

REVIEW

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# Compounding of distributions: a survey and new generalized classes

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## Abstract

Generalizing distributions is an old practice and has ever been considered as precious as many other practical problems in statistics. It simply started with defining different mathematical functional forms, and then induction of location, scale or inequality parameters. The generalization through induction of shape parameter(s) started in 1997, and the last two decades were full of such practices. But to cope with the complex situations under series and parallel structures, the art of mixing discrete and continuous started in 1998. In this article, we present a survey on compounding univariate distributions, their extensions and classes. We review several available compound classes and propose some new ones. The recent trends in the construction of generalized and compounding classes are discussed, and the need for future works are addressed.

**Keywords:** Compound G-classes, generalized classes, G-geometric class, G-Poisson class, power series class

**AMS Subject Classification:** Primary 60E05; secondary 62N05; 62F10

## 1 Introduction

The modern era on distribution theory stresses on problem-solving faced by the practitioners and applied researchers and proposes a variety of models so that lifetime data set can be better assessed and investigated in different fields. In other words, there is strong need to introduce useful models for better exploration of the real-life phenomena. Nowadays, the trends and practices in defining probability models totally differ in comparison to the models suggested before 1997. One main objective for proposing, extending or generalizing (models or their classes) is to explain how the lifetime phenomenon arises in fields like physics, computer science, insurance, public health, medical, engineering, biology, industry, communications, life-testing and many others. For example, the well-known and fundamental distributions such as exponential, Rayleigh, Weibull and gamma are very limited in their characteristics and are unable to show wide flexibility. The number of shape parameters, closed-form expressions of the cumulative distribution function (cdf), forms of the quantile function (qf), density and failure rate shapes, skewness and kurtosis features, entropy measures, estimation of stress reliability  $R = \mathbb{P}(Y < X)$ , structural properties, use of advanced mathematical functions and power series, which are available through computing platforms like Mathematica,

Maple, Matlab, Python, Ox and R, variety of estimation methods, estimation of parameters in case of censored and uncensored situations, simulation results, coping with data sets having different shapes and goodness-of-fits are some well-established characteristics that a proposed lifetime model may possess.

For complex phenomenons in reliability studies, lifetime testing, human mortality studies, engineering modeling, electronic sciences and biological surveys, the failure rate behavior can be bathtub, upside-down bathtub and other shaped but not usually monotone increasing or decreasing. Thus, in order to cope with both monotonic and non-monotonic failure rate shapes, researchers have introduced several generalizations (or G-classes) which are very flexible to study needful properties of the model and its fitness.

In the last two decades, two main generalization approaches were adopted and practiced, and have also received increased attention.

### 1.1 First generalization approach (G-classes)

The first approach deals with the shape parameter(s) induction in parent (or baseline) distribution to explore tail properties and to improve goodness-of-fits. Some well-known generalized (or G-) classes are: Marshall-Olkin-G (Marshall and Olkin 1997), exponentiated-G (Gupta et al. 1998), beta-G (Eugene et al. 2002), Kumaraswamy-G (Cordeiro and de-Castro 2011), McDonald-G (Alexander et al. 2012), ZBgamma-G (Zografos and Balakrishnan 2009; Amini et al. 2014), RBgamma-G (Ristić and Balakrishnan 2012; Amini et al. 2014), odd-gamma-G (Torabi and Montazari 2012), Kummer-beta-G (Pescim et al. 2012), beta extended Weibull-G (Cordeiro et al. 2012b), odd exponentiated generalized-G (Cordeiro et al. 2013a), truncated exponential-G (Barreto-Souza and Simas 2013), logistic-G (Torabi and Montazari 2014), gamma extended Weibull-G (Nascimento et al. 2014), odd Weibull-G (Bourguignon et al. 2014a), exponentiated-half-logistic-G (Cordeiro et al. 2014a), Libby-Novick beta-G (Cordeiro et al. 2014b; Ristić et al. 2015), Lomax-G (Cordeiro et al. 2014d), Harris-G (Batsidis and Lemonte 2015; Pinho et al. 2015), modified beta-G (Nadarajah et al. 2014b), odd generalized-exponential-G (Tahir et al. 2015), Kumaraswamy odd log-logistic-G (Alizadeh et al. 2015b), beta odd log-logistic-G (Cordeiro et al. 2016), Kumaraswamy-Marshall-Olkin-G (Alizadeh et al. 2015c), beta-Marshall-Olkin-G (Alizadeh et al. 2015a), Weibull-G (Tahir et al. 2016b), exponentiated-Kumaraswamy-G (da-Silva et al. 2016), ZBgamma-odd-loglogistic-G (Cordeiro et al. 2015a) and Tukey's  $g$ - and  $h$ -G (Jiménez et al. 2015). For more details on some well-established G-classes, the reader is referred to Tahir and Nadarajah (2015).

The mathematical properties of the Kumaraswamy-G family were studied by Nadarajah et al. (2012), Hussain (2013), and Cordeiro and Bager (2015). The failure rate and aging properties of the beta-G and ZBgamma-G models were addressed by Triantafyllou and Koutras (2014). The structural properties of the ZBgamma-G and RBgamma-G models were recently investigated by Nadarajah et al. (2015b), and Cordeiro and Bourguignon (2016).

The revolutionary idea in generalization begun with the work of Alzaatreh et al. (2013) who proposed transformed-transformer (T-X) (Weibull-X and gamma-X) family of distributions. This approach was further extended to the exponentiated T-X (Alzaghal et al. 2013), T-X{Y}-quantile based approach (Aljarrah et al. 2014), T-R{Y} (Alzaatreh et al. 2014), T-Weibull{Y} (Almheidat et al. 2015), T-gamma{Y} (Alzaatreh et al. 2016a),

T-Cauchy $\{Y\}$ (Alzaatreh et al. 2016b), Gumbel-X (Al-Aqtash 2013; Al-Aqtash et al. 2014) and logistic-X (Tahir et al. 2016).

After the wide-spread popularity of well-established exponentiated-G, Marshall-Olkin-G, beta-G, Kumaraswamy-G and McDonald-G classes, and T-X family, the transmuted-G class of distributions has received increased attention in the last decade. This class is based on the quadratic rank transmutation map (QRTM) pioneered by Shaw and Buckley (2009) and highlighted by Aryal and Tsokos (2009, 2011).

For any baseline distribution cdf  $G(x)$ , the cdf of the *transmuted-G* class (for  $|\eta| \leq 1$ ) is given by

$$F(x) = (\eta + 1) G(x) - \eta G(x)^2, \quad (1)$$

where  $\eta$  is the transmuted (or shape) parameter. For  $\eta = 0$ , the above equation reduces to the baseline distribution.

The general properties of the transmuted family were obtained by Bourguignon et al. (2016a) and Das (2015). The transmuted family was further extended as the exponentiated transmuted-G type 1 using the Lehmann alternative type 1 (LA1) class (due to Gupta et al. 1998) by Nofal et al. (2016) and Alizadeh et al. (2016a), the exponentiated transmuted-G type 2 using the Lehmann alternative type 2 (LA2) class (due to Gupta et al. 1998) by Merovci et al. (2016), and the transmuted exponentiated generalized-G by Yousof et al. (2015).

More than *fifty* transmuted distributions have been reported in the literature, which are compiled in chronological order in Table 1; see also Section 8.4.

## 1.2 Second generalization approach (compounding)

The second approach deals with the compounding of discrete models, namely the geometric, Poisson, logarithmic, binomial, negative-binomial (NB), Conway-Maxwell-Poisson (COMP) and power-series with continuous lifetime models. The basic idea of introducing compound models or families is that a lifetime of a system with  $N$  (discrete random variable) components and the positive continuous random variable, say  $Y_i$  (the lifetime of the  $i$ th component), can be denoted by the non-negative random variable  $Y = \min(Y_1, \dots, Y_N)$  (the minimum of an unknown number of any continuous random variables) or  $Z = \max(Y_1, \dots, Y_N)$  (the maximum of an unknown number of any continuous random variables), based on whether the components are in series or in parallel structure. Some useful references for the readers are Louzada et al. (2012a), Leahu et al. (2013) and Bidram and Alavi (2014).

The objectives of our article are three-fold: Firstly, we present an up-to-date account on the compounded distributions and their generalizations for the readers of modern distribution theory. Secondly, this survey will motivate the researchers to fill the gap and to furnish their work in remaining applied areas. Thirdly, this survey will also be helpful as a tutorial to the beginners of compound modeling art.

The rest of the article is organized as follows. In Section 2, two compound G-classes are reviewed to represent the distributions of the minimum and maximum of an unknown number of continuous random variables having the same parent lifetime distribution. In Section 3, fourteen existing and new compound classes for the minimum constructed from the zero-truncated geometric (ZTG), zero-truncated Poisson (ZTP), logarithmic

**Table 1** Contributed work on transmuted distributions

S.No.	Pioneer Year	Distribution	Author(s)
1.	2009	Transmuted extreme value	Aryal and Tsokos (2009)
2.	2011	Transmuted Weibull	Aryal and Tsokos (2011) Ahmad et al. (2015b) Khan et al. (2016b)
3.	2013	Transmuted log-logistic	Aryal (2013) Granzotto and Louzada (2015)
4.	2013	Transmuted Rayleigh	Merovci (2013a) Ahmad et al. (2015a)
5.	2013	Transmuted exponentiated-exponential	Merovci (2013b) Khan et al. (2015a)
6.	2013	Transmuted Fréchet	Mahmoud and Mandouh (2013)
7.	2013	Transmuted Lomax	Ashour and Eltehiwy (2013a)
8.	2013	Transmuted Lindley	Merovci (2013c)
9.	2013	Transmuted quasi-Lindley	Elbatal and Elgarhy (2013)
10.	2013	Transmuted exponentiated-Lomax	Ashour and Eltehiwy (2013b)
11.	2013	Transmuted modified inverse Weibull	Elbatal (2013a)
12.	2013	Transmuted generalized inverted exponential	Elbatal (2013b)
13.	2013	Transmuted exponentiated-modified Weibull	Ashour and Eltehiwy (2013c)
14.	2013	Transmuted generalized linear exponential	Elbatal et al. (2013)
15.	2013	Transmuted additive Weibull (AW)	Elbatal and Aryal (2013)
16.	2013	Transmuted modified Weibull (MW)	Khan and King (2013)
17.	2014	Transmuted Pareto	Merovci and Puka (2014)
18.	2014	Transmuted Maxwell	Iriarte and Astorga (2014)
19.	2014	Transmuted linear exponential	Tian et al. (2014)
20.	2014	Transmuted inverse Rayleigh (IR)	Sharma et al. (2014) Ahmad et al. (2014)
21.	2014	Transmuted generalized Rayleigh (GR)	Merovci (2014) Iriarte and Astorga (2015)
22.	2014	Transmuted inverted Weibull (IW)	Khan et al. (2014) Khan and King (2014a)
23.	2014	Transmuted generalized IW	Merovci et al. (2014) Khan and King (2014b)
24.	2014	Exponentiated transmuted Weibull	Ebraheim (2014)
25.	2014	Beta transmuted Weibull	Pal and Tiensuwan (2014)
26.	2014	Transmuted exponentiated-gamma	Hussian (2014) Lucena et al. (2015)
27.	2014	Transmuted exponentiated-Fréchet	Elbatal et al. (2014)
28.	2015	Transmuted exponential	Owoloko et al. (2015)
29.	2015	Transmuted Burr III	Abdul-Moniem (2015)
30.	2015	Transmuted Gompertz	Abdul-Moniem and Seham (2015) Khan et al. (2016c)
31.	2015	Transmuted modified IR	Khan and King (2015)
32.	2015	Generalized transmuted AW	Mansour et al. (2015a)
33.	2015	Kumaraswamy transmuted MW	Mansour et al. (2015b)
34.	2015	Transmuted generalized MW	Cordeiro et al. (2015c)
35.	2015	Kumaraswamy transmuted exponentiated MW	Al-Babtain et al. (2015)
36.	2015	Transmuted Marshall-Olkin (MO) Fréchet	Afify et al. (2015a)
37.	2015	Exponentiated transmuted GR	Afify et al. (2015b)
38.	2015	Transmuted Weibull-Lomax	Afify et al. (2015c)
39.	2015	Transmuted Dagum	Elbatal and Aryal (2015) Shahzad and Asghar (2016)
40.	2015	Transmuted exponentiated-Pareto	Luguterah and Nasiru (2015) Fatima and Roohi (2015)

**Table 1** Contributed work on transmuted distributions (*Continued*)

S.No.	Pioneer Year	Distribution	Author(s)
41.	2015	Generalized transmuted Lindley	Mansour and Mohamed (2015)
42.	2015	Transmuted Chen	Khan et al. (2015b)
43.	2015	Beta transmuted Fréchet	da-Silva et al. (2015a)
44.	2016	Transmuted Kumaraswamy	Khan et al. (2016a)
45.	2016	Transmuted generalized Lindley	Elgarhy et al. (2016)
46.	2016	Transmuted Almalki-Yuan MW	Vardhan and Balaswamy (2016)
47.	2016	Transmuted power function	Haq et al. (2016)
48.	2016	Transmuted geometric	Chakraborty and Bhati (2016)
49.	2016	Transmuted Birnbaum-Saunders	Bourguignon et al. (2016b)
50.	2016	Kumaraswamy Transmuted MO Fréchet	Yousof et al. (2016)
51.	2016	Transmuted Weibull-Pareto	Affify et al. (2016)

(Ln), zero-truncated binomial (ZTBi) and zero-truncated negative-binomial (ZTNB) distributions are described. In Section 4, we obtain the dual models for the maximum corresponding to those models discussed in Section 3. Section 5 deals with several compounding models and their extensions, which can be derived under the construction of the minimum and maximum of random variables. Sections 7 and 8 deal with other or different types of compounded models. In Section 8, we present recent trends on compounding of distributions, their *G*-classes and mixing of compounded and transmuted distributions. The main purpose of Section 9 is to briefly review general inference procedure, crude rate survival models and their inference. The paper is concluded with some remarks in Section 10.

## 2 Construction of compound *G*-classes

Suppose that a system has *N* subsystems assumed to be independent and identically distributed (i.i.d.) at a given time, where the lifetime of the *i*th subsystem is denoted by *Y<sub>i</sub>*, and that each subsystem is made of *α* parallel units, so that the system will fail if all the subsystems fail. We note that for a parallel system, the system fails only if all the subsystems fail, but for a series system, the failure of any subsystem will destroy the whole system. Further, suppose that the random variable *N* follows any discrete distribution with probability mass function (pmf)  $\mathbb{P}(N = n)$  (for  $n = 1, 2, \dots$ ). We consider that the failure times of the components  $Z_{i,1}, \dots, Z_{i,\alpha}$  for the *i*th subsystem are i.i.d. random variables having a suitable cdf, which is a function of the baseline cdf depending on a parameter vector  $\tau$ , say  $T[G(x; \tau), \alpha] = G(x; \tau)^\alpha$  (for  $x > 0$ ). In the following construction, although *α* is a positive integer called power or resilience parameter, we can consider that  $\alpha > 0$ .

If we define  $Y = \min\{Y_1, \dots, Y_N\}$ , then the conditional cdf of *Y* given *N* is

$$\begin{aligned}
 F(y | N) &= \mathbb{P}[\min(Y_1, \dots, Y_N) < y | N] = 1 - [1 - \mathbb{P}^\alpha(Z_{1,1} \leq y)]^N \\
 &= 1 - [1 - G(y; \tau)^\alpha]^N.
 \end{aligned}
 \tag{2}$$

If we define  $Z = \max\{Y_1, \dots, Y_N\}$ , then the conditional (or complementary) cdf of *Z* given *N* is

$$\begin{aligned}
 F(z | N) &= \mathbb{P}[\max(Y_1, \dots, Y_N) < z | N] = [\mathbb{P}^\alpha(Z_{n,n} \leq z)]^N \\
 &= [G(z; \tau)^\alpha]^N.
 \end{aligned}
 \tag{3}$$

The unconditional cdf of  $Y$  follows from Eq. (2) as

$$F(y) = \sum_{n=1}^{\infty} F(y|N) \mathbb{P}(N = n) = 1 - \sum_{n=1}^{\infty} [1 - G(y; \tau)^\alpha]^n \mathbb{P}(N = n). \tag{4}$$

The unconditional cdf of  $Z$  follows from Eq. (3) as

$$F(z) = \sum_{n=1}^{\infty} F(z|N) \mathbb{P}(N = n) = \sum_{n=1}^{\infty} [G(z; \tau)^\alpha]^n \mathbb{P}(N = n). \tag{5}$$

Many compound G-classes can be constructed from Eqs. (4) and (5) by choosing a discrete model with pmf  $\mathbb{P}(N = n)$ . The random variables  $Y = \min\{Y_1, \dots, Y_N\}$  and  $Z = \max\{Y_1, \dots, Y_N\}$  generate several models that arise in series and parallel systems with identical components and have many industrial and biological applications. For example, the time to the failure of a series system with an unknown number of protected components or the cancer recurrence of an individual after undergoing a certain treatment can be modeled by the generated distribution of  $Y$ . In a dual mechanism, the time to the failure of a parallel system with an unknown number of protected components or a disease manifestation, if it occurs only after an unknown number of factors have been active, can be modeled by the generated distribution of  $Z$ .

### 3 Compound G-classes

In this section, we present 14 compounded models obtained from Eq. (4). In Section 4, we present the corresponding complementary models based on Eq. (5). The list below does not include all compounded models but a large number of them and some new ones. For example, it does not describe the *exponentiated-G-Conway-Maxwell Poisson* (EGCOMP) class pioneered by Cordeiro et al. (2012a) and its special *GCOMP* model, among others. For all formulated models, we provide only their cdfs since the corresponding probability density functions (pdfs) can be determined by simple differentiation.

#### 3.1 Exponentiated G-geometric class

If  $T[G(y; \tau); \alpha] = G(y; \tau)^\alpha$  and  $N$  be a ZTG r.v. with pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$ ,  $p \in (0, 1)$ , where  $p$  is the success probability, the cdf of the exponentiated G-geometric (EGG) class can follow as

$$F_{EGG}(y) = 1 - \frac{q}{p} \sum_{n=1}^{\infty} [p(1 - G(y; \tau)^\alpha)]^n = \frac{G(y; \tau)^\alpha}{1 - p + pG(y; \tau)^\alpha}.$$

The EGG class has recently been introduced by Nadarajah et al. (2015a), and can also be called the generalized G-geometric class. For  $\alpha = 1$ , the EGG class reduces to the G-geometric (GG) class proposed by Alkarni (2012). They investigated some of its general properties.

**Remark 1** Swapping  $p$  with  $q$  in the ZTG pmf with  $p + q = 1$ , the cdf of an alternative EGG (denoted by EGGA) class is given by

$$F_{EGGA}(y) = 1 - \frac{p}{q} \sum_{n=1}^{\infty} [q(1 - G(y; \tau)^\alpha)]^n = \frac{G(y; \tau)^\alpha}{p - (p - 1)G(y; \tau)^\alpha}.$$

For  $\alpha = 1$ , the EGGA class reduces to the alternative G-geometric (GGA) class defined by Castellares and Lemonte (2016).

### 3.2 Exponentiated Kumaraswamy-G-geometric class (new)

For any baseline cdf  $G(x)$  and pdf  $g(x)$ , Cordeiro and de-Castro (2011) defined the cdf of the Kumaraswamy-G class by  $K_{a,b}(x; \tau) = 1 - [1 - G(x)^a]^b$ , where  $a > 0$  and  $b > 0$  are both shape parameters. Let  $T[G(y; \tau); \alpha] = K_{a,b}(y; \tau)^\alpha$  and  $N$  be a ZTG r.v. with pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$ , the cdf of the exponentiated Kumaraswamy-G-geometric (EKGG) class can follow as

$$F_{EKGG}(y) = 1 - \frac{q}{p} \sum_{n=1}^{\infty} \{p[1 - K_{a,b}(y; \tau)^\alpha]\}^n = \frac{K_{a,b}(y; \tau)^\alpha}{1 - p[1 - K_{a,b}(y; \tau)^\alpha]}.$$

For  $\alpha = 1$ , the EKGG class gives the new Kumaraswamy-G geometric (KGG) family.

### 3.3 McDonald-G-geometric class (new)

Alexander et al. (2012) defined the cdf of the McDonald-G class by  $M_{a,b,c}(x; \tau) = I_{G(x;\tau)^c}(a, b)$ , where  $B(a, b) = \int_0^1 w^{a-1} (1-w)^{b-1} dw$ ,  $B_z(a, b) = \int_0^z w^{a-1} (1-w)^{b-1} dw$  and  $I_z(a, b) = B_z(a, b)/B(a, b)$  are the beta function, incomplete beta function and incomplete beta function ratio, respectively.

If  $\alpha = 1$  and  $T[G(y; \tau)] = M_{a,b,c}(y; \tau)$ , and  $N$  be a ZTG r.v. with pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$ , the cdf of the McDonald-G geometric (MGG) class is given by

$$F_{MGG}(y) = 1 - \frac{q}{p} \sum_{n=1}^{\infty} \{p[1 - M_{a,b,c}(y; \tau)]\}^n = \frac{M_{a,b,c}(y; \tau)}{1 - p[1 - M_{a,b,c}(y; \tau)]}.$$

### 3.4 Beta-G-geometric class (new)

For any baseline cdf  $G(x)$ , Eugene et al. (2002) defined the cdf of the beta-G class by

$$B_{a,b}(x) = I_{G(x;\tau)}(a, b).$$

If  $\alpha = 1$  and  $T[G(y; \tau)] = B_{a,b}(y; \tau)$ , and  $N$  be a ZTG r.v. with pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$ , the cdf of the beta-G geometric (BGG) class is given by

$$F_{BGG}(y) = 1 - \frac{q}{p} \sum_{n=1}^{\infty} \{p[1 - B_{a,b}(y; \tau)]\}^n = \frac{B_{a,b}(y; \tau)}{1 - p[1 - B_{a,b}(y; \tau)]}.$$

### 3.5 Exponentiated G-Poisson class

If  $T[G(y; \tau); \alpha] = G(y; \tau)^\alpha$  and  $N$  be a ZTP r.v. with parameter  $\lambda$  defined by the pmf  $\mathbb{P}(N = n) = \lambda^n / [n! (e^\lambda - 1)]$ ,  $n = 1, 2, \dots$ , the cdf of the exponentiated G-Poisson (EGP) class can follow as

$$F_{EGP}(y) = 1 - \sum_{n=1}^{\infty} \frac{[1 - G(y; \tau)^\alpha]^n \lambda^n}{n! (e^\lambda - 1)} = \frac{1 - e^{-\lambda G(y; \tau)^\alpha}}{1 - e^{-\lambda}}.$$

The EGP class has been studied by Gomes et al. (2015). For  $\alpha = 1$ , it becomes the G-Poisson (GP) class as defined recently by Tahir et al. (2016a) by the name of the *Poisson-X* class since it was based on the T-X family.

### 3.6 Exponentiated Kumaraswamy-G-Poisson class (new)

If  $T[G(y; \tau); \alpha] = G(y; \tau)^\alpha$  and  $N$  be a ZTP r.v. with parameter  $\lambda$ , the cdf of the exponentiated Kumaraswamy-G Poisson (EKGP) class is given by

$$F_{EKGP}(y) = 1 - \sum_{n=1}^{\infty} [1 - K_{a,b}(y; \tau)^\alpha]^n \frac{\lambda^n}{n! (e^\lambda - 1)} = \frac{1 - e^{-\lambda K_{a,b}(y; \tau)^\alpha}}{1 - e^{-\lambda}}.$$

For  $\alpha = 1$ , the EKGP class reduces to the Kumaraswamy-G Poisson (KGP) family studied by Ramos et al. (2015).

**3.7 McDonald-G-Poisson class (new)**

Let  $\alpha = 1$  and  $T[G(y; \tau)] = M_{a,b,c}(y; \tau)$  and  $N$  be a ZTP r.v. with parameter  $\lambda$ , the cdf of the McDonald-G Poisson (MGP) class can be expressed as

$$F_{MGP}(y) = 1 - \frac{1}{e^\lambda - 1} \sum_{n=1}^{\infty} \frac{\{\lambda [1 - M_{a,b,c}(y; \tau)]\}^n}{n!} = \frac{1 - e^{-\lambda M_{a,b,c}(y; \tau)}}{1 - e^{-\lambda}}$$

**3.8 Beta-G-Poisson class (new)**

If  $\alpha = 1$  and  $T[G(y; \tau)] = B_{a,b}(y; \tau)$  and  $N$  be a ZTP r.v. with parameter  $\lambda$ , the cdf of the beta-G Poisson (BGP) class reduces to

$$F_{BGP}(y) = 1 - \frac{1}{e^\lambda - 1} \sum_{n=1}^{\infty} \frac{\{\lambda [1 - B_{a,b}(y; \tau)]\}^n}{n!} = \frac{1 - e^{-\lambda B_{a,b}(y; \tau)}}{1 - e^{-\lambda}}$$

**3.9 Exponentiated G-logarithmic class (new)**

If  $T[G(y; \tau); \alpha] = G(y; \tau)^\alpha$  and  $N$  be a Ln r.v. with pmf  $\mathbb{P}(N = n) = (1 - \phi)^n / [-n \ln \phi]$ ,  $\phi \in (0, 1)$ ,  $n = 1, 2, \dots$ , the cdf of the exponentiated G-logarithmic (EGLn) class is given by

$$F_{EGLn}(y) = 1 - \sum_{n=1}^{\infty} [1 - G(y; \tau)^\alpha]^n \frac{(1 - \phi)^n}{-n \ln \phi} = 1 - \frac{\ln \{1 - (1 - \phi)[1 - G(y; \tau)^\alpha]\}}{\ln \phi}$$

by noting that  $\sum_{n=1}^{\infty} Q^n/n = -\ln(1 - Q)$ .

For  $\alpha = 1$ , the EGLn class becomes the G-logarithmic (GLn) family introduced by Alkarni (2012).

**3.10 Exponentiated Kumaraswamy-G-logarithmic class (new)**

If  $T[G(y; \tau); \alpha] = G(y; \tau)^\alpha$  and  $N$  be a Ln r.v. with pmf  $\mathbb{P}(N = n) = (1 - \phi)^n / [-n \ln \phi]$ ,  $\phi \in (0, 1)$ ,  $n = 1, 2, \dots$ , the cdf of the exponentiated Kumaraswamy-G-logarithmic (EKGLn) is given by

$$\begin{aligned} F_{EKGLn}(y) &= 1 - \sum_{n=1}^{\infty} [1 - K_{a,b}(y; \tau)^\alpha]^n \frac{(1 - \phi)^n}{-n \ln \phi} \\ &= 1 - \frac{\ln \{1 - (1 - \phi)[1 - K_{a,b}(y; \tau)^\alpha]\}}{\ln \phi} \end{aligned}$$

For  $\alpha = 1$ , the EKGLn cdf is identical to the cdf of the new Kumaraswamy-G-logarithmic (KGLn) class.

**3.11 McDonald-G-logarithmic class (new)**

If  $T[G(y; \tau)] = M_{a,b,c}(y; \tau)$  and  $N$  be a Ln r.v. with pmf  $\mathbb{P}(N = n) = (1 - \phi)^n / [-n \ln \phi]$ ,  $\phi \in (0, 1)$ ,  $n = 1, 2, \dots$ , the cdf of the McDonald-G-logarithmic (MGLn) class is defined by



$$\begin{aligned}
 F_{MGLn}(y) &= 1 - \sum_{n=1}^{\infty} [1 - M_{a,b,c}(y; \tau)]^n \frac{(1 - \phi)^n}{-n \ln \phi} \\
 &= 1 - \frac{\ln \{1 - (1 - \phi)[1 - M_{a,b,c}(y; \tau)]\}}{\ln \phi}.
 \end{aligned}$$

**3.12 Beta-G-logarithmic class (new)**

If  $\alpha = 1$  and  $T[G(y; \tau)] = B_{a,b}(y; \tau)$  and  $N$  be a Ln r.v. with pmf  $\mathbb{P}(N = n) = (1 - \phi)^n / [-n \ln \phi]$ ,  $\phi \in (0, 1)$ ,  $n = 1, 2, \dots$ , the cdf of the beta-G-logarithmic (BGLn) class is given by

$$F_{BGLn}(y) = 1 - \frac{\ln \{1 - (1 - \phi)[1 - B_{a,b}(y; \tau)]\}}{\ln \phi}.$$

**3.13 Exponentiated G-binomial class (new)**

If  $T[G(y; \tau); \alpha] = G(y; \tau)^\alpha$  and  $N$  be a ZTBi r.v. with pmf given by

$$\mathbb{P}(N = k) = \frac{\binom{n}{k} \theta^k (1 - \theta)^{n-k}}{1 - (1 - \theta)^n}, \quad \theta \in (0, 1), \quad k = 1, 2, \dots, n,$$

the cdf of the exponentiated G-binomial (EGBi) class reduces to

$$F_{EGBi}(y) = \frac{1 - [1 - \theta G(y; \tau)^\alpha]^n}{1 - (1 - \theta)^n}.$$

Bakouch et al. (2012b) studied a special case of the EGBi class so-called the exponentiated-exponential binomial (EEBi) model. For  $\alpha = 1$ , the EGBi class becomes the G-binomial (GBi) class pioneered by Alkarni (2013).

**3.14 Exponentiated G-NB class (new)**

If  $T[G(y; \tau); \alpha] = G(y; \tau)^\alpha$  and  $N$  be a ZTNB r.v. with parameter  $\beta \in (0, 1)$  as the probability of success and pmf given by

$$\mathbb{P}(N = n) = \binom{s + n - 1}{n} \frac{\beta^n}{(1 - \beta)^{-s} - 1}, \quad s > 0, \quad n = 1, 2, \dots,$$

the cdf of the exponentiated G-NB (EGNB) class can follow as

$$F_{EGNB}(y) = 1 - \frac{1}{(1 - \beta)^{-s} - 1} \sum_{n=1}^{\infty} \binom{s + n - 1}{n} \{\beta [1 - G(y; \tau)^\alpha]\}^n.$$

Using the binomial expansion  $(1 - w)^{-s} - 1 = \sum_{n=1}^{\infty} \binom{s+n-1}{n} w^n$ , we obtain

$$F_{EGNB}(y) = \frac{(1 - \beta)^{-s} - \{1 - \beta [1 - G(y; \tau)^\alpha]\}^{-s}}{(1 - \beta)^{-s} - 1}. \tag{6}$$

For  $\alpha = 1$ , the EGNB class leads to the G-NB (GNB) class proposed by Percontini et al. (2013b).

**Remark 2** *i* If we replace the probability of success  $\beta$  by  $1 - \beta$  and the dispersion parameter  $s$  by  $\theta$  in (6), the cdf of an alternate form of the GNB class (denoted by EGNB1) will be

$$F_{EGNB1}(y) = 1 - \frac{\beta^\theta}{1 - \beta^\theta} \left\{ [G(y; \tau)^\alpha + (1 - \beta)\bar{G}(y; \tau)^\alpha]^{-\theta} - 1 \right\}. \tag{7}$$

For  $\alpha = 1$ , the EGNB1 class leads to the GNB family discussed by Nadarajah et al. (2013b), who also studied, as a special case of the GNB1 class, the exponential-truncated negative-binomial (ETNB) model.

- ii If we express  $\beta$  in terms of the population mean of the distribution in (6), the cdf of an alternate form of the GNB class (denoted by EGNB2) will be

$$F_{EGNB2}(y) = \frac{1 - [1 + \eta\theta G(y; \tau)^\alpha]^{-\frac{1}{\eta}}}{1 - (1 + \eta\theta)^{-\frac{1}{\eta}}}.$$

For  $\alpha = 1$ , the EGNB2 class leads to the GNB family. Louzada et al. (2012b) studied a special model of this class.

#### 4 Complementary compound G-classes

Complementary compound models are constructed by considering the maximum of a sequence of i.i.d. random variables which represents the risk time of a system having components in parallel structure. In this section, we generate from Eq. (5) the complementary G-classes of those ones presented in Section 3. Some proposed complementary G-classes are really new ones.

##### 4.1 Complementary exponentiated G-geometric class (new)

If  $T[G(z; \tau); \alpha] = G(z; \tau)^\alpha$  and  $N$  be a ZTG r.v. with geometric pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$ , the cdf of the complementary exponentiated G-geometric (CEGG) class reduces to

$$F_{CEGG}(z) = \frac{q}{p} \sum_{n=1}^{\infty} [pG(z; \tau)^\alpha]^n = \frac{(1-p)G(z; \tau)^\alpha}{1-pG(z; \tau)^\alpha}.$$

This equation is also called the complementary generalized G-geometric family. For  $\alpha = 1$ , the CEGG class becomes the complementary G-geometric (CGG) family.

**Remark 3** Swapping  $p$  with  $q$  in the ZTG pmf with  $p + q = 1$ , the cdf of an alternative CEGG (CEGGA) class will be

$$F_{CEGGA}(z) = \frac{p}{q} \sum_{n=1}^{\infty} [qG(z; \tau)^\alpha]^n = \frac{pG(z; \tau)^\alpha}{1 - (1-p)G(z; \tau)^\alpha}.$$

For  $\alpha = 1$ , the CEGGA class leads to the complementary G-geometric (CGG) family proposed by Castellares and Lemonte (2016).

##### 4.2 Complementary exponentiated Kumaraswamy-G-geometric class (new)

If  $\alpha = 1$ ,  $T[G(z; \tau)] = K_{a,b}(z; \tau)$  and  $N$  be a geometric r.v. with pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$ , the cdf of the CEKGG class is given by

$$F_{CEKGG}(z) = \frac{q}{p} \sum_{n=1}^{\infty} \{p[K_{a,b}(z; \tau)^\alpha]\}^n = \frac{q[K_{a,b}(z; \tau)^\alpha]}{1 - pK_{a,b}(z; \tau)^\alpha}.$$

For  $\alpha = 1$ , the CEKGG class leads to the complementary KGG (CKGG) class.

**4.3 Complementary McDonald-G-geometric class (new)**

If  $\alpha = 1$ ,  $T[G(z; \tau)] = M_{a,b,c}(z; \tau)$  and  $N$  be a geometric r.v. with pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$ , the cdf of the complementary McDonald-G-geometric (CMGG) class can follow as

$$F_{CMGG}(z) = \frac{q}{p} \sum_{n=1}^{\infty} \{p[M_{a,b,c}(z; \tau)]\}^n = \frac{q[M_{a,b,c}(z; \tau)]}{1 - pM_{a,b,c}(z; \tau)}.$$

**4.4 Complementary beta-G-geometric class (new)**

If  $\alpha = 1$ ,  $T[G(z; \tau)] = B_{a,b}(z; \tau)$ , and  $N$  be a geometric r.v. with pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$ , the cdf of the complementary beta-G-geometric (CBGG) class is given by

$$F_{CBGG}(z) = \frac{q}{p} \sum_{n=1}^{\infty} [pB_{a,b}(z; \tau)]^n = \frac{qB_{a,b}(z; \tau)}{1 - pB_{a,b}(z; \tau)}.$$

**4.5 Complementary exponentiated G-Poisson class (new)**

If  $T[G(z; \tau); \alpha] = G(z; \tau)^\alpha$  and  $N$  be a ZTP r.v. with parameter  $\lambda$ , the cdf of the complementary exponentiated G-Poisson (CEGP) can follow as

$$F_{CEGP}(z) = \sum_{n=1}^{\infty} [G(z; \tau)^\alpha]^n \frac{\lambda^n}{n!(e^\lambda - 1)} = \frac{e^{\lambda G(z; \tau)^\alpha} - 1}{e^\lambda - 1}.$$

For  $\alpha = 1$ , the CEGP class leads to the complementary G-Poisson (CGP) class.

**4.6 Complementary exponentiated Kumaraswamy-G Poisson class (new)**

If  $T[G(z; \tau); \alpha] = K_{a,b}(z; \tau)^\alpha$  and  $N$  be a ZTP r.v. with parameter  $\lambda$ , the cdf of the complementary exponentiated Kumaraswamy-G Poisson (CEKGP) class is given by

$$F_{CEKGP}(z) = \sum_{n=1}^{\infty} [K_{a,b}(z; \tau)^\alpha]^n \frac{\lambda^n}{n!(e^\lambda - 1)} = \frac{e^{\lambda K_{a,b}(z; \tau)^\alpha} - 1}{e^\lambda - 1}.$$

For  $\alpha = 1$ , the CEKGP class reduces to the complementary Kumaraswamy-G Poisson (CKGP) class.

**4.7 Complementary McDonald-G Poisson class (new)**

If  $\alpha = 1$  and  $T[G(z; \tau)] = M_{a,b,c}(z; \tau)$  and  $N$  be a ZTP r.v. with parameter  $\lambda$ , the cdf of the complementary McDonald-G Poisson (CMGP) class reduces to

$$F_{CMGP}(z) = \sum_{n=1}^{\infty} [M_{a,b,c}(z; \tau)]^n \frac{\lambda^n}{n!(e^\lambda - 1)} = \frac{e^{\lambda M_{a,b,c}(z; \tau)} - 1}{e^\lambda - 1}.$$

**4.8 Complementary beta G-Poisson class (new)**

If  $\alpha = 1$ ,  $T[G(z; \tau)] = B_{a,b}(z; \tau)$  and  $N$  be a ZTP r.v. with parameter  $\lambda$ , the cdf of the complementary beta G-Poisson (CBGP) class is given by

$$F_{CBGP}(z) = \frac{1}{e^\lambda - 1} \sum_{n=1}^{\infty} \frac{\{\lambda[B_{a,b}(z; \tau)]\}^n}{n!} = \frac{e^{\lambda B_{a,b}(z; \tau)} - 1}{e^\lambda - 1}.$$

**4.9 Complementary exponentiated G-logarithmic class (new)**

If  $T[G(z; \tau); \alpha] = G(z; \tau)^\alpha$  and  $N$  be a Ln r.v. with pmf given by  $\mathbb{P}(N = n) = (1 - \phi)^n / [-n \ln \phi]$ ,  $\phi \in (0, 1)$ ,  $n = 1, 2, \dots$ , the cdf of the complementary exponentiated G-logarithmic (CEGLn) class can be expressed as

$$F_{CEGLn}(z) = \frac{1}{-\ln \phi} \sum_{n=1}^{\infty} \frac{[(1 - \phi) G(z; \tau)^\alpha]^n}{n} = \frac{\ln [1 - (1 - \phi) G(z; \tau)^\alpha]}{\ln \phi},$$

by noting that  $\sum_{n=1}^{\infty} Q^n/n = -\ln(1 - Q)$ . For  $\alpha = 1$ , the CEGLn class becomes the complementary G-logarithmic (CGLn) class.

**4.10 Complementary exponentiated Kumaraswamy-G-logarithmic class (new)**

If  $T[G(z; \tau); \alpha] = K_{a,b}(z; \tau)^\alpha$  and  $N$  be a Ln r.v. with  $\mathbb{P}(N = n) = (1 - \phi)^n / [-n \ln \phi]$ ,  $\phi \in (0, 1)$ ,  $n = 1, 2, \dots$ , the cdf of the complementary exponentiated Kumaraswamy G-logarithmic of type 1 (CEKGLn1) class is given by

$$F_{CEKGLn}(z) = \sum_{n=1}^{\infty} [K_{a,b}(z; \tau)^\alpha]^n \frac{(1 - \phi)^n}{-n \ln \phi} = \frac{\ln [1 - (1 - \phi) K_{a,b}(z; \tau)^\alpha]}{\ln \phi}.$$

For  $\alpha = 1$ , the complementary Kumaraswamy-G-logarithmic (CEKGLn) class becomes the complementary G-logarithmic (CGLn) class.

**4.11 Complementary McDonald-G-logarithmic class (new)**

If  $\alpha = 1$ ,  $T[G(z; \tau)] = M_{a,b,c}(z; \tau)$  and  $N$  be a Ln r.v. with pmf  $\mathbb{P}(N = n) = (1 - \phi)^n / (-n \ln \phi)$ ,  $\phi \in (0, 1)$ ,  $n = 1, 2, \dots$ , the cdf of the complementary McDonald-G-logarithmic (CMGLn) class can be expressed as

$$F_{CMGLn}(z) = \sum_{n=1}^{\infty} [M_{a,b,c}(z; \tau)]^n \frac{(1 - \phi)^n}{-n \ln \phi} = \frac{\ln [1 - (1 - \phi) M_{a,b,c}(z; \tau)]}{\ln \phi}.$$

**4.12 Complementary beta-G-logarithmic class (new)**

If  $\alpha = 1$  and  $T[G(z; \tau)] = B_{a,b}(z; \tau)$  and  $N$  be a Ln r.v. with pmf  $\mathbb{P}(N = n) = (1 - \phi)^n / (-n \ln \phi)$ ,  $\phi \in (0, 1)$ ,  $n = 1, 2, \dots$ , the cdf of the complementary beta-G-logarithmic (CBGLn) class is given by

$$F_{CBGLn}(z) = \sum_{n=1}^{\infty} [B_{a,b}(z; \tau)]^n \frac{(1 - \phi)^n}{-n \ln \phi} = \frac{\ln [1 - (1 - \phi) B_{a,b}(z; \tau)]}{\ln \phi}.$$

**4.13 Complementary exponentiated G-binomial class (new)**

If  $T[G(z; \tau); \alpha] = G(z; \tau)^\alpha$  and  $N$  be a ZTBi r.v. with pmf

$$\mathbb{P}(N = k) = \frac{\binom{n}{k} \theta^k (1 - \theta)^{n-k}}{1 - (1 - \theta)^n}, \quad \theta \in (0, 1), \quad k = 1, 2, \dots, n,$$

the cdf of the complementary exponentiated G-binomial (CEGBi) class is given by

$$\begin{aligned} F_{CEGBi}(z) &= \frac{1}{1 - (1 - \theta)^n} \left[ \sum_{k=0}^n \binom{n}{k} [\theta G(z; \tau)^\alpha]^k (1 - \theta)^{n-k} - (1 - \theta)^n \right] \\ &= \frac{\{1 - \theta [1 - G(z; \tau)^\alpha]\}^n - (1 - \theta)^n}{1 - (1 - \theta)^n}. \end{aligned}$$

For  $\alpha = 1$ , the CEGBi class becomes the complementary G-binomial (CGBi) class.

**4.14 Complementary exponentiated G-NB class (new)**

If  $T[G(z; \tau); \alpha] = G(z; \tau)^\alpha$  and  $N$  be a ZTNB r.v. with parameter  $\beta \in (0, 1)$  with pmf

$$\mathbb{P}(N = n) = \binom{s+n-1}{n} \frac{\beta^n}{(1-\beta)^{-s}-1}, \quad s > 0, \quad n = 1, 2, \dots$$

Using the binomial expansion  $(1-w)^{-s} - 1 = \sum_{n=1}^{\infty} \binom{s+n-1}{n} w^n$ , the cdf of the complementary exponentiated G-NB (CEGNB) class reduces to

$$F_{CEGNB}(z) = \frac{[1 - \beta G(z; \tau)^\alpha]^{-s} - 1}{(1-\beta)^{-s} - 1}.$$

For  $\alpha = 1$ , the CEGNB class leads to the new complementary GNB class.

**5 Review of existing compounded models**

In this section, we review some available compounded models. In the literature, several authors have reported compounding discrete distributions, namely the ZTG, ZTP, logarithmic, ZTBi, ZTNB, zero-truncated generalized Poisson and zero-truncated power-series, with continuous lifetime models.

**5.1 Compounded models based on geometric distribution**

For the following models,  $X$  denotes the r.v. of the baseline G model.

*Exponential-geometric (EG) model.* If  $X \sim \text{Exponential}(\beta)$  have cdf  $G(x; \beta) = 1 - \exp(-\beta x)$ , then the cdf of the EG model pioneered by Adamidis and Loukas (1998) (as the first compounded model in the literature) can be determined from the GG class (Section 3.1 with  $\alpha = 1$ ) as

$$F_{EG}(y; p, \beta) = \frac{1 - e^{-\beta y}}{1 - p e^{-\beta y}}, \quad y, \beta > 0, p \in (0, 1).$$

Adamidis et al. (2005) also defined an extended EG model.

*Generalized EG (GEG) model.* If  $X \sim \text{Exponential}(\beta)$ , Silva et al. (2010) applied the LA1 class to the EG cdf in the last equation and defined the GEG cdf by

$$F_{GEG}(y; p, \beta, \gamma) = \left( \frac{1 - e^{-\beta y}}{1 - p e^{-\beta y}} \right)^\gamma, \quad y > 0,$$

where  $\gamma > 0$  is the power parameter,  $\beta > 0$  and  $p \in (0, 1)$ .

*Complementary EG (CEG) model (proposed).* If  $X \sim \text{Exponential}(\beta)$ , then the CEG cdf can be determined from the CEGGA class (Section 4.1 with  $\alpha = 1$ ) as

$$F_{CEG}(z; p, \beta) = \frac{(1-p)(1 - e^{-\beta z})}{1 - p(1 - e^{-\beta z})}.$$

*Long-term complementary EG (LCEG) model.* If  $X \sim \text{Exponential}(\beta)$ , then Louzada et al. (2012d) proposed another CEG model called the LCEG model, in a latent complementary risk framework, with cdf given by

$$F_{LCEG}(z; p, \beta, \theta) = \frac{\theta(1-p)(1 - e^{-\beta z})}{e^{-\beta z}(1-\theta) + \theta}, \quad z > 0, \theta \in (0, 1), p \in (0, 1),$$

where  $\beta > 0$  is a scale parameter,  $\theta$  is a shape parameter and  $p$  is the long-term parameter.

*Generalized complementary EG (GCEG) model.* Bidram and Nadadrajah (2016) applied the LA1 class to the CEGA cdf (given in Section 4.1 with  $\alpha = 1$ ) and obtained the GCEG cdf as

$$F_{GCEG}(z; p, \beta, \gamma) = \left[ \frac{(1-p)(1-e^{-\beta z})}{1-p(1-e^{-\beta z})} \right]^\gamma, \quad z > 0,$$

where  $\gamma > 0$  is the power parameter,  $\beta > 0$  and  $p \in (0, 1)$ .

*Complementary exponentiated-exponential geometric (CEEG) model.* If  $X \sim \text{ExponentiatedExponential}(\beta, \alpha)$  have cdf  $G(x; \beta, \alpha) = [1 - \exp(-\beta x)]^\alpha$ , then the cdf of the CEEG model, proposed by Louzada et al. (2014a), can be determined from the alternative CEEG class (Section 4.1 with  $\alpha = 1$ ) as

$$F_{CEEG}(z; p, \beta, \alpha) = \frac{p(1-e^{-\beta z})^\alpha}{1-(1-p)[1-e^{-\beta z}]^\alpha}, \quad z, \alpha, \beta > 0, p \in (0, 1).$$

*Exponentiated-CEG (ECEG) model.* Yamachi et al. (2013) generalized the CEG model  $F_{CEG}(z; p, \beta, b) = \frac{p(1-e^{-b\beta z})}{e^{-\beta z}(1-p)+p}$  introduced by Louzada et al. (2011) by applying the LA1 class and defined the ECEG cdf as

$$F_{ECEG}(z; p, \beta, b, \gamma) = \left( \frac{p(1-e^{-b\beta z})}{e^{-\beta z}(1-p)+p} \right)^\gamma,$$

where  $\gamma > 0$  is the power parameter.

*Beta-EG (BEG) model.* If  $X \sim \text{Exponential}(\beta)$ , the cdf of the BEG model proposed by Bidram (2012) and Nassar and Nada (2012) can follow from the BGG class (Section 3.4) as

$$F_{BEG}(y; p, \beta, l, m) = \frac{B_{l,m}(EG)}{1-p+pB_{l,m}(EG)},$$

where

$$B_{l,m}(EG) = \frac{1}{B(l, m)} \int_0^{\left[ \frac{1-e^{-\beta x}}{1-pe^{-\beta x}} \right]} t^{l-1} (1-t)^{m-1} dt.$$

*Exponentiated EG (EEG) model.* The cdf of the EEG model defined by Louzada et al. (2014) can be determined from the EGGA class (Section 3.1) as

$$F_{EEG}(y; p, \beta, \alpha) = \frac{(1-e^{-\beta y})^\alpha}{1-(1-p)[1-(1-e^{-\beta y})^\alpha]}, \quad y > 0, \alpha, \beta > 0, p \in (0, 1).$$

Ristić and Kundu (2016) proposed the generalized geometric extreme distribution, which is identical to the EEG model given above.

*Modified EG (MEG) model.* Let  $X \sim \text{ModifiedExponential}(\beta, \pi)$  or the MO-exponential with cdf  $G(x; \beta, \pi) = 1 - \exp(-\beta x)/[1 - (1 - \pi) \exp(-\beta x)]$ . The cdf of the modified EG (MEG) distribution given by Bordbar and Nematollah (2016) can follow from the GG class (Section 3.1 with  $\alpha = 1$ ) as

$$F_{MEG}(y; p, \beta, \pi) = \frac{1 - e^{-\beta y}}{1 - (1 - \pi)e^{-\beta y} - p\pi e^{-\beta y}}, \quad y > 0.$$

*Exponentiated-LL geometric (ELLG) and complementary exponentiated-LL geometric (CELLG) models.* If  $X \sim \text{ELL}(a, b)$  has cdf  $G(x; a, b, \alpha) = \frac{\alpha b}{a} (x/a)^{-b-1} [1 + (x/a)^{-b}]^{-\alpha-1}$ , then the cdfs of the ELLG and CELLG models proposed by Mendoza et al. (2016) can be determined from the GG and CGG classes described in Sections 3.1 and 4.1 (with  $\alpha = 1$ ), respectively, as

$$F_{ELLG}(y; p, a, b, \alpha) = \frac{[1 + (y/a)^{-b}]^{-\alpha}}{1-p \left\{ 1 - [1 + (y/a)^{-b}]^{-\alpha} \right\}}, \quad y > 0, a, b, \alpha, p \in (0, 1)$$

and

$$F_{CELLG}(z; p, a, b, \alpha) = \frac{(1-p)[1+(z/a)^{-b}]^{-\alpha}}{1-p[1+(z/a)^{-b}]^{-\alpha}}, \quad z > 0, \quad a, b, \alpha, p \in (0, 1).$$

*Generalized linear failure rate-geometric (GLFRG) model.* If  $X \sim \text{LinearExponential}(a, b)$  have cdf  $G(x; a, b) = 1 - \exp[-(ax + (b/2)x^2)]$ , then the cdf of the GLFRG model proposed by Nadarajah et al. (2014a) can follow from the EGG class (Section 3.1) as

$$F_{GLFRG}(y; p, a, b, \alpha) = \frac{[1 - e^{-(ay+(b/2)y^2)}]^\alpha}{1 - p \{1 - [1 - e^{-(ay+(b/2)y^2)}]^\alpha\}}, \quad y > 0, \quad a, b, \alpha > 0.$$

*Complementary generalized linear failure rate-geometric (CGLFRG) model.* If  $X \sim \text{LinearExponential}(a, b)$  have cdf  $G(x; a, b) = 1 - \exp[-(ax + (b/2)x^2)]$ , then the CGLFRG cdf defined by Harandi and Alamatsaz (2016) can be determined from the CEGG class (Section 4.1) as

$$F_{CGLFRG}(z; p, a, b, \alpha) = \frac{(1-p)[1 - e^{-(az+(b/2)z^2)}]^\alpha}{1 - p \{1 - [1 - e^{-(az+(b/2)z^2)}]^\alpha\}}, \quad z > 0, \quad a, b, \alpha > 0.$$

*Exponentiated power Lindley-geometric (EPLG) model.* If  $X \sim \text{ExpPowerLindley}(\alpha, \beta, \theta)$  introduced by Ashour and Eltehiwy (2015) have cdf  $G(x; \alpha, \beta, \theta) = \left[1 - \left(1 + \frac{\theta x^\beta}{\theta+1}\right) e^{-\theta x^\beta}\right]^\alpha$ , the EPLG cdf defined by Alizadeh et al. (2016b) can follow from the GG class (Section 3.1) as

$$F_{EPLG}(y; p, \alpha, \beta, \theta) = \frac{(1-p) \left[1 - \left(1 + \frac{\theta y^\beta}{\theta+1}\right) e^{-\theta y^\beta}\right]^\alpha}{1 - p \left[1 - \left(1 + \frac{\theta y^\beta}{\theta+1}\right) e^{-\theta y^\beta}\right]^\alpha}, \quad y > 0, \quad \alpha, \beta, \theta > 0.$$

*Weibull-geometric (WG) model.* If  $X \sim \text{Weibull}(a, b)$  have cdf  $G(x; a, b) = 1 - e^{-(bx)^a}$ , then the cdf of the WG model proposed by Barreto-Souza et al. (2011) can be expressed from the GG class (Section 3.1 with  $\alpha = 1$ ) as

$$F_{WG}(y; p, a, b) = \frac{1 - e^{-(by)^a}}{1 - p e^{-(by)^a}}, \quad y, a, b > 0, \quad p \in (0, 1).$$

*Exponentiated Weibull-geometric (EWG) model.* If  $X \sim \text{Weibull}(a)$  have cdf  $G(x; a) = 1 - e^{-x^a}$ , then the EWG cdf given by Chung and Kang (2014) can be obtained from the EGG cdf (Section 3.1) as

$$F_{EWG}(y; p, a, \alpha) = \frac{(1 - e^{-y^a})^\alpha}{1 - p [1 - (1 - e^{-y^a})^\alpha]}, \quad y, a, \alpha > 0, \quad p \in (0, 1).$$

*Beta-Weibull-geometric (BWG) model.* If  $X \sim \text{Weibull}(a, b)$  have cdf  $G(x; a, b) = 1 - e^{-(bx)^a}$ , then the cdf of the BWG model defined by Bidram et al. (2013) and Cordeiro et al. (2013b) can follow from the BGG cdf (given in Section 3.4) as

$$F_{BWG}(y; p, a, b, l, m) = \frac{B_{l,m(WG)}(y)}{1 - p + p B_{l,m(WG)}(y)},$$

where

$$B_{l,m(WG)}(y) = \frac{1}{B(l, m)} \int_0^{\frac{1 - e^{-(by)^a}}{1 - p e^{-(by)^a}}} t^{l-1} (1 - t)^{m-1} dt.$$

*Complementary exponentiated Weibull-geometric (CEWG) model.* If  $X$  have Weibull cdf  $G(x; a, b) = 1 - e^{-(bx)^a}$ , the cdf of the CEWG model proposed by Mahmoudi and Shiran (2012a) follows from the CEGG class (Section 4.1) as

$$F_{CEWG}(z; p, a, b, \alpha) = \frac{(1 - p)(1 - e^{-(bz)^a})^\alpha}{1 - p[1 - e^{-(bz)^a}]^\alpha}, \quad z, a, b, \alpha > 0, p \in (0, 1).$$

*Complementary Weibull-geometric (CWG) model.* If  $X \sim \text{Weibull}(a, b)$  have cdf  $G(x; a, b) = 1 - e^{-(bx)^a}$ , the cdf of the CWG model proposed by Tojeiro et al. (2014) follows from the alternative CGG class given in Section 4.1 ( $\alpha = 1$ ) as

$$F_{CWG}(z; p, a, b) = \frac{p(1 - e^{-(bz)^a})}{p + (1 - p)e^{-(bz)^a}}, \quad z, a, b > 0, p \in (0, 1).$$

*Modified Weibull-geometric (MWG) model.* If  $X \sim \text{ModifiedWeibull}(a, b, \delta)$  introduced by Sarhan and Zaindin (2009) have cdf  $G(x; a, b, \delta) = 1 - \exp[-(ax + bx^\delta)]$ , then the cdf of the MWG model defined by Wang and Elbatal (2015) can be determined from the GG class (Section 3.1,  $\alpha = 1$  and all parameters positive) as

$$F_{MWG}(y; p, a, b, \delta) = \frac{1 - e^{-(ay + by^\delta)}}{1 - p e^{-(ay + by^\delta)}}, \quad y > 0.$$

*Kummer-beta Weibull-geometric (KBWG) model.* Ashour and Wahed (2014) extended the WG model to the KBWG model. The KBWG cdf is given by

$$F_{KBWG}(y; p, a, b, c) = K \int_0^{1 - e^{-(by)^\alpha}} \frac{1 - e^{-(by)^\alpha}}{1 - p e^{-(by)^\alpha}} y^{l-1} (1 - y)^{m-1} e^{-cx} dx,$$

where  $K = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)} {}_1F_1(l; l + m; -c)$  and  ${}_1F_1$  is the Gauss hyper-geometric function.

*Additive Weibull-geometric (AWG) model.* If  $X \sim \text{AdditiveWeibull}(a, b, \gamma)$  introduced by Cordeiro et al. (2014c) have cdf  $G(x; a, b, \gamma) = 1 - e^{-(ax + bx^\gamma)}$ ,  $x, a, b > 0$  and  $\gamma \in (0, \infty) \setminus \{1\}$ , then the cdf of the AWG model defined by Elbatal et al. (2016) can be obtained from the GG class (Section 3.1,  $\alpha = 1$ ) as

$$F_{AWG}(y; p, a, b, \gamma) = \frac{1 - e^{-(ax + by^\gamma)}}{1 - p e^{-(ax + by^\gamma)}}, \quad y > 0.$$

*Lindley-geometric (LG) model.* If  $X \sim \text{Lindley}(\theta)$  have cdf  $G(x; \theta) = 1 - \left(1 + \frac{\theta x}{\theta + 1}\right) e^{-\theta x}$  (for  $x > 0, \theta > 0$ ), then the cdf of the LG model proposed by Zakerzadeh and Mahmoudi (2013) can follow from the GG class (Section 3.1,  $\alpha = 1$ ) as

$$F_{LG}(y; p, \theta) = \frac{1 - \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}}{1 - p \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}}, \quad y > 0, \theta > 0, p \in (0, 1).$$

*Exponentiated Lindley-geometric (ELG) model.* If  $X \sim \text{Lindley}(\theta)$  have cdf  $G(x; \theta) = 1 - \left(1 + \frac{\theta x}{\theta + 1}\right) e^{-\theta x}$  (for  $x > 0, \theta > 0$ ), then the cdf of the ELG model proposed by Wang (2013) follows from the EGG class (Section 3.1) as

$$F_{ELG}(y; p, \theta) = \frac{\left[1 - \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}\right]^\alpha}{1 - p + p \left[1 - \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}\right]^\alpha}.$$

## 5.2 Compounded models based on Poisson distribution

*Exponential-Poisson (EP).* If  $X \sim \text{Exponential}(\beta)$ , then the cdf of the EP model proposed by Kuş (2007) can be obtained from the GP class defined in Section 3.5 (with  $\alpha = 1$ ) as

$$F_{EP}(y; \lambda, \beta) = \frac{1 - e^{-\lambda - \lambda e^{-\beta y}}}{1 - e^{-\lambda}}, \quad y, \beta > 0, p \in (0, 1).$$



For some structural properties and applications of the EP model, the reader is referred to Kuş (2007) and Cancho et al. (2011a).

*Generalized EP (GEP) model.* If  $X \sim \text{Exponential}(\beta)$ , Barreto-Souza and Cribari-Neto (2009) obtained the GEP cdf by applying the LA1 class to the EP cdf given in the last equation

$$F_{GEP}(y; \lambda, \beta, \gamma) = \left( \frac{1 - e^{-\lambda - \lambda e^{-\beta y}}}{1 - e^{-\lambda}} \right)^\gamma, \quad y > 0,$$

where  $\gamma > 0$  is the power parameter,  $\beta > 0$  and  $p \in (0, 1)$ .

*Complementary EP (CEP) model.* If  $X \sim \text{Exponential}(\beta)$ , the cdf of the CEP model, proposed by Cancho et al. (2011a) and Rezaei and Tahmasbi (2012), can be determined from the CGP class (Section 4.5,  $\alpha = 1$ ) as

$$F_{CEP}(z; \lambda, \beta) = \frac{e^{-\lambda(1-e^{-\beta z})} - e^{-\lambda}}{1 - e^{-\lambda}}, \quad z, \beta, \lambda > 0.$$

*Modified EP (MEP) model.* If  $X \sim \text{ModifiedExponential}(\beta)$  or MO-exponential with cdf  $G(x; \beta) = \frac{1 - e^{-\beta x}}{1 - (1-p)e^{-\beta x}}$ , then the cdf of the MEP model, proposed by Preda et al. (2011), can follow from the GP class (Section 3.9,  $\alpha = 1$ ) as

$$F_{MEP}(y; \lambda, p, \beta) = \frac{1 - e^{-\lambda \left( \frac{1 - e^{-\beta y}}{1 - (1-p)e^{-\beta y}} \right)}}{1 - e^{-\lambda}}, \quad y, \beta > 0, p \in (0, 1).$$

*Exponentiated EP (EEP) model.* If  $X \sim \text{Exponential}(\beta)$ , then the cdf of the EEP model given by Ristić and Nadarajah (2014) can be obtained from the EGP class defined in Section 3.5 as

$$F_{EEP}(y; \lambda, \beta, \alpha) = \frac{1 - e^{-\lambda(1-e^{-\beta y})^\alpha}}{1 - e^{-\lambda}}, \quad y > 0, \quad \alpha, \beta, \lambda > 0.$$

*Weibull-Poisson (WP) model.* If  $X \sim \text{Weibull}(a)$  have cdf  $G(x; a, b) = 1 - e^{-bx^a}$ , then the cdf of the WP model proposed by Lu and Shi (2012) can follow from the GP class (Section 3.5,  $\alpha = 1$ ) as

$$F_{WP}(y; \lambda, a, b) = \frac{e^{\lambda e^{-by^a}} - e^\lambda}{1 - e^\lambda}, \quad y, a, b, \lambda > 0.$$

Hemmati et al. (2011) and Bereta et al. (2011) also obtained the WP or Poisson-Weibull (PW) models from the Weibull cdf  $G(y; a, b) = 1 - \exp\{-(by)^a\}$ . The cdf of their WP model is given by

$$F_{WP}(y; \lambda, a, b) = \frac{e^{\lambda e^{-(by)^a}} - e^\lambda}{1 - e^\lambda}, \quad y, a, b, \lambda > 0.$$

*Beta-Weibull Poisson (BWP) model.* If  $X \sim \text{Weibull}(a, b)$  have cdf  $G(x; a, b) = 1 - e^{-bx^a}$ , then the cdf of the BWP model, proposed by Percontini et al. (2013a), can follow from the BGP class defined in Section 3.8 as

$$F_{BEG}(y; \lambda, a, b, l, m) = \frac{e^{-\lambda [1 - B_{l,m}(EG)(y)]} - e^\lambda}{1 - e^\lambda},$$

where

$$B_{l,m}(EG)(y) = \frac{1}{B(l, m)} \int_0^{\frac{e^{\lambda e^{-by^a}} - e^\lambda}{1 - e^\lambda}} t^{l-1} (1 - t)^{m-1} dt.$$

*Complementary modified Weibull-Poisson (CMWP) model.* If  $X \sim$  ModifiedWeibull( $a, b, \delta$ ) introduced by Sarhan and Zaindin (2009) have cdf  $G(x; a, b, \delta) = 1 - \exp[-(ax + bx^\delta)]$ , then the cdf of the CMWP model proposed by Ghorbani et al. (2014) can be obtained from the GP class presented in Section 3.5 (with  $\alpha = 1$ ) as

$$F_{CMWP}(z; \lambda, a, b, \delta) = \frac{e^{\lambda[1 - e^{-(az + bz^\delta)}]} - 1}{e^\lambda - 1}, \quad z > 0, a, b, \delta, \lambda > 0.$$

*Complementary exponentiated Weibull-Poisson (CEWP) model.* If  $X \sim$  Weibull( $a, b$ ) have cdf  $G(x; a, b) = 1 - e^{-(bx)^a}$ , then the cdf of the CEWP model defined by Mahmoudi and Sepahdar (2013) can follow from the EGP class (see Section 3.5) as

$$F_{CEWP}(z; \lambda, a, b, \alpha) = \frac{e^{\lambda(1 - e^{-(bz)^a})^\alpha} - 1}{e^\lambda - 1}, \quad z, \lambda, a, b, \alpha > 0.$$

*Log-logistic generalized Weibull Poisson (LLGWP) model.* Let  $X \sim$  Weibull( $a, b$ ) have cdf  $G(x; a, b) = 1 - e^{-bx^a}$ . Oluyede et al. (2016a) first defined the log-logistic generalized Weibull (LLGW) cdf by  $F_{LLGW} = 1 - [1 + (x/s)^c]^{-1} e^{-bx^a}$  from the generalized extended Weibull family  $G(x; \theta, \psi) = 1 - B(x; \psi) \exp\{-\alpha H(x; \theta)\}$ , where  $\theta$  and  $\psi$  are both vectors, by taking  $H(x; \theta) = x^b$  and  $B(x; \psi) = 1 - [1 + (x/s)^c]$ . Then, the cdf of the LLGWP model proposed by Oluyede et al. (2016c) can follow from the GP class (Section 3.5,  $\alpha = 1$ ) as

$$F_{LLGWP}(y; \lambda, b, c, s) = \frac{1 - e^{\lambda[1 - (1 + (y/s)^c)^{-1} e^{-by^a}]} - 1}{1 - e^\lambda}, \quad y, \lambda, b, c, s > 0.$$

*Lai-modified Weibull-Poisson (LaiMWP) model.* If  $X \sim$  ModifiedWeibull( $a, b, \delta$ ) introduced by Lai et al. (2003) have cdf  $G(x; a, b, \delta) = 1 - \exp(-ax^\delta e^{bx})$ ,  $a, b, \delta > 0$ , then the cdf of the LaiMWP model proposed by Delgarm and Zadkarami (2015) can be obtained from the GP class (Section 3.5,  $\alpha = 1$ ) as

$$F_{LaiMWP}(y; \lambda, a, b, \delta) = \frac{1 - e^{-\lambda[1 - e^{-ay^\delta e^{by}}]} - 1}{1 - e^{-\lambda}}, \quad y > 0, a, b, \delta, \lambda > 0.$$

*Exponentiated Lomax-Poisson (ELoP) model.* If  $X \sim$  Lomax( $a, b$ ) have cdf  $G(x; a, b) = 1 - (1 + bx)^{-a}$ , then the cdf of the ELoP model proposed Ramos et al. (2013) can be determined from the EGP class (Section 3.5) as

$$F_{ELoP}(y; \lambda, a, b, \alpha) = \frac{e^{\lambda[1 - (1 + by)^{-a}]^\alpha} - 1}{e^\lambda - 1}, \quad y, \lambda, a, b, \alpha > 0.$$

*Complementary Poisson-Lomax (CPLo) model.* Let  $X \sim$  Lomax( $a, b$ ) have cdf  $G(x; a, b) = 1 - (1 + bx)^{-a}$ , then the cdf of the CLoP model proposed Al-Zahrani and Sagor (2014) can follow from the CGP class (Section 4.5,  $\alpha = 1$ ) as

$$F_{CLoP}(z; \lambda, a, b) = \frac{1 - e^{-\lambda[1 - (1 + bz)^{-a}]} - 1}{1 - e^{-\lambda}}, \quad z, \lambda, a, b > 0.$$

*Lindley-Poisson (LP) model.* Let  $X \sim$  Lindley( $\theta$ ) have cdf  $G(x; \theta) = 1 - \left(1 + \frac{\theta x}{\theta + 1}\right) e^{-\theta x}$ ,  $x, \theta > 0$ . The cdf of the LP model introduced by Gui et al. (2014) can be determined from the GP class (Section 3.5,  $\alpha = 1$ ) as

$$F_{LP}(y; \lambda, \theta) = \frac{1 - e^{-\lambda\left[1 - \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}\right]} - 1}{1 - e^{-\lambda}}, \quad y > 0, \theta, \lambda > 0.$$

*Complementary extended Lindley-Poisson (CExtLP) model.* If  $X \sim \text{ExtLindley}(\theta, \delta, \beta)$  proposed by Bakouch et al. (2012a) have cdf  $G(x; \theta, \delta, \beta) = 1 - \left(\frac{1+\theta+\theta x}{\theta+1}\right)^\delta e^{-(\theta x)^\beta}$ , the cdf of the CExtLP model introduced by Pararai et al. (2015b) can be determined from the CGP class (Section 4.5,  $\alpha = 1$ ) as

$$F_{CExtLP}(z; \lambda, \theta, \delta, \beta) = \frac{1 - e^{-\lambda \left[1 - \left(\frac{1+\theta+\theta z}{\theta+1}\right)^\delta e^{-(\theta z)^\beta}\right]}}{1 - e^{-\lambda}}, \quad z > 0, \theta, \delta, \beta, \lambda > 0.$$

*Poisson Birnbaum-Saunders (PBS) model.* Let  $X \sim \text{BS}(a, b)$  have cdf  $G(x; a, b) = \Phi \left[ (1/a) + (\sqrt{x/b} - \sqrt{b/x}) \right]$ ,  $x, a, b > 0$ , then the cdf of the PBS model, proposed by Hashimoto et al. (2014) in terms of long-term survivors, can be obtained from the GP class (Section 3.5,  $\alpha = 1$ ) as

$$F_{PBS}(y; \lambda, a, b) = \frac{1 - e^{-\lambda \Phi \left[ \frac{1}{a} + \left( \sqrt{\frac{y}{b}} - \sqrt{\frac{b}{y}} \right) \right]}}{1 - e^{-\lambda}}, \quad y > 0, a, b, \lambda > 0.$$

*Exponentiated-Burr XII Poisson (EBP) model.* Let  $X \sim \text{Burr}(c, k, s)$  have cdf  $G(x; c, k, s) = 1 - [1 + (x/s)^c]^{-k}$ ,  $x, c, k, s > 0$ , then the cdf of the EBP model introduced by da-Silva et al. (2015b) can be determined from the EGP class (Section 3.5) as

$$F_{EBP}(y; \lambda, c, k, s, \alpha) = \frac{1 - e^{-\lambda \left\{ 1 + [1 - (y/s)^c]^{-k} \right\}^\alpha}}{1 - e^{-\lambda}}, \quad y > 0, c, k, s, \alpha, \lambda > 0.$$

*Complementary Burr III Poisson (CB3P) model* Let  $X \sim \text{BurrIII}(c, k)$  have cdf  $G(x; c, k) = (1 + x^{-c})^{-k}$ ,  $x, c, k > 0$ , then the cdf of the CB3P model defined by Hassan et al. (2015) can follow from the CGP class (Section 4.5,  $\alpha = 1$ ) as

$$F_{CB3P}(z; \lambda, c, k) = \frac{e^{\lambda(1+z^{-c})^{-k}} - 1}{e^\lambda - 1}, \quad z, c, k, \lambda > 0.$$

The Burr III distribution is also known as the Dagum distribution (Dagum 1977), which is a very well-known model for studying income and wealth inequality data. Oluyede et al. (2016b) have recently introduced and studied the Dagum-Poisson distribution.

*Poisson generalized linear failure rate (PGLFR) model.* Let  $X \sim \text{GLFR}(a, b, \alpha)$  introduced by Sarhan and Kundu (2009) have cdf  $G(x; a, b, \alpha) = (1 - \exp[-ax - (b/2)x^2])^\alpha$ ,  $x, a, b, \alpha > 0$ , then the cdf of the PGLFR model defined by Cordeiro et al. (2015b) can be obtained from the EGP class (Section 3.5) as

$$F_{PGLFR}(y; \lambda, a, b, \alpha) = \frac{1 - e^{-\lambda \{1 - \exp[-ay - (b/2)y^2]\}^\alpha}}{1 - e^{-\lambda}}, \quad y > 0, a, b, \alpha, \lambda > 0.$$

*Complementary failure rate Poisson (CLFRP) model.* Let  $X \sim \text{LFR}(a, b, \alpha)$ , then the cdf of the CLFRP model defined by Gitifar et al. (2016) can follow from the CEGP class (Section 4.5,  $\alpha = 1$ ) as

$$F_{CLFRP}(z; \lambda, a, b) = \frac{e^{\lambda \{1 - \exp[-ay - (b/2)y^2]\}}}{e^\lambda - 1}, \quad z > 0, a, b, \lambda > 0.$$

*Complementary exponentiated power Lindley-Poisson (CEPLP) model.* If  $X \sim \text{ExpPowerLindley}(\alpha, \beta, \theta)$  introduced by Warahena-Liyanage and Pararai (2015b) have cdf  $G(x; \alpha, \beta, \theta) = \left[ 1 - \left( 1 + \frac{\theta x^\beta}{\theta+1} \right) e^{-\theta x^\beta} \right]^\alpha$  (for  $x > 0, \alpha, \beta, \theta > 0$ ), then the EPLG cdf defined by Pararai et al. (2016) can follow from the CGP class (Section 4.5) as

$$F_{CEPLP}(y; \lambda, \alpha, \beta, \theta) = \frac{e^{\lambda \left[ 1 - \left( 1 + \frac{\theta y^\beta}{\theta+1} \right) e^{-\theta y^\beta} \right]^\alpha}}{e^\lambda - 1}, \quad z > 0, \alpha, \beta, \theta, \lambda > 0.$$

### 5.3 Compounded models based on the logarithmic distribution

The following compounded models have been reported in the literature from the logarithmic discrete model.

*Exponential-logarithmic (ELn).* If  $X \sim \text{Exponential}(\beta)$  have cdf  $G(x; \beta) = 1 - \exp(-\beta x)$ , then the cdf of the exponential-logarithmic (ELn) model proposed by Tahmasbi and Rezaei (2008) can be obtained from the GLn class (Section 3.9,  $\alpha = 1$ ) as

$$F_{ELn}(y; \beta, \phi) = 1 - \frac{\ln [1 - (1 - \phi) e^{-\beta y}]}{\ln \phi}. \tag{8}$$

*Generalized ELn.* Pappas et al. (2015) applied the LA1 class to the ELn cdf given in Eq. (8), and obtained the cdf of the generalized ELn (GELn) distribution as

$$F_{GELn}(y; \beta, \phi, \gamma) = \left( 1 - \frac{\ln [1 - (1 - \phi) e^{-\beta y}]}{\ln \phi} \right)^\gamma,$$

where  $\gamma > 0$  is the power parameter.

*Chen-logarithmic (ChLn).* If  $X \sim \text{Chen}(\gamma, \beta)$  have cdf  $G(x; \gamma, \beta) = e^{\gamma(e^{x\beta} - 1)}$  introduced by Chen (2000), then the cdf of the Chen-logarithmic (ChLn) model proposed by Pappas et al. (2011) can be determined from the GLn class (Section 3.9 with  $\alpha = 1$ ) as

$$F_{ChLn}(y; \gamma, \beta, \phi) = 1 - \frac{\ln [1 - (1 - \phi) e^{\gamma(e^{y\beta} - 1)}]}{\ln \phi}.$$

*Complementary exponentiated Weibull-logarithmic (CEWLn) model.* Let  $X \sim \text{EW}(\beta, \delta, \alpha)$  have cdf  $G(x; \beta, \delta, \alpha) = \alpha \delta \beta^\delta y^\delta e^{-(\beta y)^\delta} [1 - e^{-(\beta y)^\delta}]^{\alpha-1}$ , then the cdf of the complementary exponentiated Weibull-logarithmic (CEWLn) model introduced by Mahmoudi et al. (2014) can follow from the cdf of the CGLn class given in Section 4.9 as

$$F_{CEWLn}(z; \beta, \phi, \delta, \alpha) = \frac{\ln [1 - \phi (1 - e^{-(\beta z)^\delta})^\alpha]}{\ln(1 - \phi)},$$

where  $\phi \in (0, 1)$ ,  $\alpha > 0$  and  $\delta > 0$  are shape parameters and  $\beta > 0$  is a scale parameter.

*Lomax-logarithmic (LxLn) model.* Let  $X \sim \text{Lomax}(a, b)$  have cdf  $G(x; a, b) = (1 + bx)^{-a}$ , then the cdf of the Lomax-logarithmic (LxLn) model given by Al-Zahrani and Sagor (2015) can be obtained from the GLn class given in Section 3.9 with  $\alpha = 1$  as

$$F_{LxLn}(y; a, b, \phi) = 1 - \frac{\ln [1 - (1 - \phi) (1 + by)^{-a}]}{\ln \phi}.$$

*Exponentiated Poisson-logarithmic (EPoLn) model.* Let  $X \sim \text{ExpPoisson}(\theta, \beta)$  have cdf  $G(x; a, b) = 1 - \frac{1 - \exp\{-\theta e^{-\beta x}\}}{1 - e^{-\theta}}$ , then the cdf of the exponentiated Poisson-logarithmic (EPoLn) model given by Fioruci et al. (2016) can follow from the alternative GLn cdf (Section 3.9 with  $\alpha = 1$ ) as

$$F_{EPoLn}(y; \beta, \phi, \theta) = 1 - \frac{\ln \left[ 1 - \phi \left\{ 1 - \frac{e^{-\theta e^{-\beta y}}}{1 - e^{-\theta}} \right\} \right]}{\ln(1 - \phi)}.$$

## 6 Other compounded models

(i) *Extended Weibull power series (EWPS) models.* Let  $T[G(y; \tau); \alpha] = 1 - e^{-\alpha H(y; \tau)}$  be the extended Weibull family, where  $x > 0$ ,  $\alpha > 0$  and  $H(y; \tau)$  is a non-negative monotonically increasing function depending on a parameter vector  $\tau$ . Much more than twenty known

lifetime model cdfs can be expressed in this form. Further, let  $N$  be a discrete random variable following a power series (PS) distribution (truncated at zero) with pmf

$$p_n = P(N = n) = \frac{a_n \theta^n}{C(\theta)}, n = 1, 2, \dots, \tag{9}$$

where  $a_n$  depends only on  $n$ ,  $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$  and  $\theta > 0$  is such that  $C(\theta)$  is finite. Equation (9) summarizes some power series distributions (truncated at zero) such as the Poisson, logarithmic and geometric distributions, where  $C(\theta)$  is equal to  $(e^\theta - 1)$ ,  $-\log(1 - \theta)$  and  $\theta(1 - \theta)^{-1}$ , respectively.

Silva et al. (2013) defined the EWPS cdf obtained under the current set-up from Eq. (4) as

$$F(y; \theta, \alpha, \tau) = 1 - \frac{C(\theta e^{-\alpha H(y; \tau)})}{C(\theta)}, y > 0. \tag{10}$$

Equation (10) includes as special cases the Weibull power series (WPS) class, which extends the exponential power series (EPS) family. In fact, this class includes much more than 60 ( $20 \times 3$ ) special models, some of them given by Silva et al. (2013) and others yet not investigated. In a similar context, more recently, Silva et al. (2016) defined a family by compounding the generalized gamma (GGa) and power series distributions.

(ii) *Complementary extended Weibull power series (CEWPS) models.* The CEWPS class follows by taking  $T[G(y; \tau); \alpha] = 1 - e^{-\alpha H(y; \tau)}$  and the PS distribution for the pmf. So, the cdf of the CEWPS class can be obtained from Eq. (5) as

$$F(z; \theta, \alpha, \tau) = \frac{C[\theta(1 - e^{-\alpha H(x; \tau)})]}{C(\theta)}, x > 0.$$

The CEWPS class can arise in parallel systems with identical components, which appear in many industrial and biological applications.

(iii) *Exponential-geometric range (max-min) model (New).* Let  $N$  be the ZTG r.v. with pmf of type 1  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 1, 2, \dots$  and define the range  $X_R = \max\{Y_1, \dots, Y_N | p\} - \min\{Y_1, \dots, Y_N | p\}$ . The unconditional cdf of the G-geometric range ( $R = \max - \min$ ) can be obtained as

$$F(x; p, \alpha, \tau) = \frac{(1 - p) G(x; \tau)^\alpha}{1 - p G(x; \tau)^\alpha}.$$

If  $Y \sim \text{Exp}(\beta)$  have density  $f_{Y|p}(y) = \beta e^{-\beta y}$  and  $Z|p \sim \text{Geometric}(p)$  have pmf  $\mathbb{P}(N = n) = qp^{n-1}$ ,  $n = 0, 1, \dots$ , Shahsanaei et al. (2012) proposed the unconditional cdf of the exponential-geometric range (EGR) model as

$$F(x; \beta, p) = \frac{(1 - p) (1 - e^{-\beta x})}{1 - p (1 - e^{-\beta x})}.$$

The EGR distribution is useful for modeling the time between the first failure to the last failure.

(iv) *Exponential-generalized Poisson (EGP) model.* Let  $N$  have the zero truncated generalized Poisson (ZTGP) distribution with parameters  $\lambda$  and  $\alpha$  and pmf

$$\mathbb{P}(N = n) = \frac{\lambda(\lambda + \alpha n)^{n-1} e^{-(\lambda + \alpha n)}}{(1 - e^{-\lambda}) \Gamma(n + 1)}, n = 1, 2, \dots$$

Then, Gupta et al. (2014) proposed the EGP cdf given by

$$F(y; \alpha, \beta, \lambda) = \frac{1 - e^{-(\lambda/\alpha)[W(g) + \alpha]}}{1 - e^{-\lambda}},$$

where  $g = -\alpha e^{-(\alpha+\beta\gamma)}$  and  $W(\cdot)$  is the Lambert W function defined by  $W(x)e^{W(x)} = x$ . The above cdf can be derived from the generating function of the ZTGP distribution  $M_N^*(t) = \frac{M_N(t) - P(N=0)}{1 - P(N=0)} = \frac{1 - e^{-(\lambda/\alpha)[W(g)+\alpha]} - e^{-\lambda}}{1 - e^{-\lambda}}$ , which is equivalent to the survival function  $1 - F(t; \alpha, \beta, \lambda)$ .

### 7 A different approach of compounding

The compounding of some models (continuous with continuous and discrete with discrete) are introduced such as the exponential-Weibull (Cordeiro et al. 2014c), generalized exponential-exponential (Popović et al. 2015), geometric exponential Poisson (Nadarajah et al. 2013a) and additive Weibull (Xie and Lai 1995) distributions.

(i) *Exponential-Weibull model.* Let  $W$  and  $T$  be independent random variables with exponential and Weibull distributions characterizing two series sub-system lifetimes, whose cdfs are  $G_W(z) = 1 - \exp(-\alpha z)$  and  $G_T(t) = 1 - \exp(-\beta t^\gamma)$ , respectively. Here,  $\alpha > 0$  and  $\gamma > 0$  are shape parameters and  $\beta > 0$  is a scale parameter. The distribution of the random variable  $X = \min\{W, T\}$  is called the *exponential-Weibull* (EW) distribution, whose cdf is given by

$$F(x; \alpha, \beta, \gamma) = 1 - e^{-(\alpha x + \beta x^\gamma)}, \quad x > 0. \tag{11}$$

We consider that  $\alpha > 0, \beta > 0$  and  $\gamma \in (0, \infty) \setminus \{1\}$ , which gives identifiability to the model. The mathematical properties of the EW model were investigated by Cordeiro et al. (2014c).

(ii) *Generalized exponential-exponential (GEE) model.* Gupta and Kundu (1999) introduced the two-parameter generalized exponential (GE) distribution with cdf given by  $G(x; \alpha, \lambda) = (1 - e^{-\lambda x})^\alpha$ , where  $x > 0, \alpha > 0, \lambda > 0$ . For a given  $\alpha$ , let  $Y$  be a random variable with the GE cdf with parameters  $\alpha > 0$  and  $\lambda > 0$ . Let  $\alpha$  be a random variable following an exponential distribution with parameter  $\theta > 0$ . Then, Popović et al. (2015) defined the GEE cdf (for  $x > 0$ ) by

$$F(x; \lambda, \theta) = \int_0^\infty G(x; \alpha, \lambda) \theta e^{-\theta \alpha} d\alpha = \left[ 1 - \frac{1}{\theta} \log(1 - e^{-\lambda x}) \right]^{-1}, \quad \lambda, \theta > 0.$$

(iii) *Exponentiated geometric G-Poisson (EGGP) model (New).* If  $N \sim ZTG(p)$  and  $M \sim ZTP(\lambda)$ , then the EGGP family can be determined by inserting the EGP cdf (Section 3.9) in the cdf of the EGG class (Section 3.1) as

$$F_{EGGP}(x; \alpha, \lambda, \tau, p) = \frac{e^{-\lambda[1-G(x;\tau)^\alpha]} - e^{-\lambda}}{1 - e^{-\lambda} - p(1 - e^{-\lambda[1-G(x;\tau)^\alpha]}}.$$

If  $\alpha = 1$  and  $X \sim Exp(\beta)$  in the EGGP class above, then it follows the cdf of the geometric exponential Poisson (GEP) defined by Nadarajah et al. (2013a).

(iv) *Additive Weibull (AW) model.* Suppose a system composed of two interconnected independent series sub-systems that affect the system in a different way, both following the Weibull distribution with proper parameters. Xie and Lai (1995) proposed the AW model based on the simple idea of combining the failure rates of two Weibull distributions: one has a decreasing failure rate and the other one has an increasing failure rate. The cdf of the AW model is given by  $F(t) = 1 - \exp(-at^b - ct^d)$ , where  $a > 0$  and  $c > 0$  are scale parameters and  $b > d > 0$  (or  $d > b > 0$ ) are shape parameters, which gives identifiability to the model. The interpretation of the AW model is evident. A state-of-the-art survey on the AW model can be found in Lemonte et al. (2014).

## 8 Recent trends in compounding

There are four very recent trends on compounding of distributions, which have received a great deal of attention.

### 8.1 First recent trend

**Compounding a G-class with discrete model:** The first recent trend deals with defining compound classes of lifetime distributions rather than studying a single compound model. In this technique, a non-compound G-family of distributions is compounded with a discrete model to generate a new flexible compounded class. Asgharzadeh et al. (2014) introduced the G-Poisson-Lindley (from discrete Poisson-Lindley) class of distributions by compounding the ZTPL distribution with any other continuous lifetime model. Four special models viz. Weibull Poisson-Lindley, Burr Poisson-Lindley, exponentiated-Weibull Poisson-Lindley and Dagum Poisson-Lindley were investigated. Nadarajah et al. (2015a) proposed the exponentiated G-geometric family, and reported two special models: exponentiated-Weibull geometric and exponentiated-log-logistic geometric. Ramos et al. (2015) introduced the Kumaraswamy-G Poisson family and showed that the special model Kumaraswamy-Weibull Poisson outperforms the competitors Kumaraswamy-Weibull and beta-Weibull models in studying real life data on bladder cancer. Gomes et al. (2015) proposed the exponentiated-G Poisson family and studied two special models, namely the exponentiated-Burr XII Poisson and exponentiated-Weibull Poisson. Two other compound G-classes are the G-Poisson (Alkarni and Oraby 2012) and Poisson-X (Tahir et al. 2016a).

### 8.2 Second recent trend

**Combining a continuous model with compound power series class:** For the second recent trend on compounding, one continuous lifetime model is compounded with the power series class of distributions truncated at zero. Chahkandi and Gangali (2009) first suggested compounding exponential and power series class, which exhibits decreasing failure rate. The power series class can be used to construct many compounding models with discrete distributions: Poisson, logarithmic, geometric, binomial and negative-binomial. Some well-known compound models defined from the power series class are: Weibull power series (WPS) (Morais and Barreto-Souza 2011), complementary generalized-exponential power series (CGEPS) (Mahmoudi and Jafari 2012), complementary exponentiated-Weibull power series (CEWPS) (Mahmoudi and Shiran 2012b), extended WPS (Silva et al. 2013), Kumaraswamy power series (KwPS) (Bidram and Nekouhou 2013), complementary exponential power series (CEPS) (Flores et al. 2013), Birnbaum-Saunders power series (BSPS) (Bourguignon et al. 2014b), complementary WPS (Munteanu 2014), complementary Erlang and Erlang power series (CErPS and ErPS) (Leahu et al. 2014), complementary extended WPS (Cordeiro and Silva 2014), exponentiated extended WPS (Tahmasebi and Jafari 2015a), Burr XII power series (BIIPS) (Silva and Cordeiro 2015), Lindley power series (LPS) (Warahena-Liyanage and Pararai 2015a), linear failure rate power series (LFRPS) (Mahmoudi and Jafari 2015), complementary normal power series (CNPS) (Mahmoudi and Mahmoodian 2015), complementary generalized Gompertz power series (CGGoPS) (Tahmasebi and Jafari 2015b), complementary inverse Weibull power series (CIWPS) (Shafiei et al. 2016), complementary generalized modified Weibull (CGMW) (Bagheri et al. 2016), complementary exponentiated inverse Weibull

power series (CEIWPS) (Hassan et al. 2016), generalized gamma power series (GGPS) (Silva et al. 2016), Gompertz power series (GoPS) (Jafari and Tahmasebi 2016), complementary generalized linear failure rate power series (CGLFR) (Harandi and Alamatsaz 2016), and Dagum power series (DaPS) (Oluyede et al. 2016b).

### 8.3 Third recent trend

**Combining compound G-class with the non-compound G-class:** Here, the cdf of one compound G-class or its special model is inducted into the cdf of a non-compound G-class, thus generating a new flexible G-class. Next, we propose two such classes.

(i) If the cdf of the GP class defined in Section 3.5 with  $\alpha = 1$  is inserted into the cdf of Gamma-G class, proposed by Zografos and Balakrishnan (2009), then the cdf of new class (GaGP) is obtained as

$$F_{GaGP}(y) = 1 - \frac{1}{\Gamma(a)} \Gamma(a, -\ln \bar{G}_{GP}(y)),$$

where  $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$  and  $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$  are the gamma and upper incomplete gamma functions, respectively, and  $\bar{G}_{GP}(y) = 1 - G_{GP}(y)$ .

One special model of the GaGP class, that is, the gamma-Weibull-Poisson (GaWP) was studied by Percontini et al. (2014), whose cdf is given by

$$F_{GaWP}(y) = 1 - \frac{1}{\Gamma(a)} \Gamma\left(a, \ln \left[ \frac{1 - e^\lambda}{1 - e^\lambda \bar{G}(x)} \right]\right).$$

(ii) The cdf of the new Kumaraswamy-GP (KGP) class can be obtained by inducting the cdf of the CGP model given in Section 4.5 with  $\alpha = 1$  into the cdf of the Kumaraswamy-G class, pioneered by Cordeiro and de-Castro (2011), as

$$F_{KwGP}(y) = 1 - [1 - G_{GP}^a(x)]^b = 1 - \left[ 1 - \left( \frac{1 - e^{\lambda G(x)}}{1 - e^\lambda} \right)^a \right]^b.$$

Pararai et al. (2015a) obtained the cdf of a special KwGP model called the Kumaraswamy Lindley-Poisson (KwLP) distribution.

### 8.4 Fourth recent trend

**Combining transmuted G-class with well-known compound distributions:** After receiving increased attention in the last decade, more than 50 transmuted distributions have been reported in the literature. Due to wide acceptability of transmuted G-class, a new trend has now begun by inserting the cdf of the compound G-class or distribution in the transmuted G-class cdf. Some models are given below:

(i) *Transmuted Weibull-geometric (TWG).* Merovci and Elbatal (2014b) extended the WG model introduced by Barreto-Souza et al. (2011) (given in Section 5) with the transmuted-G family and defined the cdf of the TWG model by

$$F_{TWG}(y; p, a, b, \eta) = \frac{1 - e^{-(by)^a}}{1 - p e^{-(by)^a}} \left[ 1 + \eta - \eta \left( \frac{1 - e^{-(by)^a}}{1 - p e^{-(by)^a}} \right) \right].$$

(ii) *Exponentiated transmuted-Weibull geometric (ETWG) model.* If  $X \sim \text{Weibull}(a, b)$  have cdf  $G(x; a, b) = 1 - e^{-(x/b)^a}$ , Aryal and Tsokos (2011) defined the cdf of the transmuted-Weibull distribution as

$$G_{TW}(x; a, b, \eta) = \left( 1 - e^{-(x/b)^a} \right) \left( 1 + \eta e^{-(x/b)^a} \right).$$



The cdf of the ETWG model proposed by Al-Zahrani et al. (2015) can follow from the EGG cdf defined in Section 3.1 as

$$F_{ETWG}(y; p, a, b, \eta) = \frac{(1 - e^{-(y/b)^a})^\alpha (1 + \eta e^{-(y/b)^a})^\alpha}{1 - p + p(1 - e^{-(y/b)^a})^\alpha (1 + \eta e^{-(y/b)^a})^\alpha}.$$

(iii) *Transmuted complementary Weibull-geometric (TCWG) model.* Afify et al. (2014) applied the LA1 class to the cdf of the Tojeiro et al's CWG model given in Section 5, and obtained the cdf of the TCWG model as

$$F_{TCWG}(z; p, a, b, \eta) = \frac{p(1 - e^{-(bz)^a})}{p + (1 - p)e^{-(bz)^a}} \left[ 1 + \eta - \eta \left( \frac{p(1 - e^{-(bz)^a})}{p + (1 - p)e^{-(bz)^a}} \right) \right].$$

(iv) *Transmuted complementary exponentiated Weibull-geometric (TCEWG) model.* If  $X \sim \text{ExpWeibull}(a, b, \alpha)$  have cdf  $G(x; \beta) = (1 - e^{-(bx)^\alpha})^\alpha$ , Saboor et al. (2016) used the CEWG cdf given in Section 5 and obtained the cdf of the TCEWG model as

$$F_{TCEWG}(z; p, a, b, \alpha, \eta) = \frac{(1 - p)(1 - e^{-(bz)^\alpha})^\alpha}{1 - p[1 - e^{-(bz)^\alpha}]^\alpha} \times \left[ 1 + \eta - \eta \left( \frac{(1 - p)(1 - e^{-(bz)^\alpha})^\alpha}{1 - p[1 - e^{-(bz)^\alpha}]^\alpha} \right) \right].$$

(v) *Transmuted Lindley-geometric (TLG) model.* Merovci and Elbatal (2014a) extended the LG model (described in Section 5) by proposing the TLG cdf given by

$$F_{TLG}(y; p, \theta, \eta) = \frac{1 - \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}}{1 - p \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}} \times \left[ 1 + \eta - \eta \left( \frac{1 - \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}}{1 - p \left(1 + \frac{\theta y}{\theta + 1}\right) e^{-\theta y}} \right) \right], \quad y > 0,$$

where  $\eta$  and  $\theta > 0$  are transmuted and scale parameters, respectively, and  $p \in (0, 1)$ .

(vi) *Transmuted exponential-Weibull (TEW) model.* Saboor et al. (2015) recently extended the exponentiated-Weibull model originally proposed by Cordeiro et al. (2014c). The TEW cdf is given by

$$F_{TEW}(y; \eta, \alpha, \beta, \gamma) = (\eta + 1) \left[ 1 - e^{-(\alpha x + \beta x^\gamma)} \right] - \eta \left[ 1 - e^{-(\alpha x + \beta x^\gamma)} \right]^2, \quad x > 0,$$

where  $\alpha > 0, \beta > 0$  and  $\gamma \in (0, \infty) \setminus \{1\}$ .

## 9 Estimation and inference

### 9.1 General estimation procedure

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large sample distribution theory is easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters for each model discussed in this paper from complete or censored samples by maximum likelihood.

The log-likelihood for the model parameters can be maximized either directly by using the R (`optim` function), SAS (PROC NLMIXED), Ox program (MaxBFGS sub-routine)

or by solving the nonlinear likelihood equations obtained by differentiating the log-likelihood. In the applications, we can also use the *AdequacyModel* package (version 2.0.0) available in the R programming language. It has been cited very frequently in papers related to new lifetime distributions. The package has been continuously updated and more information can be obtained from <http://cran.rstudio.com/web/packages/AdequacyModel/index.html>. It is distributed under the terms of the GNU licenses - GNU Project. An important observation is that it is not necessary to define the log-likelihood function but only the pdf and cdf of the model. The package provides some useful goodness-of-fit statistics to assess the quality of the fitted models and compare them, such as the Cramér-von Mises ( $W^*$ ) and Anderson-Darling ( $A^*$ ) statistics, Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and Kolmogorov-Smirnov (K-S) test. It is important to emphasize that we can fit several competitive models to a data set and select those which yield best fits by means of the *AdequacyModel* package.

## 9.2 Cure fraction survivor models

During the last two decades, the tendency to propose probability models to deal with survival data has increased. This increased interest has lead researchers and medical practitioners to assess correct causes and information of the disease. The survival models which are receiving increased recognition in these days are *fractional survivor models*. These models have been effectively useful in some situations of clinical, medical or biological studies, where the fractional survival (or survivor fraction) models are useful to study a cure fraction of individuals. These models are also known as *cure rate models* or *long-term survival models*. In these models, it is assumed that all units under study are susceptible to an event of interest and will eventually experience it if its follow-up is sufficiently long. However, there are situations for which a fraction of individuals is not expected to experience the event of interest, that is, those individuals are cured or insusceptible. For example, researchers may be interested in analyzing the recurrence of a disease but many individuals may never have an experience or a recurrence, therefore, a cured fraction of the population exist. In other words, the cure rate models cover the situations where the sampling units insusceptible to the occurrence of the event are of interest, and also extend understanding of time-to-event data by allowing the formulation of more accurate and informative conclusions. If the cure fraction of the population is ignored, then the results will match to standard survival analysis. The cure rate models have been used for modeling time-to-event data for cardiac failure and various types of cancers including prostate, breast, leukemia, non-Hodgkin lymphoma and melanoma. That is why, the focus of the researchers is to introduce new, extended or modified distributions which accommodate cure fraction.

In the literature, two formulations of the cure rate models have received increased attention, namely, the *mixture cure rate model* (Boag 1949; Berkson and Gage 1952) and *promotion time (or accelerated) cure rate model* (Yakovlev and Tsodikov 1996; Chen et al. 1999). Perhaps, the mixture cure rate models has received increased popularity. In mixture model, it is assumed that there is a fraction  $1 - \pi_0$  of susceptible individuals and, hence, a proportion  $\pi_0$  of cured individuals. The survivor function for the entire population is

$$S_{pop} = \pi_0 + (1 - \pi_0) S_0,$$

where  $S_0$  denotes the survival function for the non-cured group in the population. So, the flexibility and performance of the population model depends on the choice of  $S_0$ . Rodrigues et al. (2009a) unified both formulations considering the *negative binomial cure rate model*. The proper modeling of cure rate models reported in literature is with covariates, which have been proved very useful in getting improved results for the estimated cure rate. Some selected cure rate models are reported in Table 2. Further detail is beyond the scope of the paper.

### 9.3 Inference for cure fraction models

Suppose that the time-to-event is not completely observed and may be subject to right censoring. Let  $C_i$  denote the censoring time. We observe that  $Y_i = \min(T_i, C_i)$  and  $\delta_i = I(T_i < C_i)$  is such that  $\delta_i = 1$ , if  $T_i$  is a time-to-event and  $\delta_i = 0$ , if it is right censored,  $i = 1, \dots, n$ . Let  $\Theta$  denote the parameter vector of the distribution of the time-to-event. From  $n$  pairs of times and censoring indicators  $Data = (y_1, \delta_1), \dots, (y_n, \delta_n)$ , the corresponding likelihood function under uninformative censoring is

$$L(Data, \Theta) \propto \prod_{i=1}^n [f_{CRM}(Data, \Theta)]^{\delta_i} [F_{CRM}(Data, \Theta)]^{1-\delta_i},$$

where  $f_{CRM}(Data, \Theta)$  and  $F_{CRM}(Data, \Theta)$  are the density and cdf of a cure rate model (CRM), respectively.

Maximum likelihood estimation of the parameter vector  $\Theta$  is carried out through maximization of the log-likelihood function  $\ell(Data, \Theta) = \log L(Data, \Theta)$ . Details for the maximization methods are discussed in Section 9.1. Under suitable regularity conditions,

**Table 2** Some selected cure rate survival models

S.No.	Investigation	Cure rate model	Author(s)
1.	Cutaneous melanoma	COM-Poisson	Rodrigues et al. (2009b)
2.	Reducing drug abuse	Generalized exponential	Kannan et al. (2010)
3.	Malignant melanoma	Negative binomial	Cancho et al. (2011b)
4.	Malignant melanoma	Geometric Birnbaum-Saunders	Cancho et al. (2012)
5.	Prostrate Cancer	Negative binomial-beta Weibull	Ortega et al. (2012)
6.	Ovarian cancer	PE-regression	Louzada et al. (2012c)
7.	Myelomatosis (bone marrow) Leukemia (autologous marrow)	EG	Roman et al. (2012)
8.	Gastric cancer	Generalized modified Weibull	Martinez et al. (2013)
9.	Cutaneous melanoma	Destructive negative-binomial	Cancho et al. (2013a)
10.	Cutaneous melanoma	Power series	Cancho et al. (2013b)
11.	Cutaneous melanoma	COM-Poisson	Balakrishnan and Pal (2013a)
12.	Cutaneous melanoma	Negative-binomial GGa	Ortega et al. (2014)
13.	Breast cancer Hemophiliacs	Poisson Birnbaum-Saunders	Hashimoto et al. (2014)
14.	Red flour beetles	log-Weibull Negative-binomial	Louzada et al. (2015)
15.	Melanoma	Weibull Negative-binomial	Yiqi et al. (2016)
16.	Congenital malformations Hemophiliacs	Negative binomial-Weibull	Hashimoto et al. (2015)
17.	Breast carcinoma	Power series BW	Ortega et al. (2015)
18.	First calving of cows	Transmuted log-logistic	Louzada and Granzotto (2015)
19.	Cutaneous melanoma	Destructive Negative-binomial	Gallardo and Bolfarine (2016)
20.	Malignant melanoma	Negative binomial Birnbaum-Saunders	Cordeiro et al. (2016)

it can be shown that the asymptotic distribution of the maximum likelihood estimator (MLE)  $\hat{\Theta}$  is normal with mean vector  $\Theta$  and covariance matrix  $\Sigma(\hat{\Theta})$  given by

$$\widehat{\Sigma}(\hat{\Theta}) = \left[ -\frac{\partial^2 \ell(\text{Data}, \Theta)}{\partial \Theta \partial \Theta^T} \right]^{-1},$$

evaluated at  $\Theta = \hat{\Theta}$ . The required second derivatives are computed numerically.

The EM algorithm (Dempster et al. 1977) is also a very popular maximization alternative used to obtain the estimates when the model has missing data. In the literature, Balakrishnan and Pal (2012, 2013b, 2015a, 2015b), Gallardo et al. (2016), Gallardo and Bolfarine (2016) and some others have considered estimation of parameters of cure fraction survival models using the EM algorithm.

## 10 Final remarks

The need of compounding was first felt in actuarial science and later researchers of many other fields adopted this approach for lifetime and reliability modeling. We follow the two basic principles (the minimum and the maximum) used in series and parallel structure, and report more than 30 compound G-classes. In this way, the possible available compound G-classes are surveyed and using these basic principles nearly 25 new G-classes are proposed. The purpose of providing a variety of G-classes is to test flexibility of the proposed compound models to cope with the data available in complex situations. The parameters inducted in this way might be helpful in exploring phenomenon generated from real-lifetime data sets. We expect that these G-classes or generated compounded models from them will be an addition to the art of constructing useful probability models. One can imagine its motivation and usefulness in the fields which are not touched earlier. We have also briefly described the latest trends in the development of compounding technique, which portray better exposition of the strategies adopted for the researchers and practitioners. We hope to produce many more new compound G-classes from the function  $T[G(x; \tau); \alpha]$  but due to space problem we did our best to explore and present the elusive task in most tenable way by mentioning only the cumulative distributions of the classes. The remaining G-classes will appear in another article under the same series. Lastly, we offer more choices to the learners and practitioners of modeling to compare different models and to study pros and cons of old and new G-classes. The possible future projects are: (i) to propose more new compound G-classes of distributions; (ii) to review and develop bivariate compound G-classes; and (iii) to prepare a review and new developments on cure rate survival models.

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### Authors' contributions

The authors MHT and GMC with the consultation of each other carried out this work and drafted the manuscript together. Both authors read and approved the final manuscript.

### Competing interests

The authors declare that they have no competing interests.

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