+1}^{\nu} {}_\nu C_j q_e^j (1 - q_e)^{\nu - j}, \tag{4.4a}$$

$$P_e^{PPM} = q_e, \tag{4.4b}$$

where ${}_\nu C_j = \nu\,!/\{(\nu\text{-j})!\text{j}!\}$.

The most general case for the computation of q_e is one that both $\xi_1^{(1)}$ and $\xi_0^{(2)}$ are squeezed states, but in usual optical communication it is often enough to take a coherent or squeezed state as $\xi_1^{(1)}$ and the vacuum state as $\xi_0^{(2)}$. Hence we first calculate the error probability q_e for the latter two cases and compare them with the results previously obtained in [24]. Secondly we show the computation for the most general case, $\xi_1^{(1)}$ and $\xi_0^{(2)}$ squeezed, for a mathematical interest and generality, although this somehow does not fit to the assumption of our Z type transmission.

(I) Case of $\xi_1^{(1)} = |\theta >< \theta| =$ coherent state and $= |0 >< 0|$: The error probability (4.2) becomes

$$\begin{aligned} q_e &= tr_{\mathcal{H}_2}(tr_{\mathcal{K}_2} V^* |\theta >< \theta| V)|0 >< 0| \\ &= tr_{\mathcal{H}_2}(tr_{\mathcal{K}_2} |\alpha\theta >< \alpha\theta| \otimes |\beta\theta >< \beta\theta|)|0 >< 0| \\ &= tr_{\mathcal{H}_2} |\alpha\theta >< \alpha\theta||0 >< 0| \\ &= | < 0,\ \alpha\theta > |^2 \\ &= \exp(-|\alpha\theta|^2), \end{aligned}$$

which is equal to the usual result (cf. [16], [19]), but our new derivation is much simpler than old one.

(II) Case of $\xi_1^{(1)}$ = squeezed state and $\xi_0^{(2)} = |0><0|$: A squeezed state can be expressed by a unitary operator $U(z)$ ($z \in \mathbf{C}$) given in Appendix such that

$$\xi_1^{(1)} = U(z)|\gamma><\gamma|U(z)^*$$

where $|\gamma>$ is a certain coherent state. Therefore the error probability q_e is

$$\begin{aligned} q_e &= tr_{\mathcal{H}_2}(tr_{\mathcal{K}_2} VU(z)|\gamma><\gamma|U(z)^*V^*)|0><0| \\ &= trV^*(|0><0|\otimes I)VU(z)|\gamma><\gamma|U(z)^* \\ &=< U(z)\gamma,\ V^*(|0><0|\otimes I)VU(z)\gamma > \end{aligned}$$

To carry this calculation, we have to know the effect of V^* on $\mathcal{H}_2 \otimes \mathcal{K}_2$.

$$\begin{aligned} &<\gamma,\ (V^*|\gamma' > \otimes|\gamma" >) >=< V\gamma,\ (|\gamma' > \otimes|\gamma" >) > \\ &\quad =< (<\alpha\gamma|\otimes <\beta\gamma|),\ (|\gamma' > \otimes|\gamma" >) > \\ &\quad =< \alpha\gamma,\ \gamma' >< \beta\gamma,\ \gamma" > \\ &\quad = \exp\{\frac{1}{2}(-|\alpha\gamma|^2 - |\gamma'|^2 + 2\bar{\alpha}\bar{\gamma}\gamma')\}\exp\{\frac{1}{2}(-|\gamma"|^2 - |\beta\gamma|^2 + 2\bar{\beta}\bar{\gamma}\gamma")\} \\ &\quad = \exp\{-\frac{1}{2}(|\beta\gamma'|^2 + |\alpha\gamma"|^2)\} < \gamma,\ \bar{\alpha}\gamma' + \bar{\beta}\gamma" > \exp(\mathrm{Re}(\bar{\alpha}\beta\gamma'\bar{\gamma}")), \end{aligned}$$

which implies

$$\begin{aligned} V^*|\gamma' > \otimes|\gamma" > &= \exp\{-\frac{1}{2}(|\beta\gamma'|^2 + |\alpha\gamma"|^2)\} \\ &\quad \times \exp(\mathrm{Re}(\bar{\alpha}\beta\gamma'\gamma"))|\bar{\alpha}\gamma' + \bar{\beta}\gamma" > . \end{aligned}$$

Therefore q_e is

$$\begin{aligned} q_e &= \frac{1}{\pi}\int d^2w < U(z)\gamma,\ V^*(|0><0|\otimes I)Vw >< w,\ U(z)\gamma > \\ &= \frac{1}{\pi}\int d^2w \exp\{\frac{1}{2}(|\alpha|^2|\beta|^2|w|^2)\} \\ &\times < U(z)\gamma,\ |\beta|^2 w >< 0,\ \alpha w >< w,\ U(z)\gamma > \end{aligned}$$

This can be computed by the expression (A.27) given in Appendix and a Gaussian type integration:

$$\begin{aligned} < w,\ U(z)\gamma > &= < \exp(-\frac{i}{2}\varphi)w,\ U(r)\exp(-\frac{i}{2}\varphi)\gamma > \\ &= \exp\{-\frac{1}{2}(|w|^2 + |\gamma|^2)\}(\cosh r)^{-1/2} \\ &\times \exp\{\bar{w}\theta(\cosh r)^{-1} + \tanh r\{\frac{1}{2}(\exp(-i\varphi)\theta^2 - \exp(i\varphi)\bar{w}^2)\}\}, \end{aligned}$$

$$\frac{1}{\pi}\int d^2w \exp\{-|w|^2 + aw + b\bar{w} + cw^2 + d\bar{w}^2\} = \frac{1}{\sqrt{1-4cd}}\exp\{\frac{a^2d + ab + b^2c}{1-4cd}\}.$$

The result is

$$\begin{aligned} q_e =& \frac{(\cosh r)^{-1}\exp\left\{\frac{1}{2}(\gamma^2+\bar{\gamma}^2)(\tanh r) - |\gamma|^2\right\}}{\sqrt{1-(1-|\alpha|^2)^2(\tanh r)^2}} \\ &\times \exp\left(\frac{-\frac{1}{2}(1-|\alpha|^2)^2(\cosh r)^{-2}(\tanh r)(\gamma^2+\bar{\gamma}^2) + (1-|\alpha|^2)|\gamma|^2(\cosh r)^{-2}}{1-(1-|\alpha|^2)^2(\tanh r)^2}\right) \\ &= \frac{1}{\sqrt{(\cosh r)^2 - (1-|\alpha|^2)^2(\sinh r)^2}}\exp\left\{\left(\frac{1-|\alpha|^2}{(\cosh r)^2-(1-|\alpha|^2)^2(\sinh r)^2} - 1\right)|\gamma|^2\right. \\ &+ \left.\left(1 - \frac{(1-|\alpha|^2)^2}{(\cosh r)^2 - (1-|\alpha|^2)^2(\sinh r)^2}\right)\left(\frac{1}{2}(\gamma^2+\bar{\gamma}^2)(\tanh r)\right)\right\} \end{aligned}$$

which is same as the result obtained in [24]:

$$\begin{aligned} q_e = \sqrt{\tau}\exp\Big[&\{(1-\eta)\tau - 1\}|\gamma|^2 \\ &+ [1-(1-\eta)^2\tau]\{\frac{\bar{\mu}\gamma^2}{2\lambda} + \frac{\mu\bar{\gamma}^2}{2\bar{\lambda}}\}\Big] \end{aligned}$$

where $\eta = |\alpha|^2$, $\tau = \{|\lambda|^2 - (1-\eta)^2|\mu|^2\}^{-1}$ with $\lambda = \exp(i\phi)\cosh r$, $\mu = \sinh r$.

(III) Case of $\xi_1^{(1)}$ =squeezed state $U(p)|\gamma >< \gamma|U(p)^*$ and $\xi_0^{(2)}$ =squeezed state $U(q)|\sigma >< \sigma|U(q)^*$: By the similar way as the case (II),

$$\begin{aligned} q_e &= tr_{\mathcal{H}_2}(tr_{\mathcal{K}_2}VU(p)|\gamma >< \gamma|U(p)^*V^*)U(q)|\sigma >< \sigma|U(q)^* \\ &=< U(p)\gamma,\ V^*(|U(q)\sigma >< U(q)\sigma|\otimes I)VU(p)\gamma > \\ &= \frac{1}{\pi^2}d^2wd^2z < U(p)\gamma, w >< w, V^*(|U(q)\sigma >< U(q)\sigma|\otimes I)Vz >< z, U(p)\gamma > \\ &= \frac{1}{\pi^2}d^2wd^2z < U(p)\gamma, w >< \alpha w, U(q)\sigma >< \beta w, \beta z > \\ &\times < U(q)\sigma, \alpha z >< z, U(p)\gamma > \end{aligned}$$

Applying the above formula presented in the case II, we can compute this error probability q_e as

$$q_e = \frac{1}{\pi^2} < U(p)\gamma, w >< \alpha w, U(q)\sigma >< \beta w, \beta z >$$

Let $\mathbf{C}$ denote the set of all complex numbers. A **Fock representation of the Canonical Commutation Relations (CCR)** over $\mathbf{C}$ is a triple

$$\{\mathcal{H}, W, \Phi\}$$

where $\mathcal{H}$ is a Hilbert space and $W : z \in \mathbf{C} \mapsto W(z) \in U_n(\mathcal{H})$ is a map from $\mathbf{C}$ to the unitary operators on $\mathcal{H}$ such that $W(0) = \text{id}$ and

$$W(u)W(v) = \exp\{i\text{Im}\bar{u}v\}W(u+v);\ u, v \in \mathbf{C} \tag{A.1}$$

and $\Phi \in \mathcal{H}$ is a unit vector, called the Fock vacuum, satisfying

$$< \Phi, W(z)\Phi >= \exp\{-12|z|^2\};\ z \in \mathbf{C} \tag{A.2}$$

It is moreover assumed that the weak closure of the complex vector space generated by the $\{W(z) : z \in \mathbf{C}\}$ coincides with the algebra of all bounded operators on $\mathcal{H}$. This property is called **irreducibility**. Clearly any two Fock representations are canonically isomorphic. The Stone-von Neumann theorem asserts that if $\{W(z) : z \in \mathbf{C}\}$ is any irreducible family of unitary operators on a Hilbert space $\mathcal{H}$ satisfying (A.1), then it is isomorphic to the Fock representation. In particular, for any such a family, there will exist a (necessarily unique) vector Φ satisfying (A.2), i.e., a Fock vacuum for this family. A corollary of the Stone-von Neumann theorem is the following: let $T : \mathbf{C} \to \mathbf{C}$ be any real linear transformation such that

$$\text{Im}(Tu)^{-}(Tv) = \text{Im}\bar{u}v;\ \forall u, v \in \mathbf{C} \tag{A.3}$$

where "-" denotes the complex conjugate, and define

$$W_T(z) = W(Tz);\ z \in \mathbf{C} \tag{A.4}$$

Then the set $\{W_T(z) : z \in \mathbf{C}\}$ (because any T satisfying (A.3) must be invertible), hence it is irreducible. Moreover, because of (A.3), it satisfies (A.1). Hence by the Stone-von Neumann theorem, there exists a vector $\Phi_T \in \mathcal{H}$ and a unitary operator $U_T : \mathcal{H} \to \mathcal{H}$, characterized by the property:

$$U_T W(z)\Phi = W_T(z)\Phi_T;\ z \in \mathbf{C} \tag{A.5}$$

The vector $\Phi_T = U_T\Phi$ (i.e., the vacuum for the W_T representation) is called a **squeezed vector** for the W-representation. The most general operator T, satisfying (A.3), is given by the following

Proposition A.1: Let $V : \mathbf{C} \to \mathbf{R}^2$ be the isomorphism of real linear spaces characterized by

$$V(1) = (\,1\,)$$

Then a real 2×2 matrix T induces on $\mathbf{C}$ a transformation satisfying (A.3) if and only if $\det T = 1$.

Proof: A direct calculation: The identity (A.2) implies that for each $z \in \mathbf{C}$ the one parameter unitary group $\{W(tz)\}(t \in \mathbf{R})$ is strongly continuous, hence

$$W(tz) = \exp\{itB(z)\} \tag{A.7}$$

for some self-adjoint operator $B(z)$. Moreover the map $z \in \mathbf{C} \mapsto B(z)$ is real linear. The operators

$$12B(1) = q; \;\; 12B(i) = p \tag{A.8}$$

are called momentum and position operators, respectively. The condition (A.1) implies that

$$[B(u), B(v)] = 2i\mathrm{Im}\bar{u}v \tag{A.9}$$

so that, in particular

$$[q, p] = i2.$$

Finally, denoting

$$a = p - iq; \;\; a^* = p + iq \tag{A.10}$$

one has

$$[a, a^*] = 1$$

$$iB(z) = za^* - \bar{z}a; \;\; z \in \mathbf{C} \tag{A.11}$$

The vectors

$$|\theta >= W(\theta)\Phi; \;\; \theta \in \mathbf{C} \tag{A.12}$$

are called **coherent vectors**. Now let $T : \mathbf{C} \to \mathbf{C}$ be a real linear map satisfying (A.3) and let W_T, U_T, Φ_T be characterized respectively by (A.4) and (A.5). Then one has, for $z \in \mathbf{C}$:

$$W_T(z) = \exp(iB_T(z)) = \exp(za_T^* - \bar{z}a_T). \tag{A.13}$$

On the other hand, by the definition (A.4) of W_T, one also has

$$W_T(z) = W(Tz) = \exp((Tz)a^* - (Tz)^{-}a)A.14$$

$$(\, a \,)\, b$$

then for each $z \in \mathbf{C}$

$$(\, T \,)\, z$$

with

$$\bar{c} = 12([a + d] + i[c - b]); \;\; -s = 12([a - d] + i[c + b]) \tag{A.17}$$

defin

Remark: Notice that any c, s given by (A.17) satisfy

$$|c|^2 - |s|^2 = \det (a) b$$

Conversely, given $c, s \in \mathbf{C}$ such that $|c|^2 - |s|^2 = 1$, we can define a, b, c, d by (A.17) and the resulting matrix is in SL(2; $\mathbf{R}$).

Proof: Denote $\mathbf{C}^2(\mathbf{R})$ the real vector space

$$\mathbf{C}^2(\mathbf{R}) = \{\lambda (z)$$

and $V_0 : \mathbf{R}^2 \to \mathbf{C}^2(\mathbf{R})$ the isomorphism of real vector spaces characterized by

$$V_0 (1)$$

Then, if $V : \mathbf{C} \to \mathbf{R}^2$ is the isomorphism of Proposition A.1 and $z = x + iy$, one has

$$V_0 V T z = V_0 (a) b$$

Expressing x, y in terms of $z, \bar{z}$, one finds (A.16), (A.17).

Putting together Proposition (A.2) and the identity (A.14), we obtain

$$W_T(z) = \exp\{(a^*, -a) (\bar{c}) - s$$

Comparing (A.19) with (A.13) we finally find

$$a_T = ac + a^* s \qquad (A.20)$$

or equivalently

$$(a)_T$$

But from (A.5),(A.7) and (A.11), it follows that the operator U_T is characterized by the property:

$$U_T a_T U_T^* = a$$

or, in view of (A.20), by

$$U_T^* a U_T = ca + sa^* \qquad (A.22)$$

Our goal is to find the operator U_T satisfying (A.22) for given c and s satisfying (A.18). To this goal first notice that, in view of (A.18), there exist real numbers r, η, φ such that

$$c = \exp\{i\eta\} \cosh r = \exp\{i\eta\} c_r ; \;\; s = \exp\{-i\varphi\} \sinh r = \exp\{-i\varphi\} s_r \qquad (A.23)$$

Moreover, due to the identities

$$\exp\{xa^*a\}a\exp\{-xa^*a\} = \exp\{-x\}a;$$

by replacing the representation $W(z)$ with the equivalent representation $W(\exp\{i\eta\}z)$, we can always suppose that c, in (A.22), is real, i.e., $\eta = 0$ in (A.23).

Proposition A.3: Let c, s be given by (A.23) with $\eta = 0; r > 0$. Then the operator U_T, characterize by (A.22) is given by

$$U_T = \exp\{12(za^2 - \bar{z}a^{*2})\} \equiv U(z)A.25$$

Proof: Denote $D_z = 12(za^2 - \bar{z}a^{*2})$ and define

$$f(t) = \exp\{tD_z\}a\exp\{-tD_z\}$$

Then, due to the easily verified commutation relations:

$$[D_z, a] = \bar{z}a^*; \ [D_z, a^*] = za$$

one deduces the equation

$$ddt\,(\,f\,)\,(t)$$

with initial condition

$$(\,f\,)\,(0)$$

whence

$$(\,f\,)\,(t)$$

For $t = 1$, using (A.23) and the assumption $\eta = 0$, one finds

$$\exp\{D_z\}a\exp\{-D_z\} = ca + sa^*$$

so that $U_T = U(z) \equiv \exp\{D_z\}$.

Remark: Let $z = r\exp\{i\varphi\}$ and denote V_t the 1-parameter unitary group $V_t = \exp\{ita^*a\}$. Then one easily checks, using (A.24), that

$$\exp\{D_z\} = V_{-\varphi 2}\exp\{D_r\}V_{\varphi 2} \qquad (A.27)$$

So we can reduce ourselves to study the operator D_z in the case of real z. In several applications it is useful to know the matrix elements of the operator $\exp\{D_r\} = U(r)$ with respect to the coherent states in the W-representation.

Proposition A.4: In the notation (A.12),(A.23),(A.25) one has

$$< \alpha, U(r)\beta > = < 0, U(r)\beta > \exp\{\bar{\alpha}\beta c_r - \bar{\alpha}^2 s_r 2c_r\} A.28$$

Proof: Denote $f(\bar{\alpha}) = < \alpha, U(r)\beta >$. Then

$$dd\bar{\alpha} f(\bar{\alpha}) = \beta c_r (1 - \bar{\alpha} c_r \beta) f(\bar{\alpha}); \ f(0) = < 0, U(r)\beta >$$

Solving this equation, we find (A.27). Now put $f(r) = < 0, U(r)\beta >$. Then $f(0) = 1$ and

$$ddr f(r) = 12(\beta^2 c_r^2 - s_r c_r) f(r)$$

The solution of this equation is

$$f(r) = \exp \int_0^r 12(\beta^2 c_\tau^2 - s_\tau c_\tau) d\tau \qquad (A.30)$$

Keeping into account (A.23), the integral in (A.30) is easily evaluated and leads (A.29).

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