Computability in Computational Geometry

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Abstract. We promote the concept of object oriented computability in computational geometry in order to faithfully generalise the well-established theory of computability for real numbers and real functions. In object oriented computability, a geometric object is computable if it is the effective limit of a sequence of finitary objects of the same type as the original object, thus allowing a quantitative measure for the approximation. The domain-theoretic model of computational geometry provides such an object oriented theory, which supports two such quantitative measures, one based on the Hausdorff metric and one on the Lebesgue measure. With respect to a new data type for the Euclidean space, given by its non-empty compact and convex subsets, we show that the convex hull, Voronoi diagram and Delaunay triangulation are Hausdorff and Lebesgue computable.

1 Introduction

In his "Commentaries on the Difficulties in the Postulates of Euclid's Elements", Omar Khayyam, the 11th century Persian mathematician and poet, developed the first rudimentary notion of a real number. He first showed the equivalence of Euclid's notion of ratios with that of continued fractions. Then, in a stroke of genius, he defined two ratios as equal "when they can be expressed by the ratio of integer numbers with as great a degree of accuracy as we like." This idea thus contained the first notion of a real number and the germ of the concepts of computability and computation up to any precision. Three centuries later, Ghiasseddin Jamshid Kashani, another Persian mathematician, devised the first fixed point technique for computation in analysis in the beginning of the 15th century: he used a cubic polynomial in a recursive scheme to approximate the sine of 1° correctly up to 17 decimal places; see [2] for the details.

Following the formalisation of real numbers by Cantor and Dedekind in the 19th century and the development of recursion theory by Turing, Church, Gödel and Kleene in the first half of the 20th century, the concept of a computable real number was first defined by Turing in his seminal work in mid 1930's [17] and [18]. In the decades since that work, several notions of computability for real numbers and real functions have been proposed, as for example in [13], [15], [19], [16], [10], [1], which turn out to be essentially equivalent. A computable real number in all these different but equivalent approaches is in essence the limit of an effective sequence of rational numbers, and a computable real function is one which maps computable real numbers to computable real numbers in an effective way. The effective nature of the sequence of rational numbers approximating a computable number implies that each term of the sequence gives a lower and an upper bound for the real number, with the distance between the two bounds providing a quantitative measure of the approximation. Regarding a real number as an object and a rational number as a finitary object of the same type as a real number, we can say that a computable object is defined as the effective limit of two monotonic sequences of finitary objects, providing at each stage finitary lower and upper bounds for the computable object. In this sense, we say that the computability theory of real numbers and real functions is *object oriented.*

In more recent years, several attempts have been made to define the notion of computability for subsets of the Euclidean space and operations on such subsets [12], [11], [4], [7], [8], [9], [14], [3], [20], [21]. Here, there are several different approaches which give rise to a number of non-equivalent theories of computability for subsets of the Euclidean space and operations on them.

The domain-theoretic framework introduced in [7] and [8] for studying computability of subsets of Euclidean spaces and their operations is an object oriented theory for computational geometry, which faithfully generalises the object oriented computability theory of real numbers and real functions. As the membership predicate of a proper subset of a Euclidean space is undecidable on its boundary, subsets with the same boundary are identified in the domain-theoretic framework and thus any subset is represented by two disjoint open subsets: its interior and its exterior[3]. With respect to any enumeration of a countable basis of the Euclidean topology, a computable subset is one whose interior and exterior are each the union of an effective increasing sequence of the basis elements. Thus, computability of an object is defined by two effective sequences of the same type converging to the object. In a similar way, computability of all basic operations on subsets such as union, intersection, and Minkowski sum, as treated in [8], as well as the convex hull, Voronoi diagram and Delaunay triangulation, as dealt with in [9] and [14], are always defined in terms of sequences of finitary objects of similar type. For example, the computability of the convex hull of a finite number of points in the Euclidean space is defined using two effective sequences of interior and exterior convex rational polygons converging to the interior and the exterior of the convex hull of the points.

The object oriented computability provided by the domain-theoretic model provides other distinguished features:

- All basic predicates such as membership, subset inclusion and comparison as well as all basic operations are Scott continuous and computable in this model. Thus, algorithms developed in this framework are inherently robust in contrast with classical algorithms in computational geometry, which are non-robust due to the non-computability of comparison of real numbers or the membership predicate of a subset in classical geometry.
- In this model, one obtains robust algorithms for computing operations such as convex hull, Voronoi diagram and Delaunay triangulation with the same complexity of the corresponding non-robust classical algorithms.
- Since the computability of an object is defined in terms of effective sequences of finitary objects of a similar type, one can employ two quantitative measures of approximation: one using the Hausdorff metric and one using the Lebesgue measure.

Therefore, in this framework the notion of computability of a geometric object and the task of computing it up to any required precision by the user are synthesised into one paradigm, thus providing the foundation of a robust CAD system.

In this paper, we study the three notions of recursion theoretic computability, Hausdorff computability and Lebesgue computability of three basic computational geometry operations, namely the convex hull, Voronoi diagram and Delaunay triangulation, in the context of a general data type for the Euclidean space given by the domain of non-empty compact convex subsets of the space ordered by reverse inclusion.

[3] The exterior of a set is the interior of its complement.

2 The mathematical model and the new data type

The *solid domain* $(\mathbf{S}\mathbb{R}^d, \sqsubseteq)$ of $\mathbb{R}^d$ is the collection of pairs of disjoint open subsets of $\mathbb{R}^d$ partially ordered componentwise by subset inclusion: $(I, E) \sqsubseteq (I', E')$ iff $I \subseteq I'$ and $E \subseteq E'$; it is a bounded complete ω-continuous dcpo [8]. A classical geometric object, i.e. a subset $A \subseteq \mathbb{R}^d$ is represented in this model as $(A^\circ, (A^c)^\circ)$, where X° and X^c denote respectively the interior and the complement of a set X. More generally, we think of an element $(I, E) \in \mathbf{S}\mathbb{R}^d$ as a *partial solid* or *partial geometric objects* with *interior* I, *exterior* E and *boundary* $(I \cup E)^c$. An element (I, E) is maximal in $\mathbf{S}\mathbb{R}^d$ iff $I = (E^c)^\circ$ and $E = (I^c)^\circ$, which imply that I and E are regular[4]. The collection of pairs of interiors of disjoint dyadic (or rational) d-polygons forms a basis for $\mathbf{S}\mathbb{R}^d$. Any partial geometric object (I, E) can be obtained as the union of these basis elements.

Our new data type is described as follows. We assume that we have lower and upper rational bounds on the coordinates $(x_k)_{1\leq k\leq d}$ of an imprecisely given point $x \in \mathbb{R}^d$ in say n given directions, that is we have $\beta_j \leq \sum_{k=1}^{d} a_{jk}x_k \leq \gamma_j$, where $(a_{jk})_{1\leq k\leq d}$ fixes the n given directions for $1 \leq j \leq n$. We assume that the set of directions for our data type is known in advance, and is independent from the data itself. Thus, each data point x is located within a rational d-polygon, namely the intersection of the finite number of strips given by the above inequalities. In most applications, we only have the d directions of the coordinate axes, i.e. when each coordinate of an imprecisely given point is known to lie within an interval as in interval analysis, for example when the coordinates of x are given by floating point numbers. But this data type is also essential in cases when we have lower and upper bounds on some linear combination of coordinates. In Figure 1, we have shown 6 out of the 18 possible types of polygons for an imprecisely given point in $\mathbb{R}^2$, where there are precisely three directions of possible approximations: along the two coordinate axes and along the $(1, 1)$ vector, corresponding to the linear combination $x_1 + x_2$.

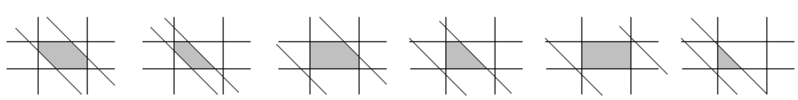

Fig. 1. Imprecise points defined by three directions $(1, 0), (0, 1), (1, 1)$

Note that the filtered intersection of a non-empty family of convex d-polygons in $\mathbb{R}^d$ is a non-empty, convex and compact subset. Our domain of computation is therefore the collection $(\mathbf{C}\mathbb{R}^d, \supseteq)$ of all non-empty, convex and compact subsets of $\mathbb{R}^d$ ordered by reverse inclusion and equipped with the Scott topology. It is a bounded complete ω-continuous domain with a countable basis given by the collection $\mathbf{P}\mathbb{R}^d$ of all rational convex d-polygons in $\mathbb{R}^d$. The map $s : \mathbb{R}^d \to \mathbf{C}\mathbb{R}^d$ with $x \mapsto \{x\}$ is a topological embedding, i.e. we can identify the maximal elements of this domain with $\mathbb{R}^d$. We also note that $(\mathbf{C}\mathbb{R}^d, \supseteq)$ is order isomorphic with a sub-domain of $\mathbf{S}\mathbb{R}^d$ by identifying a non-empty convex and compact set $A \in \mathbf{C}\mathbb{R}^d$ with $(\emptyset, A^c) \in \mathbf{S}\mathbb{R}^d$, i.e. with a geometric object whose interior is empty and its exterior is the complement of A.

In this extended abstract, we restrict ourselves to computational geometry in $\mathbb{R}^2$; our results however extend to $\mathbb{R}^d$ as we will show in the full version of the paper.

[4] An open set is *regular* if it is the interior of its closure

2.1 Computability

We assume the reader is familiar with the notion of computability for continuous domains [10], [5], [8]. Recall that given an effective structure for a bounded complete ω-continuous dcpo with respect to an enumeration of a countable basis, a *computable element* is defined as the lub of an effective increasing sequence of basis elements. A *computable function* from a bounded complete ω-continuous dcpo with an effective structure to another such domain is a map which sends computable elements to computable elements in an effective way. We fix effective structures on $\mathbf{C}\mathbb{R}^2$ and $\mathbf{S}\mathbb{R}^2$ by using, for example, an enumeration of rational convex polygons as a basis of $\mathbf{C}\mathbb{R}^2$ and an enumeration of pairs of disjoint rational polygons as a basis of $\mathbf{S}\mathbb{R}^2$. These effective structures induce effective structures on $(\mathbf{C}\mathbb{R}^2)^N$ and $(\mathbf{S}\mathbb{R}^2)^N$ for all positive integers N.

We will define the notions of Hausdorff and Lebesgue computability in the solid domain. Let d_H denote the Hausdorff distance between non-empty compact sets. We put $d_H(\emptyset, \emptyset) = 0$ and for $Y \neq \emptyset$, $d_H(\emptyset, Y) = \infty$. The notion of Hausdorff computability for a partial geometric object has been defined in [8]. Here, we define the notion of a nestedly Hausdorff computable map.

Definition 1. *Consider a computable map $f : (\mathbf{C}[-a, a]^2)^N \to \mathbf{S}[-a, a]^2$, for some $a > 0$, with $f(\hat{C}) = (f_I(\hat{C}), f_E(\hat{C}))$, where $\hat{C} = (C_1, \ldots, C_N)$ represents an ordered list of non-empty convex compact subsets of $[-a, a]^2$. Let $\{\hat{B}_i \mid i \geq 0\}$ be an enumeration of the basis of $(\mathbf{C}[-a, a]^2)^N$. Consider an arbitrary $\hat{C} = \bigsqcup_{i \geq 0} \hat{B}_{\phi(n)}$ with $d_H(\hat{C}, \hat{B}_{\phi(i)}) < 2^{-i}$, where ϕ is a total recursive function and $\langle \hat{B}_{\phi(i)} \rangle_{i \geq 0}$ is an increasing chain. We say the interior part f_I of f is* nestedly Hausdorff computable *if there exists a total recursive function ψ_1 such that*

$$d_H(\overline{(f_I(\hat{C}))}, \overline{f_I(\hat{B}_{\phi(\psi_1(i))})}) < 2^{-i} \text{ and } d_H((f_I(\hat{C}))^c, f_I(\hat{B}_{\phi(\psi_1(i))})^c) < 2^{-i}$$

where $\overline{A}$ is the closure of A and complements are with respect to $[-a, a]^2$. Similarly, the exterior part f_E of f is nestedly Hausdorff computable *if there exists a total recursive function ψ_2 such that*

$$d_H(\overline{(f_E(\hat{C}))}, \overline{f_E(\hat{B}_{\phi(\psi_2(i))})}) < 2^{-i} \text{ and } d_H((f_E(\hat{C}))^c, f_E(\hat{B}_{\phi(\psi_2(i))})^c) < 2^{-i}$$

If both f_I and f_E are nestedly Hausdorff computable then we say that f is nestedly Hausdorff computable*.*

As we will see later, the partial Delaunay triangulation map is nestedly Hausdorff computable but not Hausdorff continuous.

Proposition 1. *With the assumptions of Definition 1, suppose $f_I(\hat{C})$ and $f_I(\hat{B}_{\phi(i)})$ are regular. Then f_I is nestedly Hausdorff computable if there exists a total recursive function ψ_1 such that*

$$d_H(\partial(f_I(\hat{C})), \partial(f_I(\hat{B}_{\phi(\psi_1(i))}))) < 2^{-i},$$

where $\partial(A)$ is the boundary of A. Similarly for f_E.

Definition 2. *With the assumptions of Definition 1, we say f_I is* nestedly Lebesgue computable *if there exists a total recursive function ψ_1 such that*

$$\lambda(\overline{(f_I(\hat{C}))}, \overline{f_I(\hat{B}_{\phi(\psi_1(i))})}) < 2^{-i} \text{ and } \lambda((f_I(\hat{C}))^c, f_I(\hat{B}_{\phi(\psi_1(i))})^c) < 2^{-i}.$$

Similarly, f_E is nestedly Lebesgue computable *if there exists a total recursive function ψ_2 such that*

$$\lambda(\overline{(f_E(\hat{C}))}, \overline{f_E(\hat{B}_{\phi(\psi_2(i))})}) < 2^{-i} \text{ and } \lambda((f_E(\hat{C}))^c, f_E(\hat{B}_{\phi(\psi_2(i))})^c) < 2^{-i}$$

If both f_I and f_E are nestedly Lebesgue computable then we say f is nestedly Lebesgue computable*.*

Proposition 2. *With the assumptions of Definition 2, if the boundaries of $f_I(\hat{B}_i)$ and $f_E(\hat{B}_i)$ are continuous curves for each $i \in \omega$ and if their lengths are uniformly bounded, then f is Lebesgue computable.*

3 Convex Hull

The convex hull map for compact subsets is defined as:

$$\mathit{\Gamma} : (\mathcal{C}\mathbb{R}^2) \to \mathbf{C}\mathbb{R}^2$$

where $\mathcal{C}\mathbb{R}^2$ is the set of all non-empty compact subsets of $\mathbb{R}^2$ and $\mathbf{C}\mathbb{R}^2$ is the set of all non-empty compact convex subsets of $\mathbb{R}^2$, both with the Hausdorff metric; for any non-empty compact set C, the image $\mathit{\Gamma}(C)$ is the convex hull of C.

The partial convex hull map has type:

$$\begin{array}{rl} \mathcal{H} : (\mathbf{C}\mathbb{R}^2)^N \to & \mathbf{S}\mathbb{R}^2 \\ \hat{C} \quad\; \mapsto & (\mathcal{H}_I, \mathcal{H}_E), \end{array} \tag{1}$$

where $\hat{C} = (C_1, \ldots, C_N)$ represents an ordered list of N non-empty compact convex sets in the plane $\mathbb{R}^2$. For a given $\hat{C}$ define $R(\hat{C}) = \{\{p_1, \ldots, p_N\} \mid p_i \in C_i,\ i = 1, \ldots, N\}$ to be the collection of all possible N-element sets, each containing precisely one element from each C_i. An element $P \in R(\hat{C})$ is called a *representative set* for $\hat{C}$.

We define: $\mathcal{H}_I(\hat{C}) = \bigcap_{P \in R(\hat{C})} \mathit{\Gamma}(P)$ and $\mathcal{H}_E(\hat{C}) = \bigcup_{P \in R(\hat{C})} \mathit{\Gamma}(P)$. Thus, $\mathcal{H}_I$ is the set of points that are inside the convex hull of any representative set. Similarly, $\mathcal{H}_E$ is the set of points that are outside the convex hull of any representative set.

When $\hat{C} \in (\mathbf{C}\mathbb{R}^2)^N$ is a basis element, i.e. a list of N convex rational polygons, an algorithm has been developed [6] that computes $(\mathcal{H}_I, \mathcal{H}_E)$ as follows. Assume that there are n directions given by the unit normals d_j $(1 \le j \le n)$ with non-negative y coordinates, and ordered anti-clockwise from the positive x-axis. The unit circle is partitioned into $2n$ arcs $\widehat{d_j d_{j+1}}$ $(1 \le j \le 2n)$ with $d_{n+j} = -d_j$ for $1 \le j \le n$ and $d_{2n+1} = d_1$. Then, we have:

- $\mathcal{H}_E(\hat{C}) = \mathit{\Gamma}(\{c \mid c \text{ is a corner of } C_i,\ 1 \le i \le N\})$
- $\mathcal{H}_I(\hat{C}) = \bigcap_{j=1}^{2n} \mathit{\Gamma}(\{c_{ij} \mid 1 \le i \le N\})$,

where c_{ij} is a corner of C_i furthest away from the boundary of any half-plane containing C_i with unit normal in $\widehat{d_j d_{j+1}}$, see Figure 2. The above two expressions give an $N \log N$ algorithm to compute the interior and the exterior parts of the partial convex hull in rational arithmetic.

We will use this algorithm to prove that the partial convex hull is nestedly Hausdorff and Lebesgue computable. For a subset X of the Euclidean space and $\epsilon > 0$, we let $X^{+\epsilon} = \{y \mid d(y, X) < \epsilon\}$ and $X^{-\epsilon} = \{y \mid d(y, \overline{X^c}) > \epsilon\}$.

Lemma 1. *Suppose A is a regular compact set, then we have:*

$$\forall \epsilon > 0, \exists \delta > 0.\ A \subset \left(A^{-\delta}\right)^{+\epsilon}.$$

Proof. For any set A, we have: $\bigcup_{\delta > 0} A^{-\delta} = A^{\circ}$. Also, for any regular set A and $\epsilon > 0$:

$$\left.\begin{array}{l} \forall x \in A, \exists y \in A^{\circ}.\ d(x, y) < \epsilon \text{ (regularity)} \\ y \in A^{\circ} \Rightarrow \exists \delta > 0.\ y \in A^{-\delta} \end{array}\right\} \Rightarrow x \in \left(A^{-\delta}\right)^{+\epsilon}$$

which proves $A \subset \bigcup_{\delta > 0} \left(A^{-\delta}\right)^{+\epsilon}$. Since A is compact, $\exists \delta > 0$, such that $A \subset \left(A^{-\delta}\right)^{+\epsilon}$.

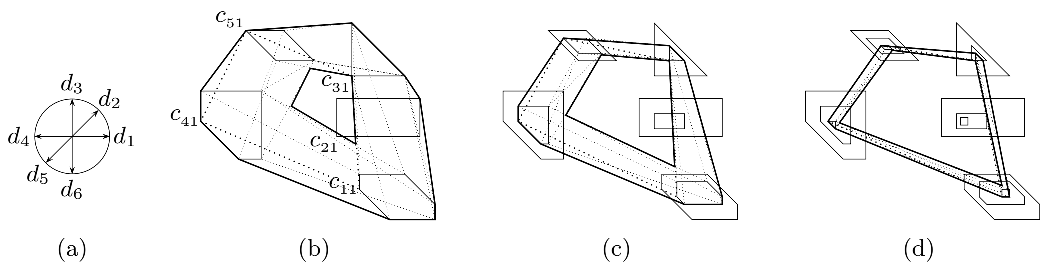

Fig. 2. (a) Three given directions have partitioned the unit circle into six arcs; (b) Partial convex hull of five partial points; (c)–(d) Partial convex hull with refined data.

Lemma 2. *Assume $P, Q \subset \mathbb{R}^2$ are two regular non-empty compact convex sets with $P \subseteq Q$. If $Q \subseteq P^{+r}$ then $\overline{P^c} \subseteq (\overline{Q^c})^{+r}$.*

Theorem 1. *The exterior part of the convex hull map is nestedly Hausdorff computable.*

Proof. Let $\hat{C} = \bigsqcup \hat{B}_i$ where $\langle \hat{B}_i \rangle$ is an effective increasing sequence of basis elements. We have $\hat{C} \subseteq \hat{B}_i$.

$$d_H(\hat{C}, \hat{B}_i) < r \Rightarrow \hat{B}_i \subseteq \hat{C}^{+r} \Rightarrow (\mathcal{H}_E(\hat{B}_i))^c \subseteq (\mathcal{H}_E(\hat{C}^{+r}))^c, \tag{2}$$

$$\hat{C} \subseteq (\mathcal{H}_E(\hat{C}))^c \Rightarrow \hat{C}^{+r} \subseteq ((\mathcal{H}_E(\hat{C}))^c)^{+r} \Rightarrow (\mathcal{H}_E(\hat{C}^{+r}))^c \subseteq ((\mathcal{H}_E(\hat{C}))^c)^{+r}. \tag{3}$$

From (2) and (3) we get $(\mathcal{H}_E(\hat{B}_i))^c \subseteq ((\mathcal{H}_E(\hat{C}))^c)^{+r}$, and by using Lemma 2, we get $\mathcal{H}_E(\hat{C}) \subseteq (\mathcal{H}_E(\hat{B}_i))^{+r}$. This shows that $d_H(\mathcal{H}_E(\hat{C}), \mathcal{H}_E(\hat{B}_i)) < r$ if $d_H(\hat{C}, \hat{B}_i) < r$. In particular, $d_H(\hat{C}, \hat{B}_i) < 2^{-i}$ implies $d_H(\mathcal{H}_E(\hat{C}), \mathcal{H}_E(\hat{B}_i)) < 2^{-i}$. □

More work is required to show the property for the interior part of the partial convex hull. For the next two results, we will work in $\mathbb{R}^d$.

Lemma 3. *For any regular compact convex set P and $r > 0$, we have: $P = (P^{+r})^{-r}$.*

Proposition 3. *Suppose A, A', B, and B' are compact convex subsets of $\mathbb{R}^d$ and $A \cap B$ and $A' \cap B'$ are regular. Then $\forall \epsilon > 0$, $\exists \delta > 0$, such that*

$$d_H(A, A') < \delta \ \& \ d_H(B, B') < \delta \Rightarrow (A \cap B) \subset (A' \cap B')^{+\epsilon} \ \& \ (A' \cap B') \subset (A \cap B)^{+\epsilon}$$

i.e. $d_H((A \cap B), (A' \cap B')) < \epsilon$.

Proof. We prove $(A \cap B) \subset (A' \cap B')^{+\epsilon}$. The other relation is similar. Let $\epsilon > 0$ be given. Using Lemma 1, there exists $\delta > 0$ such that $(A \cap B) \subset [(A \cap B)^{-\delta}]^{+\epsilon}$. Also,

$$\left. \begin{array}{r} d_H(A, A') < \delta \Rightarrow A \subset A'^{+\delta} \Rightarrow A^{-\delta} \subset (A'^{+\delta})^{-\delta} \\ A \text{ is compact and convex} \Rightarrow (A'^{+\delta})^{-\delta} = (A')^{\circ} \end{array} \right\} \Rightarrow A^{-\delta} \subset A' \Rightarrow (A \cap B)^{-\delta} \subset A'.$$

Similarly for B, we have $(A \cap B)^{-\delta} \subset B'$. Hence, $(A \cap B)^{-\delta} \subset (A' \cap B')$. Combining this with $(A \cap B) \subset [(A \cap B)^{-\delta}]^{+\epsilon}$ we get $(A \cap B) \subset (A' \cap B')^{+\epsilon}$. □

Since the interior convex hull, as discussed previously, is the intersection of a finite number of classical convex hulls, it follows from Proposition 3 that the interior part $\mathcal{H}_I$ of the partial convex hull map is a continuous map. To prove that it is nestedly Hausdorff computable, we need some preliminary results.

Theorem 2. *The map Γ is non-expansive with respect to the Hausdorff metric, i.e. $d_H(\Gamma(A),\Gamma(B)) \leq d_H(A,B)$, and therefore Hausdorff continuous.*

Proof. Assume $d_H(\Gamma(A),\Gamma(B)) > r$ for some $r>0$. Without loss of generality we can assume that:

$$\exists a \in \Gamma(A),\ \text{s.t. } b(a,r)\cap\Gamma(B)=\emptyset,$$

where $b(a,r)$ is the ball with centre a and radius r. Because both $b(a,r)$ and $\Gamma(B)$ are convex, there is a plane separating them, i.e., denoting by S^{d-1} the unit sphere with centre at the origin, we have:

$$\begin{aligned}\exists s\in S^{d-1}, u\in\mathbb{R}:\ & \Gamma(B)\subset\{x \mid s\cdot x-u\leq 0\} && (1)\\ & b(a,r)\subset\{x \mid s\cdot x-u\geq 0\} &&\\ & \Rightarrow a\in\{x \mid s\cdot x-u-r\geq 0\} &&\\ & \Rightarrow \Gamma(A)\not\subset\{x \mid s\cdot x-u-r\leq 0\} && (2)\end{aligned}$$

We can easily see that:

$$\left.\begin{aligned}(1)\Rightarrow B\subset\{x \mid s\cdot x-u\leq 0\}\\ (2)\Rightarrow A\not\subset\{x \mid s\cdot x-u-r\leq 0\}\end{aligned}\right\}\Rightarrow d_H(A,B)>r,$$

which completes the proof.

For a basis element $\hat{C}\in(\mathbf{C}\mathbb{R}^2)^N$, i.e. an ordered list of N convex rational polygons, let

$$\gamma(\hat{C}) := \inf\{\alpha \mid \alpha \text{ is the smallest angle of } \Gamma(P)\cap\Gamma(Q),\ P \text{ and } Q \text{ representative sets}\}.$$

Lemma 4. *Suppose $\hat{C}\in(\mathbf{C}\mathbb{R}^2)^N$ is a basis element. The minimum angle β of $\mathcal{H}_I(\hat{C})$ is bounded below by $\gamma(\hat{C})$. Furthermore, β is the minimum angle of $\Gamma(P)\cap\Gamma(Q)$ for a pair of representative sets $P,Q\in R(\hat{C})$ such that all the elements of P and Q are corners of the N polygons in $\hat{C}$.*

Lemma 5. *For any convex polygon P with minimum angle β and any $r>0$ such that $P^{-r}\neq\emptyset$, we have $P\subseteq(P^{-r})^{+r/\sin(\beta/2)}$.*

Theorem 3. *The interior part of the partial convex hull map is nestedly Hausdorff computable.*

Proof. Suppose $\hat{C}=\bigsqcup\hat{B}_i$, where $\langle\hat{B}_i\rangle$ is an increasing effective sequence of basis elements. We have $\hat{C}\subseteq\hat{B}_i$ and for any $r>0$,

$$d_H(\hat{C},\hat{B}_i)<r\Rightarrow\hat{B}_i\subseteq\hat{C}^{+r}\Rightarrow(\mathcal{H}_I(\hat{B}_i))^c\subseteq(\mathcal{H}_I(\hat{C}^{+r}))^c, \tag{4}$$

$$(\mathcal{H}_I(\hat{C}^{+r}))^c=\left(\bigcap_{\hat{x}\in R(\hat{C}^{+r})}\Gamma(\hat{x})\right)^c=\bigcup_{\hat{x}\in R(\hat{C}^{+r})}(\Gamma(\hat{x}))^c, \tag{5}$$

$$((\mathcal{H}_I(\hat{C}))^c)^{+r}=\left(\left(\bigcap_{\hat{y}\in R(\hat{C})}\Gamma(\hat{y})\right)^c\right)^{+r}=\left(\bigcup_{\hat{y}\in R(\hat{C})}(\Gamma(\hat{y}))^c\right)^{+r}=\bigcup_{\hat{y}\in R(\hat{C})}((\Gamma(\hat{y}))^c)^{+r}. \tag{6}$$

We show that the right hand side of (5) is a subset of the right hand side of (6).

$$\forall\hat{x}\in R(\hat{C}^{+r}),\exists\hat{y}\in R(\hat{C});\ d_H(\hat{x},\hat{y})<r\Rightarrow d_H(\Gamma(\hat{x}),\Gamma(\hat{y}))<r\Rightarrow(\Gamma(\hat{x}))^c\subseteq((\Gamma(\hat{y}))^c)^{+r}$$

Now, using (4) we have $(\mathcal{H}_I(\hat{B}_i))^c \subseteq ((\mathcal{H}_I(\hat{C}))^c)^{+r}$. A careful use of Lemma 5 shows that $d_H(\mathcal{H}_I(\hat{C}), \mathcal{H}_I(\hat{B}_i)) < r/\sin(\beta/2)$, where β is the minimum angle of $\mathcal{H}_I(\hat{C})$. Using Lemma 4, we can show that $\beta \geq \gamma(\hat{B}_{i_0})$ for the least $i_0 \geq 0$ where $\mathcal{H}_I(\hat{B}_{i_0}) \neq \emptyset$. By finding the least $k \geq 0$ such that $2^{-k} < \sin(\gamma(\hat{B}_{i_0})/2)$, the required total recursive function ρ can be taken as $\rho(i) = i + k$. □

Corollary 1. *The partial convex hull map is nestedly Lebesgue computable.*

4 Partial Perpendicular Bisector

For a point $x \in \mathbb{R}^2$ and a compact $C \in \mathbf{C}\mathbb{R}^2$, we have the following distance functions appropriate for the Voronoi diagram of partial points. Let $d_s(x, C) = \min\{|x - p| : p \in C\}$ and $d_l(x, C) = \max\{|x - p| : p \in C\}$ be, respectively, the shortest and longest distance from x to C. For two compact subsets C_1 and C_2, we define the *partial Voronoi cell* of C_1 with respect to C_2 as:

$$\mathcal{C}_{12} = \{x \mid d_l(x, C_1) < d_s(x, C_2)\} \tag{7}$$

Similarly we define $\mathcal{C}_{21}$. The *partial perpendicular bisector* of C_1 and C_2 is the remaining points of the plane, $\mathcal{B}(C_1, C_2) := (\mathcal{C}_{12} \cup \mathcal{C}_{21})^c = \{z \in \mathbb{R}^2 \mid \exists x \in P_1,\ y \in P_2;\ |z - x| = |z - y|\}$. The domain theoretic definition of the partial perpendicular bisector map is:

$$\begin{array}{l} \mathcal{B} : \mathbf{C}\mathbb{R}^2 \times \mathbf{C}\mathbb{R}^2 \to \mathbf{S}\mathbb{R}^2 \\ \quad (C_1, C_2) \qquad \mapsto (\emptyset, \mathcal{C}_{12} \cup \mathcal{C}_{21}). \end{array}$$

For basis elements $C_1, C_2 \in \mathbf{C}\mathbb{R}^2$, the boundary of $\mathcal{C}_{12} \cup \mathcal{C}_{21}$ consists of segments of parabolas and straight lines [6], see Figure 3(a).

Proposition 4. *The restriction of the partial perpendicular bisector map $\mathcal{B}$ to $\mathbf{C}[-a, a]^2$ is Hausdorff continuous for any $a > 0$.*

Proposition 5. *The partial perpendicular bisector map $\mathcal{B}$ is Scott continuous.*

Theorem 4. *The restriction of the partial perpendicular bisector map $\mathcal{B}$ to $\mathbf{C}[-a, a]^2$ is nestedly Hausdorff and Lebesgue computable for any $a > 0$.*

5 Partial Voronoi Diagram

We define the partial Voronoi map on a list $\hat{C} = (C_1, \ldots, C_N) \in (\mathbf{C}\mathbb{R}^2)^N$ of N polygons in the plane:

$$\mathcal{V} : (\mathbf{C}\mathbb{R}^2)^N \ \to \ (\mathbf{S}\mathbb{R}^2)^N,$$

with the ith component, $1 \leq i \leq N$, defined as

$$\mathcal{V}_i : \hat{C} \ \mapsto \ ((\mathcal{V}_i)_I, (\mathcal{V}_i)_E) = (\bigcap_{j \neq i} \mathcal{C}_{ij}, \bigcup_{j \neq i} \mathcal{C}_{ji}),$$

where $\mathcal{C}_{ji}$ is defined in Equation 7.

Proposition 6. *The restriction of the partial Voronoi diagram map $\mathcal{V}$ to $(\mathbf{C}[-a, a]^2)^N$ is Hausdorff continuous for any $a > 0$.*

Proposition 7. *The partial Voronoi diagram map $\mathcal{V}$ is Scott continuous.*

For a basis element $\hat{C} \in (\mathbf{C}\mathbb{R}^2)^N$, the boundaries of $\bigcap_{j \neq i} \mathcal{C}_{ij}$ and $\bigcup_{j \neq i} \mathcal{C}_{ji}$ consist of segments of parabolas and straight lines [6] as in the case of the partial perpendicular bisector.

Theorem 5. *The restriction of the partial Voronoi diagram map $\mathcal{V}$ to $(\mathbf{C}[-a, a]^2)^N$ is nestedly Hausdorff and Lebesgue computable for any $a > 0$.*

6 Partial Disc

Partial disc map has been defined by the authors in [6] as:

$$\begin{aligned} \mathcal{D}: \quad (\mathbf{C}\mathbb{R}^2)^3 &\quad\to\quad \mathbf{S}\mathbb{R}^2 \\ (C_1, C_2, C_3) &\mapsto (\mathcal{D}_I, \mathcal{D}_E), \end{aligned}$$

where $\mathcal{D}_I = \mathcal{D}_E = \emptyset$ if C_1, C_2 and C_3 are collinear, i.e. when there exists a straight line which intersects C_1, C_2 and C_3, otherwise $\mathcal{D}_I = (\bigcap\{D_{xyz} \mid x \in C_1,\ y \in C_2,\ z \in C_3\})^\circ$ and $\mathcal{D}_E = (\bigcup\{D_{xyz} \mid x \in C_1,\ y \in C_2,\ z \in C_3\})^c$, where D_{xyz} is the disc made by the circle passing through x, y, and z.

Note that $\mathcal{O}(C_1, C_2, C_3) = \{s \in \mathbb{R}^2 \mid \exists x \in C_1, y \in C_2, z \in C_3; |x - s| = |y - s| = |z - s|\}$, and hence $\mathcal{O}(C_1, C_2, C_3)$ is the locus of the centres of circles which intersect the three convex sets C_1, C_2 and C_3. We call $\mathcal{O}(C_1, C_2, C_3)$ the *partial centre* of the partial circumcircle of the three partial points.

Let $D(o_{CCF}, r_{CCF})$ denote the closed disc with centre o_{CCF} and radius r_{CCF}, which passes through the following three points: (i) the point of C_1 closest to o_{CCF}, (ii) the point of C_2 closest to o_{CCF} and (iii) the point of C_3 furthest from o_{CCF}; hence the subscript in o_{CCF}. Similarly, five other pairs of centres and radii are defined: (o_{CFC}, r_{CFC}), (o_{FCC}, r_{FCC}), (o_{FFC}, r_{FFC}), (o_{FCF}, r_{FCF}) and (o_{CFF}, r_{CFF}). Now, consider the three discs $D_1 = D(o_{FCC}, r_{FCC})$, $D_2 = D(o_{CFC}, r_{CFC})$ and $D_3 = D(o_{CCF}, r_{CCF})$ on the one hand and the three discs $D'_1 = D(o_{CFF}, r_{CFF})$, $D'_2 = D(o_{FCF}, r_{FCF})$ and $D'_3 = D(o_{FFC}, r_{FFC})$ on the other hand, Figure 3(b). As shown in [6] by the authors, the interior and the exterior of the partial disc are given by:

$$(\mathcal{D}_I, \mathcal{D}_E) = ((D_1 \cap D_2 \cap D_3)^\circ, (D'_1 \cup D'_2 \cup D'_3)^c)\,.$$

Proposition 8. *The restriction of the partial disc map $\mathcal{D}$ to $(\mathbf{C}[-a, a]^2)^3$ is Hausdorff continuous for any $a > 0$.*

Proposition 9. *The partial disc map $\mathcal{D}$ is Scott continuous.*

For a basis element $(C_1, C_2, C_3) \in (\mathbf{C}\mathbb{R}^2)^3$, the centres and radii of the six discs above can be obtained using the partial perpendicular bisectors of each pair of these three partial points [6].

Theorem 6. *The restriction of the partial disc map $\mathcal{D}$ to $(\mathbf{C}[-a, a]^2)^3$ is nestedly Hausdorff and Lebesgue computable for any $a > 0$.*

7 Partial Delaunay Triangulation

We define the *partial edge* $\mathrm{Ed}(C_1, C_2)$ of two partial points C_1 and C_2 to be the convex hull of C_1 and C_2. We also define the *partial triangle* of three partial points to be their partial convex hull. Given N partial points $C_1, \ldots, C_N \in \mathbf{C}\mathbb{R}^2$, we say that $\mathrm{Ed}(C_{i_1}, C_{i_2})$ is *legal* if there exists i_3 such that for all $j \neq i_1, i_2, i_3$ we have $C_j \subset \mathcal{D}_E(C_{i_1}, C_{i_2}, C_{i_3})$, *illegal* if there exists i_3 such that there exists $j \neq i_1, i_2, i_3$ with $C_j \subset \mathcal{D}_I(C_{i_1}, C_{i_2}, C_{i_3})$ and *indeterminate* otherwise. The *partial Delaunay triangulation* map is now defined as:

$$\begin{aligned} \mathcal{T}: \quad (\mathbf{C}\mathbb{R}^2)^N &\to \mathbf{S}\mathbb{R}^2 \\ (C_1, \ldots, C_N) &\mapsto (\mathcal{T}_I, \mathcal{T}_E), \end{aligned}$$

where $\mathcal{T}_I = \emptyset$ and

$$\mathcal{T}_E = (\bigcup\{\mathrm{Ed}(C_i, C_j) \mid \mathrm{Ed}(C_i, C_j) \text{ legal or indeterminate}\})^c.$$

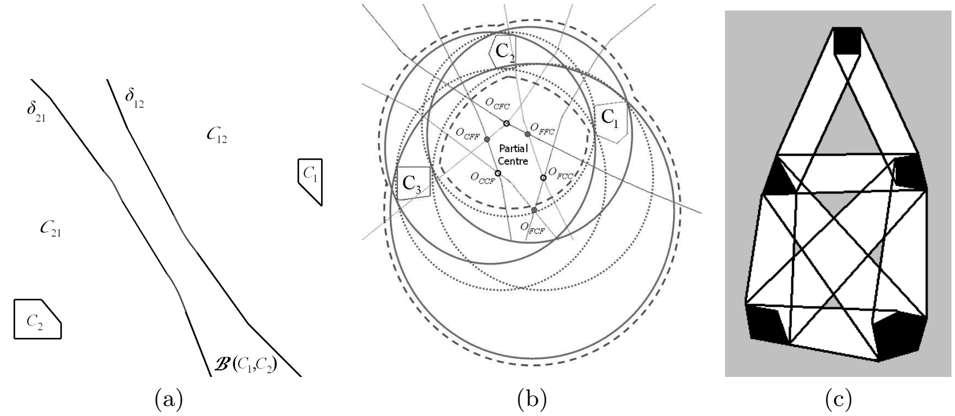

Fig. 3. (a) PPB of two polygons, (b) The interior and exterior of a partial disc, (c)The exterior of a partial Delaunay triangulation of five black polygons has been shown with gray colour. Note that there are two indeterminate and six legal edges.

We now proceed to show that the partial Delaunay triangulation map is nestedly Hausdorff computable. Since the interior is always empty, we only need to prove the computability for the exterior. In the example in Figure 3(c), the partial points are shown in black, while the exterior of the Delaunay triangulation, which is a disconnected set, is shown in gray. Note that the partial Delaunay triangulation map is not Hausdorff continuous, since an indeterminate partial edge may become illegal with an arbitrarily small non-nested perturbation of the input or partial points. The classical Delaunay triangulation map is similarly not Hausdorff continuous.

Proposition 10. *The partial Delaunay triangulation map $\mathcal{T}$ is Scott continuous.*

In [6], an incremental algorithm has been presented which computes the partial Delaunay triangulation of a set of partial points on average in $N \log N$ on non-degenerate input, generalising a similar algorithm for the classical Delaunay triangulation.

Lemma 6. *The minimum angle amongst all components of the exterior of a partial Delaunay triangulation of the set $\hat{C}$ of partial points is bounded below by the minimum interior angle of all the partial triangles with non-empty interior, each made from three partial points in $\hat{C}$.*

Theorem 7. *The partial Delaunay triangulation map $\mathcal{T}$ is nestedly Hausdorff and Lebesgue computable.*

Proof. The function $\delta = \epsilon \sin(\beta/2)$ can be used here, where β is the minimum angle as defined in Lemma 6.

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