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COMPOUND SIMPLE GAMES I: SOLUTIONS OF SUMS AND PRODUCTS

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PREFACE

The mathematical research presented in this Memorandum is concerned with solutions for certain types of games of strategy. The theory of games is of great importance because of its applicability to a variety of conflict situations — economic, political, and military.

SUMMARY

This is an investigation of the solutions of the games that are formed by combining two or more simple games, played by separate groups of individuals. A simple game is one that is completely specified by its winning coalitions. Two forms of combination are studied in detail in this Memorandum: the sum, in which a coalition wins if it wins in either component, and the product, in which a coalition wins only if it wins in both components. In both cases, relationships are established between the solutions of the compound games and those of the components.

CONTENTS

COMPOUND SIMPLE GAMES I: SOLUTIONS OF SUMS AND PRODUCTS

1. INTRODUCTION AND DEFINITIONS

An n-person game is called "simple" provided every coalition of players either can win outright or is completely defeated. A simple game can therefore be described by listing its winning coalitions. In this paper we consider the "compound" simple games that are obtained by combining two or more simple games (with different players) into a single, larger game. Two methods of combination will be treated: the product, in which a winning coalition must contain winning contingents from each component game, and the sum, in which a winning coalition need have winning strength in just one component game. Our object, in both cases, will be to establish relationships between the stable-set solutions of the compound games and those of their components.

A verbal statement of our first two theorems will help convey the nature of the results we obtain. We find that in a compound game (sum or product) there is a solution that distributes the proceeds within each component according to a pattern that would be stable (i.e., a solution) in that component alone. The amounts must be scaled down, however, to allow for what the players in the other component(s) receive. In the case of a sum, the allocation among the different components is arbitrary, but constant. In the case of a product, all possible intergroup allocations occur in the same solution.

These theorems, then, give us an easy way to construct solutions to compound games when we know solutions to the component games. It may be noted that the sums, which have many more winning coalitions than the corresponding products, have smaller (i.e., more determinate) solutions.

The later sections of this paper contain extensions and converses of these basic results, together with the solutions of some illustrative games. In a subsequent paper we shall take up a more general class of composition rules for simple games in which the components play the role of players in a supergame, and obtain somewhat similar results.

An elementary introduction to the theory and applications of simple games will be found in [4] .

Simple games. We shall denote a simple game by the symbol $\Gamma(P, \mathcal{W})$, where P is a finite set (the players) and $\mathcal{W}$ is a collection of subsets of P (the winning coalitions). We shall require of $\mathcal{W}$ that it include P and exclude O (the empty set), and that it be monotonic in the following sense:

$$(1) \qquad S \in \mathcal{W},\ T \supseteq S \rightarrow T \in \mathcal{W}.$$

In view of (1), the collection of minimal winning coalitions, denoted by $\mathcal{W}^m$, is sufficient to specify the game. If $\mathcal{W}$ has the further property that

(2) $$S \in \mathcal{W} \rightarrow P-S \notin \mathcal{W},$$

which implies (with (1)) that every two winning coalitions intersect, then the game is said to be proper. If (2) fails for some S, then the game is termed improper. The characteristic function of an improper game in the classical theory is not superadditive.

If $\mathcal{W}$ enjoys the converse property:

(3) $$S \notin \mathcal{W} \rightarrow P-S \in \mathcal{W},$$

which implies that there are no "blocking" coalitions, then the game is said to be strong. The proper strong games are "constant-sum" in the classical theory; they comprise the class of simple games that was originally defined and investigated by von Neumann and Morgenstern (see [5], Chapter X).

If we count the winning coalitions, we find that in a strong game

$$|\mathcal{W}| \geq 2^{|P|-1},$$

while in a proper game

$$|\mathcal{W}| \leq 2^{|P|-1}.$$

The fewer the winning coalitions in a game, the "weaker" it is said to be.

Sums and Products. A pair of simple games with different players can be combined into a larger game in two natural ways, related to the Boolean operations of addition and multiplication.

Let $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = O$. Then the sum

$$\Gamma(P_1, \mathcal{W}_1) \oplus \Gamma(P_2, \mathcal{W}_2)$$

is defined as the game $\Gamma(P, \mathcal{W})$ where $\mathcal{W}$ consists of all $S \subseteq P$ such that either $S \cap P_1 \in \mathcal{W}_1$ or $S \cap P_2 \in \mathcal{W}_2$. In terms of minimal winning coalitions we have

$$\mathcal{W}^m = \mathcal{W}_1^m \cup \mathcal{W}_2^m;$$

hence $|\mathcal{W}^m| = |\mathcal{W}_1^m| + |\mathcal{W}_2^m|$.

Similarly, the product

$$\Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$$

is defined as the game $\Gamma(P, \mathcal{W})$, where $\mathcal{W}$ consists of all $S \subseteq P$ such that both $S \cap P_1 \in \mathcal{W}_1$ and $S \cap P_2 \in \mathcal{W}_2$. In terms of minimal winning coalitions we have

$$\mathcal{W}^m = \{S \mid S \cap P_1 \in \mathcal{W}_1^m, \ \ S \cap P_2 \in \mathcal{W}_2^m\};$$

hence $|\mathcal{W}^m| = |\mathcal{W}_1^m| \cdot |\mathcal{W}_2^m|$.

We see that products are always weaker than the corresponding sums. In general sums are improper and products are nonstrong.[1] Addition and multiplication are commutative and associative, and yield unique decompositions of compound games. A further discussion of these operations will be found in [4].

We may remark that games in which some player v has a veto, i.e.,

$$S \in \mathcal{W} \rightarrow v \in S, \ \text{ but } \{v\} \notin \mathcal{W},$$

have as a factor the one-person game $\Gamma(\{v\}, \ \{\{v\}\})$, which we denote by B_1^v. Except for the trivial "dictator" games, these are the only simple games having outcomes that cannot be blocked by any coalition.[2]

Imputations. Let A_P denote the simplex of real nonnegative vectors x such that $\Sigma_P \, x_i = 1$. These vectors are traditionally called "imputations"

[1]This fact may explain why these operations do not appear in von Neumann and Morgenstern's account [5], which is limited to the constant-sum (i.e., strong and proper) case. (See below, p. 14.)

[2]Such games, in the general theory, are called weak, and the set of unblockable outcomes is called the core. A core outcome in a simple game is always the victory of a dictator or veto player, or a victory shared among several veto players.

when P is the set of players in a simple game. By convention, we shall say that $x_j = 0$ for any and all $j \notin P$. This is tantamount to having A_P embedded in a simplex A_Q, where Q is some large, unspecified set including all the players that might ever come under consideration.

Let us write $x(S)$ for $\Sigma_S x_i$. Let $R_S x$ be the "restriction" of x to S, thus:

$$(R_S x)_i = \begin{cases} x_i & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

The <u>barycentric projection</u> of x on A_S is then given by

$$B_S x = \frac{1}{x(S)} R_S x.$$

This is not well defined if $x(S) = 0$; otherwise it is an imputation in A_S.

Let S and T be fixed, disjoint sets, and let X and Y be subsets of A_S and A_T, respectively. Given $0 \leq \alpha \leq 1$, we define $X \underset{\alpha}{\times} Y$ to be the set of all imputations z such that

$$z = \alpha x + (1 - \alpha) y \quad \text{for some} \quad x \in X, \quad y \in Y.$$

Thus we have $X \underset{1}{\times} Y = X$ and $X \underset{0}{\times} Y = Y$. For intermediate α values $X \underset{\alpha}{\times} Y$ is a scaled-down cartesian product of X and Y. (In Fig. 1, the shaded rectangle represents the set $X \underset{1/3}{\times} Y$.)

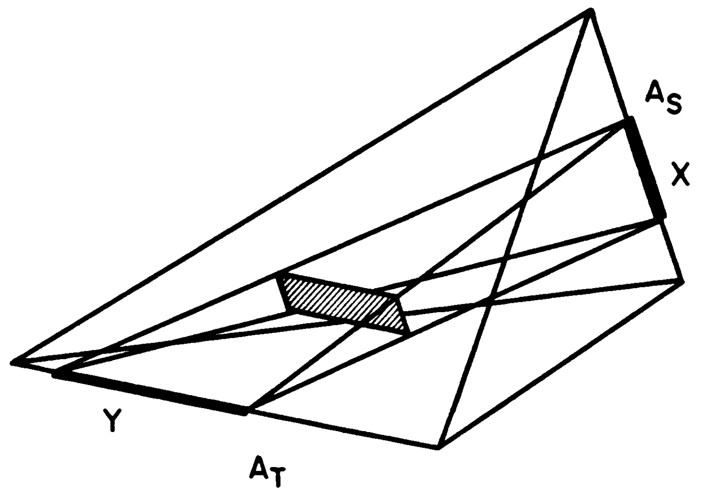

Fig. 1— Illustrating the operations $\underset{\alpha}{\Join}$ and $\Join$

Finally, we define an operation $\Join$ by

$$X \Join Y = \bigcup_{0 \leq \alpha \leq 1} X \underset{\alpha}{\Join} Y.$$

This set consists of all imputations z such that $B_S z$ and $B_T z$ are in X and Y respectively, together with the sets X and Y themselves. It is sometimes called the "join" of X and Y, being the union of all line segments running between X and Y. (If X and Y are convex, as in the figure, then $X \Join Y$ is the convex hull of $X \cup Y$.)

2. COMPOSITION OF SOLUTIONS: TWO BASIC THEOREMS

In this section we shall state and prove two theorems on the composition of solutions — one dealing with sums, the other with products.

We recall that a solution of the game $\Gamma(P, \mathcal{W})$ is a set X of imputations such that

$$X = A_P - \text{dom}\, X.$$

Here "dom X" denotes the set of all y such that, for some $x \in X$, the set $\{i \mid x_i > y_i\}$ is an element of $\mathcal{W}$. The notations "$\text{dom}_1 X$," "$\text{dom}_2 X$" will be used for domination with respect to special classes $\mathcal{W}_1$, $\mathcal{W}_2$.

Solutions are known to exist for all simple games. For example, A_S is a solution whenever $S = \mathcal{W}^m$.

The basic theorems. In the following it is always assumed that the sets P_1, P_2 have no members in common.

THEOREM 1. If X_i is a solution of $\Gamma(P_i, \mathcal{W}_i)$, $i = 1, 2$, then $X_1 \times X_2$ is a solution of $\Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$.

THEOREM 2. If X_i is a solution of $\Gamma(P_i, \mathcal{W}_i)$, $i = 1, 2$, and if $0 \leq \alpha \leq 1$, then $X_1 \underset{\alpha}{\times} X_2$ is a solution of $\Gamma(P_1, \mathcal{W}_1) \oplus \Gamma(P_2, \mathcal{W}_2)$.

The reader may now wish to read again the paragraph in Sec. 1 that gives a verbal statement of the content of these theorems. As indicated by the wording there, both theorems generalize directly to sums or products of more than two components.

If we set $\alpha = 0$ or 1 in Theorem 2, we see that any solution of a component also solves the sum. The existence of such solutions, which completely discriminate against the rest of the players, follows from the fact that any component of a sum is a "winning fraction" of the whole game, in the sense of [3] .

Note also that the solutions obtained by choosing different values of α in Theorem 2 are all proper subsets of the solution given in Theorem 1, based on the same subsolutions X_1, X_2. This is in accord with the intuitive idea that a weaker game should have a less determinate outcome.

<u>Proof of Theorem 1</u>. Let $\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$ and let $X = X_1 \times X_2$. (A) External stability. The first part of the proof is devoted to showing that $X \cup \text{dom } X = A_P$. Let y be an arbitrary point in A_P, and let $\beta_1 = y(P_1)$ and $\beta_2 = y(P_2) = 1 - \beta_1$. Let y_i denote[3] the barycentric projection of y on A_{P_i}, $i = 1, 2$, if this is well defined, i.e., if $\beta_i > 0$. There are a number of cases; our object each time will be to show that either $y \in X$ or $y \in \text{dom } X$.

[3]Henceforth we shall avoid direct reference to the components of vectors, and use subscripted symbols x_i, y_i, etc., to denote vectors also. Usually (but not always) they will be related in some stated way to their unsubscripted counterparts.

<u>Case 1</u>: $0 < \beta_1 < 1$.

<u>Case 1a</u>: $y_1 \in X_1$, $y_2 \in X_2$. Here $y \in X$.

<u>Case 1b</u>: $y_1 \notin X_1$, $y_2 \notin X_2$. Find a $z_1 \in X_1$ and a $z_2 \in X_2$ such that $y_1 \in \mathrm{dom}_1 z_1$ and $y_2 \in \mathrm{dom}_2 z_2$. Define the imputation z by

$$z = \beta_1 z_1 + \beta_2 z_2.$$

Then it is clear that $y \in \mathrm{dom}\, z$. Moreover, $z \in X$. Hence $y \in \mathrm{dom}\, X$.

<u>Case 1c</u>: $y_1 \in X_1$, $y_2 \notin X_2$. Find a $z_2 \in X_2$ such that $y_2 \in \mathrm{dom}_2 z_2$. Denote by S_2 the set of players on which $z_2 - y_2$ is positive; we have $S_2 \in \mathcal{W}_2$. Choose $\epsilon > 0$ so small that $(\beta_2 - \epsilon) z_2 - \beta_2 y_2$ is also positive on S_2. Let u_1 be any interior point of A_{P_1}, that is, a vector that is positive on P_1 and zero on P_2, and has $u(P) = 1$. Then we assert that the vector x, defined by

$$x = \beta_1 y_1 + \varepsilon u_1 + (\beta_2 - \varepsilon) z_2,$$

is an imputation, since it is a weighted average of three imputations, and that it dominates y, since $x - y$ is positive on the winning set $P_1 \cup S_2$. There are now two possibilities. If it happens that $x \in X$ then we are finished with this case, since we have $y \in \mathrm{dom}\, X$. Suppose then that $x \notin X$. Then it must be the projection on A_{P_1} that fails to be in X_1, since the other projection is in X_2 by construction. Hence we can find a $z_1 \in X_1$ such that $B_{P_1} x \in \mathrm{dom}_1 z_1$. Let $S_1 \in \mathcal{W}_1$ be the set of players on which $z_1 - B_{P_1} x$ is positive. Then the imputation z, defined by

$$z = (\beta_1 + \epsilon) z_1 + (\beta_2 - \epsilon) z_2,$$

dominates y, since $z - y$ is positive on the winning set $S_1 \cup S_2$. Moreover, $z \in X$. Hence again $y \in \mathrm{dom}\, X$.

<u>Case 1d:</u> $y_1 \notin X_1$, $y_2 \in X_2$. Like Case 1c.

<u>Case 2:</u> $\beta_1 = 0$.

<u>Case 2a:</u> $y_2 \in X_2$. Then $y \in X$.

<u>Case 2b:</u> $y_2 \notin X_2$. The argument of Case 1c may be repeated here, with the understanding (since y_1 is no longer well defined) that $\beta_1 y_1$ is to be taken equal to zero wherever it appears.

<u>Case 3:</u> $\beta_1 = 1$. Like Case 2. This completes the proof of external stability.

(B) Internal stability. It remains to show that $X \cap \mathrm{dom}\, X = O$. Suppose the contrary. Then imputations $x, y \in X$ exist with $x - y$ positive on some winning coalition S. Without loss of generality we may assume that $y(P_1) \geq x(P_1)$. But $x(P_1) \geq x(S \cap P_1) > y(S \cap P_1) \geq 0$. Hence the barycentric projections of both x and y on A_{P_1} are well defined; denote them by x_1 and y_1, respectively. As a result of our assumed inequality, the difference $x_1 - y_1$ is positive on at least the set

$S \cap P_1$, which is winning in $\Gamma(P_1, \mathcal{W}_1)$. Hence $y_1 \in \mathrm{dom}_1 x_1$. But x_1 and y_1 are both elements of the "subsolution" X_1; we have therefore reached a contradiction. This completes the proof of Theorem 1.

<u>Proof of Theorem 2</u>. Let $\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \oplus \Gamma(P_2, \mathcal{W}_2)$ and let $X^\alpha = X_1 \underset{\alpha}{\times} X_2$. (A) External stability. Let y be an arbitrary point in A_P, with β_i and y_i, $i = 1, 2$, defined as in the previous proof. Without loss of generality we may assume that $\beta_1 \leq \alpha$. Let u_1 be an arbitrary interior point of A_{P_1}, and construct an imputation z:

$$z = \beta_1 y_1 + (\alpha - \beta_1) u_1 + (1 - \alpha) x_2,$$

where x_2 is an arbitrary element of the subsolution X_2, and the term $\beta_1 y_1$ is understood to be zero if y_1 is not well defined. There are three cases to be distinguished at this juncture.

<u>Case 1</u>: $z \notin X^\alpha$. Then $\alpha > 0$, and the barycentric projection of z on A_{P_1} is well-defined; call it z_1. Then we have

$$z_1 = \frac{\beta_1}{\alpha} y_1 + \frac{\alpha - \beta_1}{\alpha} u_1 .$$

If $z_1 \in X_1$, then we would have $z \in X^\alpha$, contrary to assumption; hence $z_1 \notin X_1$. Hence $z_1 \in \mathrm{dom}_1 X_1$, and there exists $x_1 \in X_1$ such that $x_1 - z_1$ is positive over some $S_1 \in \mathcal{W}_1$. Therefore, multiplying by α, we see that $\alpha x_1 - \beta_1 y_1$ is also positive over S_1. But S_1 is also winning in the full game; hence the imputation x, defined by

$$x = \alpha x_1 + (1 - \alpha)x_2,$$

dominates y. Since $x \in X^\alpha$, we have $y \in \text{dom } X^\alpha$.

Case 2: $z \in X^\alpha$, $\beta_1 < \alpha$. Then $z - y$ is positive on the winning coalition $P_1 \in \mathcal{W}$, and we have $y \in \text{dom } X^\alpha$.

Case 3: $z \in X^\alpha$, $\beta_1 = \alpha$. Then $y_1 \in X_1$ (or is undefined). If $y_2 \in X_2$ (or is undefined) as well, then y is an element of X^α, and we are through. On the other hand, if $y_2 \notin X_2$ we can reverse the roles of P_1 and P_2 from the beginning, and the new z that will be constructed will not belong to X^α — i.e., Case 1 will apply. This completes the proof of (A).

(B) Internal stability. Suppose $x, y \in X^\alpha$, with $x - y$ positive on some winning coalition $S \in \mathcal{W}$. Then either $S \cap P_1 \in \mathcal{W}_1$ or $S \cap P_2 \in \mathcal{W}_2$; without loss of generality, assume the former. Then $\alpha = x(P_1) > 0$, so that the barycentric projections x_1 and y_1 of x and y on A_{P_1} are well defined. Obviously $x_1 - y_1$ is positive on $S \cap P_1$; hence $y_1 \in \text{dom}_1 x_1$. This contradicts the internal stability of the subsolution X_1, and completes the proof of Theorem 2.

3. THE SOLUTIONS OF SUMS: A COMPLETE CHARACTERIZATION

The first two theorems gave sufficient conditions for a set of imputations to be a solution of a compound simple game. It is natural to inquire whether the conditions are necessary as well — whether every solution is of one of the types described. The answer, curiously enough, is "yes" for sums, "no" for products. We shall consider first the case of sums.

THEOREM 3. Let X be any solution of the sum

$$\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \oplus \Gamma(P_2, \mathcal{W}_2).$$

Then there exist solutions X_1 and X_2 of $\Gamma(P_1, \mathcal{W}_1)$ and $\Gamma(P_2, \mathcal{W}_2)$, respectively, and a real number $0 \leq \alpha \leq 1$, such that $X = X_1 \underset{\alpha}{\times} X_2$.

Connection with the decomposition theory of von Neumann and Morgenstern. Taken together, Theorems 2 and 3 completely characterize the solutions of sums. There is a striking resemblance here to the main result in [5], Chapter IX, concerning solutions of "decomposable" games. They, too, are the cartesian products of the solutions of the components, although a slight modification of the solution concept (the introduction of "excess") is required at the component level to make things fit. Corresponding to our parameter α, there is a "transfer" parameter in the von Neumann–Morgenstern result which can be chosen arbitrarily between certain limits, the limits depending on the possible values for the excess in the component games.

Because of this similarity it is worth emphasizing that the two theories are actually quite separate in application. The composition of two essential simple games (in the sense of [5]) is never a simple game, since some coalitions are only half winning. On the other hand, the sum of two simple games (in our sense) is not even a game in the classical theory, since it has disjoint winning coalitions.[4]

A theoretical link between the two operations, however, is provided by an equivalence relation due to Gillies,[5] whereby the solutions of any nonsuperadditive (i.e., improper) game can be shown to be the same as the solutions of its minimum superadditive majorant, with a certain negative excess imposed. As it happens, the minimum superadditive majorants of our sums turn out to be the von Neumann–Morgenstern compositions of the same components.[6] Thus, an alternative proof of our Theorems 2 and 3 might be built around the known results in [1] and [5]. We have chosen to give direct proofs for several reasons, one being the fact that the theory of decomposable games in [5] was presented only for the constant-sum case.

Proof of Theorem 3. We shall first show that X is a cartesian product — i.e., a set of the form $X_1 \underset{\alpha}{\times} X_2$ for some $X_1 \subseteq A_{P_1}$, $X_2 \subseteq A_{P_2}$, and $0 \leq \alpha \leq 1$. Then we shall show that X_1 and X_2 are solutions of their respective subgames. Take any $x, y \in X$, and let

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[4]Note that although sums are improper, they are nevertheless useful in the analysis of proper games. For example, a game of the form $(G \oplus H) \otimes J$ may be proper. (See below, p. 20.)

[5]See [1], 66 ff.

[6]The excess that must be applied to G, the composition of $G_1, \ldots, G_m$, to make its solutions match those of $G_1 \oplus \ldots \oplus G_m$, may be taken to be any number e that satisfies $-|G|_1 < e \leq -|G|_1 + \min_i |G_i|_1$ (notation from [5], pages 368–9). This condition ensures that every nonflat coalition of G is effective everywhere in the imputation space.

x_i, y_i denote their barycentric projections on the face A_{P_i}, $i = 1, 2$, if well defined. Let u_1, u_2 be arbitrary interior points of A_{P_1}, A_{P_2}, respectively. Define $\alpha_i = x(P_i)$, $\beta_i = y(P_i)$, $i = 1, 2$, and assume without loss of generality that $\beta_1 \geq \alpha_1$. Then the imputation z, defined by[7]

$$z = \alpha_1 x_1 + \beta_2 y_2 + (\beta_1 - \alpha_1) u_2,$$

is necessarily in X, since any imputation dominating z also dominates either x or y. Similarly, the imputation z', defined by

$$z' = \alpha_1 x_1 + (\beta_1 - \alpha_1) u_1 + \beta_2 y_2,$$

is in X. But if $\beta_1 - \alpha_1$ were strictly positive, z and z' would dominate each other via the winning coalitions P_1 and P_2, in violation of the internal stability of X. Hence

$$\alpha_1 = \beta_1$$

(making z and z' the same), and we have

$$\alpha_1 x_1 + \alpha_2 y_2 \in X.$$

By symmetry,

$$\alpha_1 y_1 + \alpha_2 x_2 \in X.$$

[7] We follow our usual convention in regard to ill-defined barycentric projections.

Since x and y were arbitrary points in X, these facts show that X is of the required "cartesian" form $X_1 \underset{\alpha}{\times} X_2$, with of course $\alpha = \alpha_1$.

It remains to show that each X_i is, or can be chosen to be, a solution of its corresponding subgame $\Gamma(P_i, \mathcal{W}_i)$. If $\alpha_i = 0$ then X_i is irrelevant to X, and we can use any solution we please to satisfy the literal requirements of the theorem. On the other hand, if $\alpha_i > 0$ then X determines a unique X_i, and we must prove that the latter is a solution of $\Gamma(P_i, \mathcal{W}_i)$. As before, we do this in two parts. It will suffice to consider the case $i = 1$.

(A) External stability. Let x_1, x_2 be arbitrary elements of X_1, X_2, respectively, and take any $y_1 \in A_{P_1} - X_1$. Then the imputation y, defined by

$$y = \alpha_1 y_1 + \alpha_2 x_2,$$

is not in X, since we are assuming $\alpha_1 > 0$. Hence, some $z \in X$ dominates y. Let $S \in \mathcal{W}$ denote the set of players on which $z - y$ is positive. If $S \cap P_2$ were in $\mathcal{W}_2$, then z would dominate $\alpha_1 x_1 + \alpha_2 x_2 \in X$, violating internal stability. Therefore we must have $S \cap P_1 \in \mathcal{W}_1$. Let $z_1 = B_{P_1} z$. Then $z_1 - y_1$ is also positive on $S \cap P_1$. Hence $y_1 \in \mathrm{dom}_1 z_1$. But z_1 is clearly an element of X_1. Hence $y_1 \in \mathrm{dom}_1 X_1$.

(B) Internal stability. Suppose that $y_1 \in \mathrm{dom}_1 x_1$ for some $x_1, y_1 \in X_1$. Then we would also have $\alpha_1 y_1 + \alpha_2 x_2 \in \mathrm{dom}(\alpha_1 x_1 + \alpha_2 x_2)$

for arbitrary $x_2 \in X_2$, since $\mathcal{W}_1 \subset \mathcal{W}$. This contradicts the internal stability of X. This completes the proof of the theorem.

4. THE SOLUTIONS OF PRODUCTS: FURTHER RESULTS

The resemblance between sums and products has up to now been so close that only one more step — the product counterpart to Theorem 3 — seems to be needed to "wrap up" the solution theory of these games. Unfortunately, our best efforts in this direction produce not the hoped-for result — which would be a full-fledged converse to Theorem 1 — but only the weaker result given in Theorem 4 below. This states that _if_ a solution of a product is of the form $X_1 \times X_2$, _then_ the sets X_1 and X_2 are solutions of their respective components. Though not uninteresting in itself this contributes very little to a complete characterization of the solutions of product games.

The source of the difficulty is not hard to find. Examination of a simple three-person game will reveal a large array of solutions not predicted by Theorem 1. In what proves to be only the first move in exploring the _terra incognita_ of product-game solution theory, we establish a generalization of Theorem 1 (in two slightly different forms), in which continuous variation of the subsolutions X_1 and X_2 is permitted as a function of the parameter α. These results (Theorems 5 and 6 below) dispose of the three-person example that motivated them, but the class of solutions they generate is still far from complete in general, as an example in Sec. 6 will make clear.

A partial converse to Theorem 1.

THEOREM 4. Let X be a solution of $\Gamma(P,\mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$. If sets $X_i \subseteq A_{P_i}$, $i = 1, 2$, exist such that $X = X_1 \times X_2$, then X_1 is a solution of $\Gamma(P_1, \mathcal{W}_1)$ and X_2 is a solution of $\Gamma(P_2, \mathcal{W}_2)$.

To prove this theorem it will suffice to show that X_1 is a solution of $\Gamma(P_1, \mathcal{W}_1)$. (A) External stability. Take $y_1 \in A_{P_1} - X_1$. Then $y_1 \notin X$. Hence $y_1 \in \mathrm{dom}\, z$ for some $z \in X$. Let $S \in \mathcal{W}$ be the set of players on which $z - y_1$ is positive. Then $S \cap P_1 \in \mathcal{W}_1$. This incidentally gives us $z(P_1) > 0$ so that the barycentric projection of z on A_{P_1} is well defined; denote it by z_1. Clearly $z_1 \in X_1$. Moreover, the set on which $z_1 - y_1$ is positive must include $S \cap P_1$, since $z_1 \geq z$ on P_1. Therefore we have $y_1 \in \mathrm{dom}_1 z_1 \subseteq \mathrm{dom}_1 X_1$, as required.

(B) Internal stability. Take $y_1 \in X_1$, and suppose that $y_1 \in \mathrm{dom}_1 x_1$ for some $x_1 \in X_1$. Let $S_1 \in \mathcal{W}_1$ denote the set of players on which $x_1 - y_1$ is positive. We assert (proof below) that we can find $x_2 \in X_2$ and $S_2 \in \mathcal{W}_2$ such that x_2 is strictly positive on S_2. The imputation x, defined by

$$x = (1 - \varepsilon)x_1 + \varepsilon x_2,$$

is then in X, and for small $\varepsilon > 0$ the difference $x - y_1$ is positive on both S_1 and S_2. Hence $y_1 \in \mathrm{dom}\, X$. Since y is an element of X, this violates internal stability of X, and refutes the original assumption that $y_1 \in \mathrm{dom}_1 X_1$.

The assertion in the third sentence of (B) must still be proved. Part (A) permits us to assume that X_2 is at least externally stable. In particular, $A_{P_2} \cap \mathrm{dom}_2 X_2$ is not empty unless $X_2 = A_{P_2}$. If the latter, we may take $S_2 = P_2$ and x_2 any interior point of A_{P_2}. Otherwise, let z_2 by any point in $A_{P_2} \cap \mathrm{dom}_2 X_2$, let x_2 be a point of X_2 that dominates z_2, and let S_2 be the set on which $x_2 - z_2$ is positive. In both instances we obtain $x_2 \in X_2$ and $S_2 \in \mathcal{W}_2$ such that x_2 is strictly positive on S_2, as required. This completes the proof of the theorem.

<u>A simple example</u>. In order to motivate the next definition, let us consider a particular three-person compound game "G":[8]

$$G = \Gamma(\overline{123}, \{\overline{13}, \overline{23}, \overline{123}\}).$$

This decomposes into a product, one term of which is a sum:

$$G = (B_1^1 \oplus B_1^2) \otimes B_1^3.$$

Here B_1^i denotes the one-person simple game $\Gamma(\{i\}, \{\{i\}\})$. The solutions of G are well known. They include all the straight lines in the triangle of imputations joining the vertex $A_{\overline{3}}$ to the opposite edge $A_{\overline{12}}$, i.e., the solutions predicted by Theorem 1. They also include all other arcs running monotonically from $A_{\overline{3}}$ to $A_{\overline{12}}$, curves along which the share

[8]For brevity, we write $\overline{123}$ for $\{1, 2, 3\}$, etc.

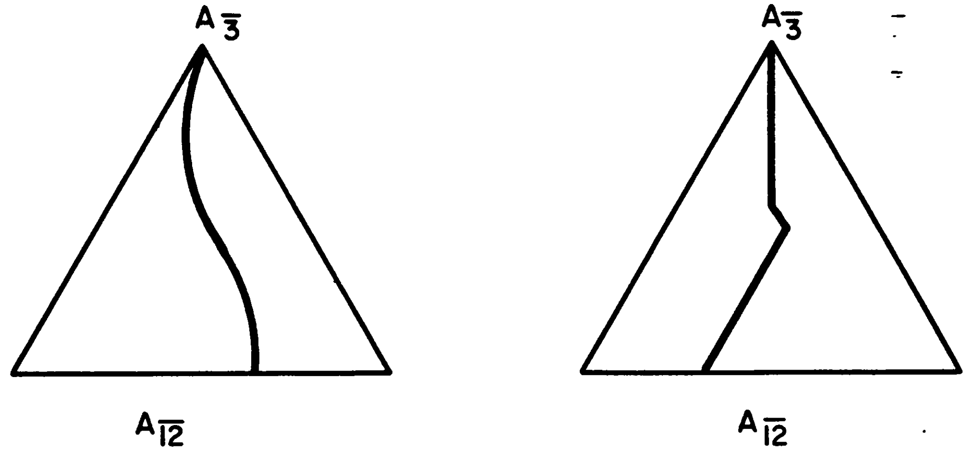

Fig. 2—Typical solutions of $(B_1^1 \oplus B_1^2) \otimes B_1^3$

of player 3 decreases steadily from 1 to 0 while the shares of players 1 and 2 increase or remain constant (see Fig. 2). There are no other solutions.

The existence of these curvilinear solutions shows that our method of combining subsolutions by means of the "join" operation has been needlessly rigid. It would seem that the relaxation indicated is to permit the sub—solutions to vary as a function of the parameter α that appears in the definition of "$\times$", but they must not be permitted to vary too rapidly. How best to control the variation when the subsolutions are sets rather than points (as above) is a rather delicate problem. The method we adopt first is not the only possible one, but it does lead to a substantial generalization of Theorem 1 and, in the process, enables us to account for all the solutions of the present example.

Semimonotonic families. A parametrized family of sets of imputations:

$$\{X(\alpha) \mid 0 \leq \alpha \leq 1\}$$

will be called semimonotonic if for every α, β, x such that $0 \leq \beta \leq \alpha \leq 1$ and $x \in X(\alpha)$, there exists a $y \in X(\beta)$ such that $\beta y \leq \alpha x$.

One example of a semimonotonic family is a constant family: $X(\alpha) \equiv X$. Another is a family that is monotonically decreasing in the sense of set-inclusion: $\alpha > \beta$ implies $X(\alpha) \subseteq X(\beta)$. A third example can be derived from an arbitrary solution of the game G just discussed, by taking $X(\alpha)$ to be the barycentric projection of the single point comprising the "α" cross-section of the solution.[9]

A generalization of Theorem 1.

THEOREM 5. Let $\{X_i(\alpha_i) \mid 0 \leq \alpha_i \leq 1\}$ be a semimonotonic family of solutions of the game $\Gamma(P_i, \mathcal{W}_i)$, for $i = 1, 2$. Then the set

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1-\alpha)$$

[9]Our terminology is less than ideal, since the vectors in each $X(\alpha)$ must be multiplied by α before anything monotonic in the ordinary sense emerges. The "semi" refers to the one-sideness of the hypothesis $\beta \leq \alpha$ (compare the definition of "monotonic" later in this section).

is a solution of the product game

$$\Gamma(P, \mathcal{W}) = \Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2).$$

The proof that follows is parallel in most respects to that of Theorem 1. (A) External stability. Take any $y \in A_P$ and define $\beta_i = y(P_i)$, $i = 1, 2$.

Case 1: $0 < \beta_1 < 1$.

Case 1a: $y_1 \in X_1(\beta_1)$, $y_2 \in X_2(\beta_2)$. Then $y \in X$.

Case 1b: $y_1 \notin X_1(\beta_1)$, $y_2 \notin X_2(\beta_2)$. Find $z_1 \in X_1(\beta_1)$ and $z_2 \in X_2(\beta_2)$ such that $y_1 \in \mathrm{dom}_1 z_1$ and $y_2 \in \mathrm{dom}_2 z_2$. Then the imputation z defined by

$$z = \beta_1 z_1 + \beta_2 z_2,$$

which is in X, clearly dominates y. Hence $y \in \mathrm{dom}\, X$.

Case 1c: $y_1 \in X_1(\beta_1)$, $y_2 \notin X_2(\beta_2)$. Find $z_2 \in X_2(\beta_2)$ such that $y_2 \in \mathrm{dom}_2 z_2$. Let $S_2 \in \mathcal{W}_2$ be the set of players on which $z_2 - y_2$ is positive. Choose $\varepsilon > 0$ so that $\beta_2 z_2 - \beta_2 y_2$ is greater than ε on S_2. Using the semimonotonic property of $\{X_2(\alpha)\}$, find z_2' in $X_2(\beta_2 - \varepsilon)$ such that $\beta_2 z_2 - (\beta_2 - \varepsilon) z_2'$ is nonnegative. This vector must be $\leq \varepsilon$ in all components, since no component of a nonnegative vector can exceed the sum of all components. It follows that we have

$$(\beta_2 - \varepsilon) z_2' > \beta_2 y_2 \text{ on } S_2.$$

Let u_1 be any interior point of A_{P_1}. Then the imputation x defined by

$$x = \beta_1 y_1 + \varepsilon u_1 + (\beta_2 - \varepsilon) z_2'$$

dominates y, because $x - y$ is positive on the winning set $P_1 \cup S_2$. If $x \in X$ then $y \in \text{dom}\, X$, and we are finished with this case. Suppose $x \notin X$. Then it must be the barycentric projection of x on A_{P_1}, namely

$$x_1 = \frac{\beta_1}{\beta_1 + \varepsilon} y_1 + \frac{\varepsilon}{\beta_1 + \varepsilon} u_1,$$

that fails to lie in the corresponding subsolution $X_1(\beta_1 + \varepsilon)$. Hence we can find $z_1 \in X_1(\beta_1 + \varepsilon)$ such that $x_1 \in \text{dom}_1 z_1$. Then it is clear that the imputation z, defined by

$$z = (\beta_1 + \varepsilon) z_1 + (\beta_2 - \varepsilon) z_2',$$

dominates y. Moreover $z \in X$. Hence $y \in \text{dom}\, X$.

<u>Case 1d</u>: $y_1 \notin X_1(\beta_1)$, $y_2 \in X_2(\beta_2)$. Like Case 1c.

<u>Case 2</u>: $\beta_1 = 0$.

<u>Case 2a</u>: $y_2 \in X_2(1)$. Then $y \in X$.

<u>Case 2b</u>: $y_2 \notin X_2(1)$. Then the argument of Case 1c may be repeated, with the usual understanding that $\beta_1 y_1$ is zero.

<u>Case 3</u>: $\beta_1 = 1$. Like Case 2. This completes part (A) of the proof.

(B) Internal stability. Suppose that for some $x, y \in X$, x dominates y — i.e., $x - y$ is positive on a winning set $S \in \mathcal{W}$. With no loss of generality we may assume that $y(P_1) \geq x(P_1)$. Define $\alpha_1 = x(P_1)$, $\beta_1 = y(P_1)$, $S_1 = S \cap P_1 \in \mathcal{W}_1$. We have

$$\beta_1 \geq \alpha_1 \geq x(S_1) > y(S_1) \geq 0;$$

hence, the projections $x_1 = B_{P_1} x$, $y_1 = B_{P_1} y$ are well defined. Moreover, $x_1 \in X_1(\alpha_1)$ and $y_1 \in X_1(\beta_1)$. Using the semimonotonicity of $\{X_1(\alpha)\}$, we can find $z_1 \in X_1(\alpha_1)$ such that $\alpha_1 z_1 \leq \beta_1 y_1$. But by our original assumption, $\alpha_1 x_1 - \beta_1 y_1$ is positive on S_1. Hence $z_1 \in \mathrm{dom}_1 x_1$, contrary to the internal stability of $X_1(\alpha_1)$. This contradiction establishes the internal stability of X, and completes the proof of the theorem.

Monotonic families and a variant of Theorem 5. The alert reader may have observed that the internal stability of the sets $X_i(1)$, $i = 1, 2$, was not used in the preceding proof. (Note that the α_1 in (B) is always less than 1, since α_2 is necessarily positive.) This observation by itself does not enlarge the scope of the theorem, since the internal stability of nearby sets $X_i(1 - \varepsilon)$ implies the internal stability of $X_i(1)$ as well, whether we want it or not. To obtain a real extension of the theorem in this direction, we must tighten some of the other hypotheses. Because of this adjustment, Theorem 6 will be independent of Theorem 5: each yields some solutions that the other misses.

A semimonotonic family $\{Y(\alpha) \mid 0 \leq \alpha \leq 1\}$ will be called <u>monotonic</u> if for every α, β, x such that $0 \leq \alpha \leq \beta \leq 1$ and $x \in Y(\alpha)$, there exists a $y \in Y(\beta)$ such that $\beta y \geq \alpha x$. (Compare the previous definition of "semimonotonic," p. 22.)

THEOREM 6. Let $\{Y_i(\alpha_i)\}$ be a monotonic family of solutions of $\Gamma(P_i, \mathcal{W}_i)$, except that $Y_i(1)$ need not be externally stable, and define

$$X_i(\alpha_i) = A_{P_i} - \mathrm{dom}_i Y_i(\alpha_i),$$

all for $i = 1, 2$. Then the set:

$$X = \bigcup_{0 \leq \alpha \leq 1} X_1(\alpha) \underset{\alpha}{\times} X_2(1-\alpha)$$

is a solution of $\Gamma(P_1, \mathcal{W}_1) \otimes \Gamma(P_2, \mathcal{W}_2)$.

Note that the sets X_i and Y_i are identical except when $\alpha_i = 1$. In the latter case, X_i is obtained from Y_i by adding the external points that Y_i fails to dominate, if any. Thus, $X_i(1)$ must be externally stable, but is perhaps not internally stable; moreover, it may contain points that are not monotonically related to the nearby sets $X_i(1 - \mathcal{E})$. (For an illustration, see Fig. 4c on p. 32.)

Theorem 6 is certainly not a major extension of the theory; it does give us, however, a first indication of the occurrence of nonsolutions in the cross sections of a solution of a product game. The proof, which we omit, entails only a slight modification of the one preceding, and should give no trouble to the aforementioned alert reader.

In the next section we shall produce specific examples that demonstrate the logical independence of Theorems 5 and 6. In one case we shall find a semimonotonic, nonmonotonic family of solutions to a certain game J. This means that games of the form $J \otimes K$ will have solutions that are given by Theorem 5 but not Theorem 6. In another case we shall find an application of Theorem 6 in which $X_i(1)$ and $Y_i(1)$ are different, and which therefore yields solutions not given by Theorem 5.

In conclusion, we point out that the word "monotonic" in Theorem 6 cannot be replaced by "semimonotonic," since this would permit the deletion of an arbitrary subset of $Y_i(1)$, and lead quickly to a counter-example. Whether the requirement of full monotonicity can be relaxed outside a neighborhood of $\alpha_i = 1$ is not known at present. If it can, then a nontrivial generalization including both theorems might result.

5. SOLUTIONS OF SOME SPECIFIC PRODUCT GAMES

The pursuit of the complexities arising in the solutions of product games is justified in part by the insights this activity provides into the still more difficult problems surrounding the solutions of general n-person games. Our results thus far in this paper (on product games) are primarily of a constructive nature. The examples to be given in this section illustrate some conspicuous limitations of our present techniques. They are intended partly to guide further work on product games and partly to stimulate efforts to find effective ways of using other general properties of the game structure to build up solutions.

Our first example is a four-person game "H":

$$H = \Gamma(P, \mathcal{W}) = \Gamma(\overline{1234}, \ \{\overline{134}, \ \overline{234}, \ \overline{1234}\}).$$

This game is a product of three components, namely $B_1^1 \oplus B_1^2$, B_1^3, and B_1^4. Denote the first by B_2^*. The factors can be associated in two essentially different ways, as follows:

$$\text{(a)} \quad H = B_2^* \otimes \left(B_1^3 \otimes B_1^4\right),$$

$$\text{(b)} \quad H = \left(B_2^* \otimes B_1^3\right) \otimes B_1^4 .$$

We shall discover that our methods give different sets of solutions to H, depending on which grouping is used.

In the first case (a) there is no difficulty in solving completely the two two-person factors. A solution of B_2^* is an arbitrary single point of $A_{\overline{12}}$, while the unique solution of $B_1^3 \otimes B_1^4$ is the whole set $A_{\overline{34}}$. Application of Theorem 5 (or 6) yields the class of product solutions exemplified in Fig. 3a. They are three-cornered surfaces spanning the tetrahedron $A_{\overline{1234}}$, made up of linear elements parallel to the edge $A_{\overline{34}}$. (The dashed lines in the figure outline a typical cross-section of

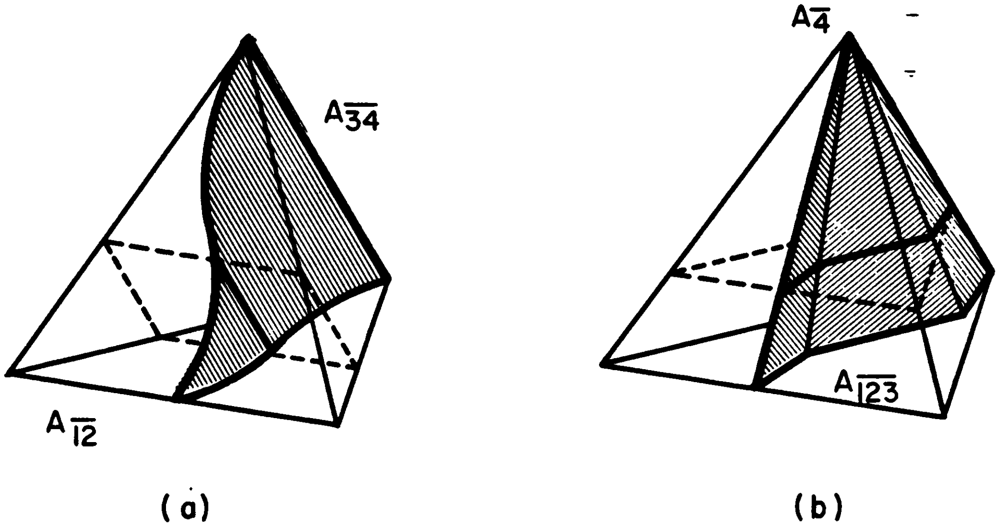

Fig. 3 — Typical solutions of $(B_1^1 \oplus B_1^2) \otimes B_1^3 \otimes B_1^4$

constant α.) The curvature of the surface is thus restricted to one dimension, and is further constrained by the monotonicity condition. Verbally, the latter requires that the payoffs to players 1 and 2 must both increase (or remain constant) if the combined payoff to players 3 and 4 goes down.

Now consider the grouping (b). It consists of the one-person game B_1^4 multiplied by a three-person game which we have already considered: the game "G" of Sec. 4 (see Fig. 2). Theorem 5 (or 6) again yields a class of three-cornered surfaces spanning the simplex $A_{\overline{1234}}$, but they are generated this time by arbitrary, monotonically-varying, monotonic curves in the cross sections parallel to the base $A_{\overline{123}}$. The curvature of the surface will in general be two-dimensional, but in Fig. 3b we have been content to depict the conical type of surface (vertex at $A_{\bar{4}}$) that results from an application of Theorem 1.

A comparison of the two classes of solutions reveals that "(b)" includes "(a)" but not vice versa. Class "(b)" in fact contains all the solutions of the game (we omit the proof). It is the failure of "(a)" to do the same that is the provocative feature of this example. The trouble seems to be that most of the solutions (including the one shown in Fig. 3b) simply do not decompose in accordance with the partition $\{\overline{12}, \overline{34}\}$. Instead, the distributions of proceeds inside the two subgames are interdependent. There seems no reason to believe that this phenomenon will not be found in other product games, where there may be no alternative arrangement of factors to save the day.

<u>A final example</u>. The four-person game that we shall call "J" is defined by

$$J = \Gamma(\overline{1234}, \{\overline{124}, \overline{134}, \overline{234}, \overline{1234}\}).$$

It differs from the preceding "H" only in the possession of one additional winning coalition. Since the game is weak (see p. 5), the "veto" player can be factored out, and we obtain the decomposition

$$J = M_3 \otimes B_1^4.$$

Here M_3 denotes the simple majority game on the set $\overline{123}$. The complete list of solutions to M_3 is well known;[10] in our present normalization it may be rendered as follows:

(a) The finite set $\left\{ \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right) \right\}$.

(b_δ) The line-segment $\{(\delta, t, 1-\delta-t) \mid 0 \leq t \leq 1-\delta\}$, for each $0 \leq \delta < \frac{1}{2}$.

(c_δ) The line-segment obtained by permuting players 1 and 2 in (b_δ), for each $0 \leq \delta < \frac{1}{2}$.

(d_δ) The line-segment obtained by permuting players 1 and 3 in (b_δ), for each $0 \leq \delta < \frac{1}{2}$.

No semimonotonic family drawn from this list can include representatives from more than one of the four categories (a) – (d). The only possible variation within such a family is in the value of δ, and semimonotonicity in fact implies monotonicity. Theorem 5 therefore generates only solutions of the types exemplified in Figs. 4a and 4b.

Theorem 6, for once, gives us something new. Setting $\delta = 1/2$ in (b), (c), or (d) above produces an internally stable set (not a solution of M_3) that is monotonically related to the solutions nearby, and is therefore

[10] See [5], page 282 ff.

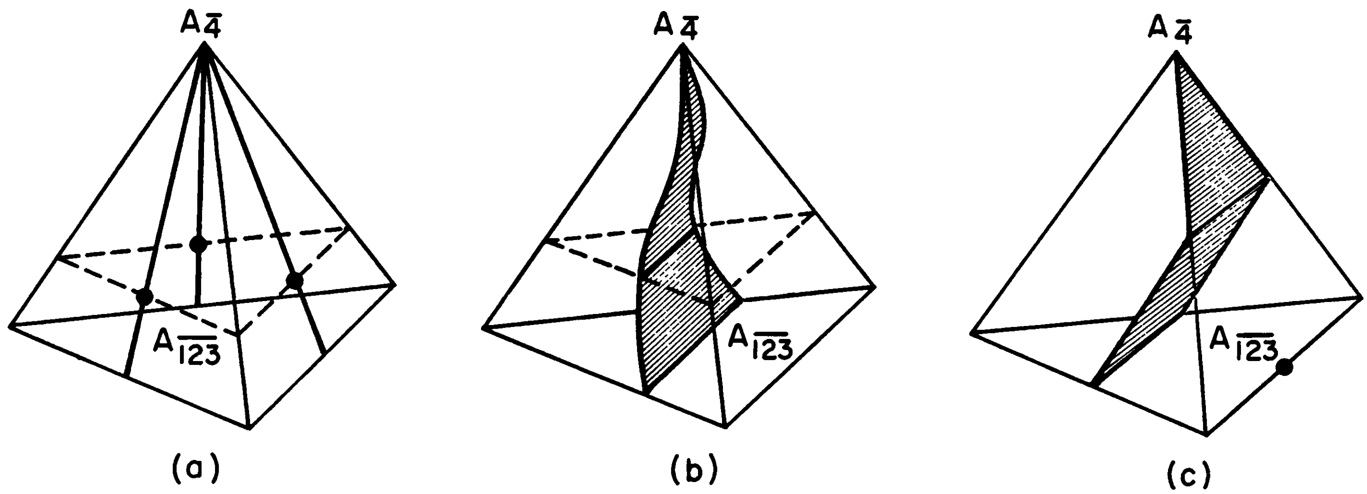

Fig. 4—Typical solutions of $M_3 \otimes B_1$

available as a "$Y_i(1)$" in the application of Theorem 6. Figure 4c illustrates one of the resulting solutions of J; its intersection with the base of the tetrahedron, we see, is internally <u>un</u>stable in the factor M_3. (The isolated point has coordinates (0, 1/2, 1/2, 0).) This is one of the examples promised at the end of Sec. 4.

Continuing, we observe that the solution of Fig. 4c, without the isolated point, can be represented as a limit of solutions like that of Fig. 4b. This fact enables us to construct a parametrized family of solutions of the four-person game J that is semimonotonic, but not monotonic. For example, we can make $X(\alpha)$ like Fig. 4c for $\alpha \leq 1/2$

and like Fig. 4b for $\alpha > 1/2$, keeping everything perfectly continuous and monotonic except for the abrupt disappearance of the isolated point at $\alpha = 1/2$. This family can then be used to produce solutions to games of the form $J \otimes K$ — solutions which Theorem 5 would predict, but not Theorem 6. This is the other example promised at the end of Sec. 4.

The game "J," innocent as it appears, actually has an enormous variety of solutions. It has been shown [2] that one may choose an entirely arbitrary closed subset of the line connecting $A_{\overline{4}}$ to the mid-point of $A_{\overline{23}}$ (or $A_{\overline{12}}$ or $A_{\overline{13}}$), and construct a solution containing this set and no other nearby points, except in the neighborhood of $A_{\overline{4}}$. Figure 5 illustrates the construction.[11]

Still other solutions exist to this game. For example, it is possible to twist a solution so as to rotate the "odd man's" role among players 1 – 3 at different levels. In particular, it is possible to make all of the categories (a) – (d) listed on page 31 appear as cross sections of the same solution. A limit to the complexity of this game may be in view, however, since the possible cross sections of its solutions appear to be restricted to the six types shown in Fig. 6, and their rotations. Three of these types represent nonsolutions of M_3: specifically "c" is internally unstable and "e" and "f" are externally unstable. All six types occur in the solution illustrated in Fig. 5: reading top to bottom the sequence is a, b, c, d, e, b, c, e, f, e.

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[11] The figure is reproduced from [2] . The arbitrary set here consists of the "dash" and two "dots" lying on the indicated median of $A_{\overline{234}}$.

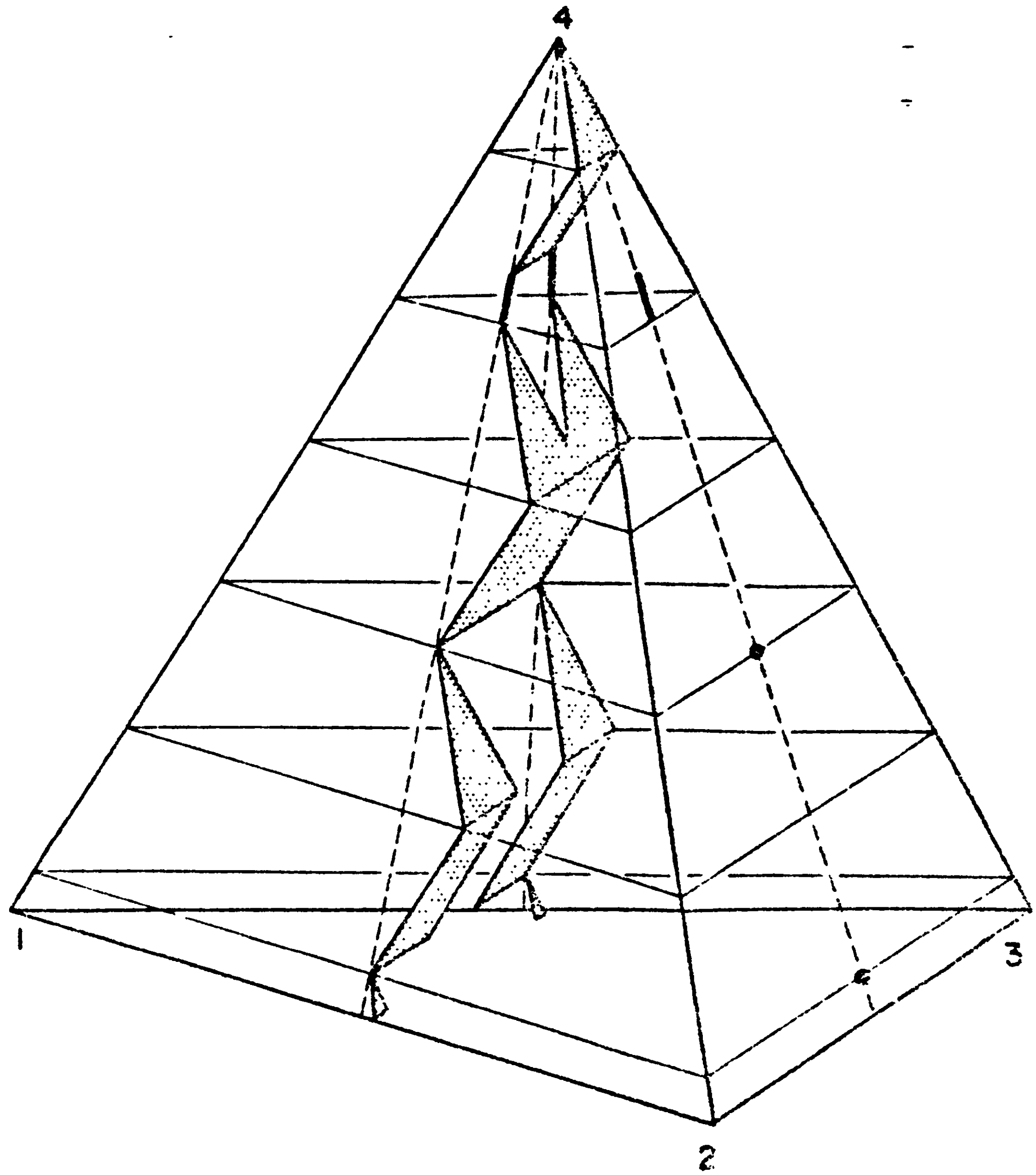

Fig. 5 — An irregular solution of $M_3 \otimes B_1$

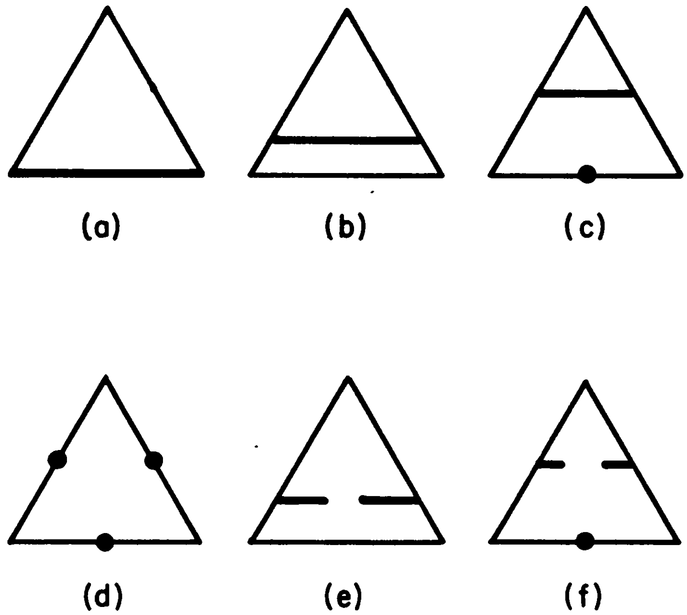

Fig. 6— Cross-section types for $M_3 \otimes B_1$

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