Computability of the Radon-Nikodym derivative

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- Let $f \in L^1(\lambda)$ be nonnegative.
- Let $\mu(A) = \int_A f \, \mathrm{d}\lambda$.
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Conversely,

Theorem (Radon-Nikodym, 1930)

For every measure $\mu \ll \lambda$ there exists $f \in L^1(\lambda)$ such that

$$\mu(A) = \int_A f \, \mathrm{d}\lambda \quad \text{for all Borel sets } A.$$

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Our problem

Is $\frac{d\mu}{d\lambda}$ computable from μ?

Theorem

On $[0,1]$*, there is a computable measure* $\mu \ll \lambda$ *(even* $\mu \leq 2\lambda$*) such that* $\frac{d\mu}{d\lambda}$ *is not* $L^1(\lambda)$*-computable.*

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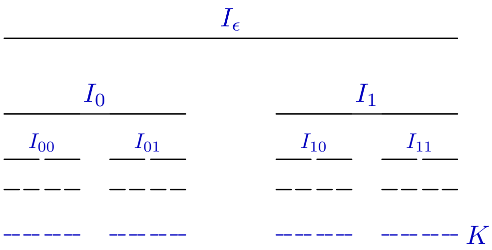
Proof.

The measure will be defined as $\mu(A) = \lambda(A|K) = \frac{\lambda(A \cap K)}{\lambda(K)}$ where $K \subseteq [0,1]$:

- is a recursive compact set,
- $\lambda(K) > 0$ is not computable (only upper semi-computable, or right-c.e.).

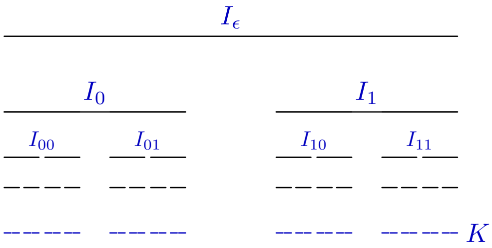


Theorem

On $[0,1]$, there is a computable measure $\mu \ll \lambda$ (even $\mu \leq 2\lambda$) such that $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ is not $L^1(\lambda)$-computable.

Proof cont'd.

There is a computable homeomorphim $\phi : \{0,1\}^{\mathbb{N}} \to K$ and μ is the push-forward $\phi_*\lambda$ of the uniform on Cantor space, so it is computable. $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \frac{1}{\lambda(K)}\mathbf{1}_K$ is not $L^1(\lambda)$-computable.



(Non-)computability of RN

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No more than the Fréchet-Riesz representation theorem.

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$$\mathrm{RN} \leq_W \mathrm{FR}.$$

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And even,

Theorem

$$\mathrm{RN} \equiv_W \mathrm{FR} \equiv_W \mathrm{EC}.$$

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- Nevertheless, $\leq_W$ is now called *Weihrauch-reducibility*.
- $f \leq_W g$ if given x, one can compute $f(x)$ applying g once.
- $f \equiv_W g$ if $f \leq_W g$ and $g \leq_W f$.

Consider two representations En and Cf of $2^{\mathbb{N}}$:

$$\begin{aligned} \mathsf{En}(p) &= \{n \in \mathbb{N} : 100^n 1 \text{ is a subword of } p\}, \\ \mathsf{Cf}(p) &= \{n \in \mathbb{N} : p_n = 1\}. \end{aligned}$$

Let $E \subseteq \mathbb{N}$:

- E is r.e. $\iff$ it is En-computable,
- E is recursive $\iff$ it is Cf-computable.

Let $\mathsf{EC} : (2^{\mathbb{N}}, \mathsf{En}) \to (2^{\mathbb{N}}, \mathsf{Cf})$ be the identity: it is not computable for these representations.

Properties of EC

- Δ^0_2 objects can be computed from one application of EC (subsets of $\mathbb{N}$, real numbers, real fonctions, points of computable metric spaces, etc.)
- Actually, for $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$,

$$f \in \Delta^0_2 \iff f \leq_W \text{EC}.$$

- Let $\mathcal{J}(X)$ be the Turing jump of $X \subseteq \mathbb{N}$: $\mathcal{J} \equiv_W \text{EC}$.
- $\text{EC} \equiv_W \lim_\mathbb{R}$.

Due to Brattka, Gherardi, Yoshikawa, Marcone (2005-2011).

Properties of EC

Used to classify mathematical theorems: to a theorem

$$\forall X \exists Y P(X, Y)$$

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- Let FR be the operator associated to the Fréchet-Riesz representation theorem (on suitable spaces):

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- Let $\mathrm{BW}_{\mathbb{R}}$ be the Bolzano-Weierstrass operator:

$$\mathrm{EC} <_W \mathrm{BW}_{\mathbb{R}}.$$

Due to Brattka, Gherardi, Yoshikawa, Marcone (2005-2011).

$\mathsf{EC} \leq_W \mathsf{RN}$

Let RN be the Radon-Nikodym operator, that maps $\mu \ll \lambda$ to $\frac{d\mu}{d\lambda} \in L^1(\lambda)$.

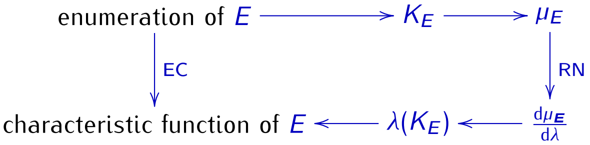
Corollary

$$\mathsf{EC} \leq_W \mathsf{RN}.$$

Proof.

Given an enumeration of $E \subseteq \mathbb{N}$:

1. construct K_E such that $\lambda(K_E) = \sum_{n \notin E} 2^{-n}$,
2. apply RN to compute $\lambda(K_E)$,
3. compute E from $\lambda(K_E)$.



□

A classical proof of the Radon-Nikodym theorem works as follows:

- apply the Fréchet-Riesz representation theorem to the continuous linear operator

$$\begin{aligned} \phi_\mu : L^2(\lambda+\mu) &\rightarrow \mathbb{R} \\ f &\mapsto \int f \, \mathrm{d}\mu. \end{aligned}$$

  It gives $g \in L^2(\lambda+\mu)$ such that for all $f \in L^2(\lambda+\mu)$,

$$\begin{aligned} \phi_\mu(f) &= \langle f, g \rangle, \\ \text{i.e. } \int f \, \mathrm{d}\mu &= \int fg \, \mathrm{d}(\lambda+\mu). \end{aligned}$$

- show that $\frac{g}{1-g}$ has the required properties for being $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$.

To compute the Radon-Nikodym derivative,

$$\begin{array}{ccc} \mu & \longrightarrow & \phi_\mu \\ \Big\downarrow \text{RN} & & \Big\downarrow \text{FR} \\ \frac{d\mu}{d\lambda} = \frac{g}{1-g} \in L^1(\lambda) & \longleftarrow & g \in L^2(\lambda + \mu) \end{array}$$

one shows that from $g \in L^2(\lambda + \mu)$ one can compute $g \in L^1(\lambda)$, knowing that $\int g \, d\lambda = 1$.

(a simple proof can be obtained using Martin-Löf randomness!)

It was proved by Brattka and Yoshikawa that on suitable spaces,

$$\mathsf{FR} \equiv_W \mathsf{EC}.$$

Hence we get

$$\mathsf{EC} \leq_W \mathsf{RN} \leq_W \mathsf{FR} \equiv_W \mathsf{EC}.$$

FR: Fréchet-Riesz
RN: Radon-Nikodym
EC: Enumeration $\mapsto$ Characteristic function