COMPRESSIBLE FLOWS WITH DEGENERATE HODOGRAPHS*

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1. Introduction. The theory of compressible perfect fluids has developed slowly because the basic equations are non-linear. Thus it has been profitable to consider special examples, such as will be studied here. The present problem originates in the study of steady, two-dimensional, isentropic, irrotational flow. If there is a biunique mapping of the physical plane onto the hodograph plane, then the equation for the velocity potential function can be linearized by a Legendre transformation [5].** This draws attention to the case in which the transformation may fail because the mapping is nowhere biunique. This suggests the problem to investigate all three-dimensional flows whose images in the hodograph space, for Cartesian coordinates and velocity components, are curves or surfaces. Such flows are sometimes said to be "lost" [10] by contrast with the nomenclature used here. By analogy with the usage in [5], flows with one- or two-dimensional hodographs will be called *simple* or *double* waves. The hodograph of a flow will be called *degenerate* when it has fewer dimensions than the original physical space.

The problem can also be motivated as follows. Among the most familiar compressible flows are Prandtl-Meyer expansion around a corner or curved wall [8]; Busemann's cylindrical or "swept-back" flow produced by superposition of plane flow and uniform flow normal to that plane [2]; Taylor-Maccoll axisymmetric flow about a cone [1, 3, 7, 11]; and Busemann's general conical flows [2]. In these examples the loci of particles of equal velocity are planes or straight lines, so their hodographs are degenerate. The question arises, whether this enumeration is exhaustive.

In this paper the flow will be assumed to be steady, isentropic, and irrotational. Characterizations of one- and two-dimensional hodographs will be developed, and generalizations will be found for the properties of the examples mentioned above. As an example the construction of flows with axisymmetric degenerate hodographs will be considered.

Some aspects of this problem have been considered by Germain [6]. M. H. Martin has also made an unpublished investigation along these lines. The construction of all axisymmetric flows with degenerate hodographs was studied by Bateman and later by Stewart [10]. Opatowski [9] has discussed very concisely the more general problem to determine those flows for which the covariant velocity components in some curvilinear coordinate system depend only on two coordinates.

2. Fundamental equations. Compressible perfect flow obeys the equations of motion

$$u_i \,\partial u_k/\partial x^i = -\rho^{-1}\,\partial p/\partial x^k \tag{2.1}$$

and the equation of continuity

$$\partial(\rho u_i)/\partial x^i = 0. \tag{2.2}$$

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**Numbers in brackets designate papers listed at the end of this note.

The x^i ($i = 1, 2, 3$) denote Cartesian coordinates in the physical space, u_i velocity components, ρ density, and p pressure. The convention that every pair of repeated sub- or superscripts implies summation over their range has been adopted. For irrotational flow

$$\partial u_i/\partial x^j = \partial u_j/\partial x^i. \tag{2.3}$$

For isentropic flow

$$p/p_0 = (\rho/\rho_0)^\gamma \tag{2.4}$$

for certain reference values p_0 and ρ_0 , and $\gamma = c_p/c_v$, the ratio of the specific heats at constant pressure and volume. Equations (2.1), (2.3), and (2.4) imply Bernoulli's equation

$$\tfrac{1}{2}u_iu_i + a^2/(\gamma - 1) = \tfrac{1}{2}c^2, \tag{2.5}$$

where

$$a^2 = dp/d\rho = \gamma p/\rho \tag{2.6}$$

is the square of the speed of sound, and the constant c is the limiting speed of flow. By (2.3) there exists a velocity potential function φ such that

$$u_i = \partial\varphi/\partial x^i. \tag{2.7}$$

By (2.1, (2.3), (2.4), and (2.6)

$$a^2\, \partial\rho/\partial x^k = -\rho u_i\, \partial u_i/\partial x^k, \tag{2.8}$$

and by (2.2), (2.7), and (2.8)

$$(a^2\delta_{ij} - u_iu_j)\, \partial^2\varphi/\partial x^i\, \partial x^j = 0, \tag{2.9}$$

where Kronecker's delta, $\delta_{ij} = 1$ (0) if $i = (\neq)\ j$, and where by (2.5)

$$a^2 = \tfrac{1}{2}(\gamma - 1)(c^2 - u_iu_i). \tag{2.10}$$

3. Degenerate Legendre transformations. The transformation $x^i \to u_i$ maps a three-dimensional region of the physical space onto an n-dimensional region of the hodograph space if and only if

$$\partial\varphi/\partial x^i = u_i = u_i(\mu), \tag{3.1}$$

the functions $\mu^\alpha(x)$ ($\alpha = 1, \cdots, n$) are independent, and n of the functions $u_i(\mu)$ are also independent. Hence the

$$\text{rank of } \|\, \partial\mu^\alpha/\partial x^i \,\| = \text{rank of } \|\, \partial u_i/\partial\mu^\alpha \,\| = n. \tag{3.2}$$

Disregard $n = 0$ (uniform flow). Then $n = 1$ or 2 for degenerate hodographs. Let

$$k = \varphi - x^iu_i \tag{3.3}$$

By (3.1) and (3.3) $\partial k/\partial x^j = -x^i(\partial u_i/\partial\mu^\alpha)(\partial\mu^\alpha/\partial x^j)$. Accordingly, the Jacobian matrix of k and μ^α has the same rank as that of μ^α alone, so $k = k(\mu)$. (3.3) becomes

$$\varphi = x^iu_i(\mu) + k(\mu), \tag{3.4}$$

and by (3.1) and (3.4) $(x^i\, \partial u_i/\partial\mu^\alpha + \partial k/\partial\mu^\alpha)\, \partial\mu^\alpha/\partial x^j = 0$. By (3.2)

$$x^i\, \partial u_i/\partial\mu^\alpha + \partial k/\partial\mu^\alpha = 0. \tag{3.5}$$

By (3.5)

$$(x^i\, \partial^2 u_i/\partial\mu^\alpha\, \partial\mu^\beta + \partial^2 k/\partial\mu^\alpha\, \partial\mu^\beta)\, \partial\mu^\beta/\partial x^i = -\partial u_i/\partial\mu^\alpha. \tag{3.6}$$

By (3.2) this implies

$$\text{rank of } \|\, x^i\, \partial^2 u_i/\partial\mu^\alpha\, \partial\mu^\beta + \partial^2 k/\partial\mu^\alpha\, \partial\mu^\beta \,\| = n. \tag{3.7}$$

Hereafter assume that $u_i(\mu)$ and $k(\mu)$ have been chosen to satisfy (3.7). Then (3.5) can be inverted to yield $\mu^\alpha(x)$.

By (3.1) a point on the hodograph is determined by setting $\mu^\alpha = \mu_0^\alpha$. The set of points in the physical space which is mapped onto $u_i(\mu_0)$ will be called its *prototype.* (3.5) implies

THEOREM 3.1: *If the coordinate axes of the physical and hodograph spaces are parallel, the prototype in the physical space of a point, P, of a one (two) dimensional degenerate hodograph, H, is contained in a plane (line) parallel to the plane (line) normal to H at P.*

So far ϕ has only been compelled to yield a degenerate map. For a compressible flow (2.9) and (2.10) must also be satisfied. In (2.9) $\partial^2\varphi/\partial x^i\, \partial x^i$ is required. By (3.1)

$$\partial^2\varphi/\partial x^i\, \partial x^i = (\partial u_i/\partial\mu^\alpha)(\partial\mu^\alpha/\partial x^i), \tag{3.8}$$

where $\partial\mu^\alpha/\partial x^i$ must be obtained from (3.6).

4. Simple waves. When $n = 1$, (3.6) to (3.8) yield

$$\partial^2\varphi/\partial x^i\partial x^i = -u_i'u_i'/(x^m u_m'' + k''), \tag{4.1}$$

where primes denote ordinary derivatives with respect to μ^1. By (2.9), (2.10), and (4.1)

$$a^2 u_i'u_i' = (u_i u_i')^2. \tag{4.2}$$

If s is arc-length measured from some point of the hodograph curve and q is speed of flow

$$(s')^2 = u_i'u_i', \tag{4.3}$$

$$q^2 = u_i u_i. \tag{4.4}$$

Now (4.2) implies

$$a^2 = \tfrac{1}{2}(\gamma - 1)(c^2 - q^2) = q^2(dq/ds)^2. \tag{4.5}$$

Construct a cone, K, with vertex, V, at the origin of the hodograph space and passing through the hodograph curve C. When K is developed onto a plane, C will be deformed into a plane curve C' to which (4.5) also applies. Hence C' is the familiar epicycloid of the Prandtl-Meyer flow around a corner. Accordingly, C will be called a *conically deformed Prandtl-Meyer epicycloid.*

THEOREM 4.1: *The hodograph of a simple wave consists of arcs of conically deformed Prandtl-Meyer epicycloids. Conversely, a sufficiently small arc of a conically deformed Prandtl-Meyer epicycloid, on which the direction of the tangent vector varies continuously, is the hodograph of a simple wave.*

For the converse, construct a velocity field with the prescribed hodograph. Suppose that for $A \leqq \mu^1 \leqq B$, $u_i = u_i(\mu^1)$ is an arc of a conically deformed Prandtl-Meyer epicycloid, the u_i being of class C^1 on AB. Let $f(\mu^1)$ be an arbitrary function continuous on AB, and let

$$x^i(\mu^1) = x_A^i + \int_A^{\mu^1} u_i(\mu)f(\mu)\, d\mu,$$

where the x_A^i are constants. To prevent the curve $x^i = x^i(\mu^1)$ from intersecting itself, decrease the interval AB, if necessary. As suggested by Theorem 3.1, through each point $x^i(\mu^1)$ construct a plane normal to $u_i'(\mu^1)$, and assign $u_i(\mu^1)$ to every point of this plane. By making the interval AB small enough, and by considering only a region close enough to the curve $x^i = x^i(\mu^1)$, a continuous single-valued velocity vector field can be obtained. Finally, by constructing in this vector field a family of streamlines close to the streamline $x^i = x^i(\mu^1)$, a stream tube, and hence a flow with the desired hodograph will be produced.

If (4.5) is interpreted as an equation of a plane curve, it is clear that *in a simple wave the flow must be supersonic.* Discontinuities in the second or higher order derivatives of $u_i(\mu^1)$ are propagated along prototype planes. Thus the Mach cone at any point, P, of a simple wave must be tangent to the prototype plane, Π, through P, and the streamline, S, through P intersects Π at the Mach angle.

The reader may verify the following assertions. (1) Sufficiently small arcs of any curve with continuous curvature can be arcs of streamlines of simple waves. (2) For a sufficiently small range of values of μ^1 any one parameter family of planes $A_m(\mu^1)x^m + B(\mu^1) = 0$ can be chosen to be the prototype planes of a simple wave, provided $A_m(\mu^1)$ and $B(\mu^1)$ are of class C^1, and provided that not all of these planes are parallel.

As an example for this section, consider a simple wave, W, in which the envelope of the prototype planes is a cylinder, S. By Theorem 3.1 the hodograph, H, of W is a plane curve. Orient axes so H lies in $u_3 =$ constant, and let $Q^2 = q^2 - u_3^2$. (4.5) becomes $\frac{1}{2}(\gamma - 1)[(c^2 - u_3^2) - Q^2] = (QdQ/ds)^2$. This defines an epicycloid obtained by shrinking the generating circles of the usual Prandtl-Meyer epicycloid by a factor $(1 - u_3^2/c^2)^{1/2}$. As indicated in Sec. 5, W is a swept-back version of Prandtl-Meyer flow.

5. Double waves. When $n = 2$ let $\Delta = \det \| x^m\, \partial^2 u_m/\partial\mu^\alpha\, \partial\mu^\beta + \partial^2 k/\partial\mu^\alpha\, \partial\mu^\beta \|$, where $\Delta \neq 0$ by (3.7). By (3.6)

$$\Delta\partial\mu^\beta/\partial x^k = (-1)^{\alpha+\beta+1}(x^m\, \partial^2 u_m/\partial\mu^{\alpha+1}\, \partial\mu^{\beta+1} + \partial^2 k/\partial\mu^{\alpha+1}\, \partial\mu^{\beta+1})\, \partial u_k/\partial\mu^\alpha, \tag{5.1}$$

where α is summed, but not β, and where $\alpha + 1$ and $\beta + 1$ are reduced mod 2. For fixed μ^α the solutions x^i of (3.5) lie on a line. Let $\nu^i(\mu)$ be parallel to this line, so

$$\nu^i\, \partial u_i/\partial\mu^\alpha = 0, \tag{5.2}$$

and let $x_0^i(\mu)$ be a particular solution of (3.5). The general solution is

$$x^i = x_0^i(\mu) + r\nu^i(\mu), \tag{5.3}$$

where the parameter r is independent of μ^α. Now (5.1) becomes

$$\Delta\partial\mu^\beta/\partial x^k = (-1)^{\alpha+\beta+1}[(r\nu^m + x_0^m)\partial^2 u_m/\partial\mu^{\alpha+1}\partial\mu^{\beta+1} + \partial^2 k/\partial\mu^{\alpha+1}\partial\mu^{\beta+1}]\partial u_k/\partial\mu^\alpha \tag{5.4}$$

for β not summed. By (2.9) and (3.7)

$$\begin{aligned}(a^2\delta^{ij} - u_iu_j)(\partial u_i/\partial\mu^\alpha)(\partial u_j/\partial\mu^\beta)(-1)^{\alpha+\beta}\\ \cdot[(r\nu^m + x_0^m)\partial^2 u_m/\partial\mu^{\alpha+1}\partial\mu^{\beta+1} + \partial^2 k/\partial\mu^{\alpha+1}\partial\mu^{\beta+1}] = 0\end{aligned} \tag{5.5}$$

with both α and β summed. Since r is independent of μ^α

$$(a^2\delta^{ij} - u_iu_j)(\partial u_i/\partial\mu^\alpha)(\partial u_j/\partial\mu^\beta)(-1)^{\alpha+\beta}\nu^m\partial^2u_m/\partial\mu^{\alpha+1}\partial\mu^{\beta+1} = 0, \tag{5.6}$$

$$(a^2\delta^{ij} - u_iu_j)(\partial u_i/\partial\mu^\alpha)(\partial u_j/\partial\mu^\beta)(-1)^{\alpha+\beta}(x_0^m\partial^2u_m/\partial\mu^{\alpha+1}\partial\mu^{\beta+1} + \partial^2k/\partial\mu^{\alpha+1}\partial\mu^{\beta+1}) = 0. \tag{5.7}$$

Now let $g_{\alpha\beta}$ be the covariant metric tensor of the hodograph surface and $b_{\alpha\beta}$ its second fundamental tensor. By definition

$$g_{\alpha\beta} = (\partial u_i/\partial\mu^\alpha)(\partial u_i/\partial\mu^\beta), \tag{5.8}$$

$$b_{\alpha\beta} = \nu^m(\partial^2u_m/\partial\mu^\alpha\partial\mu^\beta), \tag{5.9}$$

where ν^m is a unit normal to the surface, i.e.

$$\nu^i\nu^i = 1. \tag{5.10}$$

Also

$$u_k\,\partial u_k/\partial\mu^\alpha = q\,\partial q/\partial\mu^\alpha = aM\,\partial q/\partial\mu^\alpha, \tag{5.11}$$

where $M = q/a$. Write $\partial q/\partial\mu^\alpha = q_{,\alpha}$, where the subscript $_{,\alpha}$ denotes the covariant derivative with respect to μ^α and based on $g_{\alpha\beta}$. Then (5.6) becomes

$$(g_{\alpha\beta} - M^2q_{,\alpha}q_{,\beta})(-1)^{\alpha+\beta}b_{\alpha+1\ \beta+1} = 0. \tag{5.12}$$

A particular solution of (3.5) is

$$x_0^i = -(\partial k/\partial\mu^\gamma)g^{\gamma\delta}(\partial u_i/\partial\mu^\delta), \tag{5.13}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$. Since the Christoffel symbols of the first kind, based on $g_{\alpha\beta}$, are $[\alpha\beta, \gamma] = (\partial u_i/\partial\mu^\gamma)(\partial^2u_i/\partial\mu^\alpha\,\partial\mu^\beta)$, the second covariant derivative of k becomes

$$k_{,\alpha\beta} = \partial^2k/\partial\mu^\alpha\,\partial\mu^\beta - (\partial k/\partial\mu^\gamma)g^{\gamma\delta}(\partial u_i/\partial\mu^\delta)(\partial^2u_i/\partial\mu^\alpha\,\partial\mu^\beta). \tag{5.14}$$

Hence (5.7), (5.8), (5.11), and (5.14) imply

$$(g_{\alpha\beta} - M^2q_{,\alpha}q_{,\beta})(-1)^{\alpha+\beta}k_{,\alpha+1\beta+1} = 0. \tag{5.15}$$

(5.12) is a second order quasilinear partial differential equation for three functions. To determine $u_i(\mu)$ requires two more equations, which may be obtained by assigning a special form to the coefficient tensor $g_{\alpha\beta} - M^2q_{,\alpha}q_{,\beta}$. The resulting systems are classified according to the nature of the characteristic curves of their integral surfaces.

A characteristic is a curve on which the coordinate functions, their first partial derivatives, and hence the metric tensor are continuous, while the components of the second fundamental tensor may have discontinuities. Suppose $\partial u_i/\partial\mu^\alpha$ are known along $\mu^\alpha = \mu^\alpha(t)$ on $u_i = u_i(\mu)$. By (5.9) the *strip conditions* $d(\partial u_m/\partial\mu^\alpha)/dt = (\partial^2u_m/\partial\mu^\alpha\,\partial\mu^\beta)\,d\mu^\beta/dt$ imply

$$b_{\alpha\beta}\,d\mu^\beta/dt = \nu^m\,d(\partial u_m/\partial\mu^\alpha)/dt. \tag{5.16}$$

Then $b_{\alpha\beta}$ fails to be uniquely determined along $\mu^\alpha = \mu^\alpha(t)$ by (5.12) and (5.16) only if

$$(g_{\alpha\beta} - M^2q_{,\alpha}q_{,\beta})(d\mu^\alpha/dt)(d\mu^\beta/dt) = 0. \tag{5.17}$$

This defines the characteristic directions $d\mu^\alpha/dt$. By (5.14), if $u_i(\mu)$ are known, then (5.15) is a linear partial differential equation for k which also has the characteristic directions (5.17). Equations (5.12) and (5.15) will be said to be of *hyperbolic, parabolic,* or *elliptic* type wherever $\Omega = \det \| g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta} \| < 0, = 0, > 0$. Hereafter (5.12) and (5.15) will be assumed to be hyperbolic. Cohn [4] has constructed a double wave of hyperbolic type and has also given simple canonical forms for the hodograph equations for both the hyperbolic and elliptic cases.

If s_c is the arc-length of a characteristic, then (5.17) becomes

$$q^2(dq/ds_c)^2 = a^2, \tag{5.18}$$

which is identical with (4.5). Hence

THEOREM 5.1: *The characteristics of the hodographs of double waves are composed of arcs of conically deformed Prandtl-Meyer epicycloids, i.e. of one-dimensional hodographs.*

By (5.18) the component of $q_{,\alpha}$ along either characteristic is $q_{,\alpha}\, d\mu^\alpha/ds_c = \pm a/q = \pm 1/M$. Hence

THEOREM 5.2: *On the hodographs of double waves the curves of constant speed and their orthogonal trajectories bisect the angles between the characteristics.*

Curves $x^i = x^i(t)$ (other than prototype lines) in the physical space are mapped onto curves $\mu^\alpha = \mu^\alpha(t)$ on the hodograph. It is convenient to know the relation between tangent vectors of a pair of corresponding curves. By (3.5), (5.3), (5.13), and (5.14)

$$(dx^i/dt)(\partial u_i/\partial \mu^\alpha) + [r(t)b_{\alpha\beta} + k_{,\alpha\beta}](d\mu^\beta/dt) = 0 \tag{5.19}$$

for some $r(t)$. Unless $\det \| rb_{\alpha\beta} + k_{,\alpha\beta} \| = 0$, this determines $d\mu^\beta/dt$. Conversely, if the curve $\mu^\alpha = \mu^\alpha(t)$ is given on a hodograph surface, its prototype is the ruled surface

$$x^i(r, t) = x_0^i(t) + [r - A(t)]\nu^i(t), \tag{5.20}$$

where ν^i is a unit normal to the hodograph, $x_0^i(t)$ is defined by (5.13), and $A(t)$ by $dA/dt = \nu^i\, dx_0^i/dt$ (to make the curves $r =$ constant orthogonal to the rulings). For (5.20) an analog of (5.19) is

$$(\partial x^i/\partial t)(\partial u_i/\partial \mu^\alpha) + [(r - A)b_{\alpha\beta} + k_{,\alpha\beta}](d\mu^\beta/dt) = 0. \tag{5.21}$$

Since $\nu^i\, \partial x^i/\partial t = 0$, (5.21) implies

$$\partial x^i/\partial t = -(\partial u_i/\partial \mu^\alpha) g^{\alpha\beta}[(r - A)b_{\alpha\beta} + k_{,\alpha\beta}]\, d\mu^\beta/dt. \tag{5.22}$$

In general, the direction of $\partial x^i/\partial t$ will vary with r along a ruling, so (5.20) need not be developable. This raises the question, what curves on the hodograph have developable prototypes? Since $\partial^2 x^i/\partial r^2 = 0$, (5.20) will be developable if and only if

$$\det \| \partial x^i/\partial r, \quad \partial x^i/\partial t, \quad \partial^2 x^i/\partial r \partial t \| = 0. \tag{5.23}$$

Since ν^i and $\partial u_i/\partial \mu^\alpha$ are linearly independent, by (5.20) and (5.22) (5.23) is equivalent to

$$(Cb_{\alpha\beta} + Dk_{,\alpha\beta})\, d\mu^\beta/dt = 0 \tag{5.24}$$

for some $C(t)$ and $D(t)$ not both zero. By (5.12) and (5.15)

$$(g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta})(-1)^{\alpha+\beta}(Cb_{\alpha+1\beta+1} + Dk_{,\alpha+1\beta+1}) = 0. \tag{5.25}$$

Hereafter assume $Cb_{\alpha\beta} + Dk_{,\alpha\beta} \neq 0$ for some α and β. Then (5.24) and (5.25) imply (5.17), i.e. $\mu^\alpha = \mu^\alpha(t)$ is a characteristic.

Next, show that the prototypes of both families of characteristics are developable. Let $\mu^\alpha = \mu^\alpha_\epsilon(t)$ define one characteristic from each family through a given point P of the hodograph. At P, by (5.17)

$$2(g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta}) = (-1)^{\alpha+\beta}[(d\mu_1^{\alpha+1}/dt)(d\mu_2^{\beta+1}/dt) + (d\mu_1^{\beta+1}/dt)(d\mu_2^{\alpha+1}/dt)]f$$

for some $f \neq 0$. Then by (5.12) and (5.15)

$$b_{\alpha\beta}(d\mu^\alpha_{\epsilon+1}/dt)(d\mu^\beta_\epsilon/dt) = k_{,\alpha\beta}(d\mu^\alpha_{\epsilon+1}/dt)(d\mu^\beta_\epsilon/dt) = 0, \tag{5.26}$$

where ϵ is not summed. Since these have non-trivial solutions $d\mu^\alpha_{\epsilon+1}/dt$, (5.26) implies (5.24) for $\mu^\alpha = \mu^\alpha_\epsilon$ and some $C = C_\epsilon$ and $D = D_\epsilon$ not both zero.

Now investigate the relation between tangents to characteristics and unit normals n^i_ϵ to prototypes $x^i = x^i_\epsilon(r, t)$ of characteristics $\mu^\alpha = \mu^\alpha_\epsilon(t)$. By (5.20) $n^i_\epsilon v^i = n^i_\epsilon \partial x^i_\epsilon/\partial r = 0$. For some A^α_ϵ

$$n^i_\epsilon = A^\alpha_\epsilon \partial u_i/\partial \mu^\alpha. \tag{5.27}$$

Since $n^i_\epsilon \partial x^i_\epsilon/\partial t = 0$, then by (5.25) $A^\alpha_\epsilon[(r - A)b_{\alpha\beta} + k_{,\alpha\beta}]\, d\mu^\beta_\epsilon/dt = 0$. Since $x^i = x^i_\epsilon(r, t)$ is developable, it must be possible to choose A^α_ϵ so that n^i_ϵ does not vary on a ruling. Thus A^α_ϵ may be assumed independent of r, so

$$A^\alpha_\epsilon b_{\alpha\beta} d\mu^\beta_\epsilon/dt = A^\alpha_\epsilon k_{,\alpha\beta} d\mu^\beta_\epsilon/dt = 0. \tag{5.28}$$

If for some $c(t)$ and $d(t)$

$$\det \| cb_{\alpha\beta} + dk_{,\alpha\beta} \| \neq 0, \tag{5.29}$$

then by (5.26) and (5.28) $A^\alpha_\epsilon = g d\mu^\alpha_{\epsilon+1}/dt$ for some g. By (5.27)

$$n^i_\epsilon = g(\partial u_i/\partial \mu^\alpha)(d\mu^\alpha_{\epsilon+1}/dt), \tag{5.30}$$

i.e. the tangent to $\mu^\alpha = \mu^\alpha_{\epsilon+1}$ is normal to the prototype of $\mu^\alpha = \mu^\alpha_\epsilon$. These considerations and elementary calculation yield

THEOREM 5.3: *If for a double wave* $\det \| cb_{\alpha\beta} + dk_{,\alpha\beta} \| \neq 0$ *for some c and d, and if $b_{\alpha\beta}$ and $k_{,\alpha\beta}$ are linearly independent*:

(1) *The characteristics are the only curves on the hodograph with developable prototypes.*

(2) *The tangent at any point of a characteristic is normal, at points of the corresponding ruling, to the prototype of the other characteristic through those points.*

(3) *The Mach cone at any point of the prototype of a characteristic is tangent to the prototype.*

(4) *The streamlines intersect the prototypes of characteristics at the Mach angle.*

For the omitted cases, first suppose $\det \| b_{\alpha\beta} \| \equiv 0$, which includes the case $\det \| cb_{\alpha\beta} + dk_{,\alpha\beta} \| \equiv 0$. Then the hodograph is developable, so one family of lines of curvature consists of rulings. There exist b_α such that $b_{\alpha\beta} = b_\alpha b_\beta$. Suppose b_α is non-null. Let n^α be a non-trivial solution of $b_\alpha n^\alpha = 0$. Then $b_{\alpha\beta} n^\beta = 0$, so n^α is tangent to a line of curvature of curvature zero, i.e. a ruling. Since some $b_{\alpha\beta} \neq 0$, the lines of curvature of the hodograph are uniquely determined. On the other hand, by (5.12) $(g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta})(-1)^{\alpha+\beta} b_{\alpha+1} b_{\beta+1} = 0$, so $(g_{\alpha\beta} - M^2 q_{,\alpha} q_{,\beta}) n^\alpha n^\beta = 0$, and n^α is a characteristic vector. Hence one family of characteristics must consist of rulings. By Theorem

5.1 a plane characteristic is a Prandtl-Meyer epicycloid, not a straight line. Hence $b_\alpha = 0$, so $b_{\alpha\beta} = 0$, and the hodograph must be in a plane, which may be assumed to be u_3 = constant. Thus prototypes of curves on the hodograph are cylinders, with rulings parallel to the x^3-axis. These are Busemann's *cylindrical* flows. For $\mu^\alpha = u_\alpha$ (3.2) and (3.4) define the familiar Legendre transformation from the physical to the hodograph plane for plane flow, and (5.13) takes the usual form

$$(a^2\delta_{\alpha\beta} - u_\alpha u_\beta)(-1)^{\alpha+\beta}\, \partial^2 k/\partial u_{\alpha+1}\, \partial u_{\beta+1} = 0. \tag{5.31}$$

Finally, suppose det $\| b_{\alpha\beta} \| \neq 0$, but $b_{\alpha\beta}$ and $k_{,\alpha\beta}$ are linearly dependent. By (5.9) and (5.14), for $u_\alpha = \mu^\alpha$, $\partial^2 u_3/\partial u_\alpha\, \partial u_\beta$ and $\partial^2 k/\partial\, u_\alpha\, \partial u_\beta$ are also linearly dependent. Hence for some $f(u_1, u_2)$

$$\partial^2 k/\partial u_\alpha\, \partial u_\beta = f\, \partial^2 u_3/\partial u_\alpha\, \partial u_\beta\, . \tag{5.32}$$

By (5.32)$(\partial^2 u_3/\partial u_\alpha\, \partial u_\beta)(\partial f/\partial u_\gamma) = (\partial^2 u_3/\partial u_\alpha \partial u_\gamma)(\partial f/\partial u_\beta)$. Since det $\| \partial^2 u_3/\partial u_\alpha\, \partial u_\beta \| \neq 0$, then $\partial f/\partial u_\beta = 0$. (5.32) yields

$$k = B - A^i u_i\, , \tag{5.33}$$

where A^i and B are constants. By (3.5) and (5.33) all prototype lines pass through $x^i = A^i$. For each streamline S pass straight lines through $x^i = A^i$ and each point of S. The cone so constructed will be a stream sheet covered by streamlines similar to S. Accordingly such flows are said to be *conical,* a type considered by Busemann. The most familiar example is Taylor-Maccoll flow.

THEOREM 5.4: *If for a double wave* det $\| cb_{\alpha\beta} + dk_{,\alpha\beta} \| = 0$ *for all c and d, or if* $b_{\alpha\beta}$ *and* $k_{,\alpha\beta}$ *are linearly dependent:*

(1) *The flow is conical or cylindrical.*
(2) *The prototype of any curve on the hodograph is developable.*
(3) *Conclusions* (2) *to* (4) *of Theorem* 5.3 *apply to these flows.*

To every double wave that is neither cylindrical nor conical there corresponds a conical flow with the same hodograph. Such general double waves will be called *skewed conical flows.*

Reconsider the conditions for hyperbolic, parabolic, or elliptic type for (5.12) and (5.15). They are $\sin \chi >, =, < 1/M$, where χ is the angle between the velocity and the direction of a prototype line. For subsonic flow (5.12) and (5.15) must be elliptic. In sharp contrast with plane flow, they need not be hyperbolic for supersonic double waves. To see this, consider a supersonic cylindrical flow based on a subsonic plane flow. Then (5.15) or (5.31) is elliptic.

6. Double waves with axisymmetric hodographs. An important class of examples can be constructed as follows. Assume $u_i = u_i(\mu)$ is axisymmetric about the u_3-axis. The hodograph may be represented by

$$u_1 = u(t) \cos \theta, \qquad u_2 = u(t) \sin \theta, \qquad u_3 = w(t) \tag{6.1}$$

for some $u(t)$ and $w(t)$. If (6.1) is a curve, two possibilities arise. If $u = 0$, (4.5) implies $w^2 = q^2 = a^2 = (\gamma - 1)c^2/(\gamma + 1)$, so (6.1) reduces to two points. If u and w are constant, (4.5) implies $a^2 = 0$, i.e. (6.1) is the circle $q = c$, w = constant. This is a singular case of a velocity field with constant speed $q = c$ in a vacuum.

Next, suppose (6.1) is two-dimensional. Let $\mu^1 = t$, $\mu^2 = \theta$. Then by (5.8) and (5.9)

$$g_{11} = u'^2 + w'^2, \qquad g_{12} = 0, \qquad g_{22} = u^2,$$

$$b_{11} = (u'w'' - u''w')(u'^2 + w'^2)^{-1/2}, \tag{6.2}$$

$$b_{12} = 0, \qquad b_{22} = uw'(u'^2 + w'^2)^{-1/2}$$

where primes denote derivatives with respect to t. (5.12) implies

$$a^2u(w''u' - w'u'') + w'[a^2(u'^2 + w'^2) - (uu' + ww')^2] = 0. \tag{6.3}$$

This has the singular solution $q = c$, i.e. a spherical hodograph. It also has the solution w = constant, i.e. the hodograph lies in a plane and the corresponding flow is cylindrical. Hereafter, suppose w is not constant. With no loss of generality, set $t = w$. Then (6.3) becomes

$$a^2(uu'' - u'^2 - 1) + (uu' + w)^2 = 0. \tag{6.4}$$

This is a form of the differential equation for the hodographs of axisymmetric conical flows, of which Taylor-Maccoll flow or a convergent flow considered by Busemann [3] are particular examples. As stated at the end of Sec. 5, to these flows there correspond skewed conical flows with the same hodographs. To construct examples, find the function $k(w, \theta)$. By (5.14) and (6.2)

$$k_{,11} = \partial^2 k/\partial w^2 - \tfrac{1}{2}[\log (1 + u'^2)]\partial k/\partial w,$$

$$k_{,22} = \partial^2 k/\partial \theta^2 + [uu'/(1 + u'^2)]\partial k/\partial w.$$

By (5.12), (5.15), and (6.2) $b_{11}k_{,22} - b_{22}k_{,11} = 0$, so

$$\partial^2 k/\partial \theta^2 + (u/u'')\partial^2 k/\partial w^2 = 0. \tag{6.5}$$

A skewed Taylor-Maccoll flow can be constructed, *in the small*, by solving the ordinary differential equation (6.4), the *linear* partial differential equation (6.5), and finding the prototype lines (3.5). Thus, by relatively elementary processes a class of three-dimensional solutions of the non-linearized equation (2.10) can be constructed.

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