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Computability of the Spectrum of Self-Adjoint Operators[1]

Vasco Brattka
(Laboratory of Foundational Aspects of Computer Science
Department of Mathematics & Applied Mathematics
University of Cape Town, South Africa
BrattkaV@maths.uct.ac.za)

Ruth Dillhage
(Computability & Logic Group
Department of Computer Science
University of Hagen, Germany
Ruth.Dillhage@fernuni-hagen.de)

Abstract: Self-adjoint operators and their spectra play a crucial rôle in analysis and physics. For instance, in quantum physics self-adjoint operators are used to describe measurements and the spectrum represents the set of possible measurement results. Therefore, it is a natural question whether the spectrum of a self-adjoint operator can be computed from a description of the operator. We prove that given a "program" of the operator one can obtain positive information on the spectrum as a compact set in the sense that a dense subset of the spectrum can be enumerated (or equivalently: its distance function can be computed from above) and a bound on the set can be computed. This generalizes some non-uniform results obtained by Pour-El and Richards, which imply that the spectrum of any computable self-adjoint operator is a recursively enumerable compact set. Additionally, we show that the spectrum of compact self-adjoint operators can even be computed in the sense that also negative information is available (i.e. the distance function can be fully computed). Finally, we also discuss computability properties of the resolvent map.

Key Words: Computable functional analysis
Category: F.1.1, F.4.1, G.1

1 Introduction

A linear operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ over some field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ is called *self-adjoint*, if $T = T^*$ where T^* is the *adjoint operator* of T, which is the unique operator that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$. Any self-adjoint operator is *normal*, which means that $T^*T = TT^*$ holds, and for complex Hilbert spaces our results will be applicable to this larger class of operators, which also contains all *unitary operators*, i.e. operators, which satisfy $T^* = T^{-1}$. We will apply all these notions also to partial operators $T :\subseteq \mathcal{H} \to \mathcal{H}$, but in this case we additionally demand that $\mathrm{dom}(T)$ is dense in $\mathcal{H}$.

The *spectrum* $\sigma(T)$ is the set of all values $\lambda \in \mathbb{F}$ such that $\lambda - T$ has no inverse in the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators $T : \mathcal{H} \to \mathcal{H}$. In particular, all *eigenvalues* of T, i.e. all $\lambda \in \mathbb{F}$ such that there exists a non-zero $x \in \mathcal{H}$ with $Tx = \lambda x$, are elements of the spectrum. The spectrum is known to be a compact set and in absolute value it is bounded by $||T|| = \sup_{||x||=1} ||Tx||$, i.e.

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by the *operator norm* of the corresponding operator. For self-adjoint operators the spectrum is known to be real, i.e. $\sigma(T) \subseteq \mathbb{R}$ (see [Wei80] for statements regarding linear operators on Hilbert spaces).

From the perspective of computable analysis a natural question is whether the spectrum of a self-adjoint operator can be computed in some natural sense. In general, there are at least two variants of such a result, which could be of interest, a uniform and a non-uniform one:

1. (Uniform) the map $T \mapsto \sigma(T)$ is computable,
2. (Non-uniform) $\sigma(T)$ is computable for any computable T.

It is clear that any uniform result implies the corresponding non-uniform one (as computable maps map computable inputs to computable outputs). However, in general the uniform result is much stronger. What complicates things here is that the spectrum and also the operator might be computable in different senses. Regarding operators we will typically represent them in a way, which corresponds to programs (i.e. we will rather use the compact-open topology and not the operator norm topology). Regarding compact subsets, we will consider two variants of computability: one, which only includes positive information on the compact set and another one, which includes also negative information.

The main result of this paper, presented in Section 4, is that for self-adjoint operators in the real case and normal operators in the complex case we obtain the following uniform and non-uniform results:

1. (Uniform) the map $T \mapsto \sigma(T)$ is lower semi-computable (that is computable with respect to positive information on $\sigma(T)$),
2. (Non-uniform) $\sigma(T)$ is a recursively enumerable compact set for any computable T.

The non-uniform version of this result also follows from the Second Main Theorem of Pour-El and Richards [PER89, PER87]. We will also prove in Section 5 that the result cannot be strengthened to recursive compactness, since any recursively enumerable compact set can be represented as the spectrum of some computable normal operator. This is in contrast to the finite-dimensional case where the spectrum map $T \mapsto \sigma(T)$ is computable in a stronger sense [ZB01, ZB04]. However, as we will see in Section 6, the above result can also be strengthened to full computability if the operator T is additionally compact. In this case we obtain:

1. (Uniform) the map $T \mapsto \sigma(T)$ is computable (that is computable with respect to positive and negative information on $\sigma(T)$),
2. (Non-uniform) $\sigma(T)$ is a recursive compact set for any computable T.

Both main results of this paper are based on a purely classical result presented in Section 3, which characterizes the distance function of the spectrum. Finally, in Section 7 we will discuss some computability properties of the *resolvent map* $R : (\lambda, T) \mapsto (\lambda - T)^{-1}$. Roughly speaking, we will show that the resolvent map is computable whenever the spectrum is recursive.

In the Conclusions we will discuss some related results in constructive analysis. In the following Section 2 we briefly introduce some required notions from computable analysis, the Turing machine based theory of computability and complexity, which is the approach that we will use throughout this paper (see [PER89, Ko91, Wei00] for comprehensive introductions).

2 Computable Hilbert Spaces

In this section we briefly introduce the required tools from computable analysis, which we will use in the following. For a more comprehensive introduction the reader is referred to [Wei00] and the other cited references. We will not introduce notions from functional analysis here and the reader is referred to standard textbooks in this case. In the following we will discuss operators $T :\subseteq \mathcal{H} \to \mathcal{H}$ on Hilbert spaces $\mathcal{H}$ and we are in particular interested in computable Hilbert spaces, which we define below (the inclusion symbol "$\subseteq$" indicates that T might be partial). In general we assume that $\mathcal{H}$ is defined over the field $\mathbb{F}$, which might either be $\mathbb{R}$ or $\mathbb{C}$. Throughout the paper, we assume that $\mathcal{H} \neq \{0\}$.

Definition 1. A *computable Hilbert space* $(\mathcal{H}, \langle . \rangle, e)$ is a separable Hilbert space $(\mathcal{H}, \langle . \rangle)$ together with a fundamental sequence $e : \mathbb{N} \to \mathcal{H}$ (i.e. the closure of the linear span of range(e) is dense in $\mathcal{H}$) such that the induced normed space is a computable normed space.

The induced normed space is the normed space with the norm given by $||x|| := \sqrt{\langle x, x \rangle}$. A *computable normed space* is a normed space such that the metric d induced by $d(x, y) := ||x - y||$ together with the sequence $\alpha_e : \mathbb{N} \to \mathcal{H}$, defined by $\alpha_e\langle k, \langle n_0, ..., n_k\rangle\rangle := \sum_{i=0}^{k} \alpha_{\mathbb{F}}(n_i)e_i$, form a computable metric space such that the linear operations (vector space addition and scalar multiplication) become computable. Here $\alpha_{\mathbb{F}}$ is a standard numbering of $\mathbb{Q}_{\mathbb{F}}$ where $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}$ in case of $\mathbb{F} = \mathbb{R}$ and of $\mathbb{Q}_{\mathbb{F}} = \mathbb{Q}[i]$ in case of $\mathbb{F} = \mathbb{C}$. We assume that there is some $n \in \mathbb{N}$ with $\alpha_{\mathbb{F}}(n) = 0$. Without loss of generality, we can even assume that $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$. A *computable metric space* X is a separable metric space together with a sequence $\alpha : \mathbb{N} \to X$ such that range(α) is dense in X and $d \circ (\alpha \times \alpha)$ is a computable (double) sequence of reals.

If not mentioned otherwise, then we assume that all computable Hilbert spaces $\mathcal{H}$ are represented by their Cauchy representation $\delta_{\mathcal{H}}$ (of the induced computable metric space). The *Cauchy representation* $\delta :\subseteq \Sigma^\omega \to X$ of a computable metric space X is defined such that a sequence $p \in \Sigma^\omega$ represents a point $x \in X$, if it encodes a sequence $(\alpha(n_i))_{i\in\mathbb{N}}$, which rapidly converges to x, where rapid means that $d(\alpha(n_i), \alpha(n_j)) < 2^{-j}$ for all $i > j$. All computability statements with respect to Hilbert spaces are to be understood with respect to the Cauchy representation. Given representations (i.e. surjective maps) $\delta :\subseteq \Sigma^\omega \to X$ and $\delta' :\subseteq \Sigma^\omega \to Y$, a map $f :\subseteq X \to Y$ is called *computable*, if there exists a computable $F :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $f\delta(p) = \delta' F(p)$ for all $p \in \text{dom}(f\delta)$.

It is clear that the inner product of any computable Hilbert space is a computable map.

Proposition 2. *The inner product* $\langle . \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{H}, (x, y) \mapsto \langle x, y \rangle$ *of any computable Hilbert space* $\mathcal{H}$ *is computable.*

Proof. This follows from the fact that the norm $||.|| : \mathcal{H} \to \mathbb{R}$ is a computable map for any computable normed space and the fact that the inner product satisfies the polar identities

$$\langle x, y \rangle = \frac{1}{4}(||x + y||^2 - ||x - y||^2)$$

in case of $\mathbb{F} = \mathbb{R}$ and

$$\langle x, y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2)$$

for $\mathbb{F} = \mathbb{C}$. □

3 The Spectrum of Self-Adjoint Operators

In this section we will provide a formula, which characterizes the distance function of the spectrum in terms of the norm of $\lambda - T$. This is a purely classical fact and we will have to use some facts from functional analysis. The first one is a characterization of the norm of the inverse operator (the straightforward proof is left to the reader, see, for instance, Ex. 5.14 in [Wei80]).

Lemma 3. *Let X and Y be normed spaces and let $T :\subseteq X \to Y$ be a linear bounded operator. The inverse operator $T^{-1} :\subseteq Y \to X$ exists and is bounded, if and only if* $\inf\{||Tx|| : x \in \mathrm{dom}(T), ||x|| = 1\} > 0$. *In this case*

$$||T^{-1}|| = \frac{1}{\inf\{||Tx|| : x \in \mathrm{dom}(T), ||x|| = 1\}}.$$

We denote by $r(T) = \limsup_{n\to\infty} ||T^n||^{\frac{1}{n}}$ the *spectral radius* of T. It is known (see Theorem 5.17 (c) and (d) in [Wei80]) that for arbitrary operators T in case of $\mathbb{F} = \mathbb{C}$ or self-adjoint operators in case of $\mathbb{F} = \mathbb{R}$ one obtains $r(T) = \sup|\sigma(T)|$ and for normal operators in general $r(T) = ||T||$ holds (see Theorem 5.44 in [Wei80]). By combining these two results, we obtain the following lemma.

Lemma 4. *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F}$ and let $T :\subseteq \mathcal{H} \to \mathcal{H}$ be a normal operator in case of $\mathbb{F} = \mathbb{C}$ or a self-adjoint operator in case of $\mathbb{F} = \mathbb{R}$. Then*

$$||T|| = \sup|\sigma(T)| = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

The next result characterizes the norm of the inverse of $\lambda - T$ in terms of the distance to the spectrum. In general, we define for a normed space X by $\mathrm{dist}_A(x) := \inf_{a\in A} ||x - a||$ the *distance function* of a set $A \subseteq X$. Although the following result is known (see V 3.16 and 3.31 in [Kat66]), we sketch the proof for completeness.

Lemma 5. *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F}$ and let $T :\subseteq \mathcal{H} \to \mathcal{H}$ be a normal operator in case of $\mathbb{F} = \mathbb{C}$ or a self-adjoint operator in case of $\mathbb{F} = \mathbb{R}$. Then*

$$||(\lambda - T)^{-1}|| = \frac{1}{\mathrm{dist}_{\sigma(T)}(\lambda)}$$

for all $\lambda \in \mathbb{F} \setminus \sigma(T)$.

Proof. If T is normal or self-adjoint, then $\lambda - T$ shares the corresponding property and if the latter map is injective than also $(\lambda - T)^{-1}$ shares the corresponding property for $\lambda \in \mathbb{C}$ or $\lambda \in \mathbb{R}$, respectively (by Theorems 5.42 (iii) and 4.21 in [Wei80]). We obtain by Lemma 4

$$||(\lambda - T)^{-1}|| = \sup |\sigma((\lambda - T)^{-1})|$$

for all $\lambda \in \mathbb{F} \setminus \sigma(T)$. Using the continuous functional calculus or the resolvent identities (see Ex. 5.27 in [Wei80]) one can also show that

$$\sigma((\lambda - T)^{-1}) = \left\{ \frac{1}{\lambda - \lambda'} : \lambda' \in \sigma(T) \right\},$$

which altogether implies

$$||(\lambda - T)^{-1}|| = \sup \left\{ \frac{1}{|\lambda - \lambda'|} : \lambda' \in \sigma(T) \right\} = \frac{1}{\inf\{|\lambda - \lambda'| : \lambda' \in \sigma(T)\}}$$

for all $\lambda \in \mathbb{F} \setminus \sigma(T)$, which proves the claim. □

The next result is the main result of this section and it provides an equation that even holds true for all λ. It combines the observations of Lemma 3 and Lemma 5.

Proposition 6. *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F}$ and $T :\subseteq \mathcal{H} \to \mathcal{H}$ a normal operator in case of $\mathbb{F} = \mathbb{C}$ or a self-adjoint operator in case of $\mathbb{F} = \mathbb{R}$. Then*

$$\mathrm{dist}_{\sigma(T)}(\lambda) = \inf\{||(\lambda - T)x|| : x \in \mathrm{dom}(T), ||x|| = 1\}$$

for all $\lambda \in \mathbb{F}$.

Proof. Let us assume that T is normal or self-adjoint, depending on whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. In both cases $\sigma(T) \neq \emptyset$ (by Theorem 5.17 (c) and (d) in [Wei80]). Moreover, $\mathrm{dom}(T) = \mathrm{dom}(\lambda - T)$. We proof the claimed equation by a case distinction.

1. Case: $\lambda \in \mathbb{F} \setminus \sigma(T)$. By definition of the spectrum, $(\lambda - T)^{-1}$ exists in $\mathcal{B}(\mathcal{H})$. By Lemma 3 and 5 we obtain

$$\mathrm{dist}_{\sigma(T)}(\lambda) = \frac{1}{||(\lambda - T)^{-1}||} = \inf\{||(\lambda - T)x|| : x \in \mathrm{dom}(T), ||x|| = 1\}.$$

2. Case: $\lambda \in \sigma(T)$. In this case $(\lambda - T)^{-1}$ does not exist in $\mathcal{B}(\mathcal{H})$. This can have two reasons, either $(\lambda - T)^{-1}$ does not exist as a (potentially partial) linear bounded operator in which case by Lemma 3

$$\inf\{||(\lambda - T)x|| : x \in \mathrm{dom}(T), ||x|| = 1\} = 0$$

or it does exist but $\mathrm{range}(\lambda - T)$ is not dense in $\mathcal{H}$. However, the latter cannot happen for injective normal operators $\lambda - T$ as they always have a dense range (see Theorem 5.42 (i) in [Wei80]). This implies the claim since $\mathrm{dist}_{\sigma(T)}(\lambda) = 0$ for $\lambda \in \sigma(T)$ as well. □

In the next section we will exploit this proposition in order to compute the spectrum map $T \mapsto \sigma(T)$ of suitable operators.

4 Computability of the Spectrum Map

In this section we prove that the spectrum map $T \mapsto \sigma(T)$ is computable for suitable operators T in a specific sense. We will represent operators $T : \mathcal{H} \to \mathcal{H}$ as points in $\mathcal{C}(\mathcal{H}, \mathcal{H})$ by using $[\delta_{\mathcal{H}} \to \delta_{\mathcal{H}}]$, i.e. the *standard function space representation* of continuous functions $T : \mathcal{H} \to \mathcal{H}$ induced by the Cauchy representation $\delta_{\mathcal{H}}$ of $\mathcal{H}$ (see [Wei00]). This representation rather corresponds to the compact-open topology and not to the operator norm topology on $\mathcal{B}(\mathcal{H})$. However, the compact-open topology is exactly the right one for our purposes, as it captures exactly the type of information, which is available by having a program for T. Later on, we will prove that one cannot improve the result by providing the operator norm as additional input information.

Finally, since the spectrum $\sigma(T)$ is a compact subset of $\mathbb{F}$, we will introduce a representation for the hyperspace of compact subsets $\mathcal{K}_<(\mathbb{F})$ (see [BW99, BP03] for representations of hyperspaces in general).

Definition 7. We define a representation $\delta_{\mathcal{K}_<(\mathbb{F})}$ of the set $\mathcal{K}_<(\mathbb{F})$ of non-empty compact subsets of $\mathbb{F}$ by

$$\delta_{\mathcal{K}_<(\mathbb{F})}\langle p, q\rangle = K :\iff [\delta_{\mathbb{F}} \to \rho_>](p) = \mathrm{dist}_K \text{ and } \max |K| \leq \rho(q)$$

for all $p, q \in \Sigma^\omega$.

That is, roughly speaking, a name for a compact set K with respect to this representation is a name for the distance function of the set K as an upper semi-continuous function plus a name for a bound of the set. Here ρ denotes the standard Cauchy representation of the reals, $\rho_>$ denotes the upper representation (where any real number is represented by a list of all upper rational bounds) and $[\delta_{\mathbb{F}} \to \rho_>]$ denotes the standard representation for upper semi-continuous functions $\mathbb{F} \to \mathbb{R}$ (see [Wei00] for precise definitions of all these representations). Now we are prepared to prove the main result of this paper, where we tacitly assume that all the spaces are endowed with the aforementioned representations.

Theorem 8. *Let $\mathcal{H}$ be a computable Hilbert space over $\mathbb{F}$. The spectrum map*

$$\sigma :\subseteq \mathcal{C}(\mathcal{H}, \mathcal{H}) \to \mathcal{K}_<(\mathbb{F}), T \mapsto \sigma(T)$$

is computable, where $\mathrm{dom}(\sigma)$ *is the set of all normal operators $T : \mathcal{H} \to \mathcal{H}$ in case of $\mathbb{F} = \mathbb{C}$ and of all self-adjoint operators in case of $\mathbb{F} = \mathbb{R}$.*

Proof. In Theorem 5.1 of [Bra06] it has been proved that one can compute an upper bound $b \geq ||T||$ of the operator norm of a given linear bounded operator $T : \mathcal{H} \to \mathcal{H}$. By Lemma 4 it follows that $\max |\sigma(T)| = ||T|| \leq b$.

Since the representation $[\delta_{\mathbb{F}} \to \rho_>]$ admits type conversion (see Theorem 3.3.15 in [Wei00]), it suffices to show that the map

$$(T, \lambda) \mapsto \mathrm{dist}_{\sigma(T)}(\lambda)$$

is $([[\delta_{\mathcal{H}} \to \delta_{\mathcal{H}}], \delta_{\mathbb{F}}], \rho_>)$–computable. By Proposition 6 we obtain

$$\mathrm{dist}_{\sigma(T)}(\lambda) = \inf_{x \in S(0,1)} ||(\lambda - T)x||$$

where $S(0,1) = \{x \in \mathcal{H} : ||x|| = 1\}$ is the sphere with radius 1. But the sphere $S(0,1)$ is a recursive closed subset of $\mathcal{H}$ as it has a computable distance function

$$\mathrm{dist}_{S(0,1)}(x) = |\,||x|| - 1|$$

and thus there exists a computable sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$, which is dense in $S(0,1)$ (see Theorems 3.7 and 3.8 in [BP03]). The map

$$(T, \lambda, x) \mapsto ||(\lambda - T)x||$$

is easily seen to be computable by using type conversion and evaluation properties and the fact that the norm $||\ || : \mathcal{H} \to \mathbb{R}$ is computable. Since $\inf : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is $(\rho^{\mathbb{N}}, \rho_>)$–computable, it follows that

$$(T, \lambda) \mapsto \mathrm{dist}_{\sigma(T)}(\lambda) = \inf_{n\in\mathbb{N}} ||(\lambda - T)x_n||$$

is upper semi-computable, as desired. This finishes the proof. □

Theorem 8 directly implies the following non-uniform version. We recall that a non-empty compact set $K \subseteq \mathbb{F}$ is called *recursively enumerable*, if it is a computable point in $\mathcal{K}_<(\mathbb{F})$.

Corollary 9. *Let $\mathcal{H}$ be a computable Hilbert space over $\mathbb{F}$ and let $T : \mathcal{H} \to \mathcal{H}$ be a computable operator that is normal in case of $\mathbb{F} = \mathbb{C}$ and self-adjoint in case of $\mathbb{F} = \mathbb{R}$. Then the spectrum $\sigma(T)$ is a recursively enumerable compact set.*

Theorem 8 leads to a number of questions:

1. Is it possible to get more information on the spectrum under the same conditions on the input?
2. Is it possible to get more information on the spectrum in case that more input information is provided?
3. Is it possible to restrict the spectrum map to a subset of operators for which one can get more information on the spectrum?

We will deal with these questions and other topics in subsequent sections.

5 Spectral Representation of Compact Sets

In this section we will discuss the question whether the computability result on the spectrum is optimal. In particular, one can ask whether the spectrum of a self-adjoint computable operator is even a recursive compact set. That this is not the case in general, follows from the following theorem. This theorem will not be expressed for general Hilbert spaces (which is impossible as we will see) but just for the space ℓ_2, which is the set of sequences $(x_n)_{n\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}}$ bounded in the ℓ_2–norm, which is generated by the inner product $\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \rangle := \sum_{i=0}^{\infty} x_i y_i^*$. Together with the sequence $(e_n)_{n\in\mathbb{N}} \in \ell_2^{\mathbb{N}}$ of unit vectors defined by $e_{n,k} := \delta_{nk}$ with the Kronecker symbol δ we obtain a computable Hilbert space $(\ell_2, \langle . \rangle, e)$.

Theorem 10. *The spectrum map $\sigma :\subseteq \mathcal{C}(\ell_2, \ell_2) \to \mathcal{K}_<(\mathbb{F}), T \mapsto \sigma(T)$, defined for normal operators $T : \ell_2 \to \ell_2$ in case of $\mathbb{F} = \mathbb{C}$ and for self-adjoint operators T in case of $\mathbb{F} = \mathbb{R}$, is computable and it admits a multi-valued computable right inverse $\mathcal{K}_<(\mathbb{F}) \rightrightarrows \mathcal{C}(\ell_2, \ell_2)$.*

Proof. Computability of the spectrum map follows from Theorem 8 since ℓ_2 is a computable Hilbert space. We have to show that there exists a computable multi-valued right-inverse. Given the distance function dist_K of a compact set $K \subseteq \mathbb{F}$ with respect to $[\delta_{\mathbb{F}} \to \rho_>]$, we can extract a sequence $(a_k)_{k\in\mathbb{N}}$, which is dense in K (see Theorems 3.7 and 3.8 in [BP03]). Given a sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathbb{F}$, which is dense in some compact set $K \subseteq \mathbb{F}$ and given a bound $b > 0$ such that $|x| \le b$ for all $x \in K$, we can effectively determine an operator $T : \ell_2 \to \ell_2$ with $Te_i = a_i e_i$ for all $i \in \mathbb{N}$. That is, T corresponds to the infinite diagonal matrix

$$\begin{pmatrix} a_0 & 0 & 0 & \dots \\ 0 & a_1 & 0 & \dots \\ 0 & 0 & a_2 & \\ \vdots & \vdots & & \ddots \end{pmatrix}.$$

It is clear that T is linear and $||T|| = \sup_{||x||=1} ||Tx|| \le b$. The last observation also shows that we can effectively determine $T \in \mathcal{C}(\mathcal{H}, \mathcal{H})$, given $K \in \mathcal{K}_<(\mathbb{F})$ (see Theorem 4.3 (2) in [Bra03]). Moreover, T^* is just the diagonal operator induced by the complex conjugate sequence $(a_k^*)_{k\in\mathbb{N}}$ and hence T is normal and even self-adjoint in case of $\mathbb{F} = \mathbb{R}$. Moreover, we claim $\sigma(T) = K$. By Proposition 6 we obtain

$$\mathrm{dist}_{\sigma(T)}(\lambda) = \inf_{||x||=1} ||(\lambda - T)x||.$$

For our specific operator T it is not too hard to see that

$$\inf_{||x||=1} ||(\lambda - T)x|| = \inf_{k\in\mathbb{N}} |\lambda - a_k| = \mathrm{dist}_K(\lambda)$$

and thus we obtain $\sigma(T) = K$, since K and $\sigma(T)$ are both closed. □

In particular, this theorem tells us that any recursively enumerable compact set $K \subseteq \mathbb{F}$ can be represented by a normal linear bounded operator $T : \ell_2 \to \ell_2$ whose spectrum $\sigma(T)$ is just K.

Corollary 11. *A non-empty compact subset $K \subseteq \mathbb{F}$ is recursively enumerable, if and only if there exists a computable operator $T : \ell_2 \to \ell_2$, which is normal in case of $\mathbb{F} = \mathbb{C}$ or self-adjoint in case of $\mathbb{F} = \mathbb{R}$, with $\sigma(T) = K$.*

In this sense Theorem 8 is optimal. That is, under the same assumptions one cannot obtain a recursive spectrum in general. However, for finite-dimensional Hilbert spaces the situation is different. We recall that a non-empty compact set $K \subseteq \mathbb{F}$ is called *recursive*, if it is a computable point with respect to the following representation.

Definition 12. We define a representation $\delta_{\mathcal{K}(\mathbb{F})}$ of the set $\mathcal{K}(\mathbb{F})$ of non-empty compact subset of $\mathbb{F}$ by

$$\delta_{\mathcal{K}(\mathbb{F})}\langle p, q\rangle = K :\Longleftrightarrow [\delta_{\mathbb{F}} \to \rho](p) = \mathrm{dist}_K \text{ and } \max |K| \le \rho(q)$$

for all $p, q \in \Sigma^\omega$.

The following theorem follows from results in [ZB01, ZB04].

Theorem 13. *For any finite-dimensional Hilbert space $\mathcal{H}$, the spectrum map $\sigma :\subseteq \mathcal{C}(\mathcal{H},\mathcal{H}) \to \mathcal{K}(\mathbb{F}), T \mapsto \sigma(T)$, defined for all linear operators $T : \mathcal{H} \to \mathcal{H}$, is computable.*

This in turn shows that Theorem 10 cannot be generalized to arbitrary Hilbert spaces, but only to infinite-dimensional ones.

Another question regarding the optimality of Theorem 8 is whether it might help to increase the input information on the operator T. One meaningful additional input information would be the operator norm $||T||$ (not any computable operator has a computable norm). However, it does not help to add this input information since with the construction of the diagonal operator in the proof of Theorem 10 it is easy to obtain a counterexample with a computable norm.

Example 1. We use $\mathbb{F} = \mathbb{R}$. Let $a > 0$ be a left-computable but not right-computable real number and let $b > a$ be some computable real number. Then for $K = [0, a] \cup \{b\}$ there exists a computable self-adjoint operator $T : \ell_2 \to \ell_2$ such that its compact spectrum $\sigma(T) = K$ is recursively enumerable but not recursive and the operator norm $||T|| = b$ is computable.

However, in contrast to this, any suitable computable operator T with a computable spectrum has a computable norm, as we can conclude from Lemma 4 and the fact that the supremum of the recursive compact set $\sigma(T)$ under the computable map $|\ | : \mathbb{F} \to \mathbb{R}$ is computable (see Lemma 5.2.6 in [Wei00]).

Corollary 14. *Let $\mathcal{H}$ be a computable Hilbert space over $\mathbb{F}$ and $T : \mathcal{H} \to \mathcal{H}$ a computable operator, which is normal in case of $\mathbb{F} = \mathbb{C}$ or self-adjoint in case of $\mathbb{F} = \mathbb{R}$ and which has a recursive compact spectrum $\sigma(T)$. Then $||T||$ is computable.*

6 The Spectrum of Compact Operators

In this section we want to study the question whether there is any reasonable subclass of maps T for which the spectrum map $T \mapsto \sigma(T)$ becomes computable as a map with target space $\mathcal{K}(\mathbb{F})$, i.e. with full information on the spectrum. It turns out that compact operators are of this type. We recall that a linear operator $T : \mathcal{H} \to \mathcal{H}$ is called *compact*, if it maps bounded sets to relatively compact sets. This condition is equivalent to the condition that $\overline{T(S(0,1))}$ is compact, where $S(0,1) = \{x \in \mathcal{H} : ||x|| = 1\}$ denotes the unit sphere.

Before we study the spectrum of compact operators, we have to investigate some basic properties of compact operators. By $\mathcal{B}_\infty(\mathcal{H})$ we denote the set of compact operators $T : \mathcal{H} \to \mathcal{H}$. First of all, we prove that the space $\mathcal{B}_\infty(\mathcal{H})$ is a computable normed space with the operator norm and the dense subset given by the numbering

$$\alpha\langle k, \langle n_0, ..., n_k\rangle, \langle l_0, ..., l_k\rangle\rangle(x) := \sum_{i=0}^{k} \langle x, \alpha_e(n_i)\rangle \alpha_e(l_i).$$

Here we assume that $(e_i)_{i\in\mathbb{N}}$ is a computable orthonormal basis of $\mathcal{H}$ (which always exists) and α_e is the corresponding numbering of a dense subset of $\mathcal{H}$, as defined in Section 2. Then α is actually a numbering of certain finite rank operators $T_n : \mathcal{H} \to \mathcal{H}$, defined by $T_n(x) := \alpha(n)(x)$, which form a dense subset of $\mathcal{B}_\infty(\mathcal{H})$: on the one hand, the set of finite rank operators is dense in the set of compact operators with respect to the operator norm (by Theorem 6.5 in [Wei80]) and, on the other hand, any finite rank operator T can be represented as $Tx = \sum_{i=0}^{k}\langle x, x_i\rangle y_i$ with linearly independent $x_i \in \mathcal{H}$ and linearly independent $y_i \in \mathcal{H}$. Since range(α_e) is dense in $\mathcal{H}$, it follows that range(α) is dense in $\mathcal{B}_\infty(\mathcal{H})$. By $\delta_{\mathcal{B}_\infty(\mathcal{H})}$ we denote the corresponding Cauchy representation. We start with a basic observation, which helps to handle the numbering α. Note that $\langle.\rangle$ is used in two meanings: if applied to natural numbers, it stands for the canonical *Cantor pairing function*, if applied to objects in a Hilbert space, it stands for the inner product. No ambiguity is to be expected here.

Lemma 15. *Let $\mathcal{H}$ be a computable Hilbert space. There are computable functions $M : \mathbb{N} \to \mathbb{N}$ and $C : \mathbb{N} \to \mathbb{F}$ such that*

$$\alpha(n)(x) = \sum_{j=0}^{M(n)}\sum_{i=0}^{M(n)} C\langle n,i,j\rangle\langle x, e_i\rangle e_j$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{H}$ (and such that $C\langle n,i,j\rangle = 0$ for i or $j > M(n)$).

Proof. Given $n = \langle k, \langle n_0, ..., n_k\rangle, \langle l_0, ..., l_k\rangle\rangle$, it is clear that we can compute values $a_{hj}, b_{hj}, \in \mathbb{Q}_\mathbb{F}$ and $m_h, m'_h \in \mathbb{N}$ such that

$$\alpha_e(n_h) = \sum_{i=0}^{m_h} a_{hi}e_i \text{ and } \alpha_e(l_h) = \sum_{j=0}^{m'_h} b_{hj}e_j$$

for all $h = 0, ..., k$. Then M with $M(n) := \max\{m_h, m'_h : h = 0, ..., k\}$ is computable and we can assume that all yet undefined values of a_{hj}, b_{hj} are zero. A straightforward calculation using the linearity of the inner product shows that

$$\alpha(n)(x) = \sum_{h=0}^{k}\langle x, \alpha_e(n_h)\rangle\alpha_e(l_h) = \sum_{j=0}^{M(n)}\sum_{i=0}^{M(n)}\left(\sum_{h=0}^{k} a^*_{hi}b_{hj}\right)\langle x, e_i\rangle\, e_j.$$

Since by $C\langle n,i,j\rangle := \sum_{h=0}^{k} a^*_{hi}b_{hj}$ clearly a computable function C is defined, the claim follows. □

Now we prove that the space $\mathcal{B}_\infty(\mathcal{H})$ is actually a computable normed space, which will allow us to compute with compact operators easily.

Proposition 16. *Let $\mathcal{H}$ be a computable Hilbert space. Then $(\mathcal{B}_\infty(\mathcal{H}), ||\ ||, \alpha)$ is a computable normed space. In particular, the following operations are computable:*

1. $+ : \mathcal{B}_\infty(\mathcal{H}) \times \mathcal{B}_\infty(\mathcal{H}) \to \mathcal{B}_\infty(\mathcal{H}), (T, T') \mapsto T + T'$,
2. $\cdot : \mathbb{F} \times \mathcal{B}_\infty(\mathcal{H}) \to \mathcal{B}_\infty(\mathcal{H}), (\lambda, T) \mapsto \lambda T$.

Proof. We have to prove that the induced metric is computable on the dense subset. That is, given n and n', we have to compute $||\alpha(n)-\alpha(n')||$. We use the computable functions M and C from the previous lemma and we let $c_{ij} := C\langle n,i,j\rangle$, $c'_{ij} := C\langle n',i,j\rangle$ and $m := \max\{M(n), M(n')\}$. Now we obtain by the Theorem of Pythagoras and the assumption that $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis

$$\begin{aligned}||\alpha(n)-\alpha(n')||^2 &= \sup_{||x||=1}\left|\left|\sum_{j=0}^{m}\sum_{i=0}^{m}(c_{ij}-c'_{ij})\langle x,e_i\rangle e_j\right|\right|^2\\ &= \sup_{||x||=1}\sum_{j=0}^{m}\left|\sum_{i=0}^{m}(c_{ij}-c'_{ij})\langle x,e_i\rangle\right|^2\\ &= \sup_{s\in S_m}\sum_{j=0}^{m}\left|\sum_{i=0}^{m}(c_{ij}-c'_{ij})s_i\right|^2\end{aligned}$$

where $S_m := \{s=(s_0,...,s_m)\in\mathbb{F}^{m+1} : \sum_{i=0}^{m}|s_i|^2=1\}$. We note that the last equality holds since by Parseval's Identity $||x||^2=\sum_{i=0}^{\infty}|\langle x,e_i\rangle|^2$ but for our specific supremum above the Fourier coefficients $\langle x,e_i\rangle$ of x with index $i>m$ do not contribute anything to the sum. Now the last supremum is a supremum over a compact set, which can easily be computed (see[2] Corollary 6.2.5 in [Wei00]).

By definition of α it is clear that

$$\begin{aligned}&\alpha\langle k,\langle n_0,...,n_k\rangle,\langle l_0,...,l_k\rangle\rangle+\alpha\langle k',\langle n'_0,...,n'_{k'}\rangle,\langle l'_0,...,l'_{k'}\rangle\rangle\\ &=\alpha\langle k+k'+1,\langle n_0,...,n_k,n'_0,...,n'_{k'}\rangle,\langle l_0,...,l_k,l'_0,...,l'_{k'}\rangle\rangle\end{aligned}$$

and this shows that addition is computable with respect to α. Moreover, since

$$||(T+T')-(\alpha(t)+\alpha(t'))|| \leq ||T-\alpha(t)||+||T'-\alpha(t')||,$$

one can use approximations of T and T' in order to obtain an approximation of $T+T'$ of any desired quality. Altogether, this proves that addition is computable with respect to the Cauchy representation $\delta_{\mathcal{B}_\infty(\mathcal{H})}$. It is straightforward to show that the multiplication with scalars is computable as well. □

The next lemma shows that the Cauchy representation of the set of compact operators $\mathcal{B}_\infty(\mathcal{H})$ is natural in the sense that it allows to extract "programs". We will exploit this fact in the next section.

Lemma 17. *Let $\mathcal{H}$ be a computable Hilbert space. Then the injection*

$$\mathrm{inj} : \mathcal{B}_\infty(\mathcal{H}) \hookrightarrow \mathcal{C}(\mathcal{H},\mathcal{H})$$

is computable.

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[2] Since the dimension of the set S_m depends on the result of the computation, strictly speaking, one has to use here the fact that a compact image under a continuous function in ℓ_2 can be computed, which follows from Theorem 3.3 in [Wei03]. The sequence $(S_m)_{m\in\mathbb{N}}$ is a computable sequence in $\mathcal{K}(\ell_2)$, embedded in the obvious way.

Proof. Since the inner product is computable by Proposition 2, it follows from the definition of α that evaluation is computable with respect to α. But that together with the fact that $||Tx - \alpha(n)(x)|| \leq ||T - \alpha(n)|| \cdot ||x||$ implies that

$$\mathrm{ev} : \mathcal{B}_\infty(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}, (T, x) \mapsto Tx$$

is computable with respect to $\delta_{\mathcal{B}_\infty(\mathcal{H})}$. This in turn implies that the injection $\mathrm{inj} : \mathcal{B}_\infty(\mathcal{H}) \hookrightarrow \mathcal{C}(\mathcal{H}, \mathcal{H})$ is computable (see Lemma 3.3.14 in [Wei00]). □

Now we are prepared to prove the main result of this section, which shows that the spectrum map is fully computable for self-adjoint compact operators.

Theorem 18. *Let $\mathcal{H}$ be a computable Hilbert space over $\mathbb{F}$. Then the spectral map*

$$\sigma :\subseteq \mathcal{B}_\infty(\mathcal{H}) \to \mathcal{K}(\mathbb{F}), T \mapsto \sigma(T),$$

defined for all normal compact operators in case of $\mathbb{F} = \mathbb{C}$ and all self-adjoint compact operators in case of $\mathbb{F} = \mathbb{R}$, is computable.

Proof. Similarly, as in the proof of Theorem 8 it suffices to show that

$$F : \mathcal{B}_\infty(\mathcal{H}) \times \mathbb{F} \to \mathbb{R}, (T, \lambda) \mapsto \inf_{||x||=1} ||(\lambda - T)x||$$

is $([\delta_{\mathcal{B}_\infty(\mathcal{H})}, \delta_\mathbb{F}], \rho)$–computable, since by Proposition 6 we know that

$$\mathrm{dist}_{\sigma(T)}(\lambda) = \inf_{||x||=1} ||(\lambda - T)x||.$$

We first prove this fact for $T = \alpha(n)$. We use the computable functions M and C from Lemma 15 and we let $c_{ij} := C\langle n, i, j\rangle$ and $m := M(n)$. We use the fact that $c_{ij} = 0$ for all i and $j \geq m+1$ and similarly as in the proof of Proposition 16 we obtain

$$\begin{aligned}
\inf_{||x||=1} ||(\lambda - T)x||^2 &= \inf_{||x||=1} \left|\left| \sum_{j=0}^{\infty} \left(\lambda\langle x, e_j\rangle - \sum_{i=0}^{m} c_{ij}\langle x, e_i\rangle \right) e_j \right|\right|^2 \\
&= \inf_{||x||=1} \sum_{j=0}^{\infty} \left| \lambda\langle x, e_j\rangle - \sum_{i=0}^{m} c_{ij}\langle x, e_i\rangle \right|^2 \\
&= \inf_{s \in S_{m+1}} \sum_{j=0}^{m+1} \left| \lambda s_j - \sum_{i=0}^{m} c_{ij} s_i \right|^2,
\end{aligned}$$

where $S_m := \{s = (s_0, ..., s_m) \in \mathbb{F}^{m+1} : \sum_{i=0}^{m} |s_i|^2 = 1\}$. Thus, since the continuous image of the recursive compact sets S_m is computable, we can compute $\inf_{||x||=1} ||(\lambda - T)x||$ for $T = \alpha(n)$ for any given n and λ (as in the proof of Proposition 16).

For an arbitrary compact T and $||x|| = 1$ we obtain

$$||(\lambda - T)x - (\lambda - \alpha(n))x|| \leq ||T - \alpha(n)|| \cdot ||x|| = ||T - \alpha(n)||,$$

which implies that the above map F is computable. □

We say that a linear bounded operator $T : \mathcal{H} \to \mathcal{H}$ is *computably compact*, if and only if it is a computable point in $\mathcal{B}_\infty(\mathcal{H})$. We immediately obtain the following corollary of the previous theorem.

Corollary 19. *Let $\mathcal{H}$ be a computable Hilbert space over $\mathbb{F}$ and $T : \mathcal{H} \to \mathcal{H}$ a computably compact operator that is normal in case of $\mathbb{F} = \mathbb{C}$ and self-adjoint in case of $\mathbb{F} = \mathbb{R}$. Then $\sigma(T)$ is a recursive compact subset of $\mathbb{F}$.*

Here it is clear that in case of full computability of the spectrum there is no result, which corresponds to the Spectral Representation Theorem 10 of compact sets or its Corollary 11. This is because it is known that for compact operators the spectrum coincides in $\mathbb{F} \setminus \{0\}$ with the *point spectrum*, i.e. the set of eigenvalues and the spectrum is countable with 0 as only possible cluster point (see Theorem 6.7 in [Wei80]). Thus, for instance the recursive compact set $[0, 1]$ cannot be the spectrum of a computably compact operator. The fact that the spectrum is countable with only possible cluster point 0 also implies that any eigenvalue of a self-adjoint computably compact operator is computable (as any isolated point of an r.e. closed set is computable, see Proposition 3.6 in [BW99]). However, it is even known that in general the eigenvalues of a self-adjoint computable operator are computable [PER89, PER87].

7 Computability of the Resolvent Map

In this section we will briefly discuss computability properties of the *resolvent map* $R : (\lambda, T) \mapsto (\lambda - T)^{-1}$. Roughly speaking we show that the resolvent map is computable, whenever the spectrum is fully computable. We will use the following representation of the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators for this purpose (see [Bra03] for representations of linear bounded operators in general).

Definition 20. Let a representation $\delta_{\mathcal{B}(\mathcal{H})}$ of $\mathcal{B}(\mathcal{H})$ be defined by

$$\delta_{\mathcal{B}(\mathcal{H})}\langle p, q\rangle = T :\iff [\delta_{\mathcal{H}} \to \delta_{\mathcal{H}}](p) = T \text{ and } ||T|| = \rho(q)$$

for all $p, q \in \Sigma^\omega$.

Using the representation $\delta_{\mathcal{B}(\mathcal{H})}$ we can prove the following result on computability of the resolvent map.

Theorem 21. *Let $\mathcal{H}$ be a computable Hilbert space over $\mathbb{F}$ and let $\mathcal{B}_\sigma$ be a represented class of linear bounded operators $T : \mathcal{H} \to \mathcal{H}$, which are normal in case of $\mathbb{F} = \mathbb{C}$ or self-adjoint in case of $\mathbb{F} = \mathbb{R}$ and such that the maps*

1. $\mathrm{inj} : \mathcal{B}_\sigma \hookrightarrow \mathcal{C}(\mathcal{H}, \mathcal{H})$ *and*
2. $\sigma : \mathcal{B}_\sigma \to \mathcal{K}(\mathbb{F})$ *are computable.*

Then the resolvent map

$$R :\subseteq \mathbb{F} \times \mathcal{B}_\sigma \to \mathcal{B}(\mathcal{H}), (\lambda, T) \mapsto (\lambda - T)^{-1}$$

with $\mathrm{dom}(R) = \{(\lambda, T) : \lambda \in \mathbb{F} \setminus \sigma(T), T \in \mathcal{B}_\sigma\}$ *is computable as well.*

Proof. First of all, we note that the inversion map

$$\iota :\subseteq \mathcal{C}(\mathcal{H},\mathcal{H}) \times \mathbb{R} \to \mathcal{C}(\mathcal{H},\mathcal{H}), (T,b) \mapsto T^{-1},$$

defined for all (T,b) such that $T : \mathcal{H} \to \mathcal{H}$ is a bijective linear bounded operator and $||T^{-1}|| \leq b$, is computable (see Theorem 6.9 in [Bra01]). By Lemma 5

$$||(\lambda - T)^{-1}|| = \frac{1}{\mathrm{dist}_{\sigma(T)}(\lambda)}$$

holds for all $\lambda \in \mathbb{F} \setminus \sigma(T)$ and for all suitable T. If $\mathcal{B}_\sigma$ is a represented set of suitable operators such that the spectrum map σ and inj are computable, then it follows from the fact that $\mathbb{F} \times \mathcal{C}(\mathcal{H},\mathcal{H}) \to \mathcal{C}(\mathcal{H},\mathcal{H}), (\lambda, T) \mapsto \lambda - T$ is computable and

$$R(\lambda,T) = (\lambda - T)^{-1} = \iota \left(\lambda - T, \frac{1}{\mathrm{dist}_{\sigma(T)}(\lambda)}\right)$$

that the resolvent map R restricted to $\lambda \in \mathbb{F} \setminus \sigma(T)$ and $T \in \mathcal{B}_\sigma$ is computable as well. Strictly speaking, ι yields $(\lambda - T)^{-1}$ only as a point in $\mathcal{C}(\mathcal{H},\mathcal{H})$, but the additional information $||(\lambda - T)^{-1}||$, which is required to obtain $(\lambda - T)^{-1}$ as point in $\mathcal{B}(\mathcal{H})$ is available by the equation above. □

Now $\mathcal{B}_\sigma$ can be chosen, for instance, as the set of all normal or self-adjoint compact operators, depending on whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, since by Lemma 17 and Theorem 18 it follows that the requirements are satisfied. We note that in the finite-dimensional case all computable operators are automatically computably compact.

Corollary 22. *Let $\mathcal{H}$ be a computable Hilbert space over $\mathbb{F}$. Then*

$$R :\subseteq \mathbb{F} \times \mathcal{B}_\infty(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), (\lambda, T) \mapsto (\lambda - T)^{-1},$$

defined for all (λ, T) such that $\lambda \in \mathbb{F} \setminus \sigma(T)$ and $T : \mathcal{H} \to \mathcal{H}$ is a compact operator, which is normal in case of $\mathbb{F} = \mathbb{C}$ or self-adjoint in case of $\mathbb{F} = \mathbb{R}$, is computable.

Theorem 21 might have other applications besides that for compact operators. An example is the application to the set $\mathcal{B}_\sigma = \{T\} \subseteq \mathcal{C}(\mathcal{H},\mathcal{H})$ for a single computable operator $T : \mathcal{H} \to \mathcal{H}$ of suitable type and with computable spectrum.

Corollary 23. *Let $\mathcal{H}$ be a computable Hilbert space over $\mathbb{F}$ and let $T : \mathcal{H} \to \mathcal{H}$ be a computable operator, which is normal in case of $\mathbb{F} = \mathbb{C}$ and self-adjoint in case of $\mathbb{F} = \mathbb{R}$ and such that the spectrum $\sigma(T)$ is a recursive compact set. Then*

$$R(.,T) :\subseteq \mathbb{F} \to \mathcal{B}(\mathcal{H}), \lambda \mapsto (\lambda - T)^{-1}$$

with $\mathrm{dom}(R(.,T)) = \mathbb{F} \setminus \sigma(T)$ *is computable.*

It is clear that this result is optimal in the sense that a recursively enumerable spectrum would not suffice in general, as the following example shows.

Example 2. We use $\mathbb{F} = \mathbb{R}$. Let $a > 0$ be a left-computable but not right-computable real number. Then there is a self-adjoint operator $T : \ell_2 \to \ell_2$ with $\sigma(T) = [0, a]$. Let $\lambda > a$ be some computable real number. Then $\lambda \notin \sigma(T)$ and hence

$$||(\lambda - T)^{-1}|| = \frac{1}{\text{dist}_{\sigma(T)}(\lambda)} = \frac{1}{\lambda - a}$$

is not a computable real number either. Thus, $R(\lambda, T)$ is not a computable point in $\mathcal{B}(\ell_2)$.

Finally, we note that similarly, as Theorem 21 one could prove that the resolvent map $R :\subseteq \mathbb{F} \times \mathcal{B}_\sigma \to \mathcal{C}(\mathcal{H}, \mathcal{H})$ of self-adjoint operators is computable in the complex case without the requirement that the spectrum is recursive, but restricted to $\lambda \in \mathbb{C} \setminus \mathbb{R}$, since in this case an upper bound for $||(\lambda - T)^{-1}||$ can be computed using

$$||(\lambda - T)^{-1}|| \leq \left| \frac{1}{\text{Im}(\lambda)} \right|$$

(which holds true by Theorem 5.18 in [Wei80]).

8 Conclusions

In this paper we have studied questions regarding the computability of the spectrum of self-adjoint linear operators from the point of view of computable analysis. Roughly speaking, we have shown that the spectrum map of self-adjoint operators (or normal operators in the complex case) on computable Hilbert spaces is lower semi-computable and the map is even fully computable restricted to compact operators. In a certain sense our uniform results are effectivizations of corresponding topological properties, which have been studied in perturbation theory (see [Kat66]).

We have also investigated the question in which sense this result is optimal and we have seen that lower semi-computability cannot be strengthened in the general case. Finally, we have seen that the resolvent map is computable, provided the spectrum map is. The non-uniform versions of our results are generalizations of what has been proved by Pour-El and Richards [PER87, PER89]. Pour-El and Richards have also studied further questions regarding computability of eigenvalues and the sequence of eigenvalues. Uniformizations of some of these results can be found in [Dil05].

It is an interesting question how our results are related to similar results, which have been established in constructive analysis (see [BB85, BI96, Ish91]) and up to which extend our results could be obtained via the realizability interpretation of constructive analysis (see [Bau00, Lie04] for this topic). We will not be able to answer this question conclusively here, but we can give a number of pointers.

It seems that Proposition 2.6 in [BI96], which states that the approximative spectrum of a self-adjoint operator on a separable Hilbert space is separable, comes closest to our Theorem 8. In [BI96] it has also been shown that it can be proved intuitionistically that the approximative spectrum coincides with the spectrum (for self-adjoint operators), if and only if Markov's prinicple holds. It is known that Markov's principle is valid in computable analysis (see e.g.

Section 3.2.1 in [Lie04]). Another result in [BI96] is the Brouwerian Counterexample 1.1, which shows that the statement "any normable self-adjoint operator on ℓ_2 with located range has a compact approximative spectrum" implies the limited principle of omniscience. This fact is underlined by our results in Section 3, which show that the realizability interpretation of the above statement is not valid in computable analysis.

Moreover, Corollary 3.5 in [BI96] seems to be the constructive counterpart of our Theorem 18 with the slight difference that this and the aforementioned constructive results do not capture normal operators. The development of compact operators in constructive analysis (see [Ish91]) follows a different line than our approach in Section 6. Whereas for our representation of compact operators closure under addition comes for free and the fact that bounded sets are mapped to compact ones effectively, requires some efforts. This is just the other way around in the constructive approach. The results in Sections 3 and 6 demonstrate how computable analysis can import results from classical functional analysis in order to get certain shortcuts in proofs, which are not available in a rigorously intuitionistic approach.

Another interesting result, namely Theorem 2.8 in [BI96] states that for any self-adjoint operator T with bound $b > 0$ it holds that $||f(T)||$ exists for each $f \in \mathcal{C}[-b, b]$, if and only if the approximative spectrum of T is compact. Via realizability theory this result should translate to the interesting non-uniform statement that $||f(T)||$ is computable for any computable $f \in \mathcal{C}[-b, b]$, if and only if the spectrum $\sigma(T)$ is a recursive compact set (a uniform version could be obtained as well).

Finally, it would also be interesting to study the relation between perturbation theory and constructive analysis. Via a continuous realizability interpretation results from constructive analysis should yield corresponding results in perturbation theory. Under which conditions can one go the other way around? These and other questions regarding the realizability interpretation still deserve further attention.

From the computable analysis perspective, there are still a lot of open questions in spectral theory and regarding compact operators. Constructive analysis can be a rich source of inspiration here, but there are also many questions, which will probably not be answered by a realizability interpretation of existing results in constructive analysis, such as the question whether any compact computable operator is automatically computably compact or which degree of non-computability the sequence of eigenvalues has.

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