

Computability and computational complexity of the evolution of nonlinear dynamical systems

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- Computability of attractors
- Results about basins of attraction

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- Hartman-Grobman theorem
- Computability of the stable/unstable manifolds

Dynamical systems

- Much of Physics is deterministic (notable exception: quantum mechanics): the initial condition of the system + some time evolution rule (physical law) determines uniquely the evolution of the system along time.

Definition

A *dynamical system* is a triple (S, \mathbb{T}, ϕ) , where S is the state space, \mathbb{T} is a monoid which denotes the *time*, and $\phi : \mathbb{T} \times S \rightarrow S$ is the *evolution rule*, which has the following properties ($\phi_t(x) = \phi(t, x)$)

- 1 $\phi_0 : S \rightarrow S$ is the identity
- 2 $\phi_t \circ \phi_s = \phi_{t+s}$ for every $t, s \in \mathbb{T}$

In this talk we will study (ordinary) differential equations

Computability and computational complexity of ODEs

This topic was already explored in the talk presented by
Amaury Pouly

Asymptotic behavior of ODEs

- In dynamical systems theory there is a great interest in telling what happens to a system “when time goes to infinity”.
- Related problems can be found in applications (e.g. verification, control theory):
 - Given an initial point x_0 , will the trajectory starting from x_0 eventually reach some “unsafe region” (Reachability)?
 - How many attractors (“steady states”) a system has? Can we characterize these attractors? Can we compute their basins of attractions—set of points on which the trajectory will converge towards a given attractor

What about attractors?

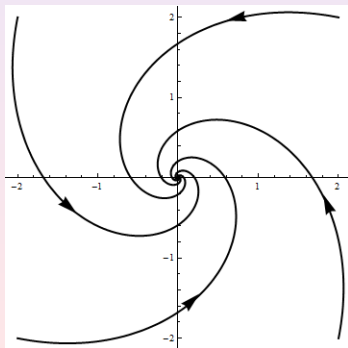
Roughly, attractors are invariant sets to which nearby trajectories converge (fragile attractors are usually dismissed). Types of attractors:

- Fixed points
- Periodic orbits (cycles)
- Surfaces, manifolds, etc.
- Strange attractors (Smale's horseshoe, Lorenz attractor, etc.): attractors with a fractal structure

Fixed points

Theorem (Graça, Zhong, 2010)

Given as input an analytic function f , the problem of deciding the number of equilibrium points of $y' = f(y)$ is undecidable, even on compact sets. However, the set formed by all equilibrium points is upper semi-computable.



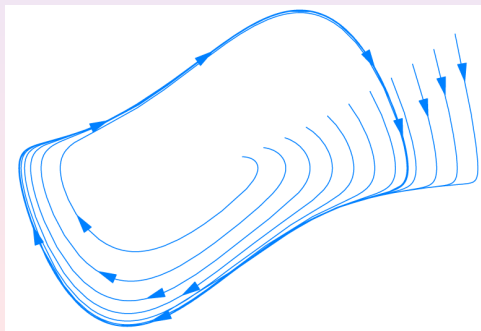
Idea of the proof

- Noncomputability arises from non-continuity of the problem of finding the number of zeros of the function f
- Nonetheless, the set consisting of all zeros of f can be upper semi-computed by discretizing the space into small squares. We can find the minimum and maximum of f over these squares and decide whether each square may have a zero.

Periodic orbits

Theorem (Graça, Zhong, 2010)

Given as input an analytic function f , the problem of deciding the number of periodic orbits of $y' = f(y)$ is undecidable (on \mathbb{R}^2), even on compact sets. However, the set formed by all hyperbolic periodic orbits is upper semi-computable.

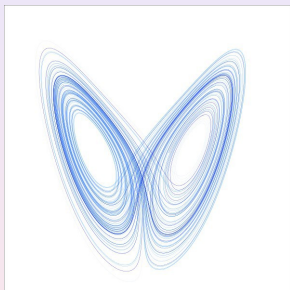


Idea of the proof

- Noncomputability arises from non-continuity problems related to the periodic orbits
- Nonetheless, the set consisting of all periodic orbits of f can be upper semi-computed by discretizing the space into small squares and by retaining only polygonal periodic orbits consisting of squares

Strange attractors

Steve Smale's 14th problem: does the Lorenz attractor exist?



$$x' = 10(y - x)$$

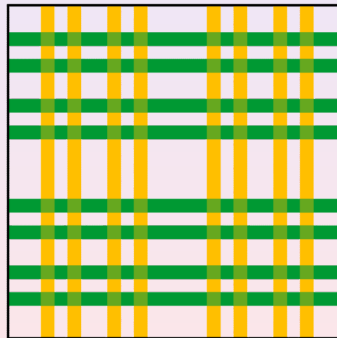
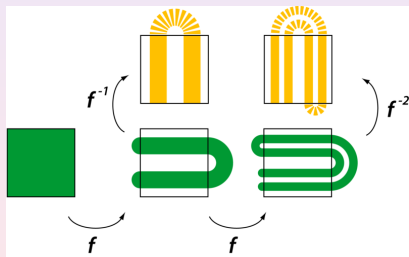
$$y' = 28x - y - xz$$

$$z' = xy - \frac{8}{3}z$$

Answer (W. Tucker, 1998): Yes! But is it computable? (open question)

Theorem (Graça, Zhong, Buescu, 2012)

The Smale Horseshoe is a computable (recursive) closed set.



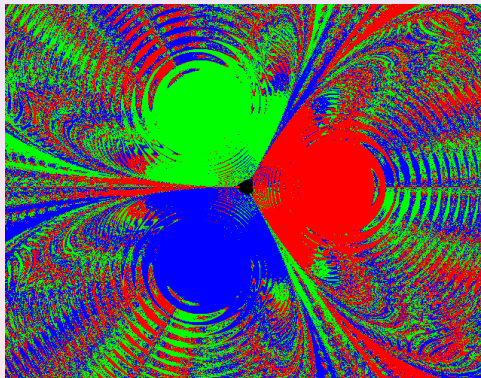
Idea of the proof

We show that the complement of Smale's horseshoe is computable by using the following fact (Zhong, 1996): An open subset $U \subseteq I$ is computable if and only if there is a computable sequence of rational open rectangles (having rational corner points) in I , $\{J_k\}_{k=0}^{\infty}$, such that

- (a) $J_k \subset U$ for all $k \in \mathbb{N}$,
- (b) the closure of J_k , \bar{J}_k , is contained in U for all $k \in \mathbb{N}$, and
- (c) there is a recursive function $e : \mathbb{N} \rightarrow \mathbb{N}$ such that the Hausdorff distance $d(I \setminus \bigcup_{k=0}^{e(n)} J_k, I \setminus U) \leq 2^{-n}$ for all $n \in \mathbb{N}$.

Basins of attraction

Problem: can we tell to which attractor a trajectory starting in a given initial point will converge?



Basins of attraction of a pendulum swinging over three magnets

- In some cases, the answer is YES (example: linear ODEs defined with hyperbolic matrices)
- There are related results on problems concerning control theory (reachability) which state that this problem is undecidable for many classes of systems
- The idea behind those undecidability proofs is to simulate Turing machines and reduce the above problem to the Halting Problem
- But to simulate Turing machines the authors need to make comparisons (e.g. if reading X , then do A , ...). This is achieved through the use of a *step function* Θ , where $\Theta(x) = 0$ if $x < 0$ and $\Theta(x) = 1$ if $x \geq 0$ or some C^k variant of the step function (e.g. integrate Θ k times).
- The above idea reduces to “gluing” different functions using a C^k joint

Theorem (Zhong, 2009)

There exists a computable C^∞ dynamical system having a computable hyperbolic equilibrium point such that its basin of attraction is recursively enumerable, but is not computable.

But what if the system is analytic?

Recall that in analytic functions, local behavior determines global behavior
 \Rightarrow no C^k gluing allowed, even if $k = +\infty$

Theorem (Graça, Zhong)

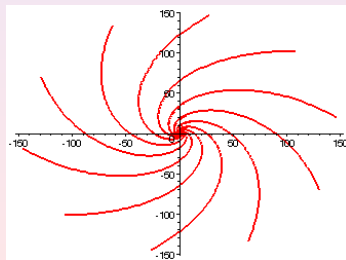
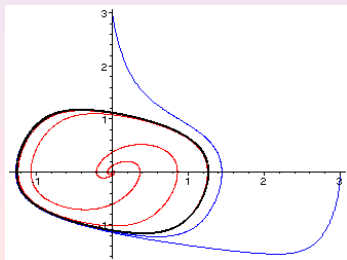
There exists a computable analytic dynamical system having a computable hyperbolic equilibrium point such that its basin of attraction is recursively enumerable, but is not computable.

Idea behind the proof

- Simulate a Turing machine with an analytic map (use interpolation techniques, and allow a certain error in the simulation—the map can still simulate a Turing machine even if the initial point and/or the dynamics are constantly perturbed. Use special techniques to keep the error under control)
- Suspend the previous map into an ODE. The classical suspension technique does not work here because it is not constructive. Instead we develop a new whole “computable” suspension technique which allows to embed a computable map into a computable ODE, under certain conditions
- The previous ODE will simulate a Turing machine and we “massage” the ODE so that the halting state corresponds to an hyperbolic fixed point (the ODE simulation of TMs is robust to perturbations)
- Then deciding which initial points will converge to the previous hyperbolic fixed point is equivalent to solving the Halting Problem

What about tools to work with dynamical systems?

- Near an hyperbolic fixed point, the flow defined by an ODE behaves in a similar way to the flow of the linearized ODE (Hartman-Grobman theorem).
- What is the connection between the computability of the original nonlinear operator and the linear operator which results from it? (asked by Pour-El & Richards in their book *Computability in Analysis and Physics*)



Theorem (Graça, Zhong, Dumas, 2012)

Near a hyperbolic equilibrium point x_0 of a nonlinear ODE $\dot{x} = f(x)$, there is a computable homeomorphism H such that $H \circ \phi = L \circ H$, where ϕ is the solution to the ODE and L is the solution to its linearization $\dot{x} = Df(x_0)x$.

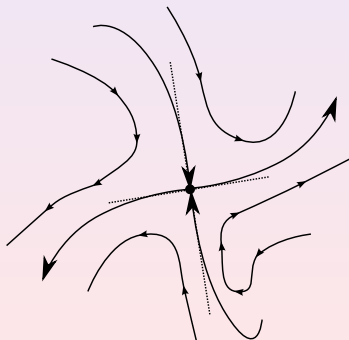
Idea behind the proof

- Problem: the classical proofs of the Hartman-Grobman theorem are not constructive
- We use the Banach fixed-point theorem (contraction mapping principle) to compute the homeomorphism by starting with a linearized version of the system and then adding nonlinear feedback
- The main problem from the classical proofs is that they use eigenvectors of $Df(x_0)$, but the process of finding eigenvectors is not computable
- We solve this problem by relying on a resolvent approach

What about stable/unstable manifolds?

Let x_0 be a hyperbolic fixed point

- Stable manifold of x_0 : set of points which trajectory will converge towards x_0 as $t \rightarrow +\infty$
- Unstable manifold of x_0 : set of points on which the trajectory will converge towards x_0 as $t \rightarrow -\infty$

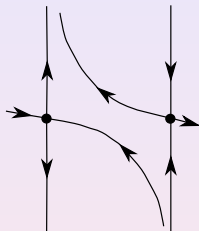


Theorem (Graça, Zhong, Buescu, 2012)

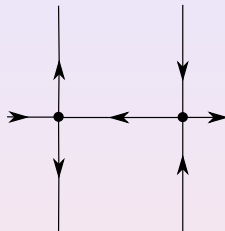
The stable and unstable manifold of an hyperbolic fixed point x_0 can be locally computed around x_0 , but not globally.

Idea of the proof

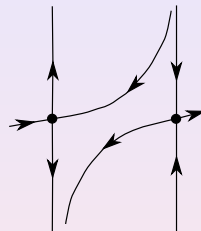
- The classical proof of the existence of the stable manifold relies on finding the eigenvectors of $Df(x_0)$ which are associated to positive eigenvalues
- However the process of finding eigenvectors is not computable because of continuity problems
- We rely on resolvents to compute directly the stable manifold, without needing to compute the eigenvectors of $Df(x_0)$
- The stable manifold is not globally computable because of continuity problems arising in the context of global bifurcations like heteroclinic bifurcations



$$\mu < 0$$



$$\mu = 0$$



$$\mu > 0$$

Thank you!