

# Compromise and synchronization in opinion dynamics

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**Abstract.** We discuss two models of opinion dynamics. We first present a brief review of the Hegselmann and Krause (HK) compromise model in two dimensions, showing that it is possible to simulate the dynamics in the limit of an infinite number of agents by solving numerically a rate equation for a continuum distribution of opinions. Then, we discuss the Opinion Changing Rate (OCR) model, which allows to study under which conditions a group of agents with a different natural tendency (rate) to change opinion can find the agreement. In the context of the this model, consensus is viewed as a synchronization process.

**PACS.** 89.65.-s Social and economic systems – 05.10.Ln Monte Carlo methods – 05.45.Xt Synchronization; coupled oscillators

Since the behavioral revolution and the birth of cybernetics, the so called ‘*soft*’ social sciences have emulated both the intellectual and methodological paradigms of the ‘*strong*’ natural sciences [1]. The certainty and stability of the Newtonian paradigm has represented for decades the cornerstone of sciences like psychology, economy and sociology, which have been largely inspired by classical mechanics and statistical thermodynamics. Clearly this trend has continued when quantum mechanics and chaos and complexity revolutions have led to a reconsideration of the relevance of the Newtonian paradigm to all natural phenomena.

In the last years, disciplines such as econophysics and sociophysics have largely demonstrated the power of *agent-based* computational models in simulating complex adaptive systems (financial markets, cultural evolution, social structures, voter communities) in which large numbers of individuals are involved in massively parallel local interactions [2,3]. In agent-based models, individuals are modeled as autonomous interacting agents with a variable degree of internal complexity and the simulations represent computational laboratories to study the evolution of a given social system under controlled experimental conditions. Of course in many cases the individual cognitive behavior is oversimplified, as for example in opinion dynamics models where human opinions are reduced to integer or real numbers [4,5]. In more complicated models, individuals are simulated by means of simple neural networks or associative memories. Also in this case, many of the simplifications adopted are somehow unrealistic. On the other hand, also the Kepler’s laws assumption of Earth as a point-mass was not realistic at all, but for the purpose of

describing celestial motion it turned out very successful. Furthermore, the aim of agent-based simulations is to provide information on averages over many people, and not on the fate of a specific person. And in this sense, despite of their simplicity, these models seem to work very well. For example, the Sznajd model prediction of the distribution of votes among candidates in Brazilian and Indian elections is encouraging, although the model is not able to predict the number of votes one specific candidate gets in one specific election [6].

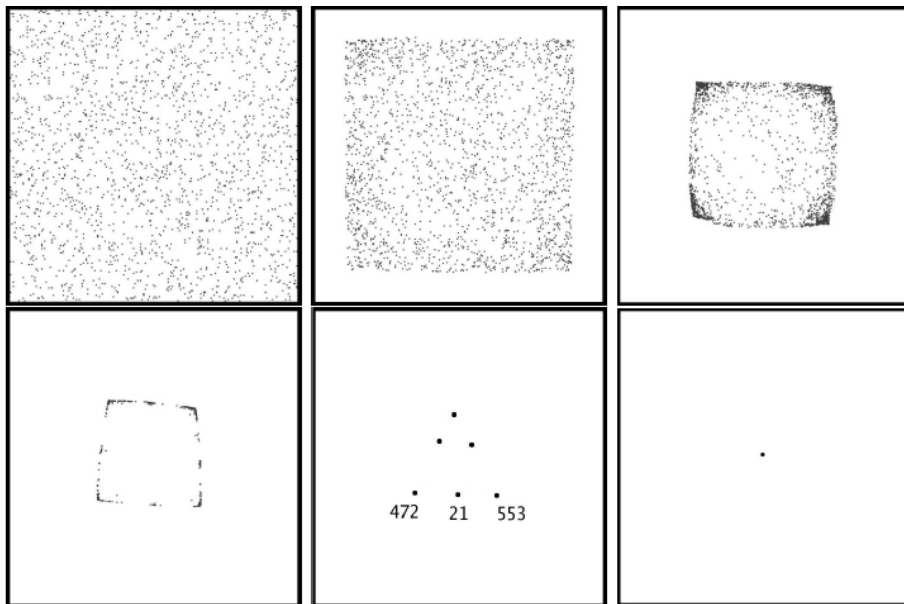
In the first part of this paper we discuss one of the well known models of opinion dynamics, the so called *compromise model* of Hegselmann and Krause [4], showing some recent result about its continuum version in a two-dimensional opinion space [7]. In the second part of the paper we discuss a new perspective in opinion dynamics based on the suggestion of a possible role of synchronization in opinion formation. By means of the so called *Opinion Changing Rate model*, a modified version of the Kuramoto model adapted to a social context, we study under which conditions a group of agents with a different natural tendency (*rate*) to change opinion can find agreement [8].

## 1 Discrete and continuum opinion dynamics in the 2-vector HK consensus model

For the sociologist Robert Axelrod “culture” is modeled as an array of features, each feature being specified by “traits”, which are expressed by numbers. The number of features or dimensions is nothing but the number of components of a vector, and two persons interact if and only

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**Fig. 1.** Sequence of snapshots of the 2D squared opinion space for the 2-vector discrete HK model with  $N = 2000$  agents and  $\epsilon = 0.25$ . The points in the upper-left panel represent different randomly distributed opinions at  $t = 0$ . In the other panels, where we show successive (but not consecutive) time steps of a Monte Carlo simulation with simultaneous update, opinions merge together in different clusters. Finally, in the lower-left panel ( $t = 12$ ) the consensus is reached and all the opinions occupy the same position. In the second-last panel, the number of opinions concentrated in the bottom clusters is also indicated (see text).

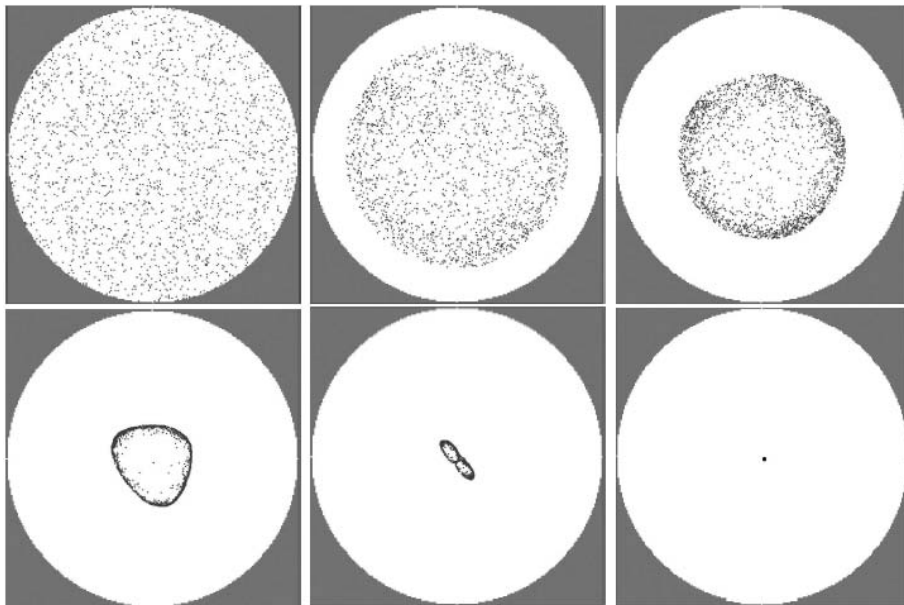
if they share at least one common feature (i.e. the same value of the corresponding vector component) [9]. In this model, two persons are culturally closer the more features they have in common, and the number of these common features is, in turn, related to the probability for the two individuals to interact. Starting from the Axelrod model, several simple agent-based models of opinion formation have been devised, mostly by physicists [3–5].

In general, a typical scalar opinion formation model starts by assigning randomly a real number (chosen in a given interval) to every agent of the system. Then the dynamics starts to act, and the agents rearrange their opinion variables, due to their interactions. At some stage, possibly, the system reaches a configuration which is stable under the dynamics. This final configuration may represent consensus, when all agents share the same opinion, polarization, when there are two main clusters of opinions (“parties”), or fragmentation, when several opinion clusters survive. However, a discussion between two persons is not simply stimulated by their common view/preference about a specific issue, but it in general depends on the global affinity of the two persons, which is influenced by several factors. So, for a more realistic modeling of opinion dynamics, one should represent the opinions/attitudes like vectors (as in the Axelrod model), and not like scalars. In this section we will focus on the 2-vector version of the Hegselmann and Krause (HK) compromise model and we will show that it is possible to simulate the discrete opinion dynamics in the limit of an infinite number of agents by solving numerically a rate equation for a continuum distribution of opinions [7].

The HK model [4] is based on the concept of bounded confidence, i.e. on the presence of a parameter  $\epsilon$ , called *confidence bound*, which expresses the compatibility among the agents in the *opinion space*. If the opinions of two agents  $i$  and  $j$  differ by less than  $\epsilon$ , their positions are close enough to allow for a discussion, which even-

tually leads to a change in their opinions, otherwise the two agents do not interact with each other. The *physical space* occupied by the agents living in a society or a community can be modeled as a graph, where the vertices represent the agents and the edges relationships between agents. So we say that two agents can eventually talk to each other if there is an edge joining the two corresponding vertices (in graph language, if the two vertices are neighbours). In the following we will consider only the general case of a society where everybody talks to everybody. The dynamics of the HK model is usually simulated by means of Monte Carlo (MC) algorithms. One chooses at random one of the agents and checks how many of its neighbours (in the physical space) are compatible, i.e. lie inside the confidence range in the opinion space. Next, the agent takes the average opinion of its compatible neighbours. The procedure is repeated by selecting at random another agent and so on. The type of final configuration reached by the system depends on the value of the confidence bound  $\epsilon$ . For a scalar opinion space  $[0, 1]$  it has been shown that consensus is reached for  $\epsilon > \epsilon_c$ , where the critical threshold  $\epsilon_c$  is strictly related to the type of graph adopted to modelize society: actually, it can take only one of two possible values,  $\epsilon_c \sim 0.2$  and  $0.5$ , depending on whether the average degree of the graph (i.e. the average number of neighbours) diverges, as in our case of a completely connected graph, or stays finite when the number of vertices goes to infinity [10]. On the other hand, the 2-vector HK model on a completely connected graph is much less studied than the 1-dimensional version. In this case the opinion space is represented by the points  $(x, y)$  of a bidimensional manifold, that in general is a square  $[0, 1] \times [0, 1]$  and the confidence range is a circle whose radius is the confidence bound  $\epsilon$ .

In Figure 1 we plot a sequence of snapshots of the opinion space for a discrete configuration of  $N = 2000$  agents and a value  $\epsilon = 0.25$ . Each point in the first snapshot



**Fig. 2.** Sequence of snapshots of the 2D circular opinion space for the 2-vector discrete HK model with  $N = 2000$  agents and  $\epsilon = 0.25$ . The dynamics is the same than in Figure 1. The different shape of the space influences the dynamical evolution of the opinions and consensus is reached in a shorter time ( $t = 6$ ) than in the case of Figure 1.

(upper-left panel) represents the opinion of one agent at time  $t = 0$ . Then the system evolves by means of simultaneous updates (i.e. all the opinions are updated at each MC time step) merging the opinions in bigger and bigger clusters until a stationary state is reached. After 12 time steps (lower-right panel) consensus is fully obtained and all the opinions lie on the same cluster. More in general, from extended numerical simulations it results [7] that the consensus threshold for the discrete 2-vector HK model is  $\epsilon_c \sim 0.24$ , a value slightly greater than that one found for the scalar model with the same topology. This value tends to the value  $\epsilon_c \sim 0.23$  when the number of agents grows.

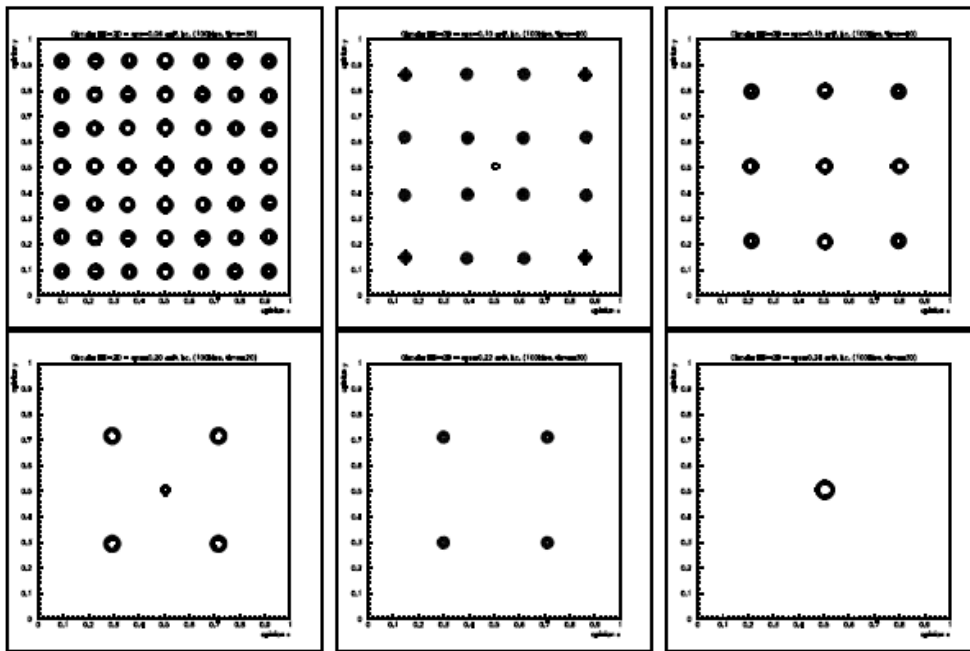
If we look at the basis of the triangle in the second-last snapshot of Figure 1, just before reaching final consensus, we can see that the two big clusters at the vertices, made by around 500 opinions and lying at a reciprocal distance greater than the confidence bound (thus a priori not interacting), are keeping in contact only by means of a small cluster of 21 agents. Such a phenomenon is very frequent in the Monte Carlo simulations of the HK model when the system is approaching consensus (in both one and two dimension); indeed, almost always consensus is reached only because of this phenomenon. This models an important feature of real social networks, i.e. the existence of the so called *connectors* which play the role of a bridge between otherwise not interacting social groups, thus ensuring the cohesiveness of the entire network [11].

Another peculiar feature of the HK model, clearly visible in the upper snapshots of Figure 1, is the fact that the dynamics always starts to act from the edges of the opinion space, where the opinion distribution is necessarily inhomogeneous, so that it is essentially the shape of the opinion space which rules the symmetry of the resulting cluster distribution. In order to better appreciate this effect, we plot in Figure 2 the temporal evolution of the same system of Figure 1, but with a circular opinion space.

In this case, even if the final configuration is the same as before, the resulting dynamics is different and, for the same value of the confidence bound, consensus is reached more quickly (six MC time steps with simultaneous update) due to the greater symmetry of the opinion space. The circular symmetry has a remarkable effect also on the consensus threshold, that in this case tends to that of the corresponding scalar HK model, i.e.  $\epsilon \sim 0.2$ . In a recent paper [7] it has been shown that the 2-vector HK model on a completely connected graph and with a squared opinion space can be described by means of a rate equation for a continuum distribution of opinions  $P(x, y, t)$ . The rate equation can be solved numerically, as already done for the scalar compromise model of Deffuant et al. [12]. The advantages of the evolution equation over discrete Monte Carlo simulations are that one can directly deal with a system with an arbitrarily large number of agents, and that the final cluster configurations for a continuum distribution are much more symmetric and regular, thus allowing a better resolution of the progressive merging of opinion clusters.

This is clearly visible in Figure 3, where we show the final configurations of the opinion space for several values of the confidence bound. In this case the squared  $(x, y)$  opinion space has been reduced to a grid of  $100 \times 100$  bins. All the simulations start from a flat distribution  $P(x, y, t = 0) = \text{const.}$  and the dynamics runs until the distribution  $P(x, y, t)$  reaches a stationary state for a given value of the confidence bound [7].

As one can see that, for small value of  $\epsilon$ , a regular lattice of clusters appears, with a squared shape inherited by the shape of the opinion space (as happened for the discrete HK model). Going on, for greater values of  $\epsilon$ , one can observe the progressive merging of the pairs of clusters with reciprocal distance less than the confidence bound radius. Finally, above the critical threshold



**Fig. 3.** Final configurations of the 2-vector HK model with a continuum distribution of opinions. From top left to bottom right:  $\epsilon = 0.08, 0.10, 0.15; 0.20, 0.22, 0.24$ .

$\epsilon_c \sim 0.23$ , consensus is completely reached. This result confirms the threshold value found with the MC simulations for a discrete dynamics of opinions in the limit of a large number of agents and encourages further applications of the rate equation technique to other opinion formation models.

## 2 The opinion changing rate model: a role for synchronization in opinion formation

Most of the opinion formation models, e.g. the HK model presented in the previous section, have the limitation of not taking into account the individual inclination to change, a peculiar feature of any social system. In fact, each one of us changes ideas, habits, style of life or way of thinking in a different way, with a different velocity. There are conservative people that strongly tend to maintain their opinion or their style of life against everything and everyone. There are more flexible people that change ideas very easily and follow the current fashions and trends. Finally, there are those who run faster than the rest of the world anticipating the others. These different tendencies can be interpreted as a continuous spectrum of different degrees of natural inclination to changes.

In a recent paper [8] we have showed how such an individual inclination to change, differently distributed in a group of people, can affect the opinion dynamics of the group itself. If we switch from the question: “Could agents with initial different opinions reach a final agreement?” into the more realistic one “Could agents with a different natural tendency to change opinion reach a final agreement?”, we can introduce a new concept, the natural opinion changing rate, that is very similar to the characteristic frequency of an oscillator. In such a way, we can

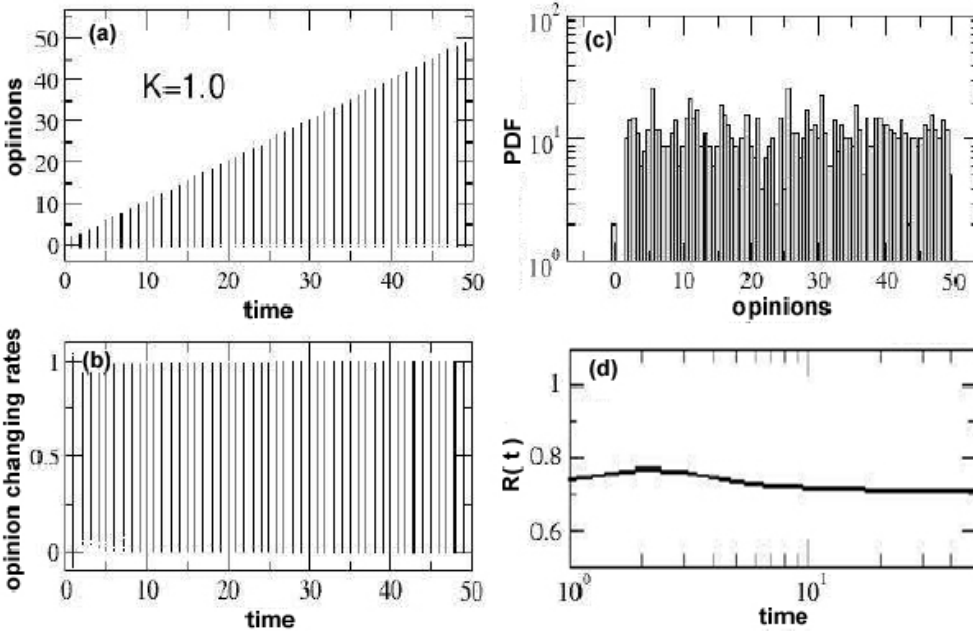
treat consensus as a peculiar kind of synchronization (frequency locking), a phenomenon which has been very well studied in different contexts by means of the Kuramoto model [13].

The Kuramoto model of coupled oscillators is one of the simplest and most successful models for synchronization. It is simple enough to be analytically solvable, still retaining the basic principles to produce a rich variety of dynamical regimes and synchronization patterns. The dynamics of the model is given by

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i) \quad i = 1, \dots, N \quad (1)$$

where  $\theta_i(t)$  is the phase (angle) of the  $i$ th oscillator at time  $t$  ( $-\pi < \theta_i(t) < \pi$ ), while  $\omega_i$  is its intrinsic frequency randomly drawn from a symmetric, unimodal distribution  $g(\omega)$  with a first moment  $\omega_0$  (typically a Gaussian distribution or a uniform one). These natural frequencies  $\omega_i$  are time-independent. The sum in the above equation is running over all the oscillators so that this is an example of a globally coupled system. The parameter  $K \geq 0$  measures the coupling strength in the global coupling term. For small values of  $K$ , each oscillator tends to run independently with its own frequency, while for large values of  $K$ , the coupling tends to synchronize (in phase and frequency) the oscillator with all the others. Kuramoto showed that the model, despite the difference in the natural frequencies of the oscillators, exhibits a spontaneous transition from incoherence to collective synchronization, as the coupling strength is increased beyond a certain threshold  $K_c$  [14].

The existence of a critical threshold for synchronization in Kuramoto model is very similar to the consensus threshold found in the majority of the opinion formation models. Of course, at variance with the phases in the Kuramoto model, in a model for opinion dynamics we do not



**Fig. 4.** OCR model for  $N = 1000$  and  $K = 1.0$  (incoherent phase). In panels (a) and (b) we report, for each time step, 1000 points corresponding respectively to the opinions and opinion changing rates of the  $N$  agents, while in panels (c) and (d) we show, respectively, the final distribution of opinions and the time evolution of the order parameter  $R$ .

need periodic opinions nor limited ones: in fact, the opinions have a very general meaning and can represent the style of life, the way of thinking or of dressing etc. Thus we do not consider periodic boundary conditions and we assume  $x_i \in ]-\infty, +\infty[ \forall i = 1, \dots, N$ . The dynamics of the OCR model is governed by the following set of differential equations [8]:

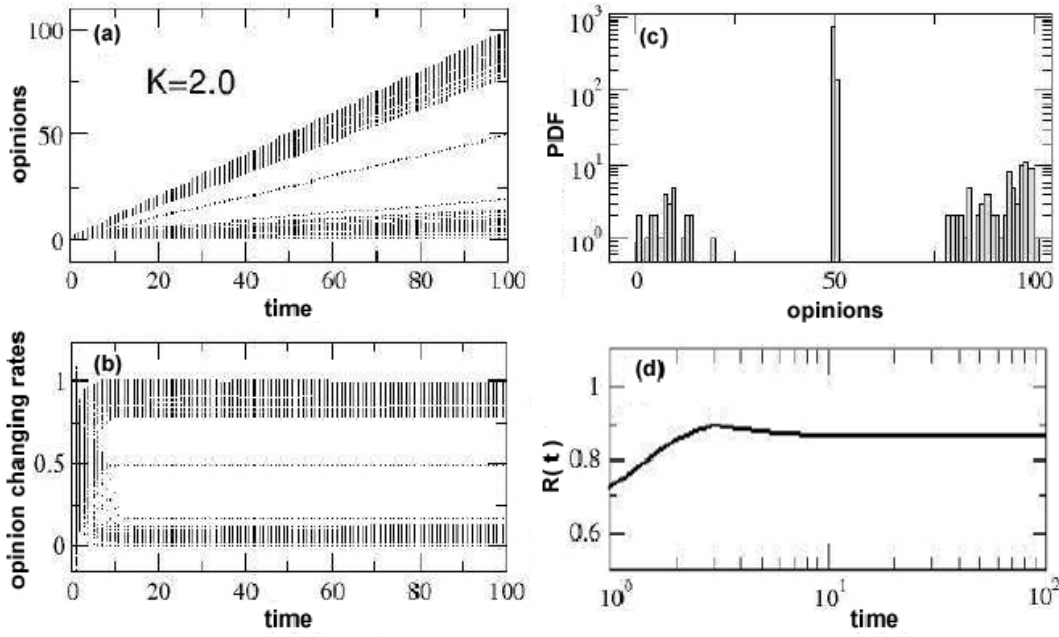
$$\dot{x}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^N \alpha \sin(x_j - x_i) e^{-\alpha|x_j - x_i|} \quad i = 1, \dots, N. \quad (2)$$

Here  $x_i(t)$  is the opinion (an unlimited real number) of the  $i$ th individual at time  $t$ , while  $\omega_i$  represents the so called *natural opinion changing rate*, i.e. the intrinsic inclination, or natural tendency, of each individual to change his opinion (corresponding to the *natural frequency* of each oscillator in the Kuramoto model). As in the Kuramoto model, also in the OCR model the  $\omega$ 's are randomly drawn from a given symmetric, unimodal distribution  $g(\omega)$  with a first moment  $\omega_0$ . Usually a uniform distribution centered at  $\omega_0$  is used. In this way one simulates the fact that in a population there are: 1) conservative individuals, that naturally tend to change their opinion very slowly, and thus are characterized by a value of  $\omega_i$  smaller than  $\omega_0$ ; 2) more flexible people, with  $\omega_i \sim \omega_0$ , that change idea more easily and follow the new fashions and trends; 3) individuals with a value of  $\omega_i$  higher than  $\omega_0$ , that run faster than the others in suggesting new ideas and insights.

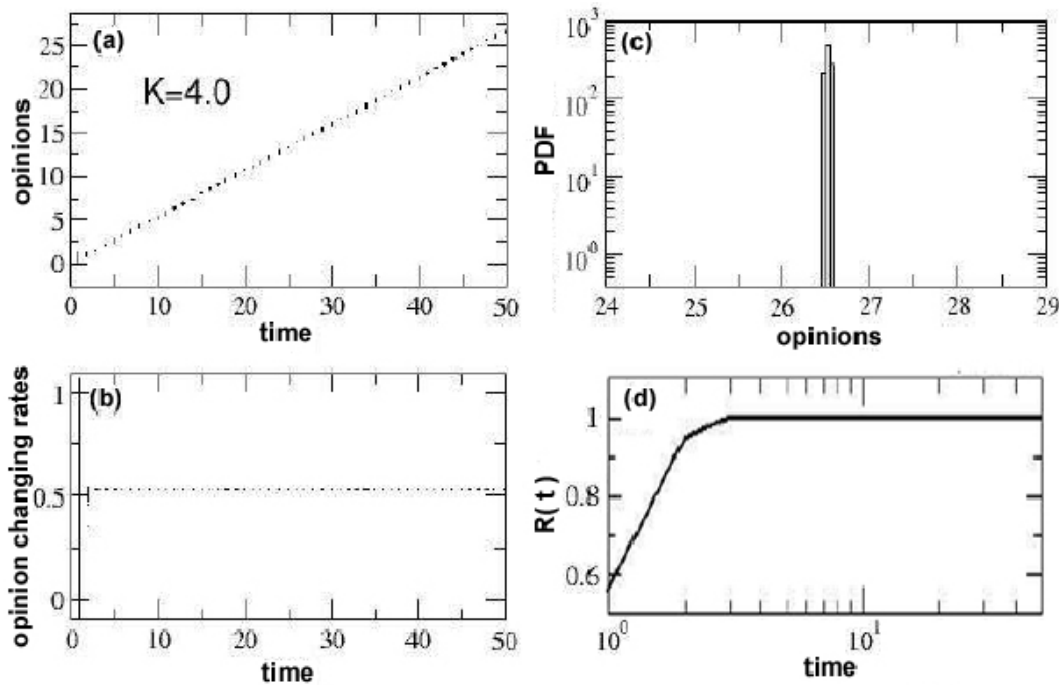
In the equation (2)  $K$ , as usual, is the coupling strength. The exponential factor in the coupling term ensures that, for reciprocal distance higher than a certain threshold, tuned by the parameter  $\alpha$ , opinions will no more influence each other. Such a requirement is inspired by the confidence bound concept discussed in the previous section. (Please note that, due to a misprinting, there is an  $\alpha$  factor missing in the coupling term of Eq. (7) of Ref.

[8].) At this point we can study the opinion dynamics of the OCR model by solving numerically the set of ordinary differential equations (2) for a given distribution of the  $\omega$ 's (natural opinion changing rates) and for a given coupling strength  $K$ . In particular, we want to find out if, as a function of  $K$ , there is a transition from an incoherent phase, in which people change opinion each one with his natural rate  $\omega_i$ , to a synchronized one in which all the people change opinion with the same rate and share a common social trend, a sort of 'public opinion'. In order to measure the degree of synchronization of the system we decided to adopt an order parameter  $R(t)$  related to the standard deviation of the opinion changing rate  $\dot{x}_j(t)$  and defined as  $R(t) = 1 - \sqrt{\frac{1}{N} \sum_{j=1}^N (\dot{x}_j(t) - \dot{X}(t))^2}$ , where  $\dot{X}(t)$  is the average over all individuals of  $\dot{x}_j(t)$ . It is easy to see that  $R = 1$  in the fully synchronized phase, where all the agents have exactly the same opinion changing rate (and very similar opinions), while  $R < 1$  in the incoherent or partially synchronized phase, in which the agents have different opinion changing rates and different opinions. The numerical simulations have been performed typically with  $N = 1000$  agents and with a uniform distribution of the initial individual opinions  $x_i(t = 0)$  in the range  $[-1, 1]$ . The natural opinion changing rates  $\omega_i$  are taken from a uniform distribution in the range  $[0, 1]$ . We fix the value of the coupling  $K$  and we let the system to evolve until a stationary (asymptotic) value  $R_\infty$  for the order parameter is obtained. In this way it is easy to recognize a Kuramoto-like transition from an incoherent phase (for  $K < K_c \sim 1.4$ ) to a partially coherent (for  $K \in [1.4, 4.0]$ ) and, finally, to a fully synchronized phase (for  $K > 4.0$ ) [8]. We now focus on the details of the dynamical evolution in each of the three phases.

In Figure 4 we analyze the case of very small coupling,  $K = 1.0$ . In the left part we show the time evolution of the opinions and of the opinion changing rates (angular



**Fig. 5.** OCR model for  $N = 1000$  and  $K = 2.0$  (partially coherent phase). Same quantities as in Figure 4.

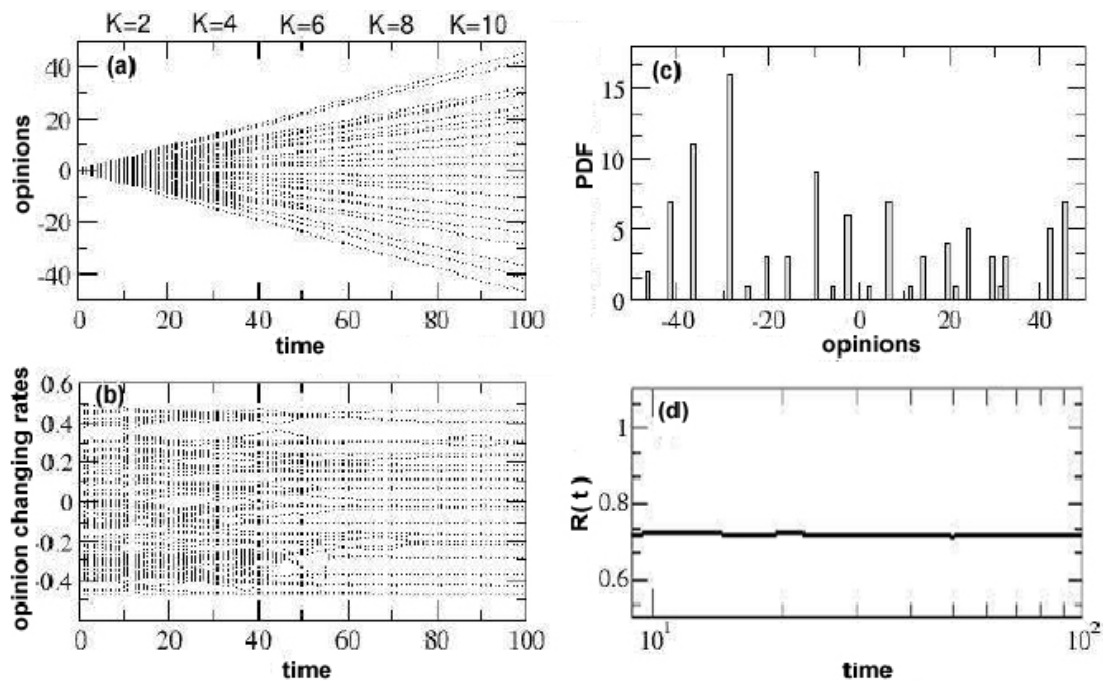


**Fig. 6.** OCR model for  $N = 1000$  and  $K = 4.0$  (synchronized phase). Same quantities as in the previous figures.

velocities or frequencies). In the right part, instead, we plot the final distribution of opinions and the order parameter time evolution. Because of the weak interactions we are in the incoherent phase and each agent tends to keep his natural opinion changing rate. It follows that the different opinions diverge in time without reaching any kind of consensus. In correspondence, the order parameter  $R$  takes the minimum possible value that, at variance with the Kuramoto model, is not zero. We could look at this case as to an ‘anarchical’ society.

In Figure 5 we plot the same quantities than before but in the case  $K = 2.0$ . The coupling is still weak but strong

enough to give rise to three different clusters of evolving opinions, each with a characteristic changing rate: the largest number of the agents, representing what we could call the “public opinion”, moves with an intermediate rate along the opinion axis, but there is a consistent group of people remaining behind them and also a group of innovative people (quicker in supply new ideas and ingenuity). From a political point of view, we could interpret this situation as a sort of ‘bipolarism’ with a large number of ‘centrists’. In this case the order parameter is larger than in the previous example, but still less than one since the opinion synchronization is only partial.



**Fig. 7.** OCR model for  $N = 100$  and a value of  $K$  increasing with a constant rate from 0.1 up to 10.1. Same quantities as in the previous figures.

Finally, in Figure 6 we report the case  $K = 4.0$ . Here the coupling is so strong that all the opinions change at the same rate and we observe a single final cluster in the opinion distribution. In this ‘dictatorial’ society all the agents think in the same way and follow the same trends and fashions. Although the *natural* frequencies of the agents are — as in the previous examples — different from each others, their opinion changing rates rapidly synchronize (frequency locking) and thus the order parameter  $R$  reaches a saturation value equal to one.

Summarizing, it has been found [8] that in order to ensure a ‘bipolarism’ — i.e. an equilibrium between conservative and progressist components — a changing society needs a level of coupling  $K$  strictly included in a narrow window ( $1.5 < K < 2.5$ ) inside the partially synchronized phase. Otherwise such an equilibrium will be broken and the final result will be anarchy or dictatorial regime. But it is worth to observe that these conclusions have been obtained for systems with fixed coupling  $K$ , simulating societies with a stable degree of interconnections among their members. Thus it is interesting to explore what happens if the coupling is let to increase its value during the dynamics, in order to simulate a society in which the interconnections between the agents increase in time, due for example to the improvement in transport or in communications.

In Figure 7 we show the results for a system in which the coupling is uniformly increased from  $K = 0.1$  to  $K = 10.1$ . The agents’ opinions initially spread freely, and then rapidly freeze in a large number of non-interacting clusters with different changing rates and variable sizes. Actually, it results that this particular cluster distribution, that could be socially interpreted as a multipolar-

ism, cannot be obtained in simulations with a constant coupling. This could suggest that the increase of interactions between the members of a society is crucial to stabilize a plurality of different non-interacting clusters of opinions (different ideologies, political parties, etc.) typical of a multipolar democracy. It seems to suggest also that a stable bipolarism is possible only in societies with a fixed degree of internal interconnections.

### 3 Conclusions

In this paper we have shown that even simple opinion formation models are able to capture many general features of real social systems. In the first part we have discussed an extension of the scalar opinion dynamics of Hegselmann-Krause model to the case in which the opinion is not just a scalar but a 2-vector with real-valued components. We investigated a community where everybody talks to everybody by means of Monte Carlo simulations of discrete opinion distributions (for squared or circular opinion space) and by solving numerically a rate equation for a continuum opinion distribution. By studying the consensus thresholds we found that the continuum case can be considered as the limit of the discrete one for a great number of agents. In the second part of the paper we have discussed a social variation of the Kuramoto model, the so called Opinion Changing Rate model (OCR). The concept of ‘*opinion changing rate*’ transforms the usual approach to opinion consensus into a synchronization problem. Similarly by the Kuramoto model, the OCR model exhibits a phase transition from an incoherent phase to a synchronized one and shows many interesting features with a clear social meaning.

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