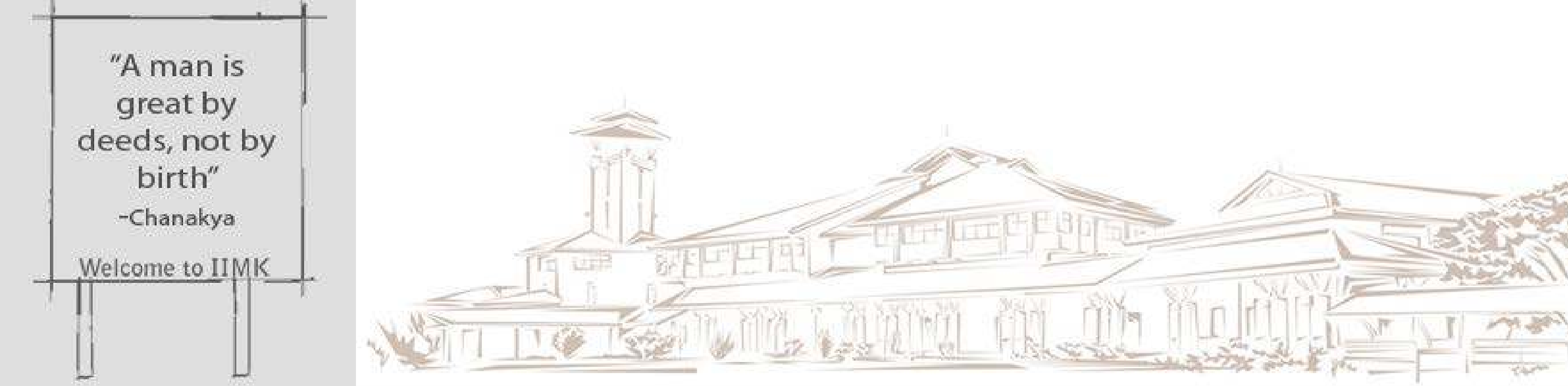

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Compounded Inverse Weibull Distributions: Properties, Inference and Applications

Jimut Bahan Chakrabarty[1]
Shovan Chowdhury[2]

[1] Research Scholar, Quantitative Methods and Operations Management Area, Indian Institute of Management Kozhikode, IIMK Campus P.O, Kerala – 673570, India, E-mail: jimutb08fpm@iimk.ac.in
[2] Associate Professor, Quantitative Methods and Operations Management Area, Indian Institute of Management Kozhikode, IIMK Campus P.O, Kerala – 673570, India, E-mail: shovanc@iimk.ac.in, Phone: +91- 495 - 2809119

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Compounded Inverse Weibull Distributions: Properties, Inference and Applications

Jimut Bahan Chakrabarty
Research Scholar of Quantitative Methods and Operations Management Area,
Indian Institute of Management; Kozhikode

Shovan Chowdhury
Associate Professor of Quantitative Methods and Operations Management Area,
Indian Institute of Management; Kozhikode

Abstract: In this paper two probability distributions are introduced compounding inverse Weibull distribution with Poisson and geometric distributions. The distributions can be used to model lifetime of series system where the lifetimes follow inverse Weibull distribution and the subgroup size being random follows either geometric or Poisson distribution. Some of the important statistical and reliability properties of each of the distributions are derived. The distributions are found to exhibit both monotone and non-monotone failure rates. The parameters of the distributions are estimated using the maximum likelihood method and the expectation-maximization algorithm. The potentials of the distributions are explored through three real life data sets and are compared with similar compounded distributions, viz. Weibull-geometric, Weibull-Poisson, exponential-geometric and exponential-Poisson distributions.

Keywords: Inverse Weibull distribution, Poisson distribution, Geometric distribution, Hazard function, Maximum likelihood estimation, EM algorithm.

1 Introduction

In reliability engineering research, inverse Weibull distribution is often used in statistical analysis of lifetime and response time data. Khan *et al* (2008) in their theoretical analysis of inverse Weibull distribution mention that numerous failure characteristics such as wear out periods and infant mortality can be modeled through inverse Weibull distribution. They mention about the wide range of areas in reliability analysis where inverse Weibull distribution model can be used successfully. Murthy *et al* (2004) mention that degradation phenomena of mechanical components such as dynamic components of diesel engine can be appropriately modeled using inverse Weibull distribution. Erto and Rapone (1984) show that inverse Weibull distribution provides a good fit for several data sets. Interpretation of inverse Weibull in the context of load strength relationship for a component was provided by Calabria and Pulcini (1994). Shafiei *et al* (2016) mention that inverse Weibull is an appropriate model for situations where hazard function is unimodal. They further mention the distribution as one of the popular distributions in complementary risk problems.

Recent literature suggests that several researchers have proposed compounding of useful lifetime distributions to model lifetime data. Adamidis and Loukas (1998) introduced a two-parameter distribution with decreasing failure rate by compounding exponential and geometric distributions. Another two parameter distribution with decreasing failure rate was introduced by Kus (2007) by compounding exponential and Poisson distributions. Tahmasbi and Rezaei (2008) also introduced a two parameter lifetime distribution with decreasing failure rate by compounding exponential and logarithmic distributions. Chahkandi and Ganjali (2009) mixed power-series and exponential distributions to arrive at a new two-parameter distribution family with decreasing failure rate. Later Barreto-Souza and Cribari-Neto (2009), Silva *et al* (2010), Barreto-Souza *et al* (2011), Hemmati *et al* (2011), Mahmoudi and Sepahdar (2013), Shafiei *et al* (2016) and Chowdhury *et al* (2016) have come up with similar studies with generalized exponential-Poisson, generalized exponential-geometric, Weibull-geometric, Weibull-Poisson, generalized Weibull-Poisson, inverse Weibull-power series and geometric-Poisson distributions respectively.

In this paper we have come up with two new three parameter distributions. The distributions are derived by compounding inverse Weibull with geometric and Poisson distributions respectively and they are applicable to minimum component lifetime of a system. Shafiei *et al* (2016) introduced the distributions pertaining to a parallel system, but to the best of our knowledge no papers so far have studied inverse Weibull geometric and inverse Weibull Poisson distributions applicable to a series system.

The rest of the paper is organized as follows. In Section 2 we introduce the distributions for series system. In Section 3 we derive and discuss the various properties pertaining to them including Rényi entropy. Parameters of the distributions are estimated by maximum likelihood method as well as expectation-maximization algorithm in section 4. We also derive the Fisher information matrix in the same section. In section 5 empirical illustrations using three real life data sets are done to explore the potentials of the distributions. In Section 6 we put down our conclusion.

2 Compounded inverse Weibull distributions

Let $X_{\text{i}}, i = 1, 2, \ldots, \text{n}$, be a random variable denoting failure times of the components of a system. Suppose X_{i} follows inverse Weibull distribution with scale parameter $\lambda > 0$ and shape parameter $\alpha > 0$. The corresponding density function is given by

$$f(x; \lambda, \alpha) = \alpha\lambda x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}}; x, \lambda, \alpha > 0. \tag{2.1}$$

The corresponding cumulative distribution function (CDF) is given by

$$F(x; \lambda, \alpha) = e^{-\lambda x^{-\alpha}}; x, \lambda, \alpha > 0. \tag{2.2}$$

If N be a random variable representing number of components, then we observe random variable Y (independent of N) representing minimum component lifetime. The random variable Y is defined by

$$Y = min(X_1, X_2, \ldots, X_N).$$

The conditional cumulative distribution function is given by

$$F_{Y|N=n}(y; \lambda, \alpha) = 1 - (1 - e^{-\lambda y^{-\alpha}})^n; y, \lambda, \alpha > 0. \tag{2.3}$$

Therefore the conditional density function can be derived as

$$f_{Y|N=n}(y; \lambda, \alpha) = n\alpha\lambda e^{-\lambda y^{-\alpha}} (1 - e^{-\lambda y^{-\alpha}})^{n-1} y^{-(\alpha+1)}; y, \lambda, \alpha > 0. \tag{2.4}$$

2.1 Inverse Weibull geometric distribution

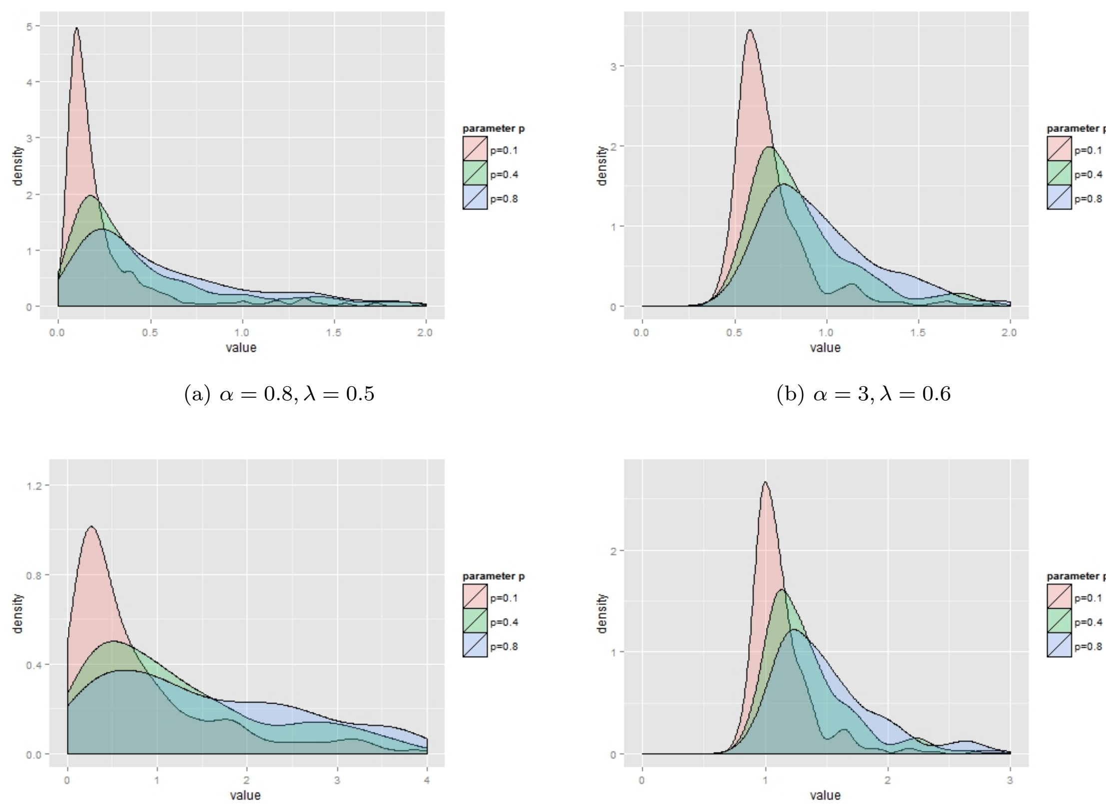
If we consider that the random variable N in (2.4) follows geometric distribution with parameter p, then the unconditional density function of Y can be derived as

$$f(y;\lambda,\alpha,p) = p\alpha\lambda y^{-(\alpha+1)} e^{-\lambda y^{-\alpha}} \left(1-(1-p)(1-e^{-\lambda y^{-\alpha}})\right)^{-2}; y,\lambda,\alpha>0, 0<p<1. \tag{2.5}$$

Hence the cumulative distribution function is given by

$$F(y;\lambda,\alpha,p) = e^{-\lambda y^{-\alpha}} \left(1-(1-p)(1-e^{-\lambda y^{-\alpha}})\right)^{-1}; y,\lambda,\alpha>0, 0<p<1. \tag{2.6}$$

Figure 1 gives the plots of IWG distribution for different parameter values.



(a) $\alpha = 0.8, \lambda = 0.5$

(b) $\alpha = 3, \lambda = 0.6$

(c) $\alpha = 0.4, \lambda = 2$

(d) $\alpha = 4, \lambda = 3$

Figure 1: Density plot for IWG distribution

The hazard function associated with IWG is given by

$$h(y;\lambda,\alpha,p) = \frac{\alpha\lambda e^{-\lambda y^{-\alpha}} y^{-(\alpha+1)}}{(1-e^{-\lambda y^{-\alpha}})\left(1-(1-p)(1-e^{-\lambda y^{-\alpha}})\right)}; y,\lambda,\alpha>0, 0<p<1. \tag{2.7}$$

We now show that the failure rate of the IWG distribution can be decreasing depending on the parameter values. A function $\eta(y) = -f'(y)/f(y)$ is defined, where f' denotes the first derivative of f.

In a straightforward manner it can be shown that

$$\eta(y) = (\alpha+1)y^{-1} + \lambda\alpha y^{-(\alpha+1)}\left(\frac{e^{-\lambda y^{-\alpha}} - \frac{p}{1-p}}{e^{-\lambda y^{-\alpha}} + \frac{p}{1-p}}\right),$$

and the first derivative is

$$\eta'(y) = -(\alpha+1)y^{-2} - \lambda\alpha(\alpha+1)y^{-(\alpha+2)}\left(\frac{e^{-\lambda y^{-\alpha}} - \frac{p}{1-p}}{e^{-\lambda y^{-\alpha}} + \frac{p}{1-p}}\right) + \frac{2p\lambda^2\alpha^2 y^{-2(\alpha+1)}e^{-\lambda y^{-\alpha}}}{(1-p)\left(e^{-\lambda y^{-\alpha}} + \frac{p}{1-p}\right)^2}.$$

If $\alpha \to 0^+$ or $\lambda \to 0^+$, then $\eta'(y) < 0 \; \forall \; y > 0$. Hence from Glaser (1980) Theorem (b) we can infer that the failure rate is decreasing. We can also see in Appendix A that for different values of parameters the failure rate can take various modal and increasing shapes.

2.2 Inverse Weibull Poisson distribution

If we consider that the random variable N in (2.4) follows Poisson distribution with parameter β, then the unconditional density function of Y can be derived as

$$f(y;\lambda,\alpha,\beta) = \beta\alpha\lambda(e^{\beta}-1)^{-1}e^{-\lambda y^{-\alpha}}e^{\beta(1-e^{-\lambda y^{-\alpha}})}y^{-(\alpha+1)}; y,\lambda,\alpha,\beta > 0. \quad (2.8)$$

Hence the cumulative distribution function is given by

$$F(y;\lambda,\alpha,\beta) = (1-e^{-\beta})^{-1}(1-e^{-\beta e^{-\lambda y^{-\alpha}}}); y,\lambda,\alpha,\beta > 0. \quad (2.9)$$

The plots of IWP distribution for different parameter values is given in Figure 2.

The hazard function associated with IWP distribution is given by

$$h(y;\lambda,\alpha,\beta) = \beta\alpha\lambda e^{-\lambda y^{-\alpha}}y^{-(\alpha+1)}\left(1-e^{-\beta(1-e^{-\lambda y^{-\alpha}})}\right)^{-1}; y,\lambda,\alpha,\beta > 0. \quad (2.10)$$

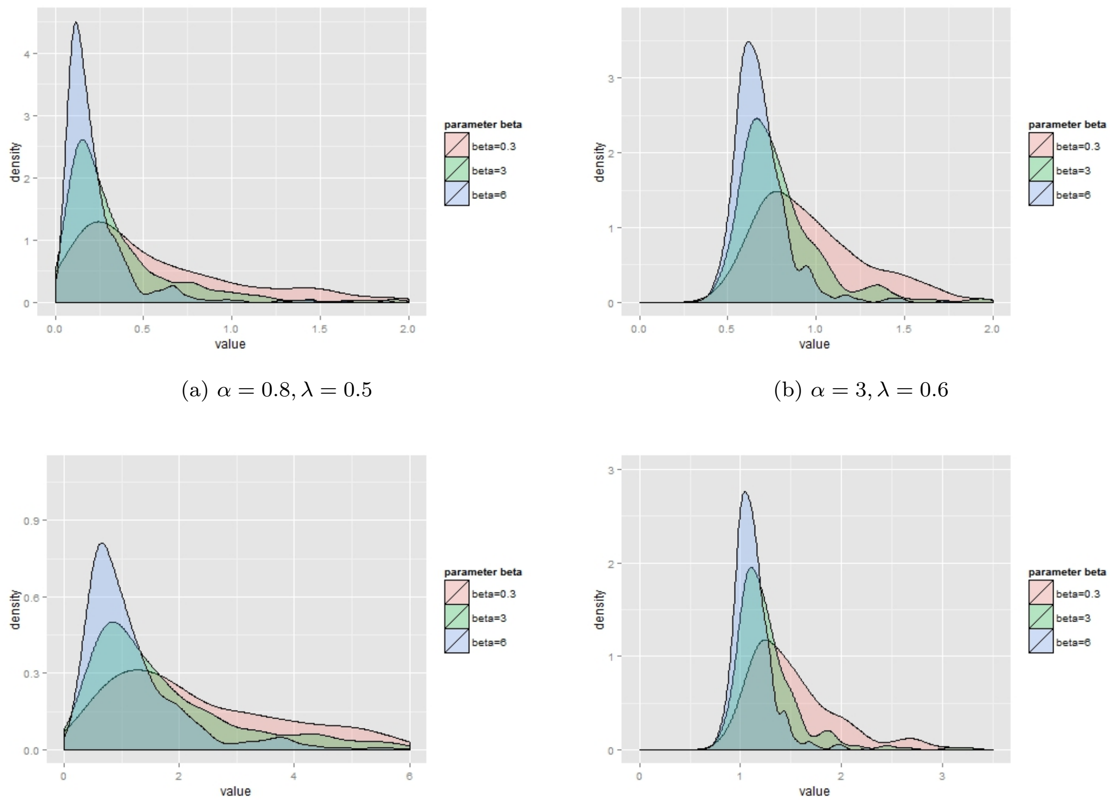
We now show that the failure rate of the IWP distribution can be decreasing depending on the parameter values. A function $\eta(y) = -f'(y)/f(y)$ is defined, where f' denotes the first derivative of f.
In a straightforward manner it can be shown that

$$\eta(y) = (\alpha+1)y^{-1} + \lambda\alpha\beta y^{-(\alpha+1)}e^{-\lambda y^{-\alpha}} - \lambda\alpha y^{-(\alpha+1)},$$

and the first derivative is

$$\eta'(y) = -(\alpha+1)y^{-2} - \beta\lambda\alpha(\alpha+1)y^{-(\alpha+2)}e^{-\lambda y^{-\alpha}} + \beta\alpha^2\lambda^2 y^{-2(\alpha+1)}e^{-\lambda y^{-\alpha}} + \lambda\alpha(\alpha+1)y^{-(\alpha+2)}.$$

Using Glaser (1980) Theorem (b) and the same argument as in 2.1 we can infer that the failure rate is decreasing. The other modal and increasing shapes of failure rate for different values of parameters are shown in Appendix A.



(a) $\alpha = 0.8, \lambda = 0.5$

(b) $\alpha = 3, \lambda = 0.6$

(c) $\alpha = 0.8, \lambda = 2$

(d) $\alpha = 4, \lambda = 3$

Figure 2: Density plot for IWP distribution

3 Properties of compounded inverse Weibull distributions

3.1 Properties of IWG distribution

3.1.1 Quantiles, moments and order statistics

The quantile function for inverse Weibull geometric distribution for minimum component lifetime can be derived from its cumulative distribution function given in (2.6). So the expression for quantile function becomes

$$Q(\pi; \alpha, \lambda, p) = \left(\log \left(\frac{\pi p}{1 - \pi(1-p)} \right)^{-\frac{1}{\lambda}} \right)^{-\frac{1}{\alpha}}. \tag{3.1}$$

The quantile function helps us deduce the median and inter-quartile range (IQR), the expressions are given by the following equations respectively

$$M = \left(\log \left(\frac{p}{1+p} \right)^{-\frac{1}{\lambda}} \right)^{-\frac{1}{\alpha}},$$

$$IQR = \left(\log\left(\frac{3p}{1+3p}\right)^{-\frac{1}{\lambda}}\right)^{-\frac{1}{\alpha}} - \left(\log\left(\frac{p}{3+p}\right)^{-\frac{1}{\lambda}}\right)^{-\frac{1}{\alpha}}.$$

The r$^{\text{th}}$ raw moment for inverse Weibull geometric distribution for minimum component lifetime can be derived from its density function given in (2.5). So the expression for r$^{\text{th}}$ raw moment becomes

$$E(Y^r) = \int_0^\infty p\alpha\lambda e^{-\lambda y^{-\alpha}} y^{-(\alpha+1-r)}\left(1-(1-p)(1-e^{-\lambda y^{-\alpha}})\right)^{-2} dy. \tag{3.2}$$

The expression for r^{th} raw moment can be rewritten in a different form as

$$E(Y^r) = \lambda\alpha p \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left({}^{i+j-1}C_j(i+1)(1-p)^i \int_0^\infty y^{-(\alpha+1-r)} e^{-\lambda(j+1)y^{-\alpha}} dy\right). \tag{3.3}$$

The Theorem below proves that the r^{th} raw moment in equation (3.3) is convergent for $\alpha > r$.

Theorem 3.1 *IWG distribution has finite moments of order r for $r < \alpha$.*

Proof. To prove the above theorem we use the following relationship

$$\int e^{ax^n} x^{-m} dx = \frac{(-1)^{z+1} a^z \Gamma(-z, -ax^n)}{n}; z = \frac{m-1}{n}, n \neq 0. \tag{3.4}$$

Using the expression (3.4) to solve (3.3) we get the following gamma functions in the solution

$$\lim_{a\to 0} \Gamma\Big(\frac{\alpha-r}{\alpha}, \lambda(j+1)a^{-\alpha}\Big)$$

and

$$\lim_{b\to \infty} \Gamma\Big(\frac{\alpha-r}{\alpha}, \lambda(j+1)b^{-\alpha}\Big).$$

So when $\alpha < r$ the expression $\frac{\alpha-r}{\alpha} < 0$, as a result both the expressions containing gamma functions become undefined. Hence moments of any order can be defined if $\alpha > r$. ∎

If $\alpha > r$ the expression for the r^{th} raw moment can be derived as

$$E(Y^r) = \lambda p \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left({}^{i+j-1}C_j(i+1)(1-p)^i(\lambda(j+1))^{\frac{r-\alpha}{\alpha}}\Gamma\Big(\frac{\alpha-r}{\alpha}\Big)\right).$$

The pdf $f_{(k)}$ of the k^{th} order statistic for a random sample $Y_1, Y_2, ...Y_n$ from the IWG distribution is given by

$$f_{(k)}(y) = nf(y)^{n-1}C_{k-1}F(y)^{k-1}(1-F(y))^{n-k}.$$

Hence the expression is

$$f_{(k)}(y) = \frac{n^{n-1}C_{k-1}\alpha\lambda p^{n-k-1} y^{-(\alpha+1)} e^{-\lambda k y^{-\alpha}}\left(1-e^{-\lambda k y^{-\alpha}}\right)^{n-k}}{\left(1-(1-e^{-\lambda k y^{-\alpha}})(1-p)\right)^{n+1}}.$$

3.1.2 Rényi entropy

Entropy is used in a wide range of situations in engineering and other applied sciences. The entropy of a random variable is a measure of uncertainty variation. The Rényi entropy is defined as $I_R(\gamma) = \frac{1}{1-\gamma}\log\{\int_{\mathbb{R}} f^{\gamma}(y)dy\}$, where $\gamma > 0$ and $\gamma \neq 1$. Using the density function of IWG distribution given in (2.5) we have

$$\int_0^{\infty} f^{\gamma}(y;\lambda,\alpha,p)dy = (\alpha\lambda p)^{\gamma}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\Big({}^{2\gamma+k-1}C_k{}^{k+i-1}C_i(1-p)^k\int_0^{\infty} y^{-\gamma(\alpha+1)}e^{-\lambda(\gamma+i)y^{-\alpha}}dy\Big).$$

If $\gamma+\gamma\alpha > 1$ then the expression for Rényi entropy becomes

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left((\lambda p)^{\gamma}\alpha^{\gamma-1}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\Big({}^{2\gamma+k-1}C_k{}^{k+i-1}C_i(1-p)^k\{\lambda(\gamma+i)\}^{\frac{1-\gamma-\gamma\alpha}{\alpha}}\Gamma\big(\frac{\gamma+\gamma\alpha-1}{\alpha}\big)\Big)\right).$$

Shannon entropy is a special case of Rényi entropy and can be obtained from $\lim_{\gamma\to1} I_R(\gamma)$.

3.2 Properties of inverse Weibull Poisson distribution

3.2.1 Quantiles, moments and order statistics

The quantile function for inverse Weibull Poisson distribution for minimum component lifetime can be derived from its cumulative distribution function given in (2.9). So the expression for quantile function becomes

$$Q(\pi;\alpha,\lambda,\beta) = \left(\log\left(\log\left(1-\pi(1-e^{-\beta})\right)^{-\frac{1}{\beta}}\right)^{-\frac{1}{\lambda}}\right)^{-\frac{1}{\alpha}}. \tag{3.5}$$

The quantile function helps us deduce the median and inter-quartile range (IQR), the expressions are given by the following equations respectively

$$M = \left(\log\left(\log\left(0.5(1+e^{-\beta})\right)^{-\frac{1}{\beta}}\right)^{-\frac{1}{\lambda}}\right)^{-\frac{1}{\alpha}},$$

$$IQR = \left(\log\left(\log\left(0.25+0.75e^{-\beta}\right)^{-\frac{1}{\beta}}\right)^{-\frac{1}{\lambda}}\right)^{-\frac{1}{\alpha}} - \left(\log\left(\log\left(0.75+0.25e^{-\beta}\right)^{-\frac{1}{\beta}}\right)^{-\frac{1}{\lambda}}\right)^{-\frac{1}{\alpha}}.$$

The r^{th} raw moment for inverse Weibull Poisson distribution for minimum component lifetime can be derived from its density function given in (2.8). So the expression for r^{th} raw moment becomes

$$E(Y^r) = \int_0^{\infty}\beta\alpha\lambda(e^{\beta}-1)^{-1}y^{-(\alpha+1-r)}e^{-\lambda y^{-\alpha}}e^{\beta(1-e^{-\lambda y^{-\alpha}})}dy. \tag{3.6}$$

The expression for r^{th} raw moment can be rewritten in a different form as

$$E(Y^r) = \frac{\alpha\lambda e^{\beta}}{e^{\beta}-1}\sum_{i=0}^{\infty}\left(\frac{(-1)^i\beta^{i+1}}{i!}\int_0^{\infty}y^{-(\alpha+1-r)}e^{-\lambda(i+1)y^{-\alpha}}dy\right). \tag{3.7}$$

The r^{th} raw moment in equation (3.7) is convergent if $\alpha > r$. The following Theorem is used to prove the same.

Theorem 3.2 *IWP distribution has finite moments of order r for $r < \alpha$.*

Proof. Using the expression (3.4) to solve (3.7) we get the following gamma functions in the solution

$$\lim_{a\to 0}\Gamma\Big(\frac{\alpha-r}{\alpha},\lambda(i+1)a^{-\alpha}\Big)$$

and

$$\lim_{b\to\infty}\Gamma\Big(\frac{\alpha-r}{\alpha},\lambda(i+1)b^{-\alpha}\Big).$$

Using the same argument as in Theorem 3.1 we find that moments of any order can be defined if $\alpha > r$. ■

If $\alpha > r$ the expression for the r^{th} raw moment can be derived as

$$E(Y^r) = \frac{\lambda e^{\beta}}{e^{\beta}-1}\sum_{i=0}^{\infty}\Big(\frac{(-1)^i\beta^{i+1}}{i!}\big(\lambda(i+1)\big)^{\frac{r-\alpha}{\alpha}}\Gamma\big(\frac{\alpha-r}{\alpha}\big)\Big).$$

The pdf $f_{(k)}$ of the k^{th} order statistic for a random sample $Y_1, Y_2, ...Y_n$ from the IWP distribution is given by

$$f_{(k)}(y) = nf(y)^{n-1}C_{k-1}F(y)^{k-1}(1-F(y))^{n-k}.$$

Hence the expression is

$$f_{(k)}(y) = n^{n-1}C_{k-1}\alpha\lambda\beta\frac{e^{\beta k}}{(e^{\beta}-1)^n}y^{-(\alpha+1)}e^{-(\lambda y^{-\alpha}+\beta e^{-\lambda y^{-\alpha}})}\big(1-e^{-\beta e^{-\lambda y^{-\alpha}}}\big)^{k-1}\big(e^{\beta(1-e^{-\lambda y^{-\alpha}})}-1\big)^{n-k}.$$

3.2.2 Rényi entropy

The Rényi entropy is defined as $I_R(\gamma) = \frac{1}{1-\gamma}\log\{\int_{\mathbb{R}} f^{\gamma}(y)dy\}$, where $\gamma > 0$ and $\gamma \neq 1$. Using the density function of IWP distribution given in (2.8) we get

$$\int_0^{\infty} f^{\gamma}(y;\lambda,\alpha,\beta)dy = (\alpha\lambda\beta)^{\gamma}e^{\beta\gamma}(e^{\beta}-1)^{-\gamma}\sum_{k=0}^{\infty}\left(\frac{(-\gamma\beta)^k}{k!}\int_0^{\infty} y^{-\gamma(\alpha+1)}e^{-(\gamma+k)\lambda y^{-\alpha}}dy\right).$$

If $\gamma + \gamma\alpha > 1$ then the expression for Rényi entropy becomes

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left(\frac{(\alpha\lambda\beta e^{\beta})^{\gamma}}{\alpha(e^{\beta}-1)^{\gamma}}\sum_{k=0}^{\infty}\frac{(-\beta\gamma)^k}{k!}\{\lambda(\gamma+k)\}^{\frac{1-\gamma\alpha-\gamma}{\alpha}}\Gamma\Big(\frac{\gamma\alpha+\gamma-1}{\alpha}\Big)\right).$$

Shannon entropy is a special case of Rényi entropy and can be obtained from $\lim_{\gamma\to 1} I_R(\gamma)$.

4 Estimation of parameters

4.1 Estimation of parameters for IWG distribution

4.1.1 Maximum likelihood estimation

Let $y = (y_1, y_2, \ldots, y_n)$ be a random sample of inverse Weibull geometric distribution with unknown parameter vector $\theta = (\lambda, \alpha, p)$. The log likelihood $L = L(\theta; y)$ for θ is

$$L = n(\log\lambda+\log\alpha+\log p)-(\alpha+1)\sum_{i=1}^{n}\log y_i-\lambda\sum_{i=1}^{n}{y_i}^{-\alpha}-2\sum_{i=1}^{n}\log\big(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\big). \quad (4.1)$$

The components of the score function $U(\theta) = (\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial p})^T$ are as follows

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} {y_i}^{-\alpha} + 2(1-p)\sum_{i=1}^{n} {y_i}^{-\alpha} e^{-\lambda {y_i}^{-\alpha}} \left(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\right)^{-1},$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \log y_i + \lambda \sum_{i=1}^{n} {y_i}^{-\alpha} \log y_i - 2\lambda(1-p)\sum_{i=1}^{n} {y_i}^{-\alpha} \log y_i e^{-\lambda {y_i}^{-\alpha}} \left(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\right)^{-1},$$

$$\frac{\partial L}{\partial p} = \frac{n}{p} - 2\sum_{i=1}^{n} (1-e^{-\lambda {y_i}^{-\alpha}})\left(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\right)^{-1}.$$

Equation $U(\theta) = 0$ is used to calculate maximum likelihood estimate (MLE).

4.1.2 EM algorithm

In order to obtain estimates from EM algorithm, we define a hypothetical complete data distribution with density function

$$f(y,n) = n\lambda\alpha p y^{-(\alpha+1)} e^{-\lambda y^{-\alpha}} \left((1-p)(1-e^{-\lambda y^{-\alpha}})\right)^{(n-1)}; y, \alpha, \lambda > 0, 0 < p < 1, n \in N. \quad (4.2)$$

Now for E-step formulation of EM cycle we require the conditional expectation of $(N|Y;\theta^{(r)})$ where $\theta^{(r)} = (\lambda^{(r)}, \alpha^{(r)}, p^{(r)})$ which gives the current estimates of θ. Using the conditional density function

$$f_{N|Y=y}(n) = n((1-p)(1-e^{-\lambda y^{-\alpha}}))^{(n-1)} \left(1-(1-p)(1-e^{-\lambda y^{-\alpha}})\right)^2; y, \alpha, \lambda > 0, 0 < p < 1, n \in N, \quad (4.3)$$

we find the conditional expectation as

$$E(N|Y) = \frac{1+(1-p)(1-e^{-\lambda y^{-\alpha}})}{1-(1-p)(1-e^{-\lambda y^{-\alpha}})}. \quad (4.4)$$

The M-step is completed using maximum likelihood estimation over θ with the missing Ns being replaced by the conditional expectation given above. The EM iteration is reduced to the following

$$p^{(r+1)} = \frac{n}{\sum_{i=1}^{n} {w_i}^{(r)}},$$

$\alpha^{(r+1)}$ and $\lambda^{(r+1)}$ can be found using the following equations respectively.

$$\frac{n}{\lambda^{(r+1)}} - \sum_{i=1}^{n} {y_i}^{-\alpha^{(r+1)}} + \sum_{i=1}^{n} \frac{({w_i}^{(r)}-1){y_i}^{-\alpha^{(r+1)}} e^{-\lambda^{(r+1)} {y_i}^{-\alpha^{(r+1)}}}}{1-e^{-\lambda^{(r+1)} {y_i}^{-\alpha^{(r+1)}}}} = 0,$$

$$\frac{n}{\alpha^{(r+1)}} - \sum_{i=1}^{n} \log y_i + \lambda^{(r+1)} \sum_{i=1}^{n} {y_i}^{-\alpha^{(r+1)}} \log y_i - \lambda^{(r+1)} \sum_{i=1}^{n} \frac{({w_i}^{(r)}-1){y_i}^{-\alpha^{(r+1)}} e^{-\lambda^{(r+1)} {y_i}^{-\alpha^{(r+1)}}}}{1-e^{-\lambda^{(r+1)} {y_i}^{-\alpha^{(r+1)}}}} = 0,$$

where

$${w_i}^{(r)} = \frac{1+(1-p^{(r)})(1-e^{-\lambda^{(r)} {y_i}^{-\alpha^{(r)}}})}{1-(1-p^{(r)})(1-e^{-\lambda^{(r)} {y_i}^{-\alpha^{(r)}}})}.$$

4.1.3 Fisher information matrix

Suppose $I(\theta)$ be the observed information matrix with elements $I_{ij} = -\partial^2 l/\partial\theta_i\partial\theta_j$, with $i, j = 1, 2, 3$. Differentiating with respect to the parameters, the elements of the symmetric, second-order observed information matrix are found as follows:

$$I_{11} = \frac{n}{\lambda^2} + 2p(1-p)\sum_{i=1}^{n}\left(\frac{{y_i}^{-2\alpha}e^{-\lambda {y_i}^{-\alpha}}}{\left(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\right)^2}\right),$$

$$I_{12} = I_{21} = -\sum_{i=1}^{n}{y_i}^{-\alpha}\log y_i - 2(1-p)\sum_{i=1}^{n}\left(\frac{{y_i}^{-\alpha}e^{-\lambda {y_i}^{\alpha}}\log y_i\left(p\lambda {y_i}^{-\alpha} - p - (1-p)e^{-\lambda {y_i}^{-\alpha}}\right)}{\left(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\right)^2}\right),$$

$$I_{22} = \frac{n}{\alpha^2} + \lambda\sum_{i=1}^{n}{y_i}^{-\alpha}(\log y_i)^2 - 2\lambda\sum_{i=1}^{n}\left(\frac{{y_i}^{-\alpha}(\log y_i)^2e^{-\lambda {y_i}^{-\alpha}}\left(p - p\lambda {y_i}^{-\alpha} + (1-p)e^{-\lambda {y_i}^{-\alpha}}\right)}{\left(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\right)^2}\right),$$

$$I_{33} = \frac{n}{p^2} - 2\sum_{i=0}^{n}\left(\frac{1-e^{-\lambda {y_i}^{-\alpha}}}{1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})}\right)^2,$$

$$I_{13} = I_{31} = 2\sum_{i=1}^{n}\frac{{y_i}^{-\alpha}e^{-\lambda {y_i}^{-\alpha}}}{\left(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\right)^2},$$

$$I_{23} = I_{32} = -2\lambda\sum_{i=1}^{n}\frac{{y_i}^{-\alpha}e^{-\lambda {y_i}^{-\alpha}}\log y_i}{\left(1-(1-p)(1-e^{-\lambda {y_i}^{-\alpha}})\right)^2}.$$

The Fisher information matrix $J_n(\theta) = E(I;\theta)$ is given by

$$J_n(\theta) = n\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

where

$$J_{11} = \frac{1}{\lambda^2} + 2p(1-p)E\left(\frac{Y^{-2\alpha}e^{-\lambda Y^{-\alpha}}}{\left(1-(1-p)(1-e^{-\lambda Y^{-\alpha}})\right)^2}\right),$$

$$J_{12} = J_{21} = -E(Y^{-\alpha}\log Y) - 2(1-p)E\left(\frac{Y^{-\alpha}e^{-\lambda Y^{\alpha}}\log Y\left(p\lambda Y^{-\alpha} - p - (1-p)e^{-\lambda Y^{-\alpha}}\right)}{\left(1-(1-p)(1-e^{-\lambda Y^{-\alpha}})\right)^2}\right),$$

$$J_{22} = \frac{1}{\alpha^2} + \lambda E(Y^{-\alpha}(\log Y)^2) - 2\lambda E\left(\frac{Y^{-\alpha}(\log Y)^2e^{-\lambda Y^{-\alpha}}\left(p - p\lambda Y^{-\alpha} + (1-p)e^{-\lambda Y^{-\alpha}}\right)}{\left(1-(1-p)(1-e^{-\lambda Y^{-\alpha}})\right)^2}\right),$$

$$J_{13} = J_{31} = 2E\left(\frac{Y^{-\alpha}e^{-\lambda Y^{-\alpha}}}{\left(1-(1-p)(1-e^{-\lambda Y^{-\alpha}})\right)^2}\right),$$

$$J_{23} = J_{32} = -2\lambda E\left(\frac{Y^{-\alpha}e^{-\lambda Y^{-\alpha}}\log Y}{\left(1-(1-p)(1-e^{-\lambda Y^{-\alpha}})\right)^2}\right),$$

$$J_{33} = \frac{1}{p^2} - 2E\left(\frac{1-e^{-\lambda Y^{-\alpha}}}{1-(1-p)(1-e^{-\lambda Y^{-\alpha}})}\right)^2.$$

Let $J(\theta) = \lim_{n\to\infty} J_n(\theta)$. Considering usual regularity conditions, it can be shown that $\hat{\theta}$ has a multivariate normal distribution as the sample size becomes large

$$\sqrt{n}(\hat{\theta} - \theta) \to MNV(0, J(\theta)^{-1}).$$

4.2 Estimation of parameters for IWP distribution

4.2.1 Maximum likelihood estimation

Let $y = (y_1, y_2, \ldots, y_n)$ be a random sample of inverse Weibull Poisson distribution with unknown parameter vector $\theta = (\lambda, \alpha, \beta)$. The log likelihood $L = L(\theta; y)$ for θ is

$$L = n\big(\log\lambda + \log\alpha - \log(e^{\beta} - 1)\big) - (\alpha+1)\sum_{i=1}^{n}\log y_i - \lambda\sum_{i=1}^{n} {y_i}^{-\alpha} + \beta\sum_{i=1}^{n}(1 - e^{-\lambda {y_i}^{-\alpha}}). \tag{4.5}$$

The components of the score function $U(\theta) = (\frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \beta})^T$ are as follows

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} {y_i}^{-\alpha} + \beta\sum_{i=1}^{n} {y_i}^{-\alpha} e^{-\lambda {y_i}^{-\alpha}},$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log y_i + \lambda\sum_{i=1}^{n} {y_i}^{-\alpha}\log y_i - \lambda\beta\sum_{i=1}^{n} {y_i}^{-\alpha} e^{-\lambda {y_i}^{-\alpha}}\log y_i,$$

$$\frac{\partial L}{\partial \beta} = -\frac{ne^{\beta}}{e^{\beta} - 1} + \sum_{i=1}^{n}(1 - e^{-\lambda {y_i}^{-\alpha}}).$$

Equation $U(\theta) = 0$ is used to calculate maximum likelihood estimate (MLE).

4.2.2 EM algorithm

In order to obtain estimates from EM algorithm, we define a hypothetical complete data distribution with density function

$$f(y, n) = \frac{\lambda\alpha\beta(e^{\beta} - 1)^{-1} y^{-(\alpha+1)} e^{-\lambda y^{-\alpha}} \big(\beta(1 - e^{-\lambda y^{-\alpha}})\big)^{(n-1)}}{(n-1)!}; y, \alpha, \beta, \lambda > 0; n \in N. \tag{4.6}$$

Now for E-step formulation of EM cycle we require the conditional expectation of $(N|Y; \theta^{(r)})$ where $\theta^{(r)} = (\lambda^{(r)}, \alpha^{(r)}, \beta^{(r)})$ which gives the current estimates of θ. Using the conditional density function

$$f_{N|Y=y}(n) = \frac{e^{-\beta(1 - e^{-\lambda y^{-\alpha}})} \big(\beta(1 - e^{-\lambda y^{-\alpha}})\big)^{(n-1)}}{(n-1)!}; y, \alpha, \beta, \lambda > 0; n \in N, \tag{4.7}$$

we find the conditional expectation as

$$E(N|Y) = \beta(1 - e^{-\lambda y^{-\alpha}}) + 1. \tag{4.8}$$

Maximum likelihood estimation over θ is used to complete M-step. The conditional expectation given above replaces the missing Ns. The EM iteration is reduced to the following equations, solving which we can find the expressions for $\lambda^{(r+1)}, \alpha^{(r+1)}$ and $\beta^{(r+1)}$ respectively.

$$\frac{n}{\lambda(r+1)} - \sum_{i=1}^{n} {y_i}^{-\alpha^{(r+1)}} + \sum_{i=1}^{n} \frac{({w_i}^{(r)} - 1){y_i}^{-\alpha^{(r+1)}} e^{-\lambda^{(r+1)} {y_i}^{-\alpha^{(r+1)}}}}{1 - e^{-\lambda^{(r+1)} {y_i}^{-\alpha^{(r+1)}}}},$$

$$\frac{n}{\alpha^{(r+1)}} - \sum_{i=1}^{n} \log y_i + \lambda^{(r+1)} \sum_{i=1}^{n} {y_i}^{-\alpha^{(r+1)}} \log y_i - \lambda^{(r+1)} \sum_{i=1}^{n} \frac{({w_i}^{(r)} - 1){y_i}^{-\alpha^{(r+1)}} e^{-\lambda^{(r+1)} {y_i}^{-\alpha^{(r+1)}}}}{1 - e^{-\lambda^{(r+1)} {y_i}^{-\alpha^{(r+1)}}}} = 0,$$

$$-\frac{n e^{\beta^{(r+1)}}}{e^{\beta^{(r+1)}} - 1} + \frac{1}{\beta^{(r+1)}} \sum_{i=1}^{n} {w_i}^{(r)} = 0,$$

where

$${w_i}^{(r)} = 1 + \beta^{(r)}(1 - e^{-\lambda^{(r)} {y_i}^{-\alpha^{(r)}}}).$$

4.2.3 Fisher information matrix

We consider $I(\theta)$ as the observed information matrix with elements $I_{ij} = -\partial^2 l/\partial\theta_i\partial\theta_j$, with $i, j = 1, 2, 3$. The following gives the second-order information matrix found by differentiating with respect to parameters:

$$I_{11} = \frac{n}{\lambda^2} + \beta \sum_{i=1}^{n} {y_i}^{-2\alpha} e^{-\lambda {y_i}^{-\alpha}},$$

$$I_{12} = I_{21} = -\sum_{i=1}^{n} {y_i}^{-\alpha} \log y_i - \beta\lambda \sum_{i=1}^{n} {y_i}^{-2\alpha} e^{-\lambda {y_i}^{-\alpha}} \log y_i + \beta \sum_{i=1}^{n} {y_i}^{-\alpha} e^{-\lambda {y_i}^{-\alpha}} \log y_i,$$

$$I_{22} = \frac{n}{\alpha^2} + \lambda \sum_{i=0}^{n} {y_i}^{-\alpha} (\log y_i)^2 - \lambda\beta \sum_{i=0}^{n} {y_i}^{-\alpha} (\log y_i)^2 e^{-\lambda {y_i}^{-\alpha}} + \lambda^2 \beta \sum_{i=0}^{n} {y_i}^{-2\alpha} (\log y_i)^2 e^{-\lambda {y_i}^{-\alpha}},$$

$$I_{13} = I_{31} = -\sum_{i=1}^{n} {y_i}^{-\alpha} e^{-\lambda {y_i}^{-\alpha}},$$

$$I_{23} = I_{32} = \lambda \sum_{i=0}^{n} {y_i}^{-\alpha} \log y_i e^{-\lambda {y_i}^{-\alpha}},$$

$$I_{33} = \frac{n e^{\beta}}{e^{\beta} - 1} - \frac{n e^{2\beta}}{(e^{\beta} - 1)^2}.$$

The Fisher information matrix $J_n(\theta) = E(I; \theta)$ is given by

$$J_n(\theta) = n \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

where

$$J_{11} = \frac{1}{\lambda^2} + \beta E\big(Y^{-2\alpha} e^{-\lambda Y^{-\alpha}}\big),$$

$$J_{12} = J_{21} = -E\big(Y^{-\alpha} \log Y\big) - \beta\lambda E\big(Y^{-2\alpha} e^{-\lambda Y^{-\alpha}} \log Y\big) + \beta\big(Y^{-\alpha} e^{-\lambda Y^{-\alpha}} \log Y\big),$$

$$J_{22} = \frac{1}{\alpha^2} + \lambda E\big(Y^{-\alpha} (\log Y)^2\big) - \lambda\beta E\big(Y^{-\alpha} (\log Y)^2 e^{-\lambda Y^{-\alpha}}\big) + \lambda^2 \beta E\big(Y^{-2\alpha} (\log Y)^2 e^{-\lambda Y^{-\alpha}}\big),$$

$$J_{13} = J_{31} = -E\big(Y^{-\alpha} e^{-\lambda Y^{-\alpha}}\big),$$

$$J_{23} = J_{32} = \lambda E\big(Y^{-\alpha} \log Y e^{-\lambda Y^{-\alpha}}\big),$$

$$J_{33} = \frac{e^{\beta}}{e^{\beta} - 1} - \frac{e^{2\beta}}{(e^{\beta} - 1)^2}.$$

Let $J(\theta) = \lim_{n\to\infty} J_n(\theta)$. Considering usual regularity conditions, it can be shown that $\hat{\theta}$ has a multivariate normal distribution as the sample size becomes large

$$\sqrt{n}(\hat{\theta} - \theta) \to MNV(0, J(\theta)^{-1}).$$

4.3 Simulation study

In order to evaluate the estimates obtained from EM algorithm an intensive simulation study is conducted for both IWG and IWP distributions. No restriction on the maximum iteration was imposed but convergence was assumed when the absolute difference between successive estimates were less than 10^{-3}. Different combinations of initial parameter values are looked at for both the distributions. The simulation for each set of parameter values were conducted using $10,000$ samples each of size $n = 50, 100, 500$ randomly generated using the IWG and IWP distributions respectively. The root-mean-square deviation (RMSD) was calculated for each set of parameters and sample size. It is found that RMSD decrease as the sample size increases. The simulation was done through software R version 3.2.2. The simulation results of IWG and IWP distributions are given in Table 1 in Appendix D.

5 Empirical Illustrations

The applicability of the proposed models is illustrated through three real life data sets. Software R version 3.2.2 is used for all the computations in this section.

The first data set (D1) used for analysis is widely popular in the literature as far as compounded distributions are concerned. The data set consists of successive failures of air-conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. The data consists of 213 observations. Adamidis and Loukas (1998) considered this data set for exponential geometric distribution, Kus (2007) used this data set to fit exponential Poisson distribution, Barreto-Souza *et al* (2011) fitted Weibull geometric distribution to this data set. While Lu and Shi (2012) considered an Weibull Poisson distribution for this data set.

The second data set (D2) consists of maintenance data with 46 observations reported on active repair times (hours) for an airborne communication transceiver by Von Alven (1964). Other than Lu and Shi (2012), the data set was also used by Chhikara and Folks (1977) to discuss inverse Gaussian distribution and Dimitrakopoulou *et al* (2007) to fit a three-parameter lifetime distribution with increasing, decreasing, bathtub, and upside down bathtub shaped failure rates which includes Weibull distribution as a special case.

The third data set (D3) consisting of 23 observations was obtained from Lawless (1982) and represents the number of revolution before failure of each of 23 ball bearings in the life tests. It was used by Dey and Kundu (2009) while trying to find a discriminating procedure for lognormal and loglogistic distribution functions while Bromideh (2012) used the data to discriminate between Weibull and lognormal distributions.

The compounded inverse Weibull distributions are fitted on the data sets mentioned above. In order to compare the results we use Weibull, inverse Weibull (IW), exponential geometric (EG), exponential Poisson (EP), Weibull geometric (WG) and Weibull Poisson (WP) distributions. The densities corresponding to the distributions are given by the following functions respectively.

$$f_1(x;\alpha,\lambda) = \alpha\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}}; x,\alpha,\lambda > 0,$$

$$f_2(x;\alpha,\lambda) = \alpha\lambda x^{-(\alpha+1)}e^{-\lambda x^{-\alpha}}; x,\alpha,\lambda > 0,$$

$$f_3(x;\lambda,p) = \lambda p e^{-\lambda x}\left(1-(1-p)e^{-\lambda x}\right)^{-2}; x,p,\lambda > 0, p < 1,$$

$$f_4(x;\lambda,\beta) = \frac{\lambda\beta}{1-e^{-\beta}}e^{-\beta-\lambda x+\beta e^{-\lambda x}}; x,\lambda,\beta > 0,$$

$$f_5(x;\lambda,\alpha,p) = p\alpha\lambda^{\alpha}x^{\alpha-1}e^{-(\lambda x)^{\alpha}}\left(1-(1-p)e^{-(\lambda x)^{\alpha}}\right)^{-2}; x,p,\alpha,\lambda > 0, p < 1,$$

$$f_6(x;\lambda,\alpha,\beta) = \frac{\lambda\alpha\beta x^{\alpha-1}}{1-e^{-\beta}}e^{-\beta-\lambda x^{\alpha}+\beta e^{-\lambda x^{\alpha}}}; x,\lambda,\alpha,\beta > 0.$$

We derive,the maximum likelihood estimates, the maximized log-likelihood $\widehat{\ell}$, the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the Kolmogorov-Smirnov distance (KS-dist) and the corresponding p value for each distribution. The method suggested by Byrd et al (1995), L-BFGS-B, is used for numerical computation of likelihood estimation. The method is used to solve large nonlinear optimization problems with described bounds using a limited memory quasi-Newton algorithm. Q-Q plot of IWG and IWP distributions for all three aforementioned data sets are shown in Appendix B and P-P plot for the same is shown in Appendix C. The conclusion drawn from the empirical illustration is that both IWG and IWP distributions can be used for data modeling. It is found that IWP distribution outperforms other models when data sets D1 and D3 are considered, in case of data set D2 it performs better than most of the models. Table 3 in Appendix E contains the results of all the aforementioned models.

6 Conclusion

In this paper we introduced IWG and IWP distribution by compounding inverse Weibull with geometric and Poisson distributions respectively. The failure rates of the new distributions are observed to be of different monotone and non monotone shapes. Various statistical and reliability properties of the distributions such as hazard rate, quantile function, r^{th} raw moment, k^{th} order statistic and entropy are stated and discussed. Estimation of parameters of both the distributions are done through maximum likelihood method as well as expectation-maximization algorithm. The Fisher information matrix for both the distributions are also provided. The simulation results show that the estimation performance is satisfactory. The two distributions are also found to be of better fit than other similar distributions in three real life data sets. We recommend the use of IWG and IWP for modeling real life maintenance data sets.

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Appendix A

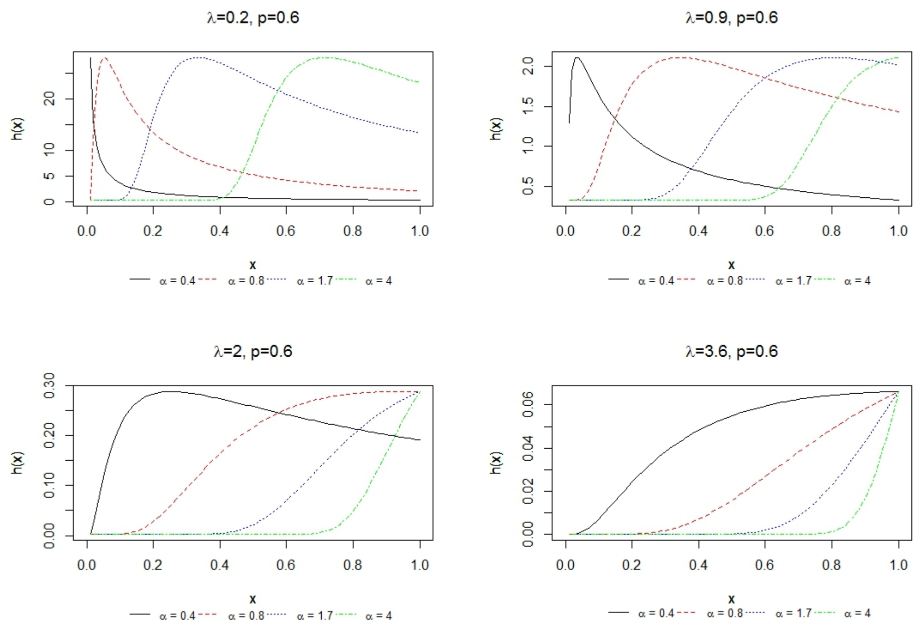

Figure 3: Density plot for IWG distribution

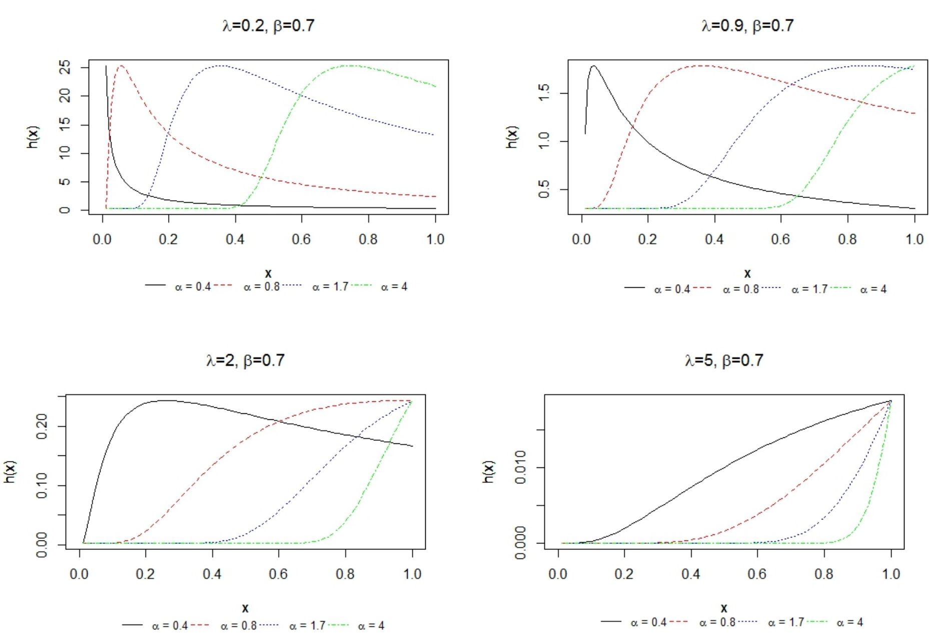

Figure 4: Density plot for IWP distribution

Appendix B

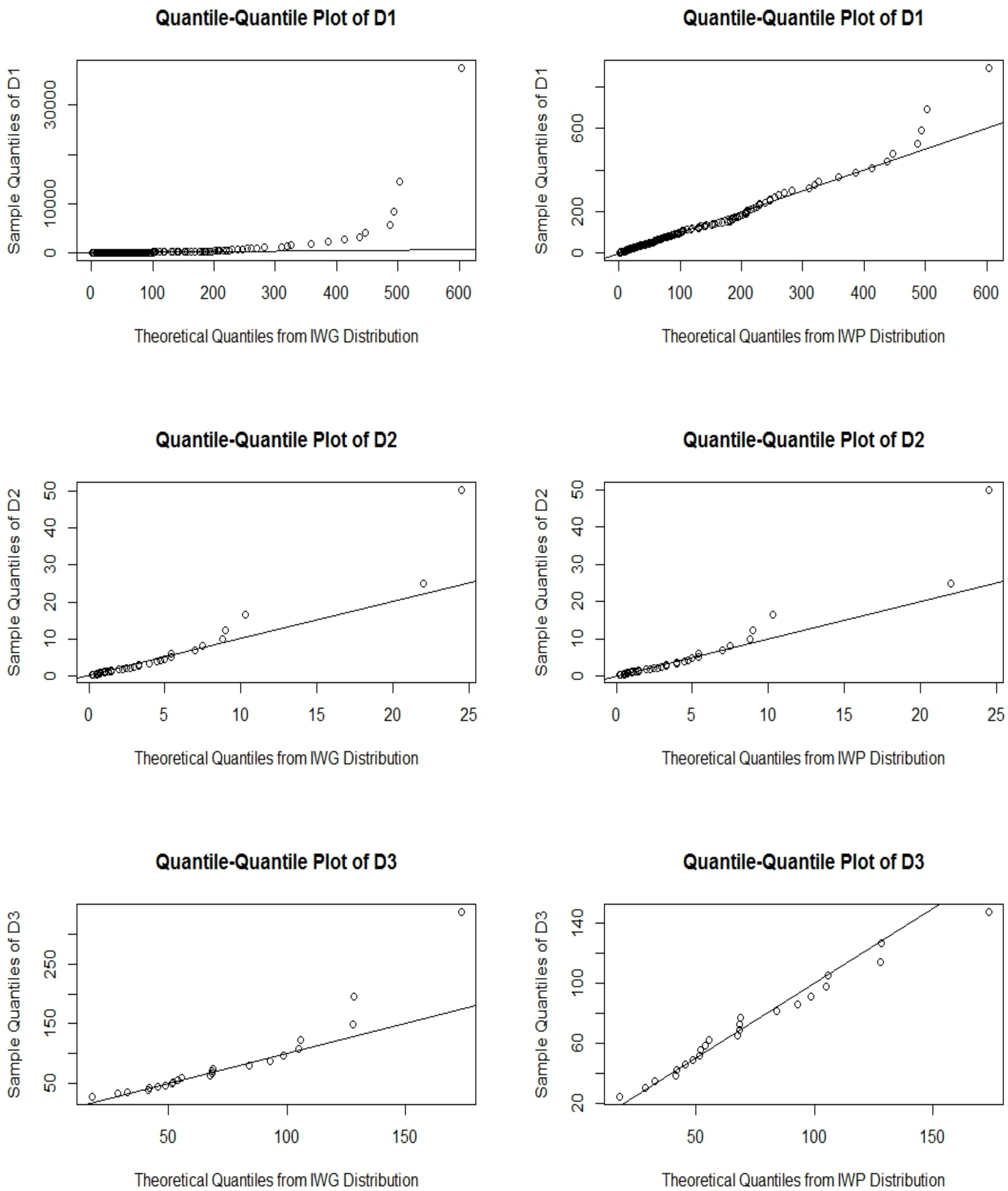

Figure 5: QQ plot of IWG and IWP distribution

Appendix C

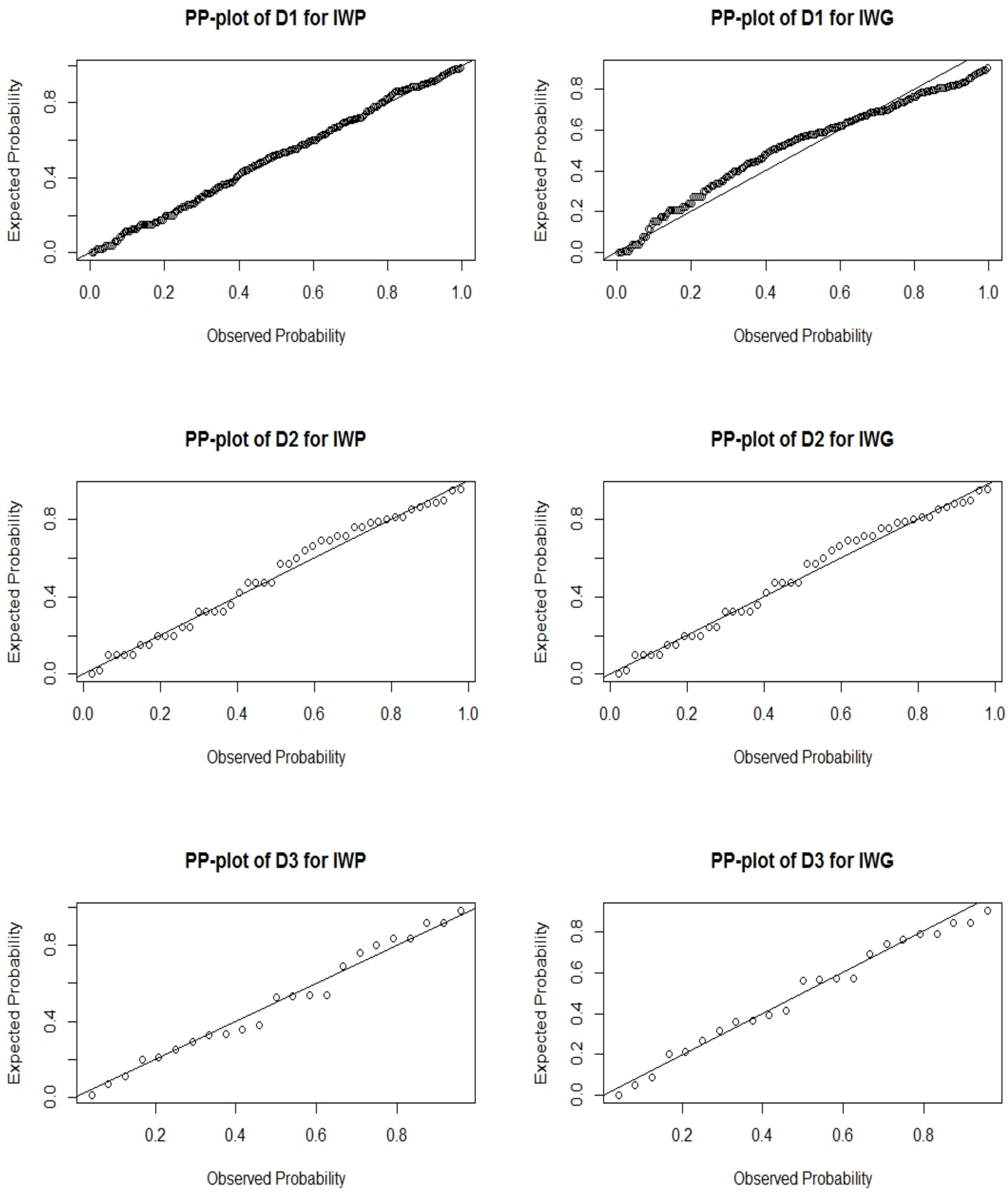

Figure 6: PP plot of IWP and IWG distribution

Appendix D

Table 1: The parameter estimates and RMSDs using EM algorithm for IWG and IWP distribution

n	IWG			IWP		
	(α,λ,p)	Parameter estimates	RMSD	(α,λ,β)	Parameter estimates	RMSD
50		0.7097,0.5954,0.2009	0.0377,0.0399,0.0093		0.7161,0.6003,0.6996	0.0740,0.0884,0.0456
100	(0.7,0.6,0.2)	0.7042,0.5982,0.2004	0.0242,0.0252,0.0066	(0.7,0.6,0.7)	0.7076,0.6003,0.6999	0.0488,0.0605,0.0323
500		0.7006,0.5999,0.2001	0.0086,0.0078,0.0029		0.7013,0.6003, 0.6999	0.0199,0.0243,0.0144
50		2.3419,1.1956,0.2009	0.2017,0.0891,0.0133		2.3586,1.2144,0.6996	0.2556,0.1524,0.0456
100	(2.3,1.2,0.2)	2.3202,1.1979,0.2004	0.0957,0.0435,0.0066	(2.3,1.2,0.7)	2.3289,1.2069,0.6999	0.1713,0.1056,0.0323
500		2.3039,1.1997,0.2001	0.0565,0.0260,0.0041		2.3061,1.2016,0.6999	0.0748,0.0456,0.0144
50		2.3355,0.5942,0.2009	0.1842,0.0636,0.0133		2.3550,0.5997,0.6996	0.2472,0.0909,0.0456
100	(2.3,0.6,0.2)	2.3161,0.5974,0.2004	0.1187,0.0422,0.0093	(2.3,0.6,0.7)	2.3264,0.5999,0.6999	0.1637,0.0632,0.0323
500		2.3024,0.5997,0.2001	0.0440,0.0158,0.0041		2.3049,0.6001,0.6999	0.0679,0.0268,0.0144
50		0.7125,1.1965,0.2009	0.0608,0.0858,0.0133		0.7134,0.5964,2.0993	0.0645,0.0719,0.0954
100	(0.7,1.2,0.2)	0.7059,1.1986,0.2004	0.0404,0.0583,0.0093	(0.7,1.2,0.7)	0.7088,1.2073,0.6999	0.0521,0.1051,0.0323
500		0.7011,1.2000,0.2001	0.0168,0.0231,0.0041		0.7018,1.2016,0.6999	0.0227,0.0452,0.0144
50		0.7168,0.6013,0.8002	0.0540,0.0654,0.0092		0.7134,0.5964,2.0993	0.0645,0.0719,0.0954
100	(0.7,0.6,0.8)	0.7080,0.6008,0.8001	0.0359,0.0450,0.0065	(0.7,0.6,2.1)	0.7062,0.5985,2.0999	0.0420,0.0481,0.0675
500		0.7014,0.6004,0.8000	0.0145,0.0182,0.0029		0.7009,0.5999,2.0999	0.0163,0.0178,0.0300
50		2.3605,1.2172,0.8002	0.2621,0.1616,0.0130		2.3517,1.2031,2.0992	0.2308,0.1177,0.0954
100	(2.3,1.2,0.8)	2.3298,1.2083,0.8001	0.1758,0.1117,0.0092	(2.3,1.2,2.1)	2.3254,1.2015,2.0999	0.1544,0.0821,0.0675
500		2.3063,1.2019,0.8000	0.0767,0.0482,0.0041		2.3052,1.2004,2.0999	0.0666,0.0355,0.0300
50		2.3571,0.6008,0.8002	0.2543,0.0952,0.0130		2.3468,0.5956,2.0993	0.2178,0.0760,0.0954
100	(2.3,0.6,0.8)	2.3275,0.6004,0.8001	0.1686,0.0662,0.0092	(2.3,0.6,2.1)	2.3221,0.5980,2.0998	0.1428,0.0521,0.0675
500		2.3052,0.6002,0.8000	0.0705,0.0282, 0.0041		2.3038,0.5998,2.0999	0.0568,0.0214,0.0300
50		0.7184,1.2175,0.8002	0.0797,0.1611,0.0130		0.7156,1.2037,2.0993	0.0700,0.1164,0.0954
100	(0.7,1.2,0.8)	0.7091,1.2086,0.8001	0.0535,0.1112,0.0092	(0.7,1.2,2.1)	0.7076,1.2018,2.0999	0.0468,0.0808,0.0675
500		0.7018,1.2018,0.8000	0.0233,0.0478,0.0041		0.7015,1.2006,2.0999	0.0199,0.0342,0.0300

Appendix E

Table 2: Maximum likelihood estimates, AIC, BIC, KS-distance and p-values obtained from the fit of each distributions

Data set	Distribution	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{p}$ / $\hat{\beta}$	$\hat{\ell}$	AIC	BIC	KS	p-value
D1(n=213)	IWG	0.73385	10.88719	0.98	-1205.680	2417.357	2427.441	0.10474	0.0187
	IWP	0.20303	10.48353	75.18689	-1175.507	2357.014	2367.098	0.03710	0.9311
	Weibull	0.92129	0.01595	-	-1177.587	2359.175	2365.897	0.05067	0.6448
	IW	0.73604	10.99672	-	-1210.316	2424.632	2431.355	0.10246	0.0228
	EG	-	0.00819	0.59028	-1175.935	2355.869	2362.592	0.04856	0.6967
	EP	-	0.00785	1.20112	-1175.814	2355.628	2362.351	0.04614	0.7549
	WG	0.96772	0.01	0.79326	-1176.811	2359.622	2369.706	0.04729	0.7277
	WP	0.97407	0.01	0.82021	-1176.281	2358.563	2368.646	0.04908	0.6840
D2(n=46)	IWG	1.00934	1.14127	0.985	-100.7034	207.4069	212.8928	0.08187	0.9174
	IWP	1.01163	1.13838	0.023	-100.7006	207.4011	212.8870	0.08131	0.9214
	Weibull	0.89858	0.33375	-	-104.4697	212.9394	216.5967	0.12044	0.5170
	IW	1.01272	1.13156	-	-100.6907	205.3814	209.0387	0.08069	0.9255
	EG	-	0.16336	0.39045	-103.2994	210.5988	214.2561	0.13549	0.3671
	EP	-	0.108	3.42619	-102.8323	209.6645	213.3218	0.12688	0.4494
	WG	1.48521	0.05342	0.0338	-100.8561	207.7123	213.1982	0.09217	0.8294
	WP	1.10112	0.09245	3.52217	-102.4637	210.9274	216.4133	0.11112	0.6210
D3(n=23)	IWG	0.72552	78.81357	0.02	-115.2595	236.5189	239.9254	0.11329	0.8974
	IWP	0.44707	32.32339	101.44462	-112.9764	231.9528	235.3593	0.11225	0.9032
	Weibull	0.91556	0.024	-	-123.3889	250.7777	253.0487	0.38644	0.0013
	IW	1.83437	1240.95337	-	-115.7833	235.5666	237.8376	0.13289	0.7632
	EG	-	0.013669	0.98	-121.5195	247.0391	249.3101	0.30761	0.0198
	EP	-	0.01375	0.026	-121.4898	246.9796	249.2506	0.30773	0.0197
	WG	2.10189	0.01222	0.99996	-113.6877	233.3754	236.7819	0.15047	0.6216
	WP	1.10333	0.01	0.01	-120.0343	246.0687	249.4751	0.32756	0.0107

Research Office

Indian Institute of Management Kozhikode

IIMK Campus P. O.,

Kozhikode, Kerala, India,

PIN - 673 570

Phone: +91-495-2809238

Email: research@iimk.ac.in

Web: https://iimk.ac.in/faculty/publicationmenu.php