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COMPRESSED POLYTOPES AND STATISTICAL DISCLOSURE LIMITATION

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Abstract. We provide a characterization of the compressed lattice polytopes in terms of their facet defining inequalities and prove that every compressed lattice polytope is affinely isomorphic to a $0/1$-polytope. As an application, we characterize those graphs whose cut polytopes are compressed and discuss consequences for studying linear programming relaxations in statistical disclosure limitation.

1. Introduction. A lattice polytope P is called compressed if every pulling triangulation of P using only the lattice points in P is unimodular. Compressed polytopes are natural to study because they represent a more inclusive class of polytopes than the unimodular polytopes (polytopes where every triangulation is unimodular). Furthermore, many naturally occurring polytopes are compressed. An important example is the Birkhoff polytope of doubly stochastic matrices as shown in [12]. In fact, the compressed nature of the Birkhoff polytope played a crucial role in the work of Diaconis and Sturmfels [5] for the statistical analysis of ranked data. Ohsugi and Hibi's paper [10] contains many other examples. In this paper, we characterize the compressed polytopes by their facet defining inequalities, extending a result from [10].

Part of our motivation for studying compressed polytopes comes from their appearance in algebraic statistics: the marginal polytopes of decomposable hierarchical models are compressed. Due to the presence of a transitive symmetry group on these marginal polytopes, the connections between compressed polytopes and certain optimization problems in statistical disclosure limitation are quite deep. As an application of the characterization of compressed polytopes, we will show that the linear programming relaxations for maximizing cell entries given marginal sums yield sharp integer bounds for all values of the marginals if and only if the marginal polytope P_Δ is compressed. Coupled with some results about compressed cut polytopes, we are able to describe some new nondecomposable families of marginals where the linear programming relaxation yields sharp integer bounds for the maximization problems.

Here is the outline for our paper. In the next section we prove the main result classifying compressed polytopes by their facet defining inequalities. We also show that every compressed polytope is affinely isomorphic to a 0/1 polytope and prove a result about pulling triangulations for highly symmetric polytopes. In Section 3, we apply the facet description of

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compressed polytopes to characterize the compressed cut polytopes. In Section 4 we explain the connection between compressed polytopes and linear optimization. Section 5 is devoted to applications of our results in statistical disclosure limitation which provides new families of marginals where linear programming yields sharp upper bounds on cell entries. These results also suggest families in which to search for large integer programming gaps [9].

REMARK 1.1. After submitting this article for publication, we discovered that a large part of Theorem 2.4 already appeared in the dissertation of Christian Haase [8]. That result was based on an unpublished proof of Francisco Santos. The applications appearing in this article are all new.

2. Characterization of compressed polytopes. In this section, we derive the characterization of the structure of the facet definining inequalities of compressed polytopes. We assume the reader is familiar with polyhedral geometry and regular subdivisions. A standard reference for this material is [14].

DEFINITION 2.1. Let P be a lattice polytope in Euclidean d-space $\boldsymbol{R}^d$ and $p_1, \dots, p_k$ an ordered list of the lattice points in P. The *pulling triangulation* $\Delta_{\mathrm{pull}}(P)$ induced by this ordering is constructed recursively as follows: If $p_1, \dots, p_k$ are affinely independent, $\Delta_{\mathrm{pull}}(P) = \{\{p_1, \dots, p_k\}\}$. Otherwise, we set

$$\Delta_{\mathrm{pull}}(P) = \bigcup_F \{\{p_1\} \cup \sigma \mid \sigma \in \Delta_{\mathrm{pull}}(F)\}$$

where the union is over all facets F of P not containing p_1, and the ordering of the lattice points in F is the ordering induced by the ordering of the lattice points in P.

DEFINITION 2.2. A triangulation Δ of a lattice polytope P is called *unimodular* if every simplex in the triangulation attains the minimal volume among all simplices formed by taking convex hulls of points in the lattice spanned by the lattice points in P.

DEFINITION 2.3. A lattice polytope P is *compressed* if every pulling triangulation of P using the lattice points in P is unimodular. If we are given a specific presentation of $P = P_A := \mathrm{conv}(A_1, \dots, A_n)$ as the convex hull of a finite set of integral points, we say that P_A is *compressed* if it is compressed with respect to the smallest lattice containing $A_1, \dots, A_n$.

Compressed polytopes were introduced by Stanley in [12] where unimodular was meant with respect to the lattice $\boldsymbol{Z}^d$. Our notion of unimodular is with respect to the smallest lattice containing the integral points in P. We say that two lattice polytopes P and Q are *lattice isomorphic* if there is an affine isomorphism which is a bijection on their lattice points. The main result of this section is the following

THEOREM 2.4. *Let $\mathcal{L}$ be a lattice and suppose that P is a lattice polytope having the irredundant linear description $P = \{\boldsymbol{x} \in \boldsymbol{R}^d \mid a_i^T \boldsymbol{x} \geq b_i, i = 1, \dots, n\}$. Then the following conditions are equivalent*:

1. *P is compressed.*

2. *For each i there is at most one nonzero real number m_i such that the set $\{\boldsymbol{x} \in \mathcal{L} \mid a_i^T \boldsymbol{x} = b_i + m_i\} \cap P$ is nonempty.*

3. *P is lattice isomorphic to an integral polytope of the form $C_n \cap L$, where C_n is the n-dimensional unit hypercube and L is an affine subspace.*

This result strengthens a result of Ohsugi and Hibi [10] who essentially proved (3) $\Rightarrow$ (1). Francisco Santos also proved (2) $\Rightarrow$ (1) but the result was never published. Subsequently, Haase included Santos' proof in his dissertation [8]. Condition (2) suggests that the term "compressed" is apt because compressed polytopes are squeezed between two hyperplanes in every facet defining direction.

PROOF. (1) $\Rightarrow$ (2). Supppose that P is compressed and that for some i there were two values $m > m'$ with $\{\boldsymbol{x} \in \mathcal{L} \mid a_i^T \boldsymbol{x} = b_i + m\} \cap P$ and $\{\boldsymbol{x} \in \mathcal{L} \mid a_i^T \boldsymbol{x} = b_i + m'\} \cap P$ nonempty. Let $p_m \in \{\boldsymbol{x} \in \mathcal{L} \mid a_i^T \boldsymbol{x} = b_i + m\} \cap P$ and $p_{m'} \in \{\boldsymbol{x} \in \mathcal{L} \mid a_i^T \boldsymbol{x} = b_i + m'\} \cap P$. Then compare the pulling triangulations with p_m first and with $p_{m'}$ first and the same ordering of the lattice points in the facet $F = \{\boldsymbol{x} \in \boldsymbol{R}^d \mid a_i^T \boldsymbol{x} = b_i\} \cap P$. Given a simplex σ in the pulling triangulation of F, the ratio of volumes $\mathrm{Vol}(\{p_m\} \cup \sigma)/\mathrm{Vol}(\{p_{m'}\} \cup \sigma) = m/m' > 1$. Hence the pulling triangulation of P with p_m first could not be unimodular, contradicting the fact that P was compressed.

(2) $\Rightarrow$ (3). Now suppose that P satisfies Condition (2) above. Since P is a lattice polytope, Condition (2) forces every lattice point in P to be a vertex since, given a facet defining inequality $a^T \boldsymbol{x} \geq b$, the largest value m such that $P \cap \{\boldsymbol{x} \in \boldsymbol{R}^d \mid a^T \boldsymbol{x} = b + m\}$ is nonempty must have $P \cap \{\boldsymbol{x} \in \mathcal{L} \mid a^T \boldsymbol{x} = b + m\}$ nonempty, as well as this set must contain a vertex of P. If there was a lattice point p in P which was not a vertex, it is in the relative interior of some face F of P of dimension greater than or equal to 1. This point p could not be in the set $P \cap \{\boldsymbol{x} \in \boldsymbol{R}^d \mid a^T \boldsymbol{x} = b + m\}$ (where m is the unique largest value where this set is nonempty) for any facet of P, $a^T \boldsymbol{x} = b$ which defines a nontrivial facet of F and, in particular, since $a^T p > b$, there must be some value $m' < m$ such that $P \cap \{\boldsymbol{x} \in \mathcal{L} \mid a^T \boldsymbol{x} = b + m'\}$ is nonempty.

Now we must show that P is affinely isomorphic to an integral polytope that is the intersection of the unit hypercube with an affine subspace. Without loss of generality, we may suppose that P does not lie in an affine subspace: if it did, we would make a unimodular change of coordinates to project to a lower dimensional space. This implies that in Condition (2) there is exactly one nonzero m_i for each i. Consider the linear transformation $\pi : \boldsymbol{R}^d \to \boldsymbol{R}^n$ given by

$$\boldsymbol{x} \mapsto ((a_1^T \boldsymbol{x} - b_1)/m_1, \ldots, (a_n^T \boldsymbol{x} - b_n)/m_n)\,.$$

The image $\pi(P)$ is a 0/1 polytope, since every vertex of P is mapped to a 0/1 vector. A point p lies in $\pi(P)$ if and only if $p \in C_n$ and p is in the affine span of the image of the vertices, because the affine transformation π sends the facet defining inequality $a_i^T \boldsymbol{x} \geq b_i$ to the inequality $y_i \geq 0$. These facts together imply that P satisfies Condition (3).

(3) $\Rightarrow$ (1). If the lattice polytope P satisfies (3) and π is the affine transformation, then P is compressed if and only if $Q = \pi(P)$ is compressed, since this transformation maps

the lattice points in P to the integer points in Q, and P and Q are otherwise isomorphic. Thus it remains to show that integral polytopes Q of the form $Q = C_n \cap \{\boldsymbol{x} : A\boldsymbol{x} = b\}$ are compressed. This result is proven in [10, Lemma 2.2]. However, we will provide a short self-contained proof of it.

Let Q be an integral polytope of the form $C_n \cap \{\boldsymbol{x} \mid A\boldsymbol{x} = b\}$. We will show that Q is compressed by induction on the dimension. If Q has dimension 0, there is nothing to show. Otherwise, suppose Q has dimension d and consider any ordering of the vertices of Q. Let p be the first vertex and construct the pulling triangulation. This is obtained by constructing the pulling triangulation of each facet of Q not containing p and coning each of these triangulations over p. The normalized volume of each simplex is the orthogonal distance from p to the facet times the volume of corresponding simplex in that facet. However, each facet has dimension $d-1$ and is of the form $C_n \cap \{\boldsymbol{x} \mid A\boldsymbol{x} = b, x_i = 0\}$ or $C_n \cap \{\boldsymbol{x} \mid A\boldsymbol{x} = b, x_i = 1\}$ for some i, and hence is compressed by induction. Thus each simplex in the pulling triangulation of each facet has normalized volume one. Further, the orthogonal distance to the corresponding facet is 1, since $p_i = 1$ when the facet is defined by the equation $x_i = 0$. So every simplex in the pulling triangulation is unimodular. Thus Q is compressed. □

Many lattice polytopes which arise in applications (in particular, the statistical applications from Section 5) possess symmetry groups that are transitive on their lattice points. From the preceding theorem we can deduce that for such polytopes either every pulling triangulation is unimodular or none are.

COROLLARY 2.5. *Suppose that P is a lattice polytope and the group of affine symmetries Γ of P is transitive on the lattice points of P. Then either P is compressed or no pulling triangulation of P is unimodular.*

PROOF. We must show that if P is not compressed, then every pulling triangulation is not unimodular. To this end we can suppose that P fails to satisfy Condition (2) in the preceding theorem. Then there exists a facet $F = \{\boldsymbol{x} \in \boldsymbol{R}^d \mid a^T\boldsymbol{x} = b\}$ of P and two nonzero reals $m > m'$ such that $\{\boldsymbol{x} \in \mathcal{L} \mid a^T\boldsymbol{x} = b + m'\} \cap P$ and $\{\boldsymbol{x} \in \mathcal{L} \mid a^T\boldsymbol{x} = b + m\} \cap P$ are nonempty. Consider any ordering of the vertices of P and the resulting pulling triangulation. After applying a suitable element $g \in \Gamma$ to this ordering, we can assume that the first point p_m in the pulling triangulation is in the set $\{\boldsymbol{x} \in \mathcal{L} \mid a^T\boldsymbol{x} = b + m\} \cap P$. Consider any other pulling triangulation which has the same order of the points in F and a point $p_{m'}$ in $\{\boldsymbol{x} \in \mathcal{L} \mid a^T\boldsymbol{x} = b + m'\} \cap P$ as the first vertex. Among the simplices in the first pulling triangulation of P are those of the form $\{p_m\} \cup \sigma$ and in the second pulling triangulation $\{p_{m'}\} \cup \sigma$, where σ is in the induced pulling triangulation of F. We see that the ratio of volumes of these simplices $\mathrm{Vol}(\{p_m\} \cup \sigma)/\mathrm{Vol}(\{p_{m'}\} \cup \sigma) = m/m'$ and hence the first pulling triangulation could not be unimodular. However, this pulling triangulation was arbitrary, so no pulling triangulation of P is unimodular. □

3. Compressed cut polytopes. As an application of our characterization of compressed polytopes, we describe those graphs G whose cut polytopes are compressed. We assume throughout that $G = (V_n, E)$ is an undirected graph with vertices $V_n = [n] := \{1, 2, \ldots, n\}$ and edges E without loops or multiple edges. Our definitions and notation come from [4], and we assume some familiarity with the basic facts about these polytopes.

DEFINITION 3.1. Let $S \subseteq V_n$. The cut semimetric on G induced by S is the 0/1 vector $\delta_G(S)$ in $\boldsymbol{R}^E$ defined by

$$\delta_G(S)_{ij} = 1 \quad \text{if } |S \cap \{i, j\}| = 1\,, \quad \text{and} \quad \delta_G(S)_{ij} = 0 \quad \text{otherwise}\,,$$

where $ij \in E$. The cut polytope of G is the 0/1 polytope

$$\mathrm{Cut}^{\square}(G) = \mathrm{conv}(\delta_G(S) | S \subseteq V_n)\,.$$

We will apply Condition (2) from Theorem 2.4 to deduce the following

THEOREM 3.2. *The cut polytope* $\mathrm{Cut}^{\square}(G)$ *of a graph G is compressed if and only if G has no K_5 minors and every induced cycle in G has length less than or equal to* 4.

A cycle in a graph is induced if there is no chord in the graph cutting across it. Equivalently, a cycle is induced if it is an induced subgraph. The proof of the theorem requires a few intermediate results.

LEMMA 3.3. *If* $\mathrm{Cut}^{\square}(G)$ *is compressed and H is obtained from G by contracting an edge, then* $\mathrm{Cut}^{\square}(H)$ *is compressed.*

PROOF. Let ij be the contracted edge. The polytope $\mathrm{Cut}^{\square}(H)$ is isomorphic to $\{\boldsymbol{x} \mid x_{ij} = 0\} \cap \mathrm{Cut}^{\square}(G)$, and hence is isomorphic to a face of $\mathrm{Cut}^{\square}(G)$. But every face of a compressed polytope is compressed. □

LEMMA 3.4. *If* $\mathrm{Cut}^{\square}(G)$ *is compressed and H is an induced subgraph of G, then* $\mathrm{Cut}^{\square}(H)$ *is compressed.*

PROOF. Let $E' \subset E$ be the union of all edges in G not incident to H together with exactly one edge which is incident to H but not contained in H (provided such an edge exists). Then $\mathrm{Cut}^{\square}(H)$ is isomorphic to $\{\boldsymbol{x} \mid x_e = 0, e \in E'\} \cap \mathrm{Cut}^{\square}(G)$, and hence is isomorphic to a face of $\mathrm{Cut}^{\square}(G)$. But every face of a compressed polytope is compressed. □

LEMMA 3.5. *The polytope* $\mathrm{Cut}^{\square}(K_5)$ *is not compressed.*

PROOF. One facet defining inequality for $\mathrm{Cut}^{\square}(K_5)$ comes by via the following hypermetric construction [4]. Let $b = (1, 1, 1, -1, -1)$ and consider the inequality

$$\sum_{1 \le i < j \le 5} b_i b_j x_{ij} \le 0\,,$$

which defines a facet of $\mathrm{Cut}^{\square}(K_5)$ called a pentagonal facet. To show that $\mathrm{Cut}^{\square}(K_5)$ is not compressed, it suffices to exhibit two sets $S, T \subset V_5$ such that

$$\sum_{1 \le i < j \le 5} b_i b_j \delta_{K_5}(S)_{ij} < \sum_{1 \le i < j \le 5} b_i b_j \delta_{K_5}(T)_{ij} < 0\,,$$

since the cut semimetrics are integral points in the cut polytope. Taking $S = \{1, 2, 3\}$ and $T = \{1, 2\}$ yields

$$-6 = \sum_{1 \le i < j \le 5} b_i b_j \delta_{K_5}(S)_{ij} < \sum_{1 \le i < j \le 5} b_i b_j \delta_{K_5}(T)_{ij} = -2 < 0\,. \qquad \square$$

The preceding three Lemmas imply that if we want to identify graphs whose cut polytopes are compressed, we may restrict attention to those graphs without K_5 minors. In general, it remains a hard open problem to give a facet description of the cut polytopes, however, in the special case of graphs without K_5 minors, a complete irredundant linear description is known.

THEOREM 3.6. *Let G be a graph without K_5 minors. Then* $\mathrm{Cut}^{\square}(G)$ *is the solution set of the following linear inequalities*:

$$0 \le x_e \le 1 \quad \textit{for} \quad e \in E\,,$$
$$\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e \le |F| - 1\,,$$

where C ranges over the induced cycles of G and F ranges over the odd subsets of C. Each of the linear inequalities of the second type is facet defining and the inequalities $0 \le x_e \le 1$ may or may not be facet defining.

Theorem 3.6 is a consequence of the decomposition theory for binary matroids. It is proven in [1] and depends on results in [11]. Thus to prove the main theorem in this section we just need to determine under what conditions these facet defining inequalities satisfy Condition (2) from Theorem 2.4. For the inequalities of type $0 \le x_e \le 1$, these always satisfy Condition (2) regardless of whether or not they are facet defining. Since the structure of the remaining facet defining inequalities only depends on the induced cycles in the graph it suffices to prove the following

LEMMA 3.7. *Let be C an induced cycle of G, and F an odd subset of C. Then the set*

$$\left\{ \boldsymbol{x} \in \mathbf{Z}^d \;\middle|\; \sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e = |F| - 1 - m \right\} \cap \mathrm{Cut}^{\square}(G)$$

is nonempty for exactly $\lfloor |C|/2 \rfloor - 1$ nonzero values of m.

PROOF. Since the value of the linear functional $\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e$ only depends on the edges in C, we can assume that $G = C$. Furthermore, the operation of switching (see [4]) shows that each such facet is equivalent (i.e., up to change of coordinates) to the facet given by $x_{12} - x_{23} - \cdots - x_{1n} \le 0$. So it suffices to prove the Lemma in this setting.

Since cut semimetrics $\delta_G(S)$ are the only integral points in $\mathrm{Cut}^{\square}(G)$, it suffices to determine what values $\delta_G(S)_{12} - \delta_G(S)_{23} - \cdots - \delta_G(S)_{1n}$ can take. Modulo 2, we have

$$\delta_G(S)_{12} - \delta_G(S)_{23} - \cdots - \delta_G(S)_{1n} = \delta_G(S)_{12} + \delta_G(S)_{23} + \cdots + \delta_G(S)_{1n} \equiv 0 \mod 2$$

so $\delta_G(S)_{12} - \delta_G(S)_{23} - \cdots - \delta_G(S)_{1n}$ must be even. Since $\delta_G(S)_{ij}$ is either a zero or a one, there are at most $\lfloor |C|/2 \rfloor - 1$ nonzero values that this expression can take. However, for each

j with $0 < j \leq \lfloor |C|/2 \rfloor$ the set $S_j = \{2i \mid i \in [j]\}$ has

$$\delta_G(S_j)_{12} - \delta_G(S_j)_{23} - \cdots - \delta_G(S_j)_{1n} = 2 - 2j\,,$$

which completes the proof. □

4. Compressed polytopes in linear optimization. Compressed polytopes are closely tied to linear integer optimization problems. In particular, we consider the following setup. Let A be an integral matrix with columns $A_1, A_2, \ldots, A_n$. We assume throughout that A is homogeneous in the sense that there is a nonzero weight vector w such that $w^T A_i = 1$ for all i. For each i consider the integer programming problem

$$\begin{gathered}\text{Maximize } x_i \text{ subject to}\\ A\boldsymbol{x} = b, \quad \boldsymbol{x} \geq 0, \quad \boldsymbol{x} \text{ integral}\,.\end{gathered}$$

For given i, A and b we denote the optimal value of the integer program by $IP_i^+(A, b)$. We say a vector b is IP-feasible if $b = A\boldsymbol{x}$ for some nonnegative integral $\boldsymbol{x}$. The corresponding linear programming relaxation drops the integrality consideration:

$$\begin{gathered}\text{Maximize } x_i \text{ subject to}\\ A\boldsymbol{x} = b, \quad \boldsymbol{x} \geq 0\,.\end{gathered}$$

We denote the optimal value of the linear programming relaxation by $LP_i^+(A, b)$. Since linear programs are considerably easier to solve than integer programs, a fundamental question in optimization is to decide what conditions guarantee that $LP_i^+(A, b) = IP_i^+(A, b)$. Let P_A be the polytope obtained by taking the convex hull of the columns of A. The pulling triangulations of P_A provide a useful sufficient condition to guarantee $LP_i^+(A, b) = IP_i^+(A, b)$.

PROPOSITION 4.1. *For fixed A and i, $LP_i^+(A, b) = IP_i^+(A, b)$ for all IP-feasible b if there exists some ordering of the columns of A with A_i first such that the pulling triangulation of P_A using only $A_1, \ldots, A_n$ is unimodular.*

PROOF. We provide a sketch of the proof which depends on some well-known results in computational algebra. Details can be found in [13, Chapter 8]. The linear programming relaxation solves the standard form integer program for all right hand sides b if an associated intial ideal of the toric ideal I_A is squarefree. The initial ideal is squarefree if and only if the corresponding regular triangulation of P_A is unimodular. In the case where the associated cost vector is the maximization of the x_i coordinate, the corresponding triangulation is a pulling triangulation of P_A with A_i first. □

The condition in Proposition 4.1 is not, however, necessary: if $LP_i^+(A, b) = IP_i^+(A, b)$ for all b, there need not exist a unimodular pulling triangulation of P_A with A_i first as the following example illustrates.

EXAMPLE 4.2. Consider the matrix A given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

This matrix has the property that $LP_1^+(A, b) = IP_1^+(A, b)$ for all IP-freasible b. Indeed, given an IP feasible b, every nonnegative vector $\boldsymbol{x}$ with $A\boldsymbol{x} = b$ has $x_1 = b_3$. On the other hand, P_A has no unimodular pulling triangulations.

This subtlety drops away if we require that $LP_i^+(A, b) = IP_i^+(A, b)$ for all IP-feasible b *and* for all i.

THEOREM 4.3. *Let A be a homogeneous matrix. Then $LP_i^+(A, b) = IP_i^+(A, b)$ for all i and all IP-feasible b if and only if P_A is compressed.*

Recall that in this context where $P_A = \text{conv}(A_1, \ldots, A_n)$ we mean that P_A is compressed with respect to the smallest lattice containing $A_1, \ldots, A_n$.

PROOF. If P_A is compressed, then any pulling triangulation with A_i first is unimodular, which implies by Proposition 4.1 that the LP optimums equal the IP optimums. Conversely, if P_A is not compressed, there is a facet defining inequality that violates Condition (2) in Theorem 2.4. We will use this violation to construct an IP feasible b such that the LP optimum for the maximization problem cannot equal the IP optimum.

Denote the violating facet by $F = \{\boldsymbol{x} \in \boldsymbol{R}^d \mid a^T\boldsymbol{x} = b\}$. Since P_A is a polytope, there is a largest real number m such that $\{\boldsymbol{x} \in \mathcal{L} \mid a^T\boldsymbol{x} = b + m\}$ is nonempty. We may suppose that $A_1 \in \{\boldsymbol{x} \in \mathcal{L} \mid a^T\boldsymbol{x} = b + m\}$. We will partition $A_2, \ldots, A_n$ in the following manner: $a^T A_i = b + m$ for $i = 2, \ldots, k$, $b < a^T A_i < b + m$ for $i = k + 1, \ldots, l$, and $a^T A_i = b$ for $i = l + 1, \ldots, n$. Let

$$K = \ker_{\boldsymbol{Z}}(A) \cap \{\boldsymbol{y} \in \boldsymbol{Z}^d \mid y_1 < 0, y_2 \leq 0, \ldots, y_k \leq 0, y_{k+1} \geq 0, y_{k+2} \leq 0 \ldots, y_l \leq 0\} .$$

Note that K is nonempty, since there exist affine dependencies among A_1, the elements of $F \cap \mathcal{L}$, and A_{k+1} (there are at least $d + 2$ points in a d dimensional lattice). Furthermore, any such affine dependency must have y_1 and y_{k+1} with opposite signs, since neither A_1 nor A_{k+1} are contained in F. Among the vectors in K, let $\boldsymbol{v} \in K$ be any such vector with v_{k+1} with the minimal value among all $\boldsymbol{v}$ in K. This minimal value is strictly greater than 1. We define the right-hand side vector b which will violate $LP_1^+(A, b) > IP_1^+(A, b)$ by

$$b = \sum_{i \mid v_i > 0} v_i A_i - A_{k+1} .$$

Clearly, b is IP-feasible, since we have expressed it as a nonnegative combination of the columns of A.

First of all, we claim that $IP_1^+(A, b) = 0$. If not, there is an improving integer vector $\boldsymbol{u} \in K$ with such that the vector $\boldsymbol{v}^+ - e_{k+1} - \boldsymbol{u}$ is nonegative and has first coordinate greater than zero. The existence of such a $\boldsymbol{u}$ violates our minimality assumption on v_{k+1} (since $u_{k+1} \leq v_{k+1} - 1$). On the other hand, the rational vector $\boldsymbol{u} = (v_{k+1})/(v_{k+1})\boldsymbol{v}$ is an improving vector such that $\tilde{\boldsymbol{v}} = \boldsymbol{v}^{+} - e_{k+1} - \boldsymbol{u}$ is a nonnegative rational vector with $A\tilde{\boldsymbol{v}} = b$ and $\tilde{v}_1 > 0$ so that $LP_1^{+}(A, b) > 0$. □

5. Applications in statistical disclosure limitation. One motivation for studying compressed polytopes comes from their relationship to certain optimization problems which arise in statistical disclosure limitation. The general problem in this area is to determine what information about individual survey respondents can be inferred from the release of partial data. This type of problem arises when government agencies like a census bureau gather information about citizens and wish to release partial data to the public for the purposes of data analysis but are required by law to maintain the privacy of citizens.

The case we consider here concerns the release of margins of a multiway contingency table. In this case, an individual cell entry is considered secure if among all nonnegative integral tables with given released marginal totals the upper and lower bounds on the cell entry are far enough apart [2, 3]. This naturally leads to standard form integer programs of the following type:

$$\text{Maximize/Minimize } x_{\mathbf{0}} \text{ subject to}$$
$$A_{\Delta}\boldsymbol{x} = b, \quad \boldsymbol{x} \geq 0\,, \quad \text{and } \ \boldsymbol{x} \text{ integral}\,,$$

where A_{Δ} is a certain $0/1$ matrix which computes the released margins b of the multiway table $\boldsymbol{x}$. A heuristic for approximating the solution to this integer program is to solve the linear programming relaxations:

$$\text{Maximize/Minimize } x_{\mathbf{0}} \text{ subject to}$$
$$A_{\Delta}\boldsymbol{x} = b \ \text{ and } \ \boldsymbol{x} \geq 0\,.$$

A fundamental problem in this area is to determine under what conditions the linear programming relaxation is equal to the true integer value. We will focus here on the maximization problem. To state our results, we first need to establish notation for the contingency table problems of interest. Here $\boldsymbol{x}$ denotes a $d_1 \times d_2 \times \cdots \times d_n$ multiway contingency table. The particular collection of margins of this table which are released are encoded by a simplicial complex Δ on the n-element set $[n]$.

Each facet $S \in \Delta$ corresponds to a released margin. Computing a collection of marginals of a multiway table is a linear transformation. The matrix, represented in the standard basis, which encodes this linear transformation is denoted by A_{Δ}. Note that the size of the matrix A_{Δ} and problems related to linear programming relaxations depend on Δ *and* the integer vector $d = (d_1, d_2, \ldots, d_n)$ though we suppress the dependence on d when we use the notation A_{Δ}. We use the notation P_{Δ} to denote the convex hull of the columns of the matrix A_{Δ}. From the previous section, we deduce the following basic fact.

COROLLARY 5.1. *The linear programming relaxation solves the integer programs* $IP_{\mathbf{0}}^{+}(A_{\Delta}, b) = LP_{\mathbf{0}}^{+}(A_{\Delta}, b)$ *for all marginals* b *if and only if the marginal polytope* P_{Δ} *is compressed.*

PROOF. Because of the transitive symmetry group on the vertices of P_Δ, $IP_{\mathbf{0}}^{+}(A_\Delta, b) = LP_{\mathbf{0}}^{+}(A_\Delta, b)$ for the **0** cell entry if and only if this holds for all cell entries. Then by Theorem 4.3 this holds if and only if P_Δ is compressed. □

Thus we are led to study the following general problem:

PROBLEM 5.2. *Characterize the pairs (Δ, d) of simplicial complexes Δ and integer vectors $d = (d_1, \ldots, d_n)$ such that P_Δ is compressed.*

It seems a challenging problem to classify such marginals in general, since it would require the knowledge of many families of facet defining inequalities of the marginal polytopes. There is very little known about these facet defining inequalities in general. In the remainder of this section, we provide some constructions for producing compressed marginal polytopes. As a corollary, we deduce that the marginal polytopes of decomposable models are compressed. We also provide a complete characterization of compressed marginal polytopes in two restricted cases.

There are a few standard operations on simplicial complexes that send compressed marginal polytopes to compressed marginal polytopes.

PROPOSITION 5.3. *Suppose that the pair (Δ, d) has P_Δ compressed.*

1. *If $\Delta' \subset \Delta$ is an induced subcomplex and d' the correspond integer vector, then the pair (Δ', d') has $P_{\Delta'}$ compressed.*
2. *If $d' \leq d$ coordinatate-wise, then the pair (Δ', d') with $\Delta' = \Delta$ has $P_{\Delta'}$ compressed.*

PROOF. In both cases $P_{\Delta'}$ is isomorphic to a face of P_Δ. However, the faces of compressed polytopes are compressed. □

PROPOSITION 5.4. *Suppose that the pair (Δ, d) has the marginal polytope P_Δ compressed. Let Δ' be the new simplicial complex on $[n+1]$ obtained from Δ by $\Delta' = \{\{n+1\} \cup F \mid F \in \Delta\}$ and $d' = (d_1, \ldots, d_n, d_{n+1})$, where d_{n+1} is any positive integer. Then the pair (Δ', d') has a compressed marginal polytope $P_{\Delta'}$.*

PROOF. The marginal polytope $P_{\Delta'}$ is isomorphic to the direct join of d_{n+1} copies of P_Δ. But the direct join of compressed polytopes is compressed, since any triangulation of the direct join is obtained by taking the direct join of the induced triangulations of the pieces. The direct join of two unimodular triangulations is unimodular. □

DEFINITION 5.5. A simplicial complex Δ is called *reducible* with decomposition (Δ_1, S, Δ_2) if

1. Δ_1 and Δ_2 are induced subcomplexes of Δ,
2. $S \subset [n]$,
3. $\Delta_1 \cup \Delta_2 = \Delta$, and
4. $\Delta_1 \cap \Delta_2 = 2^S$.

A simplicial complex is called *decomposable* if Δ is reducible and each of Δ_1 and Δ_2 is either decomposable or a simplex.

Given a reducible simplicial complex Δ with decomposition (Δ_1, S, Δ_2) together with the integer vector d, denote by d^1 and d^2 the induced vectors with indices corresponding to the nodes of Δ_1 and Δ_2, respectively.

PROPOSITION 5.6. *If Δ is reducible and the pairs (Δ_1, d^1) and (Δ_2, d^2) have compressed marginal polytopes, then the marginal polytope P_Δ is compressed.*

PROOF. For reducible models Δ, the marginal polytopes are given by

$$P_\Delta = P_{\Delta_1} \times P_{\Delta_2} \cap \{(\boldsymbol{x}, \boldsymbol{y}) \mid \pi_1(\boldsymbol{x}) = \pi_2(\boldsymbol{y})\},$$

where π_1 and π_2 are the S-marginal maps of $\boldsymbol{x}$ and $\boldsymbol{y}$, repsectively. In particular, the set of facet defining inequalities of P_Δ is just the union of the facet defining of P_{Δ_1} and P_{Δ_2}. Since P_{Δ_1} and P_{Δ_2} are compressed, these facet defining inequalites satisfy Condition (2) of Theorem 2.4. But this implies that they also satisfy Condition (2) of Theorem 2.4 with respect to P_Δ as well. This implies that P_Δ is compressed. □

COROLLARY 5.7. *If Δ is decomposable then P_Δ is compressed.*

PROOF. If $\Delta = 2^{[n]}$ then P_Δ is a simplex. Thus, if Δ is decomposable, P_Δ is compressed by applying Proposition 5.6 and induction on the number of facets of Δ. □

The preceding propositions provide methods for producing compressed marginal polytopes from smaller compressed marginal polytopes. However, these results are far from giving a complete characterization of all pairs (Δ, d) such that the marginal polytopes are compressed. In the remainder of this section, we provide characterizations of compressed marginal polytopes in two settings where we place "extremal" conditions on Δ, or d or both.

PROPOSITION 5.8. *Let Δ be the boundary of an $n-1$ simplex. Then P_Δ is compressed if and only if for at most two i, $d_i > 2$ or $n = 3$ and up to symmetry $d = (3, 3, d_3)$.*

PROOF. In the case where for at most two i, $d_i > 2$, it is known that P_Δ is a unimodular polytope (e.g., [13, Chapter 14]) and hence is compressed. The case where $n = 3$ and $d = (3, 3, d_3)$, the complete facet description of this polytope is known (e.g., [6]) and one verifies that the facet defining inequalities in this case satisfy Condition (2) of Theorem 2.4. Direct computation using Polymake [7] shows that Condition (2) of Theorem 2.4 fails in the case $n = 3$, $d = (3, 4, 4)$ and $n = 4$, $d = (2, 3, 3, 3)$. These results together with Proposition 5.3 imply that P_Δ is compressed in no other cases. □

The cut polytopes from Section 3 are intimately tied to the marginal polytopes we are interested in, in the special case where $d = (2, 2, \ldots, 2)$ and all facets of Δ are 0 or 1-dimensional. In this case Δ is a graph and we have the following well-known result (see [4]):

LEMMA 5.9. *Given a graph Δ and $d = (2, 2, \ldots, 2)$, there is a lattice isomorphism of the marginal polytope P_Δ to the cut polytope $\mathrm{Cut}^\square(\tilde{\Delta})$, where $\tilde{\Delta}$ is the graph obtained from Δ by adding a new vertex v and all edges from v to the nodes of Δ.*

The lattice isomorphism in the preceding Lemma is known as the covariance mapping. Then we can deduce:

THEOREM 5.10. *Let Δ be a graph and $d = (2, 2, \ldots, 2)$. Then P_Δ is compressed if and only if Δ is free of K_4 minors and every induced cycle in Δ has length less than or equal to* 4.

PROOF. The graph $\tilde{\Delta}$ is free of K_5 minors and has all induced cycles of length less than or equal to four if and only if Δ has no K_4 minors and all induced cycles of length less than or equal to four. Thus, this is a direct consequence of Theorem 3.2 characterizing the compressed cut polytopes. □

In these cases we can in fact say more: even though the size of the integer program seems exponential in n the number of nodes in the simplicial complex, in the case where P_Δ is compressed we can solve the corresponding linear program (and hence the integer program) in polynomial time.

COROLLARY 5.11. *Suppose that $d = (2, 2, \ldots, 2)$ and Δ is a graph that is free of K_4 minors and has every induced cycle of length less than or equal to four. Then the IP-maximum value $IP_i^+(A_\Delta, b)$ can be computed in polynomial time in n and the bit complexity of b.*

PROOF. Since $IP_i^+(A, b) = LP_i^+(A, b)$ for these graphs, it suffices to show that the linear program can be solved in polynomial time. However, the problem of maximizing a coordinate is polynomial time equivalent to determining if a point lies in P_Δ. For graphs without K_4 minors, the containment problem can be decided in polynomial time as illustrated in [4]. □

In general, we would like to understand how far the linear programming relaxations can be from the true integer programming values for these optimization problems in statistical disclosure limitation. This leads to the study of the integer programming gap [9]. A natural question to ask is: How does the failure of Condition (2) in Theorem 2.4 relate to the integer programming gap? A natural family of marginal polytopes where this problem could be explored is the family of cycles.

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