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Compression in the Earth

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Summary

Using Birch's development of finite-strain theory for the Earth, a simple formula is derived expressing the compression f in terms of the pressure and incompressibility at internal points of the Earth. The formula is free from certain limitations in an earlier formula used by Birch, and is not restricted to the mantle. The formula is applied to determining values of f for six representative Earth models, and preferred estimates of f are set down for the Earth. The results are compared with Birch's earlier estimates of f for the mantle. The computed values of f do not significantly exceed 0·13 in the mantle or 0·20 in the core. Across the mantle–core boundary, f is likely to be nearly continuous; any sudden change is likely to be less than about 5 per cent. The results are subject to the reliability of Birch's form of equation of state for the internal regions of the Earth and, as in Birch's method, neglect possible effects of phase changes.

1. Introduction

This paper is concerned with the application of finite-strain theory to estimate the distribution of the compression f in the Earth, f being connected with the dilatation θ by

$$1+2f=(1-\tfrac{1}{3}\theta)^2; \tag{1}$$

thus $f=-\varepsilon$, where ε is the strain.

Let ρ, k, p and g be the density, incompressibility, pressure and gravitational intensity at the point P at depth z below the Earth's surface, and let

$$\phi = \alpha^2-4\beta^2/3, \tag{2}$$

where α and β are the seismic bodily wave velocities at the point P. The subscript zero will denote values which variables relating to the material at P take when p and f are reduced to zero.

In an adaptation of Murnaghan's theory of finite strain, Birch (1952) derived a formula expressing f in terms of g, ϕ and $z-z_0$ at points P in particular regions of the Earth, z_0 depending on the chemical composition at P. The derivation assumes g constant, which holds approximately in the mantle, and in applications z_0 has to be separately determined for regions of different composition. Birch used the formula to estimate f inside the mantle. In respect of the core, he stated only that f is not expected to exceed 0·3 anywhere in the Earth.

The present paper uses an alternative formula for determining f. The formula, though innate in Birch's work, is simpler than the one Birch has used, with the further advantages that it does not assume g constant and does not require values of z_0 to be estimated. Thus the formula, in addition to being more accurate and simply applicable in the mantle, is also applicable to the core, subject only to the reliability of the finite-strain theory used.

The new formula will be applied to estimating distributions of f throughout the interiors of a representative set of Earth models. The numerical results provide a check on Birch's calculations for the mantle and give new detail for the core.

Birch's formula, equation (9) below, does not contain k or ρ explicitly, and its form may possibly be accounted for as an endeavour to estimate f using data on the distributions of α and β in the Earth, without recourse to data on the distribution of ρ. Our formula, equation (13) below, requires explicit knowledge of p and k, and hence implicitly of ρ, at points where it is applied. However, equation (9) involves a measure of dependence on ρ through the presence of g, and, further, the simplification in taking g constant in the mantle puts restrictions on the allowable density distributions; thus equation (9) is not wholly independent of the distribution of ρ.

Birch's discussion takes account of temperature terms, but he finally ignores them as being fairly small in effect. In consequence, we shall ignore temperature effects from the outset, and make no discrimination between adiabatic and isothermal incompressibilities.

In order to compare Birch's approach to estimating f with the approach in this paper, it will be convenient first to outline Birch's derivation of his formula. This will be done in Section 2. The alternative formula will be presented and discussed in Section 3, and applied to give numerical determinations of f in Section 4.

2. Outline of derivation of Birch's formula for f

In a region of uniform composition and phase, assuming the usual hydrostatic conditions, we have

$$d\rho/dp = \rho/k, \tag{3}$$

$$dp/dz = g\rho. \tag{4}$$

Using equation (1) and $\rho v = \rho_0 v_0$, where v is an element of volume containing P, gives

$$\rho = \rho_0(1+2f)^{3/2}. \tag{5}$$

Birch's finite-strain development led him to an equation of state equivalent to

$$p = 3k_0 f(1+2f)^{5/2}. \tag{6}$$

The equations (2)–(6) yield

$$\begin{aligned} g\,dz &= k\rho^{-2}d\rho \\ &= 3(k/\rho)_0(1+7f)\,df, \end{aligned} \tag{7}$$

$$\begin{aligned} \int_{z_0}^{z} g\,dz &= 3\phi_0 f(1+3{\cdot}5f) \\ &= 3\phi f(1+3{\cdot}5f)(1+2f)^{-1}(1+7f)^{-1}. \end{aligned} \tag{8}$$

The derivation of equation (8) from equations (2)–(6) is exact. Birch then used the approximation g = constant to arrive at his formula for estimating f, namely,

$$g(z-z_0) = 3\phi f(1+3{\cdot}5f)(1+2f)^{-1}(1+7f)^{-1}. \tag{9}$$

In applying equation (9), Birch took $z_0 = 6$ and 100 km for the ranges $33 < z < 600$, $600 < z < 2900$ km, respectively, and took $g = 1000\,\mathrm{cm/s^2}$ throughout the mantle. This procedure involved some additional simplification since equation (9) has been derived on the assumption of chemical homogeneity. Thus, strictly, inside regions such as $C(400 < z < 1000$ km) and D'' ($2700 < z < 2900$ km), where there is evidence of significant inhomogeneity, z_0 ought to be treated as a continuously varying function of z.

3. Alternative formula for f

A much simpler formula for f can be readily derived as follows (or otherwise). By equations (5) and (6),

$$\rho^{-1}d\rho = 3(1+2f)^{-1}df, \tag{10}$$

$$p^{-1}dp = \{f^{-1}+5(1+2f)^{-1}\}\,df, \tag{11}$$

whence

$$p\rho^{-1}d\rho/dp = 3f(1+7f)^{-1}. \tag{12}$$

Then equation (3) gives

$$3f(1+7f)^{-1} = pk^{-1}, \tag{13}$$

or

$$f = p(3k-7p)^{-1}. \tag{14}$$

The formulae (13) and (14) come exactly from equation (3), (5) and (6) and do not assume g constant. Chemical homogeneity was assumed in deriving equations (13) and (14), but (13) and (14) apply at points P inside inhomogeneous as well as homogeneous regions. This is because dJ and dp in equations (10) and (11) relate, not to variations with respect to the depth z, but to variations in which the composition of the material at P is kept unvaried.

Thus, apart from the possible need for corrections due to phase changes (in this respect, there is no real difference between Birch's approach and the present approach), equations (13) and (14) can be applied to all parts of the Earth's interior without the complication of having to estimate z_0 anew whenever the composition changes, and without restriction to the mantle.

It needs to be noted that where the material present at any level in the Earth has undergone significant phase transformation, ρ_0 and k_0 as given in equations (5) and (6) and subsequent equations are not the actual 'uncompressed' density and incompressibility, but are parameters which may be taken to be defined by equation (5), (6) and (14) and which relate to an artificial uncompressed state in which reversions in phase are ignored. The quantity f as given by equation (14) has to be correspondingly interpreted. These points have not always been appreciated in arguments involving ρ_0, k_0 and ϕ_0. It seems fairly well established that phase changes occur between about 400 and 1000 km depth. If their accumulated contribution to the densities should reach a sufficient magnitude, the total compressions inside the mantle below 400 km could be significantly greater than the values of f in Table 1 (including the values given by Birch). In the event of the outer core being a phase transformation of the material of the lower mantle, the core values of f would, of course, be seriously underestimated. It is intended in a later paper to examine the possibility of setting limits to any corrections to Table 1 that phase changes in the Earth may require.

Birch's equation (8) can of course be readily deduced starting from equation (13). Thus, by equation (13),

$$3k\rho^{-2}d\rho = pf^{-1}(1+7f)\,\rho^{-2}d\rho$$
$$= 9k_0\,\rho_0^{-1}(1+7f)\,df, \tag{15}$$

on using equations (5) and (6). The relation (15) leads, as in the case of equation (7), to equation (8). Hence Birch's approach differs essentially from the present approach only in the step from equation (8) to (9) and his simplified estimates of z_0 and g.

4. Values of the compression in the interiors of a representative selection of Earth models

The formula (14) was applied to determine values of the compression f from values of p and k in the interiors of six Earth models. The selected models provide a sufficiently representative coverage to indicate the order of uncertainty in the computed values of f arising from all sources except possible errors in the finite-strain theory used and possible allowances for phase changes.

The Earth models A and B (Bullen 1965), though obsolete because they assume the value 0·3335 MR^2 for the Earth's moment of inertia I, are included for reference because so many other calculations have been based on them and also for sake of comparison with Birch's values of f.

Features of the other models that are relevant to the present calculations are:

Model A'' (*Bullen & Haddon* 1967*a*). Differs from Model A only in that $I = 0{\cdot}3309\,MR^2$ and (as with model A'—Bullen 1965) the central density, σ say, is kept to the minimum (in model A'', $\sigma = 12{\cdot}51$ g/cm^3). Since no account is taken in Model A'' of complexities in the lower core, values of f are not given below 4982 km depth (the same applies with model A). Complexities associated with the region D'' (2700–2898 km depth) are also disregarded.

Model B_1 (*Bullen & Haddon* 1967*b*). Takes $I = 0{\cdot}3309\,MR^2$, assumes k continuous at the mantle–core boundary (taken at 2878 km depth) and smoothly varying with p everywhere below 1000 km depth, and takes account of complexities in the region D'' and the lower core. In model B_1, it is assumed that the seismic S velocity β, where not zero, follows a similar pattern of variation to that of the P velocity α; in consequence $\sigma = 15{\cdot}18$ g/cm^3.

Model B_2 (*Bullen & Haddon* 1967*b*). Differs from model B_1 only in that the distribution of β in the lower core is selected so as to yield the minimum value of σ (13·00 g/cm^3 on the assumptions here taken).

Model HB_1 (*Bullen & Haddon* 1967*c*). Takes account of free Earth oscillation data, has $I = 0{\cdot}3309\,MR^2$, has more complicated layering than the other models above 1000 km depth, but does not take account of complexities in the region D'' and in the lower core. Model HB_1 has $\sigma = 12{\cdot}46$ g/cm^3, but computed values of f are not reliable below 4982 km depth and are not given. The depth of the mantle–core boundary is 2878 km.

Table 1 shows the computed values of f for the six models and also Birch's earlier estimates of f. Birch's values agree closely with the model A values except at 600 km depth, where inhomogeneity causes Birch's estimate of his z_0 to be less reliable. Since Birch's data are closely similar to those used in constructing model A, this agreement is satisfactory.

Table 1

Estimates, based on values of p and k in various Earth models, of the compression f in the Earth's interior

Model depth (km)	Birch (1952)	A	B	A''	B_1	B_2	HB_1	Optimum estimate
100	0·0090	0·009	0·008	0·0088	0·0088	0·0088	0·0094	0·009
200	0·0175	0·018	0·017	0·0176	0·0175	0·0176	0·0174	0·0175
400	0·0323	0·033	0·032	0·0325	0·0309	0·0311	0·0286	0·029
600	0·0382	0·034	0·037	0·0348	0·0351	0·0349	0·0341	0·034
800	0·0397	0·040	0·045	0·0410	0·0423	0·0417	0·0413	0·041
1000	0·0490	0·049	0·055	0·0494	0·0501	0·0496	0·0499	0·050
1200	0·0590	0·059	0·065	0·0600	0·0615	0·0610	0·0603	0·060
1400	0·0688	0·068	0·074	0·0693	0·0710	0·0704	0·0697	0·070
1600	0·0775	0·077	0·082	0·0784	0·0798	0·0793	0·0787	0·079
1800	0·0853	0·085	0·090	0·0865	0·0890	0·0874	0·0869	0·087
2000	0·0930	0·093	0·098	0·0942	0·0959	0·0951	0·0947	0·095
2200	0·1007	0·101	0·106	0·1025	0·1041	0·1034	0·1029	0·103
2400	0·1088	0·109	0·114	0·1106	0·1123	0·1117	0·1110	0·111
2600	0·1167	0·117	0·121	0·1192	0·1209	0·1203	0·1198	0·120
Base of mantle	0·1271*	0·138	0·134	0·1411	0·1325	0·1318	0·1399	0·132
Top of core		0·152	0·134	0·135	0·133	0·132	0·132	0·132
3200		0·174	0·156	0·157	0·158	0·157	0·157	0·157
3600		0·185	0·171	0·170	0·171	0·170	0·170	0·170
4000		0·195	0·181	0·180	0·182	0·180	0·179	0·180
4400		0·205	0·191	0·189	0·192	0·190	0·187	0·190
4561		0·204	0·191	0·189	0·193	0·190	0·187	0·190
4711		0·202	0·192	0·188	0·195	0·190	0·187	0·190
4982		0·197	0·192	0·185	0·195	0·191	0·183	0·19
5121			0·190		0·196	0·191		0·19
6371			0·182		0·195	0·191		0·19

* Value at 2800 km depth.

For all models, the greatest value yielded for f at the base of the mantle is 0·141, the value in model A''. In models, including A'', where no allowance is made for the steepened density gradient in the region D'' (Bullen 1949), the value of f at the base of the mantle is, however, on this count, likely to be overestimated by about 0·008. Thus, if the effects of phase changes can be ignored, f is not expected to exceed 0·13 significantly anywhere in the mantle.

For model A, in which I has the old value of $0{\cdot}3335\,MR^2$, the results show a discontinuous increase in f across the mantle–core boundary, corresponding to the discontinuous decrease in k at this boundary in model A. In all the other models, the value of f at the top of the core is less than or equal to the value at the bottom of the mantle.

For all the models, the computed f are less than 0·20 throughout the core.

The final column of Table 1, and Fig. 1, give the author's preferred estimates of f at the various levels, subject to the stated assumptions. For $100 < z < 2600$ km, the model HB_1 values are preferred because of the account taken in HB_1 of recent evidence from free Earth oscillations on complexities in the upper mantle layering. For this range of depth, the values for models HB_1, B_1 and B_2 mostly agree fairly closely. From 2600 km depth to the centre, the model B_1 and B_2 values are preferred because of the special attention given to complexities in the region D'' and the lower core. Model B_2 is preferred to B_1 because it has a central density of 13 g/cm^3, in contrast to 15·2 g/cm^3 for B_2.

The table includes values at special depths below 4400 km which correspond to discontinuities in α or $d\alpha/dz$ as assigned by various authors.

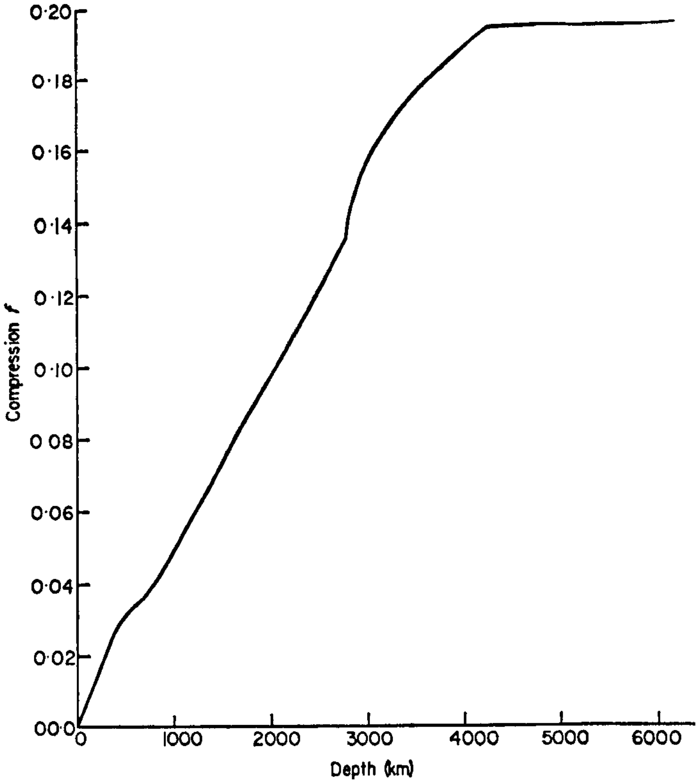

FIG. 1. Preferred values of the compression f in the Earth's interior.

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