Logical Methods in Computer Science
Vol. 10(2:8)2014, pp. 1–28
www.lmcs-online.org

Submitted Dec. 10, 2013
Published Jun. 12, 2014

COMPUTABILITY OF 1-MANIFOLDS

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ABSTRACT. A semi-computable set S in a computable metric space need not be computable. However, in some cases, if S has certain topological properties, we can conclude that S is computable. It is known that if a semi-computable set S is a compact manifold with boundary, then the computability of ∂S implies the computability of S. In this paper we examine the case when S is a 1-manifold with boundary, not necessarily compact. We show that a similar result holds in this case under assumption that S has finitely many components.

1. Introduction

A closed subset of $\mathbb{R}^m$ is computable if it can be effectively approximated by a finite set of points with rational coordinates with arbitrary precision on an arbitrary bounded region of $\mathbb{R}^m$. A compact subset S of $\mathbb{R}^m$ is semi-computable if we can effectively enumerate all rational open sets which cover S. Each compact computable set is semi-computable. On the other hand, there exist semi-computable sets which are not computable.

Hence the implication

$$S \text{ semi-computable } \Rightarrow S \text{ computable} \tag{1.1}$$

does not hold in general and the question arises whether there are some conditions under which it does hold. A motivation for this question lies in the fact that semi computable subsets of $\mathbb{R}^m$ are exactly compact co-computably enumerable sets. A closed subset of $\mathbb{R}^m$ is called co-computably enumerable (co-c.e.) if its complement can be effectively covered by open balls. Furthermore, co-c.e. sets are exactly the sets of the form $f^{-1}(\{0\})$, where $f : \mathbb{R}^m \to \mathbb{R}$ is a computable function. So the question under what conditions (1.1) holds is related to the question under what conditions the set of all zero-points of a computable function $f : \mathbb{R}^m \to \mathbb{R}$ is computable.

It is known that there exists a computable function $f : \mathbb{R} \to \mathbb{R}$ which has zero-points and all of them lie in $[0, 1]$, but none of them is computable [13]. This means that $f^{-1}(\{0\})$ is a nonempty semi-computable set which contains no computable point. In particular, $f^{-1}(\{0\})$ is not computable. Since each nonempty computable set contains computable points, this shows that there exist semi-computable sets which are "far away from being computable".

2012 ACM CCS: [**Theory of computation**]: Continuous mathematics—Topology—Geometric topology.
Key words and phrases: computable metric space, computable set, semi-computable set, co-c.e. set, 1-manifold with boundary.

LOGICAL METHODS IN COMPUTER SCIENCE
DOI:10.2168/LMCS-10(2:8)2014

However, it turns out that under certain assumptions implication (1.1) does hold. In particular, it has been proved in [9] that (1.1) holds whenever $S \subseteq \mathbb{R}^m$ is a topological sphere (i.e. homeomorphic to the unit sphere $S^n \subseteq \mathbb{R}^{n+1}$ for some n) or S is homeomorphic to the closed unit ball $B^n \subseteq \mathbb{R}^n$ for some n (i.e. S is an $n-$cell) by a homeomorphism $f : B^n \to S$ such that $f(S^{n-1})$ is a semi-computable set. Furthermore, by [6], these results hold not just in $\mathbb{R}^m$, but also in any computable metric space which is locally computable. Results related to (1.1) can also be found in [1], [4], [8] and [11].

Recently, the results for topological spheres and cells with semi-computable boundary spheres have been generalized in [7] where it was proved that in any computable metric space implication (1.1) holds if S is a compact manifold with boundary such that the boundary ∂S is computable. In other words, if S is a compact manifold with boundary and if S is semi-computable, then

$$\partial S \text{ computable } \Rightarrow S \text{ computable.} \tag{1.2}$$

The notion of a semi-computable set coincides with the notion of a compact co-c.e. set in a computable metric space which has compact closed balls and the effective covering property. Therefore, in such a computable metric space, if S is a compact manifold with boundary and if S is co-c.e., then (1.2) holds.

In this paper we observe the case when S is a 1-manifold, not necessarily compact, and we examine what can be said in this case in view of implication (1.2). We first have to find some appropriate generalization of the notion of a semi-computable compact set. The idea is that this new notion be a generalization to those sets S which may not be compact, but such that $S \cap B$ is compact for each closed ball B. We will say that S is semi-computable compact on closed balls or semi-c.c.b. if $S \cap B$ is semi-computable, uniformly for each closed rational ball B in the ambient space.

Our main result will be this: if S is a 1-manifold with boundary in a computable metric space and if S is semi-c.c.b. and S has finitely many components, then (1.2) holds. We will also show that (1.2) does not hold in general (without the assumption that S has finitely many components).

It will turn out that in a computable metric space which has compact closed balls and the effective covering property the notions of a semi-c.c.b. set and a co-c.e. set coincide. Therefore, in such a computable metric space we will have that if S is a 1-manifold with boundary, S is co-c.e. and S has finitely many components, then (1.2) holds.

The main step in the proof of our main result is to prove the following: if S is homeomorphic to $[0, \infty\rangle$ by a homeomorphism which maps 0 to a computable point or S is homeomorphic to $\mathbb{R}$, then S is computable if it is semi-c.c.b. (Here $[0, \infty\rangle$ denotes the set of all nonnegative real numbers.) Moreover, we will prove the following: if S is such a set and $S \cup F$ is semi-c.c.b., where F is closed and disjoint with S, then S is a computably enumerable set, which means that we can effectively enumerate all open rational balls which intersect S. This will be the key result and it will easily imply the main result for 1-manifolds.

In order to prove this, the central notion will be the notion of a chain and we will rely on techniques from [4].

It should be mentioned here that a semi-c.c.b. 1-manifold with boundary (with finitely many components) need not be computable if its boundary is not computable. An example for this we already have in the compact case: in each $\mathbb{R}^m$ there exists a line segment which is semi-computable, but not computable [9] (of course, at least one endpoint of such a line segment is not computable). However, it is interesting to mention that this example does not mean that the computability of the boundary is necessary for the computability of the entire manifold. By [9], there exists a computable arc in $\mathbb{R}^2$ with noncomputable endpoints, hence the computability of 1-manifold with boundary does not imply the computability of its boundary.

Regarding the computability of a manifold, we can notice that this does not mean that the manifold can be parameterized by a computable function. Namely, by [9], there exists a computable arc S in $\mathbb{R}^2$ with computable endpoints such that there exists no computable bijection $f : [0, 1] \to S$.

In Section 2 we give necessary definitions and some basic facts. In Section 3 we define semi-c.c.b. sets. In Section 4 we introduce chains and we develop certain techniques which we will need later. In Section 5 we prove that S is computably enumerable if $S \cup F$ is semi-c.c.b., where F is a closed set disjoint with S and S is a topological ray with computable endpoint (Theorem 5.2). In Section 6 we prove the same under assumption that S is a topological line (Theorem 6.2). Finally, in Section 7 we get that each semi-c.c.b. 1-manifold with boundary which has finitely many components is computable if its boundary is semi-c.c.b. (Theorem 7.3). This in particular means that each semi-c.c.b. (boundaryless) 1-manifold which has finitely many components is computable. In Section 7 we will actually prove this: if M is a 1-manifold with boundary and if both M and ∂M are semi-c.c.b., then each component of M is computably enumerable (Theorem 7.2).

Let us mention that the uniform version of the result for 1-manifolds (Theorem 7.3) does not hold in general. Namely, by Example 7 in [4], there exists a sequence (S_i) of topological circles in $\mathbb{R}^2$ such that S_i is uniformly semi-computable, but not uniformly computable. Moreover, each S_i is contained in the compact set $[0, 1] \times [0, 1]$.

2. Basic notions and techniques

If X is a set, let $\mathcal{P}(X)$ denote the set of all subsets of X.

For $m \in \mathbb{N}$ let $\mathbb{N}_m = \{0, \dots, m\}$. For $n \geq 1$ let

$$\mathbb{N}_m^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{N}_m\}.$$

We say that a function $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ is **computable finitely valued** or **c.f.v.** if the function $\overline{\Phi} : \mathbb{N}^{k+n} \to \mathbb{N}$ defined by

$$\overline{\Phi}(x, y) = \chi_{\Phi(x)}(y),$$

$x \in \mathbb{N}^k$, $y \in \mathbb{N}^n$ is computable (i.e. recursive), where $\chi_S : \mathbb{N}^n \to \{0, 1\}$ denotes the characteristic function of $S \subseteq \mathbb{N}^n$, and if there exists a computable function $\varphi : \mathbb{N}^k \to \mathbb{N}$ such that

$$\Phi(x) \subseteq \mathbb{N}^n_{\varphi(x)}$$

for all $x \in \mathbb{N}^k$.

Proposition 2.1.

(1) *If $\Phi, \Psi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ are c.f.v. functions, then the function $\mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$, $x \mapsto \Phi(x) \cup \Psi(x)$ is c.f.v.*

(2) *If $\Phi, \Psi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ are c.f.v. functions, then the sets $\{x \in \mathbb{N}^k \mid \Phi(x) = \Psi(x)\}$ and $\{x \in \mathbb{N}^k \mid \Phi(x) \subseteq \Psi(x)\}$ are decidable.*

(3) *Let* $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ *and* $\Psi : \mathbb{N}^n \to \mathcal{P}(\mathbb{N}^m)$ *be c.f.v. functions. Let* $\Lambda : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^m)$ *be defined by*
$$\Lambda(x) = \bigcup_{z \in \Phi(x)} \Psi(z),$$
$x \in \mathbb{N}^k$. *Then* Λ *is a c.f.v. function.*

(4) *Let* $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ *be c.f.v. and let* $T \subseteq \mathbb{N}^n$ *be c.e. Then the set* $S = \{x \in \mathbb{N}^k \mid \Phi(x) \subseteq T\}$ *is c.e.* □

2.1. **Computable metric spaces.** A function $F : \mathbb{N}^k \to \mathbb{Q}$ is called **computable** if there exist computable functions $a, b, c : \mathbb{N}^k \to \mathbb{N}$ such that
$$F(x) = (-1)^{c(x)} \frac{a(x)}{b(x)+1}$$
for each $x \in \mathbb{N}^k$. A number $x \in \mathbb{R}$ is said to be **computable** if there exists a computable function $g : \mathbb{N} \to \mathbb{Q}$ such that $|x - g(i)| < 2^{-i}$ for each $i \in \mathbb{N}$ [14].

By a **computable** function $\mathbb{N}^k \to \mathbb{R}$ we mean a function $f : \mathbb{N}^k \to \mathbb{R}$ for which there exists a computable function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that
$$|f(x) - F(x, i)| < 2^{-i}$$
for all $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$.

Proposition 2.2.

(1) *If* $f, g : \mathbb{N}^k \to \mathbb{R}$ *are computable, then* $f + g, f - g : \mathbb{N}^k \to \mathbb{R}$ *are computable.*

(2) *If* $f, g : \mathbb{N}^k \to \mathbb{R}$ *are computable functions, then the set* $\{x \in \mathbb{N}^k \mid f(x) > g(x)\}$ *is c.e.* □

A tuple (X, d, α) is said to be a **computable metric space** if (X, d) is a metric space and $\alpha : \mathbb{N} \to X$ is a sequence dense in (X, d) (i.e. a sequence the range of which is dense in (X, d)) such that the function $\mathbb{N}^2 \to \mathbb{R}$,
$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$
is computable (we use notation $\alpha = (\alpha_i)$).

If (X, d, α) is a computable metric space, then a sequence (x_i) in X is said to be **computable** in (X, d, α) if there exists a computable function $F : \mathbb{N}^2 \to \mathbb{N}$ such that
$$d(x_i, \alpha_{F(i,k)}) < 2^{-k}$$
for all $i, k \in \mathbb{N}$. A point $a \in X$ is said to be **computable** in (X, d, α) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $d(a, \alpha_{f(k)}) < 2^{-k}$ for each $k \in \mathbb{N}$.

The points $\alpha_0, \alpha_1, \dots$ are called **rational points**. If $i \in \mathbb{N}$ and $q \in \mathbb{Q}$, $q > 0$, then we say that $B(\alpha_i, q)$ is an (open) **rational ball**. Here, for $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball of radius r centered at x, i.e. $B(x, r) = \{y \in X \mid d(x, y) < r\}$. By $\widehat{B}(x, r)$ (for $x \in X$ and $r \geq 0$) we will denote the corresponding closed ball $\{y \in X \mid d(x, y) \leq r\}$.

If $B_1, \dots, B_n$, $n \geq 1$, are open rational balls, then the union $B_1 \cup \dots \cup B_n$ will be called a **rational open set**.

Example 2.3. If $\alpha : \mathbb{N} \to \mathbb{R}^n$ is a computable function (in the sense that the component functions of α are computable) whose image is dense in $\mathbb{R}^n$ and d is the Euclidean metric on $\mathbb{R}$, then $(\mathbb{R}^n, d, \alpha)$ is a computable metric space. A sequence (x_i) is computable in this computable metric space if and only if (x_i) is a computable sequence in $\mathbb{R}^n$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$ is a computable point in this space if and only if $x_1, \dots, x_n$ are computable numbers.

2.2. **Effective enumerations.** Let (X, d, α) be a computable metric space. Let $q : \mathbb{N} \to \mathbb{Q}$ be some fixed computable function whose image is $\mathbb{Q} \cap \langle 0, \infty \rangle$ and let $\tau_1, \tau_2 : \mathbb{N} \to \mathbb{N}$ be some fixed computable functions such that $\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbb{N}\} = \mathbb{N}^2$. Let $(\lambda_i)_{i\in\mathbb{N}}$ be the sequence of points in X defined by $\lambda_i = \alpha_{\tau_1(i)}$ and let $(\rho_i)_{i\in\mathbb{N}}$ be the sequence of rational numbers defined by $\rho_i = q_{\tau_2(i)}$. For $i \in \mathbb{N}$ we define

$$I_i = B(\lambda_i, \rho_i), \ \widehat{I_i} = \widehat{B}(\lambda_i, \rho_i).$$

The sequences (I_i) and $(\widehat{I_i})$ represent effective enumerations of all open rational balls and all closed rational balls.

Let $\sigma : \mathbb{N}^2 \to \mathbb{N}$ and $\eta : \mathbb{N} \to \mathbb{N}$ be some fixed computable functions with the following property: $\{(\sigma(j, 0), \dots, \sigma(j, \eta(j))) \mid j \in \mathbb{N}\}$ is the set of all finite sequences in $\mathbb{N}$ (excluding the empty sequence), i.e. the set $\{(a_0, \dots, a_n) \mid n \in \mathbb{N},\ a_0, \dots, a_n \in \mathbb{N}\}$. We use the following notation: $(j)_i$ instead of $\sigma(j, i)$ and $\overline{j}$ instead of $\eta(j)$. Hence

$$\{((j)_0, \dots, (j)_{\overline{j}}) \mid j \in \mathbb{N}\}$$

is the set of all finite sequences in $\mathbb{N}$. For $j \in \mathbb{N}$ let

$$[j] = \{(j)_i \mid 0 \le i \le \overline{j}\}. \tag{2.1}$$

For $j \in \mathbb{N}$ we define

$$J_j = \bigcup_{i\in[j]} I_i.$$

Then (J_j) is an effective enumeration of all rational open sets.

Note that the function $\mathbb{N} \to \mathcal{P}(\mathbb{N})$, $j \mapsto [j]$ is c.f.v. (Proposition 2.1(3)). Also note that any finite nonempty subset of $\mathbb{N}$ equals $[j]$ for some $j \in \mathbb{N}$.

Corollary 2.4. *Let $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N})$ be a c.f.v. function such that $\Phi(x) \neq \emptyset$ for each $x \in \mathbb{N}^k$. Then there exists a computable function $\varphi : \mathbb{N}^k \to \mathbb{N}$ such that $\Phi(x) = [\varphi(x)]$ for each $x \in \mathbb{N}^k$.*

Proof. For each $x \in \mathbb{N}^k$ there exists $j \in \mathbb{N}$ such that $\Phi(x) = [j]$. Since the set of all (x, l), $x \in \mathbb{N}^k$, $l \in \mathbb{N}$, for which $\Phi(x) = [j]$ holds is decidable by Proposition 2.1(2), for each $x \in \mathbb{N}^k$ we can effectively find $j \in \mathbb{N}$ such that $\Phi(x) = [j]$. □

2.3. **Formal properties.** In Euclidean space $\mathbb{R}^n$ we can effectively calculate the diameter of the finite union of rational balls. However, in a general computable metric space the function $\mathbb{N} \to \mathbb{R}$, $j \mapsto \operatorname{diam}(J_j)$, need not be computable. This is the reason that we are going to use the notion of the formal diameter.

Let (X, d) be a metric space and $x_0, \dots, x_k \in X$, $r_0, \dots, r_k \in \mathbb{R}_+$. The **formal diameter** associated to the finite sequence $(x_0, r_0), \dots, (x_k, r_k)$ is the number $D \in \mathbb{R}$ defined by

$$D = \max_{0\le v,w\le k} d(x_v, x_w) + 2 \max_{0\le v\le k} r_v.$$

It follows from this definition that $\operatorname{diam}(B(x_0, r_0) \cup \cdots \cup B(x_k, r_k)) \leq D$.

Let (X, d, α) be a computable metric space. We define the function $\operatorname{fdiam} : \mathbb{N} \to \mathbb{R}$ in the following way. For $j \in \mathbb{N}$ the number $\operatorname{fdiam}(j)$ is the formal diameter associated to the finite sequence

$$\left(\lambda_{(j)_0}, \rho_{(j)_0}\right), \dots, \left(\lambda_{(j)_{\overline{j}}}, \rho_{(j)_{\overline{j}}}\right).$$

Clearly $\operatorname{diam}(J_j) \leq \operatorname{fdiam}(j)$ for each $j \in \mathbb{N}$.

Let $i, j \in \mathbb{N}$. We say that I_i and I_j are **formally disjoint** if

$$d(\lambda_i, \lambda_j) > \rho_i + \rho_j.$$

Note that we define this as a relation between the numbers i and j, not the sets I_i and I_j.

Let $i, j \in \mathbb{N}$. We say that J_i and J_j are **formally disjoint** if I_k and I_l are formally disjoint for all $k \in [i]$ and $l \in [j]$. Clearly, if J_i and J_j are formally disjoint, then $J_i \cap J_j = \emptyset$.

We will also say that I_i and J_j are **formally disjoint** if I_i and I_l are formally disjoint for each $l \in [j]$. Note that formal disjointness of I_i and J_j implies $\widehat{I_i} \cap J_j = \emptyset$.

Let $i, m \in \mathbb{N}$ and $a \in X$. We say that I_i is **formally contained** in $B(a, m)$ and write $I_i \subseteq_F B(a, m)$ if $d(\lambda_i, a) + \rho_i < m$ (again, this as a relation between i, a and m, not between I_i and $B(a, m)$). Clearly, if $I_i \subseteq_F B(a, m)$, then $I_i \subseteq B(a, m)$. For $j \in \mathbb{N}$ we write

$$J_j \subseteq_F B(a, m)$$

if $I_i \subseteq_F B(a, m)$ for each $i \in [j]$. If $J_j \subseteq_F B(a, m)$, then $J_j \subseteq B(a, m)$.

In the same way we define that I_i is formally contained in I_m ($I_i \subseteq_F I_m$) and that J_j is formally contained in I_m ($J_j \subseteq_F I_m$).

Proposition 2.5. (1) *The function* $\operatorname{fdiam} : \mathbb{N} \to \mathbb{R}$ *is computable.*
(2) *The sets* $\{(i, j) \in \mathbb{N}^2 \mid I_i$ *and* J_j *are formally disjoint*$\}$ *and* $\{(i, j) \in \mathbb{N}^2 \mid J_i$ *and* J_j *are formally disjoint*$\}$ *are c.e.*
(3) *If* a *is a computable point, then the set* $\{(j, m) \mid J_j \subseteq_F B(a, m)\}$ *is c.e.*
(4) *The set* $\{(j, m) \in \mathbb{N}^2 \mid J_j \subseteq_F I_m\}$ *is c.e.* □

Proof. For (1) and (2) see [7, Proposition 2.4]. Let us prove (3). Let

$$\Omega = \{(j, m) \mid J_j \subseteq_F B(a, m)\} \text{ and } \Gamma = \{(i, m) \mid I_i \subseteq_F B(a, m)\}.$$

Let $\Phi : \mathbb{N}^2 \to \mathcal{P}(\mathbb{N}^2)$ be defined by $\Phi(j, m) = [j] \times \{m\}$. Then

$$(j, m) \in \Omega \Leftrightarrow \Phi(j, m) \subseteq \Gamma.$$

We have that Φ is c.f.v. So if we prove that Γ is c.e., we will have that Ω is c.e. (Proposition 2.1). However, the fact that Γ is c.e. follows from Proposition 2.2 since

$$(i, m) \in \Gamma \Leftrightarrow d(\lambda_i, a) + \rho_i < m.$$

In the same way we get (4). □

The following simple lemma will be very useful to us later.

Lemma 2.6. *Let* $m \in \mathbb{N}$ *and let* $x \in I_m$. *Then there exists* $\varepsilon > 0$ *with the following property: if* $j \in \mathbb{N}$ *is such that* $x \in J_j$ *and* $\operatorname{fdiam}(j) < \varepsilon$, *then* $J_j \subseteq_F I_m$.

Proof. We have $d(\lambda_m, x) < \rho_m$ and therefore there exists $r > 0$ such that

$$d(\lambda_m, x) + r < \rho_m.$$

Let $\varepsilon = \frac{r}{2}$. Suppose $j \in \mathbb{N}$ is such that $x \in J_j$ and $\text{fdiam}(j) < \varepsilon$. Let $i \in [j]$. Then $\rho_i < \text{fdiam}(j) < \varepsilon$ and $d(x, \lambda_i) \leq \text{diam}(J_j) \leq \text{fdiam}(j) < \varepsilon$. We have

$$d(\lambda_m, \lambda_i) + \rho_i \leq d(\lambda_m, x) + d(x, \lambda_i) + \rho_i < d(\lambda_m, x) + 2\varepsilon = d(\lambda_m, x) + r < \rho_m.$$

So $d(\lambda_m, \lambda_i) + \rho_i < \rho_m$ and $I_i \subseteq_F I_m$. Hence $J_j \subseteq_F I_m$. □

2.4. **Computable sets.** Let (X, d, α) be a computable metric space. A closed subset S of (X, d) is said to be **computably enumerable** in (X, d, α) if

$$\{i \in \mathbb{N} \mid S \cap I_i \neq \emptyset\}$$

is a c.e. subset of $\mathbb{N}$. A closed subset S of (X, d) is said to be **co-computably enumerable** in (X, d, α) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$X \setminus S = \bigcup_{i \in \mathbb{N}} I_{f(i)}.$$

We say that S is a **computable set** in (X, d, α) if S is a computably enumerable and a co-computably enumerable set ([2, 15]).

Let (X, d, α) be a computable metric space. We say that K is a **semi-computable compact set** in (X, d, α) if K is a compact set in (X, d) and if the set $\{j \in \mathbb{N} \mid K \subseteq J_j\}$ is c.e. We say that K is a **computable compact set** if K is a semi-computable compact set and K is computably enumerable.

3. Ambient space and c.c.b. sets

A computable metric space (X, d, α) has the **effective covering property** if the set

$$\{(i, j) \in \mathbb{N}^2 \mid \widehat{I}_i \subseteq J_j\}$$

is computably enumerable ([2]). Euclidean space $\mathbb{R}^n$ (example 2.3) has the effective covering property (see e.g. [4]).

A computable metric space which has the effective covering property and in which each closed ball is compact has a property which turns out to be important if we want to get that some set is computable. The property is this: if S is compact and co-c.e., then we can effectively enumerate all rational open sets which cover S. In other words, if a compact set is co-c.e., then it is semi-computable compact.

We have mentioned the result from [7] regarding the computability of co-c.e. compact manifolds. In [7] the following is proved:

Fact 3.1. If a computable metric space has the effective covering property and compact closed balls, then each co-c.e. compact manifold in this space with computable boundary is computable.

However, this result is just a consequence of the following result which is also proved in [7]:

Fact 3.2. In any computable metric space any compact manifold which is semi-computable compact and whose boundary is computable compact is computable compact.

Note that in Fact 3.2 we have the stronger assumptions (and the stronger conclusion) on the sets, but there are no assumptions on the ambient space. Since the notions of a co-c.e. set and a semi-computable compact set coincide for compact sets in computable metric space with the effective covering property and compact closed balls, the Fact 3.2 is clearly a generalization of Fact 3.1.

In this paper we examine 1-manifolds, the sets which are not compact in general. We will have the result that if a 1-manifold with finitely many components is co-c.e. and its boundary is computable, then this manifold is computable. However, we will need for this result the assumption that the ambient space has the effective covering property and compact closed balls. We would like to find some analogue of the notion of a semi-computable set for noncompact sets so that, in the same manner as in the case of compact manifolds, we can remove the assumptions on the computable metric space. Of course, we want that the new result which holds in general computable metric spaces be the generalization of the previous result for co-c.e. sets in the computable metric spaces with effective covering property and compact closed balls. And this will be true if this analogue of semi-computability coincides with the the notion of a co-c.e. set in these special computable metric spaces.

That a set S is semi-computable compact means that we can effectively enumerate all rational open sets which cover S. The idea for a generalization of this notion is to observe a set S which may not be compact, but such that the intersection $S \cap B$ is compact for each closed ball B in the ambient space and furthermore such that we can effectively (and uniformly) enumerate all rational open sets which cover $S \cap B$ for each closed ball B.

Let (X, d, α) be a computable metric space. Let $S \subseteq X$. We say that S is **c.c.b.** (or **computable compact on closed balls**) if the following holds:

(1) $S \cap \widehat{B}(x, r)$ is a compact set for all $x \in X$ and $r > 0$;
(2) the set $\{(i, j) \in \mathbb{N}^2 \mid \widehat{I}_i \cap S \subseteq J_j\}$ is c.e.;
(3) S is computably enumerable.

If S is a set which satisfies conditions (1) and (2), then we will say that S is **semi-c.c.b**. Hence S is c.c.b. if and only if S is semi-c.c.b. and computably enumerable. Note that semi-c.c.b. sets (and c.c.b. sets) are closed (this follows from (1)).

Let (X, d, α) be a computable metric space. Then X is semi-c.c.b. in (X, d, α) if and only if (X, d, α) has compact closed balls and the effective covering property. For example, $\mathbb{R}^n$ is semi-c.c.b. (and also c.c.b.) in the computable metric space from Example 2.3. Hence semi-c.c.b. sets (and also c.c.b. sets) need not be compact.

On the other hand, we now show that each semi-computable compact set is semi-c.c.b. In other words, the notion of a semi-c.c.b. set generalizes the notion of a semi-computable compact set.

Proposition 3.3. *Let (X, d, α) be a computable metric space. Let S be a semi-computable compact set in this space. Then S is semi-c.c.b.*

Proof. We have to show that the set $\{(i, j) \in \mathbb{N}^2 \mid \widehat{I}_i \cap S \subseteq J_j\}$ is c.e.

Suppose $i, j \in \mathbb{N}$ are such that $\widehat{I}_i \cap S \subseteq J_j$. Let $x \in S \setminus J_j$. Then $x \notin \widehat{I}_i$ and therefore there exists some $k_x \in \mathbb{N}$ such that $x \in I_{k_x}$ and such that I_i and I_{k_x} are formally disjoint. The set $S \setminus J_j$ is closed, hence compact (since S is compact) and this implies that there exist $n \in \mathbb{N}$ and $x_0, \dots, x_n \in S \setminus J_j$ such that $S \setminus J_j \subseteq I_{k_{x_0}} \cup \dots \cup I_{k_{x_n}}$. It follows

$$S \subseteq J_j \cup I_{k_{x_0}} \cup \dots \cup I_{k_{x_n}}.$$

Therefore, there exists $l \in \mathbb{N}$ such that

$$S \subseteq J_j \cup J_l \text{ and } I_i \text{ and } J_l \text{ are formally disjoint.} \tag{3.1}$$

On the other hand, suppose that (3.1) holds for some $i, j, l \in \mathbb{N}$. Then $\widehat{I}_i \cap J_l = \emptyset$ and therefore $\widehat{I}_i \cap S \subseteq J_j$. Hence we have the following conclusion: $\widehat{I}_i \cap S \subseteq J_j$ if and only if there exists $l \in \mathbb{N}$ such that (3.1) holds.

The function $\mathbb{N}^2 \to \mathcal{P}(\mathbb{N})$, $(j,l) \mapsto [j] \cup [l]$ is c.f.v., therefore by Corollary 2.4 there exists a computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $[j] \cup [l] = [\varphi(j,l)]$ for all $j, l \in \mathbb{N}$. Hence $J_j \cup J_l = J_{\varphi(j,l)}$ for all $j, l \in \mathbb{N}$ and using the fact that S is semi-computable we conclude that the set $\{(j,l) \in \mathbb{N}^2 \mid S \subseteq J_j \cup J_l\}$ is c.e. This implies that the set of all $(i,j,l) \in \mathbb{N}^3$ such that (3.1) holds is c.e. (Proposition 2.5) and we conclude that the set $\{(i,j) \in \mathbb{N}^2 \mid \widehat{I}_i \cap S \subseteq J_j\}$ is c.e. □

Note that semi-computable compact sets are exactly those semi-c.c.b. sets which are compact. (If S is a compact set, then $S \subseteq \widehat{I}_{i_0}$ for some $i_0 \in \mathbb{N}$, so if S is semi-c.c.b, the set $\{j \in \mathbb{N} \mid \widehat{I}_{i_0} \cap S \subseteq J_j\}$ is c.e. This set clearly equals $\{j \in \mathbb{N} \mid S \subseteq J_j\}$.)

Now we show that semi-c.c.b. sets are co-c.e. First we have the following property of semi-c.c.b. sets.

Proposition 3.4. *Let (X, d, α) be a computable metric space. Let S be a semi-c.c.b. set. Then the set*

$$\Omega = \{i \in \mathbb{N} \mid \widehat{I}_i \cap S = \emptyset\}$$

is c.e.

Proof. We may assume that $S \neq \emptyset$. Let $\Gamma = \{(i,j) \in \mathbb{N}^2 \mid \widehat{I}_i \cap S \subseteq J_j\}$.

Suppose $i \in \Omega$. Then $\widehat{I}_i \cap S = \emptyset$ which implies $\widehat{I}_i \neq X$ and therefore there exists $j \in \mathbb{N}$ such that I_i and J_j are formally disjoint. Clearly $\widehat{I}_i \cap S \subseteq J_j$, hence $(i,j) \in \Gamma$.

Conversely, let us take $j \in \mathbb{N}$ such that I_i and J_j are formally disjoint and $(i,j) \in \Gamma$. Then $\widehat{I}_i \cap J_j = \emptyset$. But we have $\widehat{I}_i \cap S \subseteq J_j$ and this can only be true if $\widehat{I}_i \cap S = \emptyset$. Hence $i \in \Omega$.

We have the following conclusion:

$$i \in \Omega \iff \text{there exists } j \in \mathbb{N} \text{ such that } (i,j) \in \Gamma \text{ and } I_i \text{ and } J_j \text{ are formally disjoint.}$$

The fact that Γ is c.e. and Proposition 2.5 imply that Ω is c.e. □

Proposition 3.5. *Let (X, d, α) be a computable metric space. Let S be a semi-c.c.b. set. Then S is co-c.e.*

Proof. Let $x \in X \setminus S$. Since S is closed, we have $B(x,r) \subseteq X \setminus S$ for some $r > 0$. Take a rational point a and a positive rational number λ so that $\lambda < \frac{r}{2}$ and $x \in B(a,\lambda)$. Then $\widehat{B}(a,\lambda) \cap S = \emptyset$. The conclusion is this: for each point $x \in X \setminus S$ there exists $i \in \mathbb{N}$ such that $x \in I_i$ and $\widehat{I}_i \cap S = \emptyset$.

Let $\Omega = \{i \in \mathbb{N} \mid \widehat{I}_i \cap S = \emptyset\}$. It follows from the previous fact that

$$X \setminus S = \bigcup_{i \in \Omega} I_i.$$

However Ω is c.e. by Proposition 3.4 and this means that S is co-c.e. □

In general, a co-c.e. set need not be semi-c.c.b, even if it is compact. Moreover, even a singleton set need not be semi-computable compact if it is co-c.e. To see this, note first the following: if (X, d, α) is a computable metric space and $x \in X$ such that $\{x\}$ is semi-computable compact, then x is a computable point. Namely, for each $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that

$$\{x\} \subseteq J_j \text{ and } \operatorname{fdiam}(j) < 2^{-k}. \tag{3.2}$$

Since the set of all $(k, l) \in \mathbb{N}^2$ for which (3.2) holds is c.e., there exists a computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that (3.2) holds for each $k \in \mathbb{N}$ and $l = \varphi(k)$. Recall $J_j = I_{(j)_0} \cup \cdots \cup I_{(j)_{\bar{j}}}$ and $I_{(j)_0} = B(\lambda_{(j)_0}, \rho_{(j)_0})$. So we have $d(x, \lambda_{(j)_0}) < 2^{-k}$ for each $k \in \mathbb{N}$ and x is computable point.

By Example 3.2. in [5] there exists a computable metric space (X, d, α) and a point $x \in X$ such that $\{x\}$ is co-c.e., but x is not a computable point. Therefore $\{x\}$ is not a semi-computable compact set.

We have mentioned that in a computable metric space which has the effective covering property and compact closed balls a set is semi-computable compact if and only if it is compact and co-c.e. Now we prove a more general result.

Proposition 3.6. *Let (X, d, α) be a computable metric space. Suppose (X, d, α) has the effective covering property and compact closed balls. Let $S \subseteq X$. Then S is co-c.e. if and only if S is semi-c.c.b.*

Proof. We have to prove that if S is co-c.e., then S is semi-c.c.b. Suppose S is co-c.e. It is easy to conclude that then there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $J_{f(k)} \subseteq J_{f(k+1)}$ for each $k \in \mathbb{N}$ and

$$X \setminus S = \bigcup_{k \in \mathbb{N}} J_{f(k)}.$$

Let $i, j \in \mathbb{N}$ and suppose that $\widehat{I}_i \cap S \subseteq J_j$. It follows that the set $\widehat{I}_i \setminus J_j$ is contained in $X \setminus S$. The set $\widehat{I}_i \setminus J_j$ is compact and therefore there exists $k \in \mathbb{N}$ such that $\widehat{I}_i \setminus J_j \subseteq J_{f(k)}$ and consequently

$$\widehat{I}_i \subseteq J_j \cup J_{f(k)}. \tag{3.3}$$

On the other hand, if (3.3) holds for some $i, j, k \in \mathbb{N}$, then $\widehat{I}_i \cap S \subseteq J_j$ (since $S \cap J_{f(k)} = \emptyset$). Hence $\widehat{I}_i \cap S \subseteq J_j$ if and only if there exists $k \in \mathbb{N}$ such that (3.3) holds. The set of all $(i, j, k) \in \mathbb{N}^3$ such that (3.3) holds is c.e. (we can find a computable function $\varphi : \mathbb{N}^2 \to \mathbb{N}$ such that $J_j \cup J_{f(k)} = J_{\varphi(i,k)}$ for all $i, k \in \mathbb{N}$ as in the proof of Proposition 3.3 and (X, d, α) has the effective covering property). Therefore the set of all $(i, j) \in \mathbb{N}^2$ such that $\widehat{I}_i \cap S \subseteq J_j$ is c.e., which means that S is semi-c.c.b. □

An immediate consequence of the previous proposition is the fact that in a computable metric space which has the effective covering property and compact closed balls a set S is computable (closed) if and only if S is c.c.b.

4. Chains

If S is a semi-c.c.b. set in a computable metric space and if a is a rational point, then for a given $n \in \mathbb{N}$ we can effectively enumerate all rational open sets which contain $S \cap \widehat{B}(a, n)$. In general, the problem is that we do not know, for a given rational open set $U = B_1 \cup \dots \cup B_m$ which contains $S \cap \widehat{B}(a,n)$, which of these rational balls $B_1, \dots, B_m$ intersects S.

If we can somehow, for given $n \in \mathbb{N}$ and $\varepsilon > 0$, effectively find a rational open set $U = B_1 \cup \dots \cup B_m$ which contains $S \cap \widehat{B}(a,n)$, such that each of the rational balls $B_1, \dots, B_m$ has the diameter less then ε and such that each of these balls intersects S, then we have that S is computable. Namely, we only have to prove that S is computably enumerable (since S is semi-c.c.b. by assumption). And if $i \in \mathbb{N}$, then it is not hard to see that I_i intersects S if and only if I_i (formally) contains some of the balls $B_1, \dots, B_m$ for some $n \in \mathbb{N}$ and $\varepsilon > 0$.

In order to effectively get, for given $n \in \mathbb{N}$ and $\varepsilon > 0$, such a rational open set U, we will use the notion of a chain.

Let X be a metric space. A finite sequence $C_0, \dots, C_m$ of nonempty open subsets of X is said to be a **chain** in X if $C_i \cap C_j = \emptyset$ for all $i, j \in \{0, \dots, m\}$ such that $|i - j| > 1$ (see [3, 10]). We say that C_i $(0 \le i \le m)$ is a **link** of the chain $C_0, \dots, C_m$. If $\mathcal{A} = (A_0, \dots, A_m)$ is a finite sequence of nonempty bounded subsets of X, we define

$$\operatorname{mesh}(\mathcal{A}) = \max_{0 \le i \le m} \operatorname{diam}(A_i).$$

If ε is a positive real number and $\mathcal{C}$ is a chain, we say that $\mathcal{C}$ is an $\varepsilon-$chain if $\operatorname{mesh}(\mathcal{C}) < \varepsilon$.

Let (X, d, α) be a computable metric space. For $l \in \mathbb{N}$ let $\mathcal{H}_l$ be the finite sequence of sets $J_{(l)_0}, \dots, J_{(l)_{\bar{l}}}$. Furthermore, for $j, p, q \in \mathbb{N}$ let $\mathcal{H}_l^{p \le q}$ be the finite sequence of sets $J_{(l)_p}, \dots, J_{(l)_q}$ if $p \le q$, otherwise let $\mathcal{H}_l^{p \le q}$ denote the empty sequence. Clearly $\mathcal{H}_l = \mathcal{H}_l^{0 \le \bar{l}}$.

Let the function fmesh : $\mathbb{N} \to \mathbb{R}$ be defined by

$$\operatorname{fmesh}(l) = \max_{0 \le j \le \bar{l}} \operatorname{fdiam}((l)_j),$$

$l \in \mathbb{N}$.

Let $l \in \mathbb{N}$. We say that $\mathcal{H}_l$ is a **formal chain** if $J_{(l)_i}$ and $J_{(l)_j}$ are formally disjoint for all $i, j \in \{0, \dots, \bar{l}\}$ such that $|i - j| > 1$.

Let $a \in X$ and $l, p, q, m \in \mathbb{N}$. We say that $\mathcal{H}_l^{p \le q}$ is **formally contained** in $B(a, m)$ if $J_{(l)_i} \subseteq_F B(a, m)$ for each $i \in \mathbb{N}$ such that $p \le i \le q$.

Proposition 4.1. *Let (X, d, α) be a computable metric space.*

(1) *The function* fmesh : $\mathbb{N} \to \mathbb{R}$ *is computable.*
(2) *The set $\{l \in \mathbb{N} \mid \mathcal{H}_l$ is a formal chain$\}$ is c.e.*
(3) *If a is a computable point, then the set*

$$\Gamma = \{(l, p, q, m) \mid \mathcal{H}_l^{p \le q} \text{ formally contained in } B(a, m)\}$$

is c.e. □

Proof. For (i) and (ii) see Proposition 5.4. in [7].

For the proof of (iii), let $\Omega = \{(j, m) \mid J_j \subseteq_F B(a, m)\}$. By Proposition 2.5 Ω is c.e. Let $\Phi : \mathbb{N}^4 \to \mathcal{P}(\mathbb{N}^2)$ be defined by

$$\Phi(l, p, q, m) = \{((l)_i, m) \mid p \le i \le q\}.$$

Then Φ is c.f.v. (Proposition 2.1(3)) and $(l, p, q, m) \in \Gamma$ if and only if $\Phi(l, p, q, m) \subseteq \Omega$. Now Γ is c.e. by Proposition 2.1(4). □

Let $j, l \in \mathbb{N}$, We say that J_j and $\mathcal{H}_l$ are **formally disjoint** if J_j and J_i are formally disjoint for each $i \in [l]$. The following Lemma is an easy consequence of Proposition 2.5(ii).

Lemma 4.2. *Let (X, d, α) be a computable metric space. Then the set of all $(j, l) \in \mathbb{N}^2$ such that J_j and $\mathcal{H}_l$ are formally disjoint is c.e.* □

In the similar way we define that I_i is formally disjoint with $\mathcal{H}_l^{p \leq q}$ and the statement similar to Lemma 4.2 also holds.

If $\mathcal{A} = (A_0, \dots, A_m)$ is a finite sequence of sets, then by $\bigcup \mathcal{A}$ we denote the union $A_0 \cup \dots \cup A_m$. If $\mathcal{A}$ is the empty sequence, we take $\bigcup \mathcal{A} = \emptyset$. Let S be a set. We say that $\mathcal{A}$ **covers** S if $S \subseteq \bigcup \mathcal{A}$.

Lemma 4.3. *Let (X, d, α) be a computable metric space. Then there exists a computable function $\zeta : \mathbb{N}^3 \to \mathbb{N}$ such that*

$$\bigcup \mathcal{H}_l^{p \leq q} = J_{\zeta(l,p,q)}$$

for all $l, p, q \in \mathbb{N}$ such that $p \leq q$.

Proof. Let $l, p, q \in \mathbb{N}$ be such that $p \leq q$. We have

$$\bigcup \mathcal{H}_l^{p \leq q} = \bigcup_{j=p}^{q} J_{(l)_j} = \bigcup_{j=p}^{q} \left(I_{((l)_j)_0} \cup \dots \cup I_{((l)_j)_{\overline{(l)_j}}} \right).$$

Let $\Lambda : \mathbb{N}^3 \to \mathcal{P}(\mathbb{N})$ be defined by

$$\Lambda(l, p, q) = \bigcup_{j=p \text{ or } p \leq j \leq q} \left\{ ((l)_j)_0, \dots, ((l)_j)_{\overline{(l)_j}} \right\}.$$

Then Λ is c.f.v. by Proposition 2.1(3). Clearly, we have

$$\bigcup \mathcal{H}_l^{p \leq q} = \bigcup_{i \in \Lambda(l,p,q)} I_i. \tag{4.1}$$

for all $l, p, q \in \mathbb{N}$ such that $p \leq q$. Since $\Lambda(l, p, q) \neq \emptyset$ for all $l, p, q \in \mathbb{N}$ (condition $j = p$ in the definition of Λ ensures this), there exists a computable function $\zeta : \mathbb{N} \to \mathbb{N}$ such that $\Lambda(l, p, q) = [\zeta(l, p, q)]$ for all $l, p, q \in \mathbb{N}$. This means that

$$\bigcup_{i \in \Lambda(l,p,q)} I_i = J_{\zeta(l,p,q)} \tag{4.2}$$

for all $l, p, q \in \mathbb{N}$. Comparing (4.1) and (4.2) we see that ζ is the desired function. □

Proposition 4.4. *Let (X, d, α) be a computable metric space. Suppose S is a semi-c.c.b. set.*

(1) *The set*

$$\Gamma = \left\{ (i, l, p, q) \in \mathbb{N}^4 \mid \mathcal{H}_l^{p \leq q} \text{ covers } S \cap \widehat{I}_i \right\}$$

is c.e.

(2) *Let a be a rational point. The sets*

$$\Omega = \left\{ (n, l, p, q) \in \mathbb{N}^4 \mid \mathcal{H}_l^{p \leq q} \text{ covers } S \cap \widehat{B}(a, n) \right\}$$

$$\Omega' = \left\{ (n, l, p, q, u) \in \mathbb{N}^5 \mid S \cap \widehat{B}(a, n) \subseteq \bigcup \mathcal{H}_l^{p \leq q} \cup J_u \right\}$$

are c.e.

Proof. Let ζ be the function from Lemma 4.3.

(i) For all $i, l, p, q \in \mathbb{N}$ we have

$$(i, l, p, q) \in \Gamma \Leftrightarrow (S \cap \widehat{I}_i \subseteq J_{\zeta(l,p,q)} \text{ and } p \leq q) \text{ or } (p > q \text{ and } S \cap \widehat{I}_i = \emptyset).$$

That Γ is c.e. as the union of two c.e. sets follows now from Proposition 3.4 and the fact that S is semi-c.c.b.

(ii) Let us first notice that the set $\{j \in \mathbb{N} \mid a \in J_j\}$ is c.e. This follows from the fact that

$$a \in J_j \Leftrightarrow \exists i \in \mathbb{N} \text{ such that } a \in I_i \text{ and } i \in [j]$$

and $\{i \in \mathbb{N} \mid a \in I_i\}$ is c.e. since

$$a \in I_i \Leftrightarrow d(a, \lambda_i) < \rho_i$$

(we use here Proposition 2.2). It follows easily now that the set $\{(l, p, q) \in \mathbb{N}^3 \mid a \in \bigcup \mathcal{H}_l^{p \leq q}\}$ is c.e.

Since a is a rational point, there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\widehat{B}(a, n) = \widehat{I}_{f(n)}$ for each $n \in \mathbb{N}$ such that $n \geq 1$. Note that $\widehat{B}(a, 0) = \{a\}$.

Let us observe the case $a \in S$. Then we have

$$(n, l, p, q) \in \Omega \Leftrightarrow (\mathcal{H}_l^{p \leq q} \text{ covers } S \cap \widehat{I}_{f(n)} \text{ and } n \geq 1) \text{ or } (a \in \bigcup \mathcal{H}_l^{p \leq q} \text{ and } n = 0).$$

The fact that Γ is c.e. implies that Ω is c.e.

Let us now observe the case $a \notin S$. Then we have

$$(n, l, p, q) \in \Omega \Leftrightarrow (\mathcal{H}_l^{p \leq q} \text{ covers } S \cap \widehat{I}_{f(n)} \text{ and } n \geq 1) \text{ or } n = 0$$

and it follows that Ω is c.e. In the same way we get that Ω' is c.e. □

Suppose (X, d) is a metric space and S is an arc in this space (a continuous injective image of the segment $[0, 1]$). Then for each $\varepsilon > 0$ there exists an $\varepsilon-$ chain in (X, d) which covers S. We will need an effective version of this fact.

Let (X, d, α) be a computable metric space. Let $A \subseteq X$, $j \in \mathbb{N}$ and $r \in \mathbb{R}$, $r > 0$. We write $\langle A, j, \lambda \rangle$ to denote the following fact:

$$A \subseteq J_j \text{ and } (I_i \cap A \neq \emptyset \text{ and } \rho_i < \lambda \text{ for each } i \in [j]).$$

Note that $\langle A, j, \lambda \rangle$ and $\lambda \leq \lambda'$ implies $\langle A, j, \lambda' \rangle$.

Proposition 4.5. *Let (X, d, α) be a computable metric space and let $f : [0, r] \to X$ be a continuous injection, where $r > 0$. Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$, $n_0 \geq 1$, such that for each $n \geq n_0$ there exist numbers $j_0, \dots, j_{n-1} \in \mathbb{N}$ such that*

(1) $\langle f([\frac{i}{n} r, \frac{i+1}{n} r]), j_i, \varepsilon \rangle$ *for each* $i \in \{0, \dots, n-1\}$*;*
(2) J_{j_i} *and* $J_{j_{i'}}$ *are formally disjoint for all* $i, i' \in \{0, \dots, n-1\}$ *such that* $|i - i'| > 1$*;*
(3) $\mathrm{fdiam}(j_i) < \varepsilon$ *for each* $i \in \{0, \dots, n-1\}$.

Before we prove this proposition, we need some facts.

Lemma 4.6. *Let (X, d, α) be a computable metric space. Let A and B be compact, nonempty and disjoint subsets of X. Then*

(1) *For each $\varepsilon > 0$ there exists $j \in \mathbb{N}$ such that $\langle A, j, \varepsilon \rangle$.*
(2) *For each $\varepsilon > 0$ there exists $\lambda > 0$ such that $\lambda < \varepsilon$ and if $j, j' \in \mathbb{N}$ and $A' \subseteq A$ and $B' \subseteq B$ are such that*

$$\langle A', j, \lambda \rangle \text{ and } \langle B', j', \lambda \rangle,$$

then J_j and $J_{j'}$ are formally disjoint.

Proof. Let $\mathcal{U} = \{B(\alpha_i, r) \mid i \in \mathbb{N},\ r \in \mathbb{Q}^+,\ r < \varepsilon\}$. Then $\mathcal{U}$ is an open cover of (X, d) (since α is a dense sequence in (X, d)). The set A is compact and therefore there exist $U_1, \dots, U_n \in \mathcal{U}$ such that $A \subseteq U_1 \cup \cdots \cup U_n$. We may assume that $U_j \cap A \neq \emptyset$ for each $j \in \{1, \dots, n\}$. Choose $j_1, \dots, j_n \in \mathbb{N}$ so that $U_k = I_{j_k}$ and $\rho_{j_k} < \varepsilon$ for each $k \in \{1, \dots, n\}$. Let $l \in \mathbb{N}$ be such that $[l] = \{j_1, \dots, j_n\}$. Then $\langle A, l, \varepsilon \rangle$.

(ii) Since A and B are compact, nonempty and disjoint, we have $d(A, B) > 0$. Let

$$\lambda = \min\left\{\varepsilon, \frac{d(A, B)}{4}\right\}.$$

Suppose $j, j' \in \mathbb{N}$ and $A' \subseteq A$, $B' \subseteq B$ are such that $\langle A', j, \lambda \rangle$ and $\langle B', j', \lambda \rangle$. Let $i \in [j]$ and $i' \in [j']$. We claim that

$$d(\lambda_i, \lambda_{i'}) > \rho_i + \rho_{i'}. \tag{4.3}$$

Since $\langle A', j, \lambda \rangle$, we have $\rho_i < \lambda$ and $I_i \cap A' \neq \emptyset$. Therefore there exists $a \in A$ such that $d(a, \lambda_i) < \rho_i$, hence $d(a, \lambda_i) < \lambda$. Similarly, $\rho_{i'} < \lambda$ and there exists $b \in B$ such that $d(b, \lambda_{i'}) < \lambda$.

We have

$$\rho_i + \rho_{i'} + 2\lambda < 4\lambda \leq d(A, B) \leq d(a, b) \leq d(a, \lambda_i) + d(\lambda_i, \lambda_{i'}) + d(\lambda_{i'}, b) <$$
$$< \rho_i + \rho_{i'} + d(\lambda_i, \lambda_{i'}) < 2\lambda + d(\lambda_i, \lambda_{i'})$$

Hence $\rho_i + \rho_{i'} + 2\lambda < 2\lambda + d(\lambda_i, \lambda_{i'})$ and (4.3) follows. The conclusion: J_j and $J_{j'}$ are formally disjoint. □

Lemma 4.7. *Let (X, d, α) be a computable metric space and let $A_1, \dots, A_n$ be compact nonempty sets in this space. Let $\varepsilon > 0$. Then there exist $j_1, \dots, j_n \in \mathbb{N}$ such that*

$$\langle A_1, j_1, \varepsilon \rangle,\ \dots,\ \langle A_n, j_n, \varepsilon \rangle \tag{4.4}$$

and such that for all $p, q \in \{1, \dots, n\}$ the following holds:

$$(A_p \cap A_q = \emptyset \implies J_{j_p} \text{ and } J_{j_q} \text{ are formally disjoint}).$$

Proof. Let

$$C = \{(p, q) \in \{1, \dots, n\} \times \{1, \dots, n\} \mid A_p \cap A_q = \emptyset\}.$$

For each $(p, q) \in C$ by Lemma 4.6 there exists $\lambda_{(p,q)} > 0$ such that $\lambda_{(p,q)} < \varepsilon$ and such that $\langle A_p, j, \lambda_{(p,q)} \rangle$ and $\langle A_q, j', \lambda_{(p,q)} \rangle$ implies that J_j and $J_{j'}$ are formally disjoint. Let

$$\lambda = \min\left\{\lambda_{(p,q)} \mid (p, q) \in C\right\}.$$

By Lemma 4.6 there exist $j_1, \dots, j_n \in \mathbb{N}$ such that

$$\langle A_1, j_1, \lambda \rangle,\ \dots,\ \langle A_n, j_n, \lambda \rangle. \tag{4.5}$$

If $p, q \in \{1, \dots, n\}$ are such that $A_p \cap A_q = \emptyset$, then $(p, q) \in C$ and $\lambda \leq \lambda_{(p,q)}$. Therefore $\langle A_p, j_p, \lambda \rangle$ and $\langle A_q, j_q, \lambda \rangle$ implies $\langle A_p, j_p, \lambda_{(p,q)} \rangle$ and $\langle A_q, j_q, \lambda_{(p,q)} \rangle$ and this implies that J_j and $J_{j'}$ are formally disjoint. And (4.4) clearly follows from $\lambda < \varepsilon$ and (4.5). □

Lemma 4.8. *Let (X, d, α) be a computable metric space. Let $A \subseteq X$, $j \in \mathbb{N}$ and $r > 0$ be such that $\langle A, j, r\rangle$. Then* $\mathrm{fdiam}(j) < 4r + \mathrm{diam}\ A$.

Proof. Let $i, i', i'' \in [j]$ be such that

$$\mathrm{fdiam}(j) = d(\lambda_i, \lambda_{i'}) + 2\rho_{i''}. \tag{4.6}$$

Since $B(\lambda_i, \rho_i) \cap A \neq \emptyset$ there exists $a \in A$ such that $d(\lambda_i, a) < \rho_i$. Also, there exists b such that $d(\lambda_{i'}, b) < \rho_{i'}$. Now

$$d(\lambda_i, \lambda_{i'}) \leq d(\lambda_i, a) + d(a, b) + d(b, \lambda_j) < \rho_i + \rho_{i'} + \mathrm{diam}\ A < 2r + \mathrm{diam}\ A.$$

Using $\rho_{i''} < r$ and (4.6) we get $\mathrm{fdiam}(j) < 4r + \mathrm{diam}\ A$. □

Let us now prove Proposition 4.5. Since f is uniformly continuous, there exists $n_0 \in \mathbb{N}$, $n_0 \geq 1$, such that $\mathrm{diam}(f([\frac{i}{n}r, \frac{i+1}{n}r])) < \frac{\varepsilon}{2}$ for all $i \in \{0, \dots, n-1\}$ and $n \geq n_0$.

Fix $n \geq n_0$. Let

$$A_i = f\left(\left[\frac{i}{n}r, \frac{i+1}{n}r\right]\right)$$

for $i \in \{0, \dots, n-1\}$. By Lemma 4.7 there exist $j_0, \dots, j_{n-1} \in \mathbb{N}$ such that $\langle A_i, j_i, \frac{\varepsilon}{8}\rangle$ for each $i \in \{0, \dots, n-1\}$ and such that J_{j_i} and $J_{j_{i'}}$ are formally disjoint for all $i, i' \in \{0, \dots, n-1\}$ such that $|i - i'| > 1$. Finally, for each $i \in \{0, \dots, n-1\}$ we have $\mathrm{diam}\, A_i < \frac{\varepsilon}{2}$ and $\langle A_i, j_i, \frac{\varepsilon}{8}\rangle$ and it follows from Lemma 4.8 that $\mathrm{fdiam}(j_i) < \varepsilon$. □

5. Co-c.e. topological rays

A metric space R is said to be a **topological ray** if R is homeomorphic to $[0, \infty\rangle$. If $f : [0, \infty\rangle \to R$ is a homeomorphism, then we say that $f(0)$ is an **endpoint** of R.

In this section we prove that a semi-c.c.b. set R must be c.c.b. if R is a topological ray with a computable endpoint. Actually, we will prove a more general fact: if R is a topological ray with computable endpoint and if $R \cup F$ is semi-c.c.b, where F is a closed set disjoint with R, then R is computably enumerable.

The first fact that we need here is that for such an R the following holds: if $f : [0, \infty\rangle \to R$ is a homeomorphism, then $f(t)$ "converges to infinity" as t converges to infinity. (In particular, R is unbounded.) The following proposition gives a precise description of this property.

Proposition 5.1. *Let (X, d) be a metric space. Let R be a subset of X such that $R \cap B$ is a compact set for each closed ball B in (X, d) and such that there exists a homeomorphism $f : [0, \infty\rangle \to R$. Then for each closed ball B there exists $t_0 \in [0, \infty\rangle$ such that $f(t) \notin B$ for each $t \geq t_0$.*

Proof. Suppose the opposite. Then there exists a closed ball B such that for each $t_0 \in [0, \infty\rangle$ there exists $t \geq t_0$ such that $f(t) \in B$. Therefore there exists a sequence $(t_n)_{n \in \mathbb{N}}$ in $[0, \infty\rangle$ such that

$$t_n \geq n \text{ and } f(t_n) \in B$$

for each $n \in \mathbb{N}$. Then clearly $f(t_n) \in R \cap B$ for each $n \in \mathbb{N}$ and since $R \cap B$ is compact, there is a subsequence $(t_{n_i})_{i \in \mathbb{N}}$ of (t_n) such that the sequence $(f(t_{n_i}))$ converges to a point in $R \cap B$, hence it converges to a point in R. However, since f is homeomorphism (and $f^{-1} : R \to [0, \infty\rangle$ is continuous), the sequence (t_{n_i}) converges to some point in $[0, \infty\rangle$, which is impossible since this sequence is clearly unbounded ($i \leq n_i \leq t_{n_i}$ for each $i \in \mathbb{N}$). □

Note that the previous proposition does not hold without the assumption that R is compact on closed balls. For example, if (X, d) is the real line with the Euclidean metric and $R = [0, 1\rangle$, then R is homeomorphic to $[0, \infty\rangle$, but R is clearly bounded.

Suppose R is a semi-c.c.b. topological ray with computable endpoint in some computable metric space. How to prove that R is c.c.b., i.e. how to prove that R is c.e.? Let a be some fixed rational point which is close to the endpoint of R. We want, for given $n, k \in \mathbb{N}$, to effectively find finitely many rational open sets $C_0, \dots, C_l$ whose diameters are less then 2^{-k} and such that these sets cover $R \cap \widehat{B}(a, n)$ and each of these sets intersects R. If we can do this, the fact that R is c.e. will easily follow. Informally, we can imagine that the image of that part of R which lies in $\widehat{B}(a, n)$ becomes sharper and sharper as k tends to infinity.

So how to get such sets $C_0, \dots, C_l$? Let $f : [0, \infty\rangle \to R$ be a homeomorphism. By Proposition 5.1 there exists $t_0 > 0$ such that $f(t)$ leaves $\widehat{B}(a, n)$ after $t = t_0$. (See Figure 1. The blue curve is $f([0, t_0])$. The black circle is the boundary of $\widehat{B}(a, n)$.)

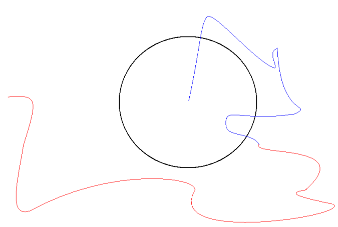

Figure 1.

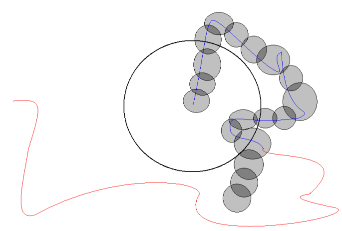

Figure 2.

Therefore, $R \cap \widehat{B}(a, n)$ is contained in $f([0, t_0])$ and this implies that there exists a rational $2^{-k}-$chain $C_0, \dots, C_l$ which covers $R \cap \widehat{B}(a, n)$ and such that $f(0) \in C_0$. (Figure 2.) These conditions are semi-decidable by results from Section 4 and therefore we can effectively find such a sequence of sets. However, we do not have the condition that each of these sets intersects R and the question is does this follow from the conditions that we have? The answer is no, as Figure 2 shows (the bottom three links do not intersect R). So the question is what additional conditions to require on the chain $C_0, \dots, C_l$ so that these conditions are semi-decidable and so that they imply that each of the sets $C_0, \dots, C_l$ intersects R?

The idea is to proceed in the following way. Since $f([0, t_0])$ is compact, there exists $m \in \mathbb{N}$ such that $f([0, t_0]) \subseteq \widehat{B}(a, m)$. (See Figure 3. The green circle is the boundary of $\widehat{B}(a, m)$.) Now we can cover $R \cap \widehat{B}(a, m)$ (in the same way as we covered $R \cap \widehat{B}(a, n)$) by a $2^{-k}-$chain $C_0, \dots, C_p$ such that $f(0) \in C_0$ (Figure 4). Again, some of the sets $C_0, \dots, C_l$ may not intersect R (the last three in Figure 4). However, it will be possible to conclude that for some $p \in \{0, \dots, l-1\}$ the links $C_0, \dots, C_p$ cover $\widehat{B}(a, n)$ and they are all formally contained in $B(a, m)$ (these conditions are semi-decidable). This altogether will imply that each of the links $C_0, \dots, C_p$ intersects R. (In Figure 4 $C_0, \dots, C_m$ are the links between blue links, including blue links.)

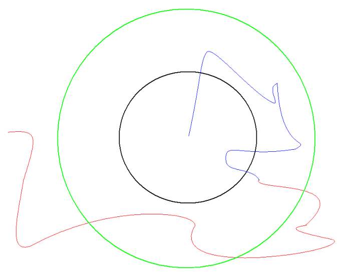

Figure 3.

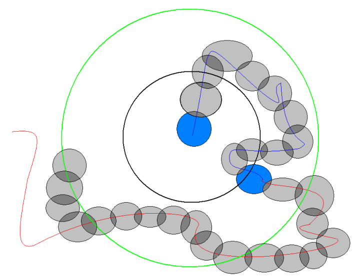

Figure 4.

The described procedure is applied in the proof of the following theorem.

Theorem 5.2. *Let (X, d, α) be a computable metric space. Let R be a subset of X which is, as a subspace of (X, d), a topological ray whose endpoint is computable. Suppose F is a closed set in (X, d) which is disjoint with R and such that $R \cup F$ is semi-c.c.b. Then R is a computably enumerable (closed) set.*

Proof. Let $f : [0, \infty\rangle \to R$ be a homeomorphism. Let a be some rational point such that $d(a, f(0)) < 1$. For each $n \in \mathbb{N}$ let

$$R_n = R \cap \widehat{B}(a, n) \text{ and } F_n = F \cap \widehat{B}(a, n).$$

Let $n, k \in \mathbb{N}$. By Proposition 5.1 there exists $r > 0$ such that $f(x) \notin \widehat{B}(a, n)$ for each $x \geq r$. The set $f([0, r])$ is compact and therefore there exists $m \in \mathbb{N}$, $m \geq 1$ such that $f([0, r]) \subseteq B(a, m)$. Again, by Proposition 5.1 there exists $r' > r$ such that $f(x) \notin \widehat{B}(a, m)$, for each $x \geq r'$.

Note that

$$R_n \subseteq f([0, r]) \subseteq R_m \subseteq f([0, r']). \tag{5.1}$$

Let

$$D = \max \{d(a, f(x)) \mid x \in [0, r]\}.$$

Since $f([0, r]) \subseteq B(a, m)$, we have $D < m$. Let

$$\mu = \frac{m - D}{2}.$$

By Lemma 4.6 there exists $\lambda > 0$ such that if $j, j' \in \mathbb{N}$ and $A \subseteq f([0, r'])$, then

$$(\langle F_m, j, \lambda\rangle \text{ and } \langle A, j', \lambda\rangle) \Longrightarrow J_j \text{ and } J_{j'} \text{ are formally disjoint.} \tag{5.2}$$

Let $\varepsilon = \min\{\mu, \lambda\}$. Let $u \in \mathbb{N}$ be such that $\langle F_m, u, \lambda\rangle$.

Using Proposition 4.5 we get $n' \geq 1$ and $j_0, \dots, j_{n'-1} \in \mathbb{N}$ so that

(1) $\langle f([\frac{i}{n'} r', \frac{i+1}{n'} r']), j_i, \varepsilon\rangle$ for each $i \in \{0, \dots, n' - 1\}$;
(2) J_{j_i} and $J_{j_{i'}}$ are formally disjoint for all $i, i' \in \{0, \dots, n' - 1\}$ such that $|i - i'| > 1$;
(3) $\operatorname{fdiam}(j_i) < \min\{2^{-k}, \varepsilon\}$ for each $i \in \{0, \dots, n' - 1\}$.

It follows from (5.2) that J_u and J_{j_i} are formally disjoint for each $i \in \{0, \dots, n' - 1\}$.

Let $\ell \in \mathbb{N}$ be such that

$$((\ell)_0, \dots, (\ell)_{\overline{\ell}}) = (j_1, \dots, j_{(n'-1)}).$$

Then $\mathcal{H}_\ell$ is a formal chain which covers $f([0, r'])$, $\operatorname{fmesh}(\ell) < 2^{-k}$ and J_u and $\mathcal{H}_\ell$ are formally disjoint. Note that $f(0) \in J_{(\ell)_0}$.

Since $R_m \subseteq f([0, r'])$, we have $R_m \subseteq \cup \mathcal{H}_\ell$. Furthermore, since $r \in [0, r']$, there exists $p \in \{0, \dots, \overline{\ell}\}$ such that $f(r) \in J_{(\ell)_p}$. The property (1) above ensures that $f([0, r]) \subseteq \cup \mathcal{H}_\ell^{0 \leq p}$ and each link of $\mathcal{H}_\ell^{0 \leq p}$ intersects $f([0, r])$. It follows $R_n \subseteq \bigcup \mathcal{H}_\ell^{0 \leq p}$.

We claim that $\mathcal{H}_\ell^{0 \leq p}$ is formally contained in $B(a, m)$. To see this, let us take $i \in \{0, \dots, p\}$. We want to prove that $J_{(\ell)_i}$ is formally contained in $B(a, m)$.

It would be enough to prove that $I_{k'}$ is formally contained in $B(a, m)$ for each $k' \in [(\ell)_i]$. So let $k' \in [(\ell)_i]$. Since $J_{(\ell)_i}$ intersects $f([0, r])$, there exists $b \in J_{(\ell)_i}$ such that $b \in f([0, r])$. Note that

$$d(\lambda_{k'}, b) \leq \operatorname{diam}(J_{(\ell)_i}) \leq \operatorname{fdiam}((\ell)_i) \text{ and } \rho_{k'} \leq \operatorname{fdiam}((\ell)_i).$$

Also note that

$$\operatorname{fdiam}((\ell)_{\mathrm{i}}) \leq \operatorname{fmesh}(\ell) < \varepsilon \leq \mu. \tag{5.3}$$

Therefore

$$d(\lambda_{k'}, a) + \rho_{k'} \leq d(\lambda_{k'}, b) + d(a, b) + \rho_{k'} \leq D + 2 \operatorname{fdiam}((\ell)_i) < D + 2\mu = m.$$

Hence $d(\lambda_{k'}, a) + \rho_{k'} < m$ and this means that $I_{k'}$ is formally contained in $B(a, m)$.

Finally, note that $p < \overline{\ell}$. Otherwise, we would have $p = \overline{\ell}$. It is clear from the construction of the chain $\mathcal{H}_\ell$ (property (1)) that $f(r') \in J_{(\ell)_{\overline{\ell}}}$. Hence $f(r')$ would belong to $J_{(\ell)_p}$. However $f(r') \notin \widehat{B}(a, m)$ which would contradict the fact that $J_{(\ell)_p}$ is (formally) contained in $B(a, m)$.

We have the following conclusion. For each $n, k \in \mathbb{N}$ there exist $\ell, m, p, u \in \mathbb{N}$ such that

(1) $\mathcal{H}_\ell$ is a formal chain;
(2) J_u and $\mathcal{H}_\ell$ are formally disjoint;
(3) $f(0) \in J_{(\ell)_0}$;
(4) $R_n \cup F_n \subseteq \bigcup \mathcal{H}_\ell^{0 \leq p} \cup J_u$;
(5) $R_m \cup F_m \subseteq \bigcup \mathcal{H}_\ell \cup J_u$;
(6) $\mathcal{H}_\ell^{0 \leq p}$ is formally contained in $B(a, m)$;
(7) $p < \overline{\ell}$ and $m \geq 1$;
(8) $\operatorname{fmesh}(\ell) < 2^{-k}$.

Let

$$T = \left\{(n, k, m, \ell, p, u) \in \mathbb{N}^6 \mid \text{for } n, k, m, \ell, p, u \text{ properties (1)–(8) hold}\right\}.$$

Using Proposition 4.1, Lemma 4.2, Proposition 4.4 and Proposition 2.2 we conclude that T is c.e. as the intersection of c.e. sets. (Recall that $f(0)$ is computable point, and if c is some computable point, then it is straightforward to see that the set $\{j \in \mathbb{N} \mid c \in J_j\}$ is c.e.)

We have shown that for all $n, k \in \mathbb{N}$ there exist $m, \ell, p, u \in \mathbb{N}$ such that $(n, k, m, \ell, p, u) \in T$. Therefore there exist computable functions $\widetilde{m}, \widetilde{\ell}, \widetilde{p}, \widetilde{u} : \mathbb{N}^2 \to \mathbb{N}$ such that

$$(n, k, \widetilde{m}(n, k), \widetilde{\ell}(n, k), \widetilde{p}(n, k), \widetilde{u}(n, k)) \in T$$

for all $n, k \in \mathbb{N}$ (Single-Valuedness Theorem).

Let $n, k \in \mathbb{N}$. Let $m = \widetilde{m}(n, k)$, $\ell = \widetilde{\ell}(n, k)$, $p = \widetilde{p}(n, k)$ and $u = \widetilde{u}(n, k)$. Then for n, k, m, ℓ, p, u properties (1)–(8) hold. Now we want to prove that each link of the chain $\mathcal{H}_\ell^{0 \leq p}$ intersects R.

Notice first that there exists $t \in [0, \infty\rangle$ such that $d(a, f(t)) = m$. Otherwise, $B(a, m)$ and $X \setminus \widehat{B}(a, m)$ would be disjoint open sets whose union contain R. However, each of these sets intersects R, which follows from $d(a, f(0)) < 1$ and Proposition 5.1 and so we would have that the topological ray R is disconnected, which is impossible.

The set $\{t \in [0, \infty\rangle \mid d(a, f(t)) = m\}$ is a closed and nonempty subset of $[0, \infty\rangle$ and therefore it has a minimal element. Let t_0 be that element. Then $d(a, f(t)) < m$ for each $t \in [0, t_0\rangle$ (if $f(t) > m$ for some $t \in [0, t_0\rangle$, then connectedness of $f([0, t])$ implies that $d(a, f(s)) = m$ for some $s \in [0, t]$ which is impossible since $s < t_0$). Hence $f([0, t_0]) \subseteq R_m$. It follows from property (5) that

$$f([0, t_0]) \subseteq \bigcup \mathcal{H}_\ell \cup J_u.$$

However $f([0, t_0]) \cap \bigcup \mathcal{H}_\ell \neq \emptyset$ by (3) and $\bigcup \mathcal{H}_\ell \cap J_u = \emptyset$ by (2). The fact that $f([0, t_0])$ is connected now gives

$$f([0, t_0]) \subseteq \bigcup \mathcal{H}_\ell.$$

Therefore $f(t_0) \in J_{(\ell)_v}$ for some $v \in \{0, \dots, \overline{\ell}\}$. But now the property (6) implies that $p < v$. (If $v \leq p$, then $J_{(l)_v}$ is (formally) contained in $B(a, m)$ which is impossible since $f(t_0) \in J_{(\ell)_v}$ and $f(t_0) \notin B(a, m)$.)

Finally, let us prove that each link of the chain $\mathcal{H}_\ell^{0 \leq p}$ intersects R. Suppose that there exists $i \in \{0, \dots, p\}$ such that $J_{(\ell)_i} \cap R = \emptyset$. Then $i \neq 0$ (since $f(0) \in J_{(\ell)_0}$), hence $0 < i < v$. Now

$$U = J_{(\ell)_0} \cup \dots \cup J_{(\ell)_{i-1}} \text{ and } V = J_{(\ell)_{i+1}} \cup \dots \cup J_{(\ell)_{\overline{\ell}}}$$

are open disjoint sets which cover $f([0, t_0])$ and each of these sets intersects $f([0, t_0])$ ($f(0) \in J_{(\ell)_0}$, $f(t_0) \in J_{(\ell)_v}$). This is impossible since $f([0, t_0])$ is connected.

So we have proved that $J_{(\ell)_i} \cap R \neq \emptyset$ for each $i \in \{0, \dots, p\}$. Another fact regarding the chain $\mathcal{H}_\ell^{0 \leq p}$ that we want to verify is this: if $s \in [0, \infty\rangle$ is such that $f([0, s]) \subseteq B(a, n)$, then $f(s)$ lies in some link of $\mathcal{H}_\ell^{0 \leq p}$.

But if s is such that $f([0, s]) \subseteq B(a, n)$, then $f([0, s]) \subseteq R_n$ and now (4), together with the fact that $f([0, s])$ is connected, gives $f([0, s]) \subseteq \bigcup \mathcal{H}_\ell^{0 \leq p}$. In particular $f(s)$ lies in some link of $\mathcal{H}_\ell^{0 \leq p}$.

We have the following conclusion: for all $n, k \in \mathbb{N}$

(1) the formal diameter of each link of the chain $\mathcal{H}_{\widetilde{\ell}(n,k)}^{0 \leq \widetilde{p}(n,k)}$ is less than 2^{-k};

(2) each link of the chain $\mathcal{H}_{\widetilde{\ell}(n,k)}^{0 \leq \widetilde{p}(n,k)}$ intersects R;

(3) if $s \in [0, \infty\rangle$ is such that $f([0, s]) \subseteq B(a, n)$, then $f(s)$ lies in some link of $\mathcal{H}_{\widetilde{\ell}(n,k)}^{0 \leq \widetilde{p}(n,k)}$.

Note the following: if $c \in R$, then $c = f(s)$ for some $s \in [0, \infty\rangle$ and there exists $n \in \mathbb{N}$ such that $f([0, s]) \subseteq B(a, n)$. Then $f(s)$ (i.e. the point c) lies in some link of $\mathcal{H}_{\widetilde{\ell}(n,k)}^{0 \leq \widetilde{p}(n,k)}$ for each $k \in \mathbb{N}$.

Let $i \in \mathbb{N}$. Suppose $I_i \cap R \neq \emptyset$. Let $c \in I_i \cap R$. Using Lemma 2.6 we conclude that there exist $n, k \in \mathbb{N}$ such that c belongs to some link of $\mathcal{H}_{\widetilde{\ell}(n,k)}^{0 \leq \widetilde{p}(n,k)}$ which is formally contained in I_i. So there exists $w \in \mathbb{N}$ such that

$$w \leq \widetilde{p}(n, k) \text{ and } J_{(\widetilde{\ell}(n,k))_w} \subseteq_F I_i. \tag{5.4}$$

On the other hand, if (5.4) holds for some $n, k, w \in \mathbb{N}$, then I_i intersects R because $J_{(\widetilde{\ell}(n,k))_w}$ does. Hence $I_i \cap R \neq \emptyset$ if and only if there exist $n, k, w \in \mathbb{N}$ such that (5.4) holds. It follows from Proposition 2.5(4) that $\{i \in \mathbb{N} \mid I_i \cap R \neq \emptyset\}$ is c.e. and this means that R is c.e. □

Corollary 5.3. *Let (X, d, α) be a computable metric space and let R be a semi-c.c.b. set in this space. Suppose R is a topological ray whose endpoint is computable. Then R is c.c.b.* □

6. Co-c.e. topological lines

We will say that L is a **topological line** if L is a metric space homeomorphic to $\mathbb{R}$.

While we may imagine topological rays as arcs which have one endpoint in infinity, a topological line can be thought of as an arc whose both endpoints are in infinity. And while for computability of a semi- c.c.b. topological ray we needed the assumption that its endpoint is computable, in the case of a semi-c.c.b. topological line naturally we will have no such assumption. Hence we will prove that each semi-c.c.b. topological line is c.c.b. Actually, as in the case of topological rays, we will have a more general result.

First, we have a proposition similar to Proposition 5.1 which says that, under certain assumption, both tails of a topological line "converge to infinity".

Proposition 6.1. *Let (X, d) be a metric space. Let L be a subset of X such that $L \cap B$ is a compact set for each closed ball B in (X, d) and such that there exists a homeomorphism $f : \mathbb{R} \to L$. Then for each closed ball B there exists $t_0 \in [0, \infty\rangle$ such that $f(t) \notin B$ for each $t \geq t_0$ and $t \leq -t_0$.*

Proof. The set $f([0, \infty\rangle)$ is closed in L. Therefore for each closed ball B in (X, d) the set $f([0, \infty\rangle) \cap B$ is closed in L and consequently in $L \cap B$ which is compact. Hence $f([0, \infty\rangle) \cap B$ is compact. Similarly, $f(\langle -\infty, 0]) \cap B$ is compact for each closed ball B in (X, d). Now we apply Proposition 5.1 on homeomorphisms $[0, \infty\rangle \to f([0, \infty\rangle)$, $x \mapsto f(x)$, and $[0, \infty\rangle \to f(\langle -\infty, 0])$, $x \mapsto f(-x)$. □

Theorem 6.2. *Let (X, d, α) be a computable metric space. Let L be a subset of X which is, as a subspace of (X, d), a topological line. Suppose F is a closed set in (X, d) which is disjoint with L and such that $L \cup F$ is semi-c.c.b. Then L is a computably enumerable set.*

Proof. Let $f : \mathbb{R} \to L$ be a homeomorphism. Let a be some rational point such that $d(a, f(0)) < 1$. For each $n \in \mathbb{N}$ let

$$L_n = L \cap \widehat{B}(a, n) \text{ and } F_n = F \cap \widehat{B}(a, n).$$

Let $\delta > 0$ be such that $f([-\delta, \delta]) \subseteq B(a, 1)$. (Such a number exists since f is continuous.)

Now choose $A, B, C \in \mathbb{N}$ and $k_0 \in \mathbb{N}$ so that $f(-\delta) \in I_A$, $f(\delta) \in I_B$, $f(0) \in I_C$, $\rho_A < \frac{2^{-k_0}}{4}$, $\rho_B < \frac{2^{-k_0}}{4}$, $\rho_C < \frac{2^{-k_0}}{4}$ and

$$2^{-k_0} < \min \{d(I_A, f([0, +\infty\rangle)), d(I_B, f(\langle -\infty, 0])), d(I_C, F)\}. \tag{6.1}$$

Let $n, k \in \mathbb{N}$. By Proposition 6.1 there exists $r > 0$ such that $f(x) \notin \widehat{B}(a, n)$ for each $x \in \mathbb{R}$ such that $x \geq r$ or $x \leq -r$. Since $f([-r, r])$ is compact, there exists $m \in \mathbb{N}$, $m \geq 1$ such that $f([-r, r]) \subseteq B(a, m)$. By Proposition 6.1 there also exists $r' > r$ such that $f(x) \notin \widehat{B}(a, m)$, whenever $x \geq r'$ or $x \leq -r'$.

We have

$$L_n \subseteq f([-r, r]) \subseteq L_m \subseteq f([-r', r']). \tag{6.2}$$

Let

$$D = \max \{d(a, f(x)) \mid x \in [-r, r]\}.$$

Then $D < m$ since $f([-r, r]) \subseteq B(a, m)$. Let

$$\mu = \frac{m - D}{2}.$$

By Lemma 4.6 there exists $\lambda > 0$ such that if $j, j' \in \mathbb{N}$ and $G \subseteq f([-r', r'])$, then

$$(\langle F_m, j, \lambda\rangle \text{ and } \langle G, j', \lambda\rangle) \Longrightarrow J_j \text{ and } J_{j'} \text{ are formally disjoint.} \tag{6.3}$$

Let $\varepsilon = \min\{\mu, \lambda, 2^{-(k+k_0+3)}\}$. Let $u \in \mathbb{N}$ be such that

$$\langle F_m, u, \varepsilon\rangle. \tag{6.4}$$

Let $g : [0, 2r'] \to X$ be the function defined by

$$g(t) = f(t - r'),$$

$t \in [0, 2r']$.

Applying Lemma 4.5 to g, we get numbers $n' \geq 1$ and $j_0, \dots, j_{n'-1} \in \mathbb{N}$ such that

(1) $\langle g([i\frac{2r'}{n'}, (i+1)\frac{2r'}{n'}]), j_i, \varepsilon\rangle$ for each $i \in \{0, \dots, n'-1\}$;
(2) J_{j_i} are $J_{j_{i'}}$ formally disjoint for all $i, i' \in \{0, \dots, n'-1\}$ such that $|i - i'| > 1$;
(3) $\operatorname{fdiam}(j_i) < \varepsilon$ for each $i \in \{0, \dots, n'-1\}$.

We can choose n' so that $\frac{2r'}{n'} < \min\{\frac{r'-r}{2}, \frac{r}{2}\}$. Let $\ell \in \mathbb{N}$ be such that

$$((\ell)_0, \dots, (\ell)_{\overline{\ell}}) = (j_0, \dots, j_{(n'-1)}).$$

Then $\mathcal{H}_\ell$ is a formal chain and $\operatorname{fmesh}(\ell) < \varepsilon$. It clearly covers $g([0, 2r'])$, i.e. $f([-r', r'])$. Hence $L_m \subseteq \bigcup \mathcal{H}_\ell$. And by (6.3) J_u and $\mathcal{H}_\ell$ are formally disjoint.

Let $D' = \frac{2r'}{n'}$. Let us choose numbers $p, q, e \in \mathbb{N}$ so that

(4) $-r + r' \in [pD', (p+1)D']$;
(5) $r' \in [eD', (e+1)D']$;
(6) $r + r' \in [qD', (q+1)D']$;

Note that $f(-r) \in J_{j_p}$, $f(0) \in J_{j_e}$ and $f(r) \in J_{j_q}$.

We claim that $p < e < q < \overline{\ell}$. It holds

$$pD' \leq r' - r \leq (e+1)D' - 2D' < eD'.$$

Dividing by D' we get $p < e$. Also

$$eD' \leq r' + r - r \leq (q+1)D' - r < (q+1)D' - 2D' < qD'$$

and we get $e < q$.

Let us prove that $q < \overline{\ell}$. First we have

$$(q+1)D' < qD' + \frac{r' - r}{2} \leq r + r' + \frac{r' - r}{2} = \frac{3r' + r}{2} < 2r'.$$

Hence $(q+1)D' < 2r'$. By definition of ℓ it holds $\overline{\ell} = n' - 1$. Now

$$qD' = (q+1)D' - D' < 2r' - D' = (n'-1)D' = \overline{\ell}D'$$

and it follows $q < \overline{\ell}$.

We claim that I_A and $\mathcal{H}_\ell^{e \leq \overline{\ell}}$ are formally disjoint. Suppose the opposite. Then there exists $i \in \{e, \dots, \overline{\ell}\}$ such that I_A and $J_{(\ell)i}$ are not formally disjoint. Therefore there exists $j \in [(\ell)_i]$ such that

$$d(\lambda_A, \lambda_j) \leq \rho_A + \rho_j.$$

Note that by the construction of $\mathcal{H}_\ell$ each link of the chain $\mathcal{H}_\ell^{e\leq\bar{\ell}}$ intersects $f([0,\infty\rangle)$. Therefore there exists $y \in J_{(\ell)_i} \cap f([0,\infty\rangle)$. Now

$$\begin{aligned}
d(I_A, f[0,\infty\rangle) &\leq d(f(-\delta), y)\\
&\leq d(f(-\delta),\lambda_A) + d(\lambda_A,\lambda_j) + d(\lambda_j, y)\\
&\leq 2\rho_A + \rho_j + \operatorname{diam}(J_{(\ell)_i}) < 2\frac{2^{-k_0}}{4} + 2\varepsilon < 2^{-k_0}
\end{aligned}$$

which contradicts (6.1). Hence, I_A and $\mathcal{H}_\ell^{e\leq\bar{\ell}}$ are formally disjoint. In the same way we get that I_B and $\mathcal{H}_\ell^{0\leq e}$ are formally disjoint and also, using (6.4), that I_C and J_u are formally disjoint.

From the definition of numbers p and q we deduce that

$$[-r,r] \subseteq \bigcup_{p\leq i\leq q} [iD'-r', (i+1)D'-r']$$

which gives

$$f([-r,r]) \subseteq \bigcup_{p\leq i\leq q} f([iD'-r',(i+1)D'-r']) = \bigcup_{p\leq i\leq q} g([iD',(i+1)D']) \subseteq \bigcup \mathcal{H}_\ell^{p\leq q}.$$

Hence

$$L_n \subseteq \bigcup \mathcal{H}_\ell^{p\leq q}.$$

Finally, let us prove that $\mathcal{H}_\ell^{p\leq q}$ is formally contained in $B(a,m)$.

Let $i \in \{p,\dots,q\}$. To prove that $J_{(\ell)_i}$ is formally contained in $B(a,m)$ let us first prove that $J_{(\ell)_i}$ intersects $f([-r,r])$. Since

$$g([iD',(i+1)D']) \subseteq J_{(\ell)_i}$$

it suffices to see that

$$[iD'-r',(i+1)D'-r'] \cap [-r,r] \neq \emptyset.$$

For $i=p$ this intersection contains $-r$ and for $i=q$ it contains r. If $p<i$, then $p+1\leq i$ and $(p+1)D'-r' \leq iD'-r'$ which implies $-r \leq iD'-r'$. In the same way get that $i<q$ implies $(i+1)D'-r' \leq r$. Hence if i is between p and q, then the segment $[iD'-r',(i+1)D'-r']$ is contained in $[-r,r]$.

Now we proceed in the same way as in the proof of Theorem 5.2. We take $k' \in [(\ell)_i]$ and we want to prove that $I_{k'}$ is formally contained in $B(a,m)$.

Since $J_{(\ell)_i}$ intersects $f([-r,r])$, there exists $b \in J_{(\ell)_i}$ such that $b \in f([-r,r])$. Then

$$d(\lambda_{k'}, b) \leq \operatorname{diam}(J_{(\ell)_i}) \leq \operatorname{fdiam}((\ell)_i) \text{ and } \rho_{k'} \leq \operatorname{fdiam}((\ell)_i).$$

Also note that

$$\operatorname{fdiam}((\ell)_{\mathrm{i}}) \leq \operatorname{fmesh}(\ell) < \varepsilon \leq \mu. \tag{6.5}$$

Therefore

$$d(\lambda_{k'},a) + \rho_{k'} \leq d(\lambda_{k'},b) + d(a,b) + \rho_{k'} \leq D + 2\operatorname{fdiam}((\ell)_i) < D + 2\mu = m.$$

Hence $d(\lambda_{k'},a) + \rho_{k'} < m$ and $I_{k'}$ is formally contained in $B(a,m)$.

The conclusion: for all $n,k \in \mathbb{N}$ there exist $m,\ell,p,q,e,u \in \mathbb{N}$ such that

(1) $\mathcal{H}_\ell$ is a formal chain;
(2) $\mathcal{H}_\ell$ and J_u are formally disjoint;
(3) $L_n \cup F_n \subseteq \cup\mathcal{H}_\ell^{p\leq q} \cup J_u$;

(4) $L_m \cup F_m \subseteq \bigcup \mathcal{H}_\ell \cup J_u$;
(5) $\mathcal{H}_\ell^{p \leq q}$ is formally contained in $B(a,m)$;
(6) $p < e < q < \overline{\ell}$, $m \geq 1$;
(7) $\text{fmesh}(\ell) < 2^{-(k+k_0+3)}$;
(8) I_A and $\mathcal{H}_\ell^{e \leq \overline{\ell}}$ are formally disjoint;
(9) I_B and $\mathcal{H}_\ell^{0 \leq e}$ are formally disjoint;
(10) I_C and J_u are formally disjoint.

Let T be the set of all $(n,k,m,\ell,p,q,e,u) \in \mathbb{N}^8$ such that properties (1)–(10) hold. As in the proof of Theorem 5.2 we conclude that T is c.e. and we also conclude that there exists a computable function $\varphi : \mathbb{N}^2 \to \mathbb{N}^6$ such that

$$(n,k,\varphi(n,k)) \in T \tag{6.6}$$

for all $n,k \in \mathbb{N}$. This concludes the first part of the proof of Theorem 6.2.

In the second part we prove that the existence of a such function φ implies that L is c.e.

Suppose we have $n,k,m,\ell,p,q,e,u \in \mathbb{N}$ such that properties (1)–(10) hold. We also assume that $n \geq 1$. We want to prove that each link of $\mathcal{H}_\ell^{p\leq q}$ intersects L. For $i \in \{0,\dots,\overline{\ell}\}$ let $C_i = J_{(\ell)_i}$. Hence

$$\mathcal{H}_\ell = (C_0,\dots,C_{\overline{\ell}}).$$

First we prove the following: if $t,s \in \mathbb{R}$ are such that $t \leq 0 \leq s$, then
(1) $f([t,s]) \subseteq \widehat{B}(a,m)$ implies $f([t,s]) \subseteq \bigcup \mathcal{H}_\ell$;
(2) $f([t,s]) \subseteq \widehat{B}(a,n)$ implies $f([t,s]) \subseteq \bigcup \mathcal{H}_\ell^{p\leq q}$.

If $f([t,s]) \subseteq \widehat{B}(a,m)$, then $f([t,s]) \subseteq L_m$, this and (4) imply

$$f([t,s]) \subseteq \bigcup \mathcal{H}_\ell \cup J_u$$

and $\bigcup \mathcal{H}_\ell$ and J_u are disjoint by (2). Since $f([t,s])$ is connected, it must be entirely contained in one of these sets. But this cannot be J_u since $f(0) \in f([t,s])$ and $f(0)$ belongs to I_C which is disjoint with J_u by (10). Hence $f([t,s]) \subseteq \bigcup \mathcal{H}_\ell$. In the same way we prove (2).

Since $f([-\delta,\delta]) \subseteq B(a,1) \subseteq B(a,m)$, there exist $\alpha,\beta \in \{0,\dots,\overline{\ell}\}$ such that $f(-\delta) \in C_\alpha$ and $f(\delta) \in C_\beta$.

As in the proof of Theorem 5.2, we conclude that there exist $s_0,t_0 \in \mathbb{R}$ such that $s_0 < 0 < t_0$, $d(a,f(s_0)) = d(a,f(t_0)) = m$ and $f(t) \in B(a,m)$ for each $t \in \langle s_0,t_0\rangle$. It follows that there exist $v,w \in \{0,\dots,\overline{\ell}\}$ such that $f(s_0) \in J_{(\ell)_v}$, $f(t_0) \in J_{(\ell)_w}$.

We claim that $p-1 \leq \alpha < e$ and $e < \beta \leq q+1$.

First, let us prove $p \leq \alpha+1$. Suppose the opposite. Then $\alpha+1 < p < q$. The link C_α is then disjoint with each of the links $C_p, \dots, C_q$. However $f([-\delta,\delta]) \subseteq B(a,n)$ since $n \geq 1$, therefore $f(-\delta) \in C_p \cup \dots \cup C_q$ and, by definition of α, $f(-\delta) \in C_\alpha$. A contradiction. Hence, $p \leq \alpha+1$.

Let us prove $\alpha < e$. Suppose the opposite. Then $\alpha \geq e$, hence the link C_α is one of the links $C_e,\dots,C_{\overline{\ell}}$ and $f(-\delta) \in C_\alpha$. On the other hand, $f(-\delta) \in I_A$ and this now contradicts (8). So $\alpha < e$ and altogether

$$p-1 \leq \alpha < e.$$

In the same way we get

$$e < \beta \leq q+1.$$

Now we claim that $v < p$ and $q < w$. Let us prove $v < p$.

Suppose $p \leq v$. This implies $q < v$. Otherwise we have $v \leq q$, which together with $p \leq v$ means that C_v is one of the links of the chain $\mathcal{H}_\ell^{p\leq q}$. But this chain is formally contained in $B(a, m)$, hence $C_v \subseteq B(a, m)$. This is impossible since $f(s_0) \in C_v$.

Hence $p < q < v$. So $q + 1 \leq v$ which together with $\beta \leq q + 1$ gives $\beta \leq v$. But $\beta \neq v$ because $\beta = v$ would imply

$$d(f(s_0), f(\delta)) < \text{diam } C_v < 2^{-k_0},$$

and this is impossible by (6.1). Therefore $\beta < v$.

We also have

$$f([s_0, -\delta]) \cap C_\beta = \emptyset. \tag{6.7}$$

Otherwise, there exists $y \in f([s_0, -\delta]) \cap C_\beta$ and

$$d(I_B, f(\langle -\infty, 0])) \leq d(f(\delta), f(\langle -\infty, 0])) \leq d(f(\delta), f([s_0, -\delta])) \leq$$
$$\leq d(f(\delta), y) \leq \text{diam } C_\beta < 2^{-k_0}$$

which again contradicts (6.1). Hence (6.7) holds.

Let U and V be defined by

$$U = \bigcup_{0\leq i\leq \beta-1} C_i, \quad V = \bigcup_{\beta+1\leq i\leq \bar{\ell}} C_i.$$

Since $\mathcal{H}_\ell$ covers $f([s_0, t_0])$ and (6.7) holds,

$$f([s_0, -\delta]) \subseteq U \cup V. \tag{6.8}$$

We have $f(-\delta) \in C_\alpha$ and $\alpha < e < \beta$, hence

$$f([s_0, -\delta]) \cap U \neq \emptyset. \tag{6.9}$$

Furthermore, $f(s_0) \in C_v$ and $\beta < v$, so

$$f([s_0, -\delta]) \cap V \neq \emptyset. \tag{6.10}$$

Finally, (1) implies that $C_i \cap C_{i'} = \emptyset$ whenever $i, i' \in \{0, \dots, \bar{\ell}\}$ are such that $i < \beta < i'$. Hence

$$U \cap V = \emptyset. \tag{6.11}$$

From (6.8), (6.9), (6.10) and (6.11) it follows that $f([s_0, -\delta])$ is not connected. A contradiction.

So we have proved that $v < p$. In the same way we get $q < w$. Hence

$$v < p < q < w.$$

It is easy to conclude from this that each link of the chain $\mathcal{H}_\ell^{p\leq q}$ intersects L. Namely, let $i \in \mathbb{N}$ be such that $p \leq i \leq q$. Then $v < i < w$. Suppose that $C_i \cap L = \emptyset$. Then

$$U = C_0 \cup \cdots \cup C_{i-1} \text{ and } V = C_{i+1} \cup \cdots \cup C_{\bar{\ell}}$$

are disjoint sets, their union covers $f([s_0, t_0])$ and each of these sets intersects $f([s_0, t_0])$ because $f(s_0) \in C_v \subseteq U$ and $f(t_0) \in C_w \subseteq V$. This contradicts the fact that $f([s_0, t_0])$ is connected.

Hence each link of the chain $\mathcal{H}_\ell^{p\leq q}$ intersects L (under the assumption that $n \geq 1$).

Let $\widetilde{m}, \widetilde{\ell}, \widetilde{p}, \widetilde{q}, \widetilde{e}, \widetilde{u} : \mathbb{N}^2 \to \mathbb{N}$ be the component functions of the function φ from (6.6).

If $c \in L$, then $c \in f([-t, t])$ for some $t \geq 0$. Choose $n \in \mathbb{N}$, $n \geq 1$, so that $f([-t, t]) \subseteq B(a, n)$. Then for each $k \in \mathbb{N}$ some link of the chain $\mathcal{H}_{\widetilde{\ell}(n,k)}^{\widetilde{p}(n,k)\leq \widetilde{q}(n,k)}$ contains c.

Let $i \in \mathbb{N}$. As in the proof of Theorem 5.2 we conclude that $I_i \cap L \neq \emptyset$ if and only if there exist $n, k, w \in \mathbb{N}$ such that

$$\widetilde{p}(n,k) \leq w \leq \widetilde{q}(n,k),\ n \geq 1 \text{ and } J_{(\widetilde{\ell}(n,k))_w} \subseteq_F I_i.$$

Therefore L is c.e. □

Corollary 6.3. *Let (X, d, α) be a computable metric space and let L be a semi-c.c.b. set in this space. Suppose L is a topological line. Then L is c.c.b.* □

7. 1-manifolds

A 1-**manifold with boundary** is a second countable Hausdorff topological space X in which each point has a neighborhood homeomorphic to $[0, \infty\rangle$. The **boundary** ∂X of X consists of those points $x \in X$ for which every homeomorphism between a neighborhood of x and $[0, \infty\rangle$ maps x to 0. Therefore, each point of $X \setminus \partial X$ has a neighborhood in X which is homeomorphic to $\mathbb{R}$. If $\partial X = \emptyset$, then we simply say that X is a 1-**manifold**.

If X and Y are topological spaces and $f : X \to Y$ a homeomorphism and if X is a 1-manifold with boundary, then Y is also and $\partial Y = f(\partial X)$.

For example, $\mathbb{R}$ and the unit circle S^1 in $\mathbb{R}^2$ are 1-manifolds, while $[0, \infty\rangle$ and $[0, 1]$ are 1-manifolds with boundary, $\partial[0, \infty\rangle = \{0\}$, $\partial[0, 1] = \{0, 1\}$. Each topological line is a 1-manifold and if R is a topological ray and a is its endpoint, then R is a 1-manifold with boundary and $\partial R = \{a\}$. Furthermore, if S is an arc with endpoints a and b, then S is a manifold with boundary and $\partial S = \{a, b\}$. Note the following: if a subspace M of some topological space X is a manifold with boundary, then the boundary of M in general differs from the topological boundary of M in X.

Since a is a computable point if and only if $\{a\}$ is c.c.b., Theorem 5.2 means that a semi-c.c.b. topological ray is c.c.b. if its boundary is c.c.b. The natural question arises whether this holds for each 1-manifold, i.e. if M is a semi-c.c.b. 1-manifold in a computable metric space, does the implication

$$\partial M \text{ c.c.b.} \Rightarrow M \text{ c.c.b.} \tag{7.1}$$

hold? The answer is no, implication (7.1) fails to be true in general.

To see this, let S be a c.e. subset of $\mathbb{N}$ which is not computable. The fact that S is c.e. implies that the set $T = \mathbb{N} \setminus S$ is co-c.e. in $\mathbb{R}$. Therefore $T \times \mathbb{R}$ is co-c.e. in $\mathbb{R}^2$. Let $M = T \times \mathbb{R}$. Since $T \subseteq \mathbb{N}$, we have that M is a 1-manifold. That M is not computable in $\mathbb{R}^2$ can be deduced from the fact that T is not computable in $\mathbb{N}$. Of course M is semi-c.c.b. by Proposition 3.6 and we conclude that (7.1) does not hold (note that $\partial M = \emptyset$).

However, we will show later that (7.1) holds under additional assumption that M has finitely many components.

It is known (see e.g. [12]) that if X is a connected 1-manifold with boundary, then X is homeomorphic to $\mathbb{R}$, $[0, \infty\rangle$, $[0, 1]$ or S^1. (Here S^1 denotes the unit circle in $\mathbb{R}^2$.) Hence topological lines, topological rays, arcs and topological circles are all connected 1-manifolds.

It is easy to conclude that if X is a 1-manifold with boundary, then each component of X is also a 1-manifold with boundary and $x \in X$ belongs to the boundary of X if and only if x belongs to the boundary of some component of X.

Theorem 7.1. *Let (X, d, α) be a computable metric space. Suppose M is a semi-c.c.b. set which is a 1-manifold with boundary. Let K be a component of M.*

(1) *If K is a topological line or a topological circle, then K is c.e.*
(2) *If K is a topological ray with computable endpoint or an arc with computable endpoints, then K is c.e.*

Proof. Let $x \in K$. Then x has a neighborhood in M which is homeomorphic to $[0, \infty\rangle$. Hence x has a neighborhood in M which is connected and which therefore is contained in K. This means that x belongs to some set which is open in M and is contained in K. So the conclusion is that K is open in M.

Let $F = M \setminus K$. Then F is closed in M, but since M as a semi-c.c.b. set is closed in (X, d), we have that F is closed in (X, d). Hence F is closed, disjoint with K and $F \cup K$ is semi-c.c.b. Now Theorem 5.2 and Theorem 6.2 imply that K is c.e. if K is a topological ray with computable endpoint or a topological line.

Suppose now that K is a topological circle or an arc with computable endpoints. Then K is compact and since it is disjoint with F (which is closed), there exist $i_0, \dots, i_n \in \mathbb{N}$ such that

$$K \subseteq \widehat{I}_{i_0} \cup \dots \cup \widehat{I}_{i_n} \subseteq X \setminus F.$$

Then we have

$$K = K \cap (\widehat{I}_{i_0} \cup \dots \cup \widehat{I}_{i_n}) = (K \cup F) \cap (\widehat{I}_{i_0} \cup \dots \cup \widehat{I}_{i_n}) = (M \cap \widehat{I}_{i_0}) \cup \dots \cup (M \cap \widehat{I}_{i_n}).$$

So for $j \in \mathbb{N}$ the following equivalence holds:

$$K \subseteq J_j \Leftrightarrow M \cap \widehat{I}_{i_0} \subseteq J_j, \dots, M \cap \widehat{I}_{i_n} \subseteq J_j.$$

From this and the fact that M is semi-c.c.b. we conclude that K is semi-computable compact set. Hence K is a compact manifold with computable boundary and therefore, by [7], K is a computable compact set. In particular, K is c.e. □

As we have seen, if M is a 1-manifold with boundary such that M is semi-c.c.b. and ∂M is c.c.b., then M need not be c.c.b. Since M is already semi-c.c.b., this means that M need not be computably enumerable. However, although M is not necessarily computably enumerable, each component of M is computably enumerable.

Theorem 7.2. *Let (X, d, α) be a computable metric space. Let M be a 1-manifold with boundary in this space and suppose M and ∂M are semi-c.c.b. Then each component of M is computable enumerable.*

Proof. In view of Theorem 7.1 it suffices to prove that each point in ∂M is computable. Let $x \in \partial M$. Then x has a neighborhood N in M such that there exists a homeomorphism $f : N \to [0, \infty\rangle$ such that $f(x) = 0$. It is clear from this that x is the only point in N which belongs to the boundary of M. It follows that $B(x, r) \cap \partial M = \{x\}$ for some $r > 0$ and we conclude from this that $\widehat{I}_i \cap \partial M = \{x\}$ for some $i \in \mathbb{N}$. Since ∂M is semi-c.c.b., $\widehat{I}_i \cap \partial M$ is clearly semi-computable compact set, hence $\{x\}$ is semi-computable and consequently x is a computable point. □

Since the union of finitely many c.e. sets in (X, d, α) is a c.e. set, we have the following theorem.

Theorem 7.3. *Let (X, d, α) be a computable metric space. Let M be a subset of X which is, as a subspace of (X, d), a 1-manifold with boundary which has finitely many components. Suppose M and ∂M are semi-c.c.b. Then M is c.c.b.* □

Corollary 7.4. *Let (X, d, α) be a computable metric space. Let M be a 1-manifold in this space and suppose M has finitely many components and M is semi-c.c.b. Then M is c.c.b.* □

The following theorem is an immediate consequence of Theorem 7.3 and Proposition 3.6.

Theorem 7.5. *Let (X, d, α) be a computable metric space which has compact closed balls and the effective covering property. Let M be a 1-manifold with boundary in this space such that M has finitely many components. Suppose M and ∂M are co-c.e. Then M is computable.* □

Corollary 7.6. *Let (X, d, α) be a computable metric space which has compact closed balls and the effective covering property. Let M be a 1-manifold in this space and suppose M has finitely many components and M is co-c.e. Then M is computable.* □

Finally, let us mention that Theorem 7.5 and Corollary 7.6 do not hold in a general computable metric space. In [5] an example of a computable metric space (X, d, α) can be found in which there exist a co-c.e. arc with computable endpoints which is not computable and a co-c.e. topological circle which is not computable. Moreover, we can find such (X, d, α) so that (X, d, α) has compact closed balls and we can also find such (X, d, α) so that (X, d, α) has the effective covering property (but of course not with both of these properties at the same time).

Acknowledgements

The authors are grateful to anonymous referees for their useful suggestions and corrections.

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