Computability Results Used in Differential Geometry

Barbara F. Csima* and Robert I. Soare†

June 30, 2006

Contents

*The first author was partially supported by Canadian NSERC Discovery Grant 312501.

†The second author was partially supported by National Science Foundation Grants DMS 98-02619 and 00-99556. Both authors presented some of these results in invited talks to special sessions of the American Mathematical Society

Abstract

Topologists Nabutovsky and Weinberger discovered how to embed computably enumerable (c.e.) sets into the geometry of Riemannian metrics modulo diffeomorphisms. They used the complexity of the settling times of the c.e. sets to exhibit a much greater complexity of the depth and density of local minima for the diameter function than previously imagined. Their results depended on the existence of certain sequences of c.e. sets, constructed at their request by Csima and Soare, whose settling times had the necessary dominating properties. Although these computability results had been announced earlier, their proofs have been deferred until this paper.

Computably enumerable sets have long been used to prove *undecidability* of mathematical problems such as the word problem for groups and Hilbert's Tenth Problem. However, this example by Nabutovsky and Weinberger is perhaps the first example of the use of c.e. sets to demonstrate specific *mathematical or geometric complexity* of a mathematical structure such as the depth and distribution of local minima.

1 Introduction

1.1 Computability and Differential Geometry

1.1.1 Astonishing Richness of Riemannian Metrics

In the book from his Porter lectures, topologist and geometer Shmuel Weinberger [2005] explained the significance of the present interaction between computability and differential geometry on the space Riem(M) of Riemannian metrics (modulo diffeomorphisms) on certain smooth, compact manifolds M, a space which is of interest to a variety of mathematicians and physicists. Nabutovsky and Weinberger describe in their paper [2003] on fractals *"the astonishing richness of the space of Riemannian metrics on a smooth manifold, up to reparametrization."* This "astonishing richness" of the space depends on two main parts: constructing a sequence of c.e. sets with a certain complexity of the settling time functions; and embedding c.e. sets into the geometric space using diverse results of mathematics, group theory, differerential topology, differential geometry, and other items.

1.1.2 Computably Enumerable Sets and Unsolvability

First, Nabutovsky and Weinberger expanded a long tradition of embedding c.e. sets into mathematical objects; unsolvability of the word problem and triviality problem for finitely presented groups; associating such a group with the fundamental group of a manifold; the unsolvability of the homeomorphism problem for manifolds, and much more. However, instead of merely obtaining unsolvable problems in a new area of mathematics, they linked sequences of computably enumerable (c.e.) sets W_n, and their settling times to the geometry of this space. Remarkably they related the halting time of the Turing machine enumerating W_n to the *depth and distribution of local minima* for certain functions on the space such as the diameter function as explained in §1.4. (Previously, embeddding c.e. sets into a given mathematical structure had been used primarily to show *undecidability* of some associated theory, not the mathematical or geometric *complexity* of that structure.)

1.1.3 C. E. Sets and Geometric Complexity

Second, Weinberger asked Soare to prove a specific result about sequences of computably enumerable sets so that the complexity of halting times of the Turing machines could be transferred into the geometric complexity of the local minima. Soare constructed the required sequence of c.e. sets. Later Weinberger asked for a sequence with a stronger property in order to simplify the geometric part of the proof. Csima [2003] constructed this stronger sequence. In the present paper we prove the Main Theorem 5.1 which builds a sequence of c.e. sets $\{A_n\}_{n\in\omega}$ which combines these two results and further generalizes them by making the sequence decrease in Turing degree as well. We give more details of the Nabutovsky-Weinberger results and their relation to c.e. sets and degrees in §1.4.

1.1.4 C.E. Turing Degrees and Depth of Minima

The main domination property requested by Weinberger of Soare and Csima was that the settling time of A_n dominates that of A_{n+1}, denoted by $A_n >_{st} A_{n+1}$, and defined below in Definition 1.5. However, the original results of Nabutovsky and Weinberger used the Sacks density theorem as we explain in §1.4.1. This is because their full results, quoted in §1.4.2 and explained further in Soare [2004, §9], relate the Turing *degree* of a c.e. set (not merely the structure of the c.e. set itself) to the depth and distribution of the local minima of the diameter function on $A\ell(M)$. Therefore, it is of interest when constructing the sequence $\{A_n\}_{n\in\omega}$ to consider *both* partial orderings $A_n >_{st} A_{n+1}$ and $A_n >_{\mathrm{T}} A_{n+1}$ and this is accomplished in our Main Theorem 5.1 developed by Csima and Soare jointly.

In an upcoming paper, Csima and Shore [ta] show that any partial ordering embeds into the $>_{st}$ ordering. This result also implies the required result for the Nabutovsky Weinberger work, though it does not control Turing degree.

1.1.5 Settling Time Reducibility and *ibT*-Reducibility

In addition, the partial ordering $>_{st}$ and its underlying ordering, *identity bounded Turing reducibility* ($>_{ibT}$), were of interest to Weinberger and had previously been studied in computability theory and in Kolmogorov complexity.[1] The special case of *identity bounded Turing reducibility* ($B \leq_{ibT} A$) if $A \leq_{bT} B$ with $h(x)$ the identity, has been explicitly identified in Soare [2004], but the concept has often been used in the literature for decades. For example, the standard permitting method in Soare [1987, p. 85] to build a simple set B computable in a noncomputable c.e. set A arranges that x enters B only when some $y \leq x$ enters A. This ensures not only that $B \leq_{\mathrm{T}} A$, but in fact that $B \leq_{ibT} A$, although this is not always explicitly mentioned. (See Definition 2.1 for bT and ibT.)

1.1.6 Kolmogorov Complexity and *ibT*-reducibility

The *ibT*-reducibility is also used in Kolmogorov complexity. Lewis and Barmpalias begin their papers [ta] and [ta2] with a definition of ibT and in [ta1] they explain, “This gives a reducibility which is complexity sensitive and which, in particular, preserves most notions of randomness for binary strings.” They go on to relate *ibT* to the “computably Lipschitz” ($B \leq_{cl} A$) condition where the bound is $h(x) = x + c$ for some constant c called ($B \leq_{sw} A$), a linear condition studied by Downey, Hirschfeldt, and LaForte, who wrote in *Randomness and Reducibility* [2004, p. 5],

[1] In the literature bT-reducibility has also been called weak-truth-table(wtt) reducibility, viewed as a weakening of tt-reducibility which is already a strengthening of Turing reducibility, whereas bT reaches the same destination in one step from Turing reducibility.

> "We begin with sw-reducibility, which has some nice features but also some shortcomings. It is related to a reducibility [*ibT*] recently studied by Soare [2004] and Csima [2003] in connection with computability-theoretic notions arising from the work of Nabutovsky and Weinberger [2003] in differential geometry. Informally, sw-reducibility says that there is a natural way, with little compression, to produce the bits of one real from another. It agrees with Solovay reducibility on *strongly* c.e. reals, but is in general different. Recently Yu and Ding have proven a number of interesting results about sw-reducibility, one of which is that there is no maximum sw-degree of c.e. reals. ..."

The concept of *sw*-reducibility is natural and useful, as studied in Downey, Hirschfeldt, and LaForte [2004, §2], and as they explain on page 6 of §2.

1.1.7 Other Papers on Settling Time and *ibT*

Csima [2003] developed new results about these reducibilities and related them to earlier work in the subject by R.W. Robinson and others. Some of these results are given in §2. Further properties of the $>_{st}$ ordering will be given in Csima and Shore [ta].

1.2 Background Sources

All the computability results related to differential geometry and announced previously in Soare [2004], Csima [2003], Nabutovsky and Weinberger [2003], and Weinberger [2005] are contained in this paper. The notation and background results on computability theory can be found in Soare [1987] and the forthcoming book Soare [cta] on computability theory and applications.

Soare [2004] gives a account of the Nabutovsky-Weinberger results in [NW, 2000] and [NW, 2003] and provides a background for logicians of the topology, number theory, and differential geometry needed to understand these results. There in §9 the Nabutovsky-Weinberger results are stated and very brief sketches of some of the main ideas of the proofs are given. Weinberger [2005] gives a very interesting description of the main mathematical areas needed to understand the proof including logic. This provides much more explanation and intuition into the geometry and topology than the original papers by Nabutovsky and Weinberger [2000] and [2003]. However, the present paper can be read without knowledge of the geometry because the results here are purely computability theoretic, and deal with:

(1) properties of the ordering $>_{st}$ in §2; and (2) the Main Theorem 5.1 on the existence of dominating sequences.

1.3 Notation

We use the notation of Soare [cta] which is an update of that in Soare [1987], defining any new notation not in Soare [1987]. The following notation appears not only in Soare [cta], but also in Cooper [2004] and is becoming standard.

Definition 1.1 (i) Let P_e be the e^{th} Turing program and let φ_e denote the e^{th} partial computable (p.c.) function, that computed by P_e.

(ii) We let $\widehat{P}_e$ be the e^{th} oracle Turing program and $\Phi_e^A(x) = y$ the partial function computed by $\widehat{P}_e$ with A on the oracle tape on input x if it halts with output y. We call Φ_e the *Turing reduction (Turing functional)* defined by oracle Turing program $\widehat{P}_e$.

(iii) Corresponding to $\Phi_e^A(x) = y$ is the *use function* $\varphi_e^A(x) = u$, where u is the maximum argument in the characteristic function of A scanned (used) during the computation.

Convention 1.2 The p.c. function $\varphi_e(x)$ in part (i) should *never* be confused with the A-computable use function $\varphi_e^A(x)$ in part (iii). The latter *always* has a supersript of the corresponding oracle set A, but the former *never* does.

1.3.1 Quantifiers, Domination, and Setting Time

Definition 1.3 (i) $(\exists^\infty x)\, R(x)$ abbreviates $(\forall y)(\exists z > y)\, R(z)$, which is read, "there exist infinitely many x such that $R(x)$."

(ii) $(\forall^\infty x)\, R(x)$ abbreviates $(\exists y)(\forall z > y)\, R(z)$, which is read, "for almost every x we have $R(x)$," and is often written as "$(a.e.\ x)$" in text, while $(\forall^\infty x)$ is used in displayed equations. (These quantifiers are dual to each other because $(\forall^\infty x)\, R(x)$ holds iff $\neg(\exists^\infty x)\neg R(x)$.)

(iii) For a set A (and similarly for a function f) we define two restrictions,

$$A\restriction x = \{\, y \in A \ :\ y < x \,\} \quad \text{and} \quad A\upharpoonright\!\!\upharpoonright x = \{\, y \in A \ :\ y \leq x \,\}.$$

(iv) For strings $\sigma, \tau \in 2^\omega$ we write $\sigma \subset \tau$ if σ is an proper initial segment of τ, and write $\sigma < \tau$ if $\sigma \subset \tau$ or if

$$(\exists i)\, [\ \sigma(i) < \tau(i) \quad \& \quad \sigma\restriction i \ =\ \tau\restriction i\ ].$$

Definition 1.4 (i) Function g *dominates* function f, written $f <^* g$, if

$$(\forall^\infty x)\,[\, f(x) < g(x)\,].$$

(ii) f *escapes (domination by)* g if $f \not<^* g$, *i.e.,*

$$(\exists^\infty x)\,[\, g(x) \leq f(x)\,].$$

(iii) An infinite *set* $A = \{a_0 < a_1 < \cdots\}$ dominates or escapes g according as its principal function p_A does, where $p_A(n) = a_n$.

1.3.2 Modulus Function and Settling Time Ordering

Definition 1.5 (i) For every computably enumerable c.e. set W_e define *modulus function,* as in Soare [cta], which is also called in this paper and in Soare [2004] the *settling function,*

$$m_e(x) = (\mu s)\,[\, W_{e,s}\upharpoonright x = W_e\upharpoonright x\,] \tag{1}$$

with respect to the canonical enumeration $\{W_{e,s}\}_{e,s\in\omega}$. If A is a c.e. set with enumeration $\{A_s\}_{s\in\omega}$ understood, we write the modulus

$$m_A(x) = (\mu s)\,[\, A_s\upharpoonright x = A\upharpoonright x\,].$$

(ii) A c.e. set A with enumeration $\{A_s\}_{s\in\omega}$ *settling-time dominates* a c.e. set B with enumeration $\{B_s\}_{s\in\omega}$ (written $A >_{st} B$) if

$$(\forall \text{ computable } f)(\forall^\infty x)[\ m_A(x)\ >\ f(m_B(x))\ ]. \tag{2}$$

Andre Nies showed that (ii) is independent of the choice of enumerations, which we generalize in §2.

(iii) A uniformly c.e. sequence $\{A_n\}_{n\in\omega}$ of c.e. sets is a *settling-time dominating sequence* if

$$(\forall n)[\ A_n\ >_{st}\ A_{n+1}\ ]. \tag{3}$$

(iv) If g is a computable function, then the sequence $\{A_n\}_{n\in\omega}$ is a *settling-time g-dominating sequence* if place of (2) we have

$$(\forall n)(\forall \text{ computable } f)(\forall^\infty x)\,[\ m_{A_n}(x)\ >\ f(m_{A_{n+1}}(g(x)))\ ]. \tag{4}$$

1.4 The Nabutovsky-Weinberger Results

1.4.1 The Sacks Density Theorem and Infinite Injury

Before seeing the Soare result, Nabutovsky and Weinberger had used the Sacks density theorem as they explain in [2003] Theorem 11.1 page 25. They used the fact that if A and B are c.e. sets with enumerations which have modulus functions $m_A(x)$ and $m_B(x)$ and if $A <_{\mathrm{T}} B$, then for every computable function f the modulus function $m_B(x)$ escapes $f \circ m_A(x)$, *i.e.*,

$$(\exists^\infty x)\,[\, f \circ m_A(x) \leq m_B(x)\,], \tag{5}$$

because otherwise $B \leq_{\mathrm{T}} A$. Using this and the Sacks density theorem which they cite in [2003] Theorem 11.1 page 25 they can get an infinite sequence $\{A_n\}_{n\in\omega}$ with the weak escape ordering of (5). From this they concluded that the associated basins (local minima) corresonding to A_n were infinitely often much deeper than those corresponding to A_{n+1} even when the latter is composed with any arbitrary computable function f. This produces a settling time escape sequence in place of our stronger settling-time dominating sequence of Definition 1.5.

This surprising connection to the structure of local minima in differential geometry is probably the first application of an existing theorem in computability theory (the Sacks Density Theorem) which was applied to obtain structural results in differential geometry (as opposed to merely undecidability results). It was also the first application in differential geometry of any theorem on c.e. degrees proved with the infinite injury method. The advantage of the Sacks density theorem is that they did not have to explicitly construct the sequence $\{A_n\}_{n\in\omega}$, but rather they could apply the Sacks theorem infinitely many times to get a sequence off the shelf. The disadvantage is that it had each settling time exceed the next only infinitely often, not for almost all x.

1.4.2 Deeper Local Minima Almost Everywhere

In spite of partial success with the Sacks density theorem and local minima greater on *infinitely many* arguments, Nabutovsky and Weinberger wanted the sequence $\{A_n\}_{n\in\omega}$ to have the property that the settling time of each was much more complex for *almost every* argument, not merely infinitely many, so they turned to Soare. Later to simplify their proof they asked the question which Csima anwered, improving the Soare sequence. Nabutovsky and Weinberger cited the result of Soare on dominating settling-time sequences as they write in Nabutovsky-Weinberger [2003] "Section 11. C.E.

Sets," on page 24. On page 26 of [2003] they write "Theorem (R. Soare)", and they are even more explicit.

In [2003] "Section 12. First Fractal Properties of Met(M)" in Theorem 0.1 (Rigorous version) on page 27 they again cite Soare's c.e. sets and then they state the following remarkable theorem which relates the notions of "dense" and "deep" to c.e. degrees. Nabutovsky and Weinberger write the following.

> "In order to use Soare's c.e. sets β_i explained in the previous section we need the following stronger c.e. set version of our Theorem 0.1 (and which is, in fact, the version we proved):"

> "THEOREM 0.1 (Rigorous version). Let M be a closed smooth manifold of dimension $n > 4$. Let S be any c.e. set. Let T denote the halting function of a Turing machine τ enumerating S. There exist a constant $c(n) > 0$, depending only on n, and increasing unbounded computable functions f and g, $(f < g)$, such that for all sufficiently large x, the number of local minima of the diameter, D, on $Al(M)$, such that the value of the diameter does not exceed x and of depth between $f(T([x]))$ and $g(T([x]))$ is at least $[exp(c(n)x^n)]$. Moreover, these $f(T([x]))$-deep local minima form a $g(T([x]))$-dense in the path metric subset of $D^{-1}((0,x]) \subset Al(M))$. These minima are $C^{1,\alpha}$-smooth Riemannian structures on M for any $\alpha \in (0,1)$."

This means that for every suitable n-manifold there are infinitely many local minima of the diameter functional on the subset $A\ell(M)$ of Met(M). Moreover, there is a constant $c(n)$ depending only on n such that for every c.e. degree α the local minima of depth at least α are α-dense in the path metric on $A\ell(M)$, and the number of α-deep local minima where the diameter does not exceed d is not less than $\exp(c(n)d^n)$. For further explanation see Soare [2004].

1.4.3 Fractals

Using the results stated more precisely in §1.5 and explained in §1.4.4 Nabutovsky and Weinberger observed the fractal nature of the local minima. That is, A_n determines an infinite sequence of "basins" (local minima), A_{n+1} determines an infinite sequence of much smaller basins coming off of them, and the latter contain still smaller basins coming off the sides of them, and so on, where the relative size of one set of basins to the next exceeds any computable function. The fact that the second set of basins comes off the first comes from the relatively close distribution of the local minima explained in Soare [2004, §9]. In the preface of [2003] Nabutovsky and Weinberger wrote,

> "In particular, we will see that there are large 'basins' that have topology, and are repeated infinitely often within the space (and even, in some sense, 'all over the space'). On the other hand, the structure is rather more complicated than what is usually associated with fractals. There seem to be infinitely many different sorts of basins with different geometries from each other."

For a discussion of the fractal nature of the results see Weinberger [2005] Part IV, *The Large Scale Fractal Geometry of Reimannian Moduli Space.* The dependence of the depth and distribution of the local minima on the settling function for a c.e. set and its degree were described in §1.4.2 and is further described in Soare [2004].

1.4.4 Computably Enumerable Degrees

The reason to consider c.e. *degrees* in Theorem 5.1 as well as modulus functions of c.e. sets in Corollary 1.7 is twofold. First Nabutovsky and Weinberger used the Sacks density theorem on c.e. degrees as a kind of first theorem to obtain the weaker sequence which is the settling-time *escaping sequence* (not dominating) as explained in §1.4.1.

Second Nabutovsky and Weinberger thought in terms of *degrees* not just sets in their applications to geometry. (See the α-dense and α-deep basins mentioned above for a c.e. degree α.) In their final draft of [2003] the computability theorem which they quote and apply is the increasing c.e. *degree* version, proved by Csima and Soare as Theorem 5.1. Nabutovsky and Weinberger [2003] §11 p. 26 wrote,

> " Theorem. There exists an infinite strictly increasing of c.e. degrees d_i and an infinite sequence of c.e. sets $\beta_i \in d_i$ with the following property: Let for any $i = 1, 2, \ldots, T_i$ be an arbitrary Turing machine enumerating β_i and h_i be its halting function. ...."

Nabutovsky and Weinberger *never* cite or apply the nondegree version Corollary 1.7 in either [2000] or [2003]. For more on the relation of c.e. degrees to the geometric properties see Nabutovsky and Weinberger [2003] of Soare [2004].

1.5 A Precise Statement of Our Main Theorem

To apply their results in differential geometry [2000], [2003] Nabutovsky and Weinberger asked Soare to construct a dominating sequence which he did, as described in Soare [2004]. To simplify their proof Weinberger then asked for a g-dominating sequence which Csima [2003] constructed. The proofs of the results cited in Soare [2004, §8] and also cited and used by Nabutovsky and Weinberger [2000], [2003] have been deferred to this paper. Combining these methods with those to ensure strictly decreasing Turing degree ($A_n >_{\mathrm{T}} A_{n+1}$). Csima and Soare obtained the Main Theorem 5.1 proved in §5 and restated here as Theorem 1.6.

Theorem 1.6 (Main Theorem) *For every computable function g there is a g-dominating sequence $\{A_n\}_{n\in\omega}$ such that $A_n >_{\mathrm{T}} A_{n+1}$ for every n.*

Corollary 1.7 *For every computable function g there is a g-dominating sequence $\{A_n\}_{n\in\omega}$.*

Because the proof for the general case of the theorem and even the corollary can become quite cumbersome in notation, we develop the main ideas through a series of special cases. In §3 we consider a Lachlan game between two players which captures the dynamic relationship $A >_{st} B$. We prove that if B is infinite and $A >_{st} B$, then A is high, but not necessarily complete. In §4 we build three c.e. sets A, B, C such that $A >_{st} B >_{st} C$ and $A >_{\mathrm{T}} B >_{\mathrm{T}} C$, and in §5 we extend this to an infinite sequence to prove the Main Theorem 5.1.

2 Examining the Ordering

We first examine basic properties of the ordering $<_{st}$ on c.e. sets.

Definition 2.1 (i) A set A is *bounded Turing reducible* to a set B ($A \leq_{bT} B$) if $A \leq_T B$ and there is a computable function $h(x)$ and a Turing reduction $A = \Phi_e^B$ with use function $\varphi_e^B(x) \leq h(x)$.

(ii) Set A is *identity bounded Turing reducible* to B ($A \leq_{ibT} B$) if $A \leq_{bT} B$ with $h(x) = x$, namely $A = \Phi_e^B$ with $\varphi_e^B(x) \leq x$ for all x.

Properties of the reducibilities bT and ibT are developed in Chapter 5 of Soare [cta], and here in §1.1.5 and §1.1.6. The following generalizes Nies's observation that the notion of $A >_{st} B$ does not depend on the particular enumerations of A and B.

Theorem 2.2 *Let A and B be c.e. sets, B infinite, with enumerations $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ respectively.*

(i) Suppose $A \leq_{bT} B$, with use bounded by h. Then there is a strictly increasing computable function f such that

$$(\forall x)[\, m_A(x) \leq f \circ m_B(h(x))\,].$$

(ii) Suppose $A \leq_{ibT} B$. Then there is a strictly increasing computable function f such that

$$(\forall x)[\, m_A(x) \leq f \circ m_B(x)\,].$$

Proof. Note that (ii) follows immediately from (i), so we prove (i). Say $A = \Phi_j^B$, with $\varphi_j^B(x) \leq h(x)$ for all x. We assume without loss of generality that h is non-decreasing and that $h(x) \geq x$ for all x. Let $f(0) = 0$. Let $f(n)$ be the max of $f(n-1)+1$ and t_n, where t_n is obtained as follows.

Find $s_n > n$ such that some z_n enters W_i at s_n. Let $t_n > s_n$ be the least t such that

$$A_t \upharpoonright z_n = \{y \leq z_n : \Phi_{j,t}^{B_t}(y) = 1\} \tag{6}$$

Note that t_n must exist since $A = \Phi_j^B$.

Verification: Suppose $n = m_B(h(x))$. Then since $s_n > n$, we have $z_n > h(x)$. Also, since $t_n > n$, we have $B_{t_n} \upharpoonright h(x) = B \upharpoonright h(x)$.

So by (6) and since $x \leq h(x) < z_n$, we have

$$A_{t_n} \upharpoonright x = \{y \leq x : \Phi_{j,t_n}^{B_{t_n}}(y) = 1\} \tag{7}$$

since $B_{t_n} \upharpoonright h(x) = B \upharpoonright h(x)$ and $\varphi_j^B(y) \leq h(y)$,

$$A_{t_n} \upharpoonright x = \{y \leq x : \Phi_j^B(y) = 1\} \tag{8}$$

$$= A \upharpoonright x \tag{9}$$

That is, $m_A(x) \leq t_n \leq f(n) = f(m_B(h(x)))$. ■

Theorem 2.3 *Let A, B, and C be c.e. sets with enumerations $\{A_s\}_{s\in\omega}$, $\{B_s\}_{s\in\omega}$, and $\{C_s\}_{s\in\omega}$, respectively. If $A \leq_{ibT} B$ and $B <_{st} C$ then $A <_{st} C$.*

Proof. We wish to show that for g computable, for a.e. x, $g(m_A(x)) < m_C(x)$. Note that it suffices to show this for g strictly increasing, so we assume this w.l.o.g. By Theorem 2.2, there is a f strictly increasing function such that for all x, $m_A(x) \leq f(m_B(x))$. So for all x, $g(m_A(x)) \leq g(f(m_B(x)))$. Since $B <_{st} C$, we have $g(f(m_B(x))) < m_C(x)$ for a.e. x. So for a.e. x, $g(m_A(x)) < m_C(x)$, as desired. ∎

Theorem 2.4 *Let A, B, and C be c.e. sets with enumerations $\{A_s\}_{s\in\omega}$,$\{B_s\}_{s\in\omega}$, and $\{C_s\}_{s\in\omega}$, respectively. If $A <_{st} B$ and $B \leq_{ibT} C$ then $A <_{st} C$.*

Proof. We wish to show for g computable, for a.e. x, $g(m_A(x)) < m_C(x)$. By Theorem 2.2, there is exists f strictly increasing such that for all x, $m_B(x) \leq f(m_C(x))$. Since $A <_{st} B$, $f(g(m_A(x))) < m_B(x)$ for a.e. x. Now since f is strictly increasing, this gives $g(m_A(x)) < m_C(x)$ for a.e. x as desired. ∎

Corollary 2.5 *$<_{st}$ is well-defined on ibT-degrees.*

Corollary 2.6 (Nies) *$<_{st}$ is well-defined on c.e. sets. That is, it is independent of the particular enumeration. So if $\{A_s\}_{s\in\omega}$ and $\{\tilde{A}_s\}_{s\in\omega}$ are enumerations of the same c.e. set $A = \tilde{A}$, $\{B_s\}_{s\in\omega}$ and $\{\tilde{B}_s\}_{s\in\omega}$ are enumerations of the same c.e. set $B = \tilde{B}$, and $A <_{st} B$, then $\tilde{A} <_{st} \tilde{B}$.*

Proof. If $A = \tilde{A}$ then $A \equiv_{ibT} \tilde{A}$, similarly for B. ∎

The same does not hold for Turing degrees, as is a consequence of the next theorem.

Theorem 2.7 *Given $A >_{st} B$, B infinite, there exists $C \equiv_T A$ such that $C \not>_{st} B$.*

Proof. We wish to construct a c.e. set $C \equiv_T A$ and a computable function g such that $g \circ m_B(x) \geq m_C(x)$ for infinitely many x. First we define a computable function f as follows:

$$f(x) = \max\{\ \{0\} \bigcup \{s \leq x \mid (\exists y \leq x)[y \in B_s - B_{s-1}]\ \}\ \}.$$

Note that f is a nondecreasing, unbounded computable function with $m_B(x) \geq f(x)$ for all x. Hence it suffices to build g computable, nondecreasing such that there are infinitely many x with $g \circ f(x) \geq m_C(x)$. We now construct g and C.

Construction:

Stage 0: For $k \in \omega$, set $x_k^0 = k$, $g_0 = \emptyset$.

Stage $s+1$: If there is some $k \in A_{s+1} - A_s$, enumerate x_k^s into C. Let $x_k^{s+1} = \mu y[f(y) \notin \text{dom} g_s]$. For $l > k$ let $x_l^{s+1} = x_k^{s+1} + (l-k)$. For $l \leq k$ let $x_l^{s+1} = x_l^s$. Let $g_{s+1}(x) = g_s(x)$ for all $x \in \text{dom} g_s$, and let $g_{s+1}(x) = s+1$ for all $0 \leq x \leq f(x_k^s)$ where g has not yet been defined.

If no element enters A at stage $s+1$, set $x_l^{s+1} = x_l^s$ for all l, and set $g_{s+1} = g_s$.

Let $g = \cup_s g_s$. Note that g is defined on progressively larger intervals.

Verification:

Let $x_k = \lim_{s\to\infty} x_k^s$. This limit is always finite since x_k^s is only redefined if some $l \leq k$ enters A.

Lemma 2.8 $A \equiv_T C$.

Proof. Note that $k \in A_{s+1} - A_s$ iff $x_k^s \in C_{s+1} - C_s$.

Suppose we want to know whether $y \in C$. A computes the stage s by which $A_s \upharpoonright y = A \upharpoonright y$. Then

$$y \in C \iff (\exists t \leq s)(\exists k \leq y)[\, k \in A_t - A_{t-1} \ \wedge\ y = x_k^t \,].$$

Suppose we want to know whether $k \in A$. As x_k^s is only redefined if some $x_l^s \leq x_k^s$ enters C, C computes the stage s by which $C_s \upharpoonright x_k^s = C \upharpoonright x_k$. Then $k \in A$ iff $(\exists t \leq s)[x_k^t \in C_t - C_{t-1}]$. ■

Lemma 2.9 $C \not>_{st} B$.

Proof. Claim: For all $k \in \omega$, $m_C(x_k) \leq g \circ m_B(x_k)$. Indeed, let $s = m_C(x_k)$. Note that $s = m_A(k)$. Suppose that $l < k$ is such that $l \in A_s - A_{s-1}$ (a unique such l exists). Then since $k > l$, $f(x_k^{s+1}) \notin \text{dom} g_s$ and so $g(f(x_k^{s+1})) > s$. As f and g are non-decreasing, and since $x_k \geq x_k^{s+1}$, we must have $g(f(x_k)) \geq g(f(x_k^{s+1})) \geq s$.

So $m_C(x_k) = s \leq g \circ f(x_k) \leq g \circ m_B(x_k)$, as desired. ■

■

Corollary 2.10 *$<_{st}$ is not well-defined on Turing degrees.*

In Csima and Shore [ta] it is shown that $<_{st}$ is not well-defined on 1-1 degrees.

3 Building $A >_{st} B$

3.1 The game for $A >_{st} B$

We wish to build two sets, $A >_{st} B$, in such a way that we can extend the method to build a whole chain of such. The idea is that we have control over both A and B, and we wish to ensure that A settles much more slowly than B. Indeed, we require that for each partial computable function φ_j, if φ_j is total then

$$(\forall^\infty x)\,[\; m_A(x) \;>\; \varphi_j(m_B(x))\;].$$

Viewing this as a game, where we win if we can build $A >_{st} B$, the only tools the opponent can use against us are the partial computable functions φ_j. We win by enumerating members into B very sparsely, so that for every x that enters B, there are many $x_j < x$ ready to enter A at a later stage, to guard against various possibly total functions φ_j. To illustrate the method, we prove the following theorem.

Theorem 3.1 *For any computable function g there exist infinite c.e. sets A and B such that for any computable function f*

$$(\forall^\infty x)\,[\; m_A(x) > f(m_B(g(x)))\;].$$

Proof. We first assume g is strictly increasing. For each $j \in \omega$ we meet the requirements:

$$P_j: \quad \varphi_j \text{ total} \implies (\forall^\infty x)[\; m_A(x) > \varphi_j(m_B(g(x)))\;].$$

We construct the sets A and B by stages as follows.

Stage s: Let α_s be greater than any number mentioned so far in the construction. Enumerate $g(\alpha_s + s)$ into B at stage s. If at a later stage t, $\varphi_{i,t}(s)\downarrow$ for some $i \le s$, enumerate $\alpha_s + i$ into A at stage t. We think of $\alpha_s + i$ as the "guard" ready to enter A if φ_i converges on s.

Verification:

Suppose f is any computable function. Then $f = \varphi_j$ for some j. Let t be the stage by which $B{\Uparrow}g(\alpha_j) = B_t{\Uparrow}g(\alpha_j)$.

Suppose x is such that $m_B(g(x)) \ge t$. This is true for a.e. x. To show that $m_A(x) > f(m_B(g(x)))$, it suffices to show that for each $y \le g(x)$ which enters B at a stage $s \geq t$, there is some $z \leq x$ that enters A at a stage greater than $f(s)$. Suppose $y \leq g(x)$ enters B at a stage $s \geq \max\{t, j\}$. Then $y = g(\alpha_s + s)$ by construction. Now $\alpha_s + j$ is enumerated into A at the stage where $\varphi_j(s)$ converges, which exists since φ_j is total. This stage is greater than $f(s)$ since $f(s) = \varphi_j(s)$ and $\varphi_j(s)$ must be less than the stage at which it converged. Note that $g(\alpha_s + s) = y \leq g(x)$, so since g is increasing $\alpha_s + s \leq x$. So $\alpha_s + j < x$, and so a number less than x was enumerated into A at a stage greater than $f(s)$. Thus $m_A(x) > \varphi_j(m_B(g(x))) = f(m_B(g(x)))$. ■

To show that the theorem holds for arbitrary computable g, we make use of the following definition.

Definition 3.2 For any computable function h, define h^* as follows. $h^*(0) = h(0)$, $h^*(n+1) = \max\{h(n+1), h^*(n)+1\}$. Then h^* is computable, increasing, and $(\forall x)[h^*(x) \geq h(x)]$.

Now suppose g is any computable function. Then by the above, there exist c.e. sets A and B so that for any computable f, for a.e. x, $m_A(x) > f(m_B(g^*(x)))$. In particular, for any computable f, for a.e. x we have

$$m_A(x) > f^*(m_B(g^*(x))) \geq f^*(m_B(g(x))) \geq f(m_B(g(x)))$$

So the result holds for arbitrary computable g. ■

This is just the basic game for the condition $A >_{st} B$, and in applications we shall use extensions of the game to control the Turing degree of A.

3.2 Dominant functions and e-dominant sets

We use the following definitions from Soare [1987] pages 208 and 214.

Definition 3.3 (i) A function f is *dominant* if it dominates every computable function g, namely $f(x) > g(x)$ for all but finitely many x.

(ii) A c.e. set A is *e-dominant* if $A = W_e$ for some e such that its settling function $m_e(x)$ is dominant.

By Martin's theorem (see Soare [1987, p. 208]) any e-dominant set A is high, namely $A' \equiv_T 0''$. (See the results on e-dominant sets by R.W. Robinson in Soare [1987, p. 214].)

Theorem 3.4 *If A and B are c.e. sets such that $A >_{st} B$ and B is infinite then A is e-dominant, and hence high.*

Proof. Let $\{A_s\}$ and $\{B_s\}$ be enumerations of A and B. Since B is infinite, we may choose an infinite computable subset $C \subseteq B$ such that $C = \{c_0 < c_1 < \ldots\}$. For all $k \in \omega$ let s_k be the stage at which c_k enters B. Note that the function $h(k) = s_k$ is computable, and we may choose C so that $h(k)$ is increasing. Let $\{\varphi_e\}_{e\in\omega}$ be an effective listing of all partial computable functions. If $\varphi_e(y)\downarrow$ for all $y \leq c_{k+1}$, then define $\psi_e(s_k) = \max\{\varphi_e(y) \mid y \leq c_{k+1}\}$. For $s_{k-1} \leq s < s_k$, define $\psi_e(s) = \psi_e(s-1)$. Suppose φ_e is total. Then so is ψ_e, so since $A >_{st} B$ there is some N such that

$$(\forall x \geq N)[\, m_A(x) \geq \psi_e(m_B(x))\,].$$

Choose $x \geq N$. Let k be such that $c_k \leq x < c_{k+1}$. Note that $m_B(c_k) \geq s_k$ and that ψ_e is non-decreasing. Then

$$m_A(x) \geq m_A(c_k) \geq \psi_e(m_B(c_k)) \geq \psi_e(s_k) \geq \varphi_e(x).$$

Thus m_A dominates φ_e. ∎

This method is well suited to force an infinite computable set into A but has limitations. To make ψ_e of any use one must threaten to make it total, namely arrange that the values are defined in order. This is not so well suited to more delicate coding, such as trying to improve the preceding theorem by showing that A is complete. In fact, this is not possible.

Theorem 3.5 *There are c.e. sets A and B such that $A >_{st} B$, B is infinite and computable, and A is incomplete ($A <_{\mathrm{T}} K$).*

Proof. This is an immediate consequence of the Main Theorem 5.1. ∎

4 Strictly increasing Turing degrees

Theorem 4.1 *There are c.e. sets A and B such that $A >_{st} B$, B is infinite, and $A >_T B$.*

Proof. In addition to meeting the requirements to make $A >_{st} B$, we must meet requirements to ensure $A \not\leq_T B$. That is we must meet the requirements

$$N_e : \quad A \neq \Phi_e^B.$$

This is not difficult. Since we control B, we just spread out B more, so that there are enough guards ready to enter A both for the sake of the P_e as in

Theorem 3.1, and also to diagonalize against the possibly total Φ_e^B. Again, since we control B, it is no problem to hold B on a segment in order to preserve a disagreement between A and Φ_e^B. ■

The above case was not enough to give a proper feel for constructing an entire sequence as in the main theorem, since there was no pressure to enumerate into B. Suppose now that we want to build three sets, $A >_{st} B >_{st} C$, with strictly increasing Turing degrees. Not worrying about the Turing degrees for a moment, it is easy to see that to build $A >_{st} B >_{st} C$, we just need to spread out C sufficiently so that there are sparse enough guards ready to enter B for some $P_e^{B>_{st}C}$, so that when these guards enter B there are guards ready to enter A for some $P_e^{A>_{st}B}$. To have $B \not\leq_T C$ is also easy, as before, since it is not a problem to spread out C more, and to hold C for the sake of keeping a computation. However, it is no longer so easy to ensure $A \not\leq_T B$. We can still spread out B (and so also C) to have enough witnesses to enter A in order to diagonalize, however, we can no longer simply hold B to maintain a disagreement, since the requirements $P_e^{B>_{st}C}$ may want us to enumerate into B. So now we must use a priority argument, noting that each φ_e is either total or is not.

Theorem 4.2 *There are c.e. sets A, B, and C such that $A >_{st} B >_{st} C$, $A >_T B >_T C$, and C is infinite.*

Proof. For each $e \in \omega$ we meet the requirements:

$$\begin{array}{llcl}
P_e^{A>_{st}B}: & \varphi_e \text{ total} & \Longrightarrow & (\forall^\infty x)[\, m_A(x) \geq \varphi_e(m_B(x))\,] \\
P_e^{B>_{st}C}: & \varphi_e \text{ total} & \Longrightarrow & (\forall^\infty x)[\, m_B(x) \geq \varphi_e(m_C(x))\,] \\
N_e^{A>_TB}: & A \neq \Phi_e^B & & \\
N_e^{B>_TC}: & B \neq \Phi_e^C & &
\end{array}$$

At each stage, we will enumerate a large number into C. We will also appoint numbers that may enter A and B, in the future, should certain events come to pass. If a number is appointed as a guard or witness to enter a set, it will only have one possible reason for doing so. We will enumerate numbers into A and B at stage $s+1$ if they had been appointed at a prior stage and the event they were waiting for has come to pass.

At stage $s+1$, we will enumerate into C, so as to make C infinite. We will choose a number c large enough so that there are $s+1$ many numbers less than c eligible to enter B for the sake of the $P_e^{B>_{st}C}$ for $e \leq s$, labeled

$c^B_0, ..., c^B_s$. A number b will be "eligible to enter B" if there are $s+1$ many numbers less than b ready to enter A for the sake of the $P_e^{A>_{st}B}$ for $e \leq s$, labeled $b^A_0, ..., b^A_s$. We will also appoint $s+1$ many witnesses eligible to enter B for the sake of $N_e^{B>_T C}$ for $e \leq s$, labeled $[B]_0^{s+1}, ..., [B]_{s+1}^{s+1}$, and 2^{s+1} many witnesses ready to enter A for the sake of $N_e^{A>_T B}$, labeled $[A]_\tau^{s+1}$ where τ is a string in 2^{e-1}, and $e \leq s$.

At stage s, α_s will be a string of length s guessing at which functions are total. For $\beta \in 2^\omega$ we call s a β-stage if $\beta \restriction s \subset \alpha_s$. Let $\alpha_0 = \emptyset$, so 0 is a β-stage for all β. For $s > 0$, we define $\alpha_s(i)$ by induction on i for $0 \leq i < s$ as follows. Let $\alpha_s(i) = 0$ iff φ_i increased its length of totality since t_s where $t_s = \max\{t \mid t < s \;\&\; t \text{ is an } \alpha_s\restriction i\text{-stage}\}$.

For $\beta \in 2^{<\omega}$, let $r_B(\beta, 0) = 0$. Let $R_B(\gamma, s) = \max\{r_B(\beta, s) \mid \beta \leq \gamma\}$.

Stage $s+1$: Enumerate c into C.

Action for "$P^{B>_{st}C}$" requirements: If $\varphi_{e,s+1}(s^C_d)\downarrow$, where d was enumerated into C at stage s^C_d, then if $e \leq s^C_d$, and $d^B_e > R_B(\alpha_s \restriction e, s)$, then if not already done so, enumerate d^B_e into B, and set $r_B(\beta, s+1) = 0$ for those $\beta > \alpha_s\restriction e$ for which $d^B_e < r_B(\beta, s)$.

Action for "$P^{A>_{st}B}$" requirements: If $\varphi_e(s^B_b)\downarrow$ at stage $s+1$, where b was enumerated into B at stage s^B_b, then if there was some b^A_e assigned, enumerate b^A_e into A.

Action for "$N^{B>_T C}$" requirements: Choose the least e such that there is some $[B]^k_e \notin B_s$, $[B]^k_e > R_B(\alpha_s\restriction e, s)$, and $(B \Uparrow [B]^k_e = \Phi^C_e \Uparrow [B]^k_e)[s]$. For that e, choose the least such k, and enumerate $[B]^k_e$ into B. Set $r_B(\beta, s+1) = 0$ for those $\beta > \alpha_s\restriction e$ for which $[B]^k_e < r_B(\beta, s)$.

Action for "$N^{A>_T B}$" requirements: Choose the least $e \leq s$ such that $r_B(\alpha_s\restriction e, s) = 0$ and there is $[A]^k_{\alpha_s\restriction e}$ with $(A \Uparrow [A]^k_{\alpha_s\restriction e} = \Phi^B_e \Uparrow [A]^k_{\alpha_s\restriction e})[s]$, and φ_j "spent" for all $j < e$ with $\alpha_s(j) = 0$. By φ_j being "spent", we mean that for every $c^B_j \leq \varphi^B_e([A]^k_{\alpha_s\restriction e})$, if $c^B_j > r_B(\alpha_s \restriction j, s)$, then $\varphi_j(s^C_c)\downarrow$ and c^B_j has already been enumerated into B. That is, requirement $P_j^{B>_{st}C}$ will not cause a change in B below the use. If such e exists, choose the least such k for that e, enumerate $[A]^k_{\alpha_s\restriction e}$ into A, set $r_B(\alpha_s\restriction e, s+1) = \varphi^B_e([A]^k_{\alpha_s\restriction e})$, and $r_B(\beta, s+1) = 0$ for those $\beta > \alpha_s\restriction e$ for which $[A]^k_{\alpha_s\restriction e} < r_B(\beta, s)$.

Set $r_B(\beta, s+1) = r_B(\beta, s)$ for all β for which it has not been defined.

Verification:

The $P_e^{A>_{st}B}$ are met. For any b enumerated into B after stage e, a b^A_e was assigned. And so if $\varphi_e(s^B_b)\downarrow$ at stage s, $b^A_e < b$ was enumerated into A at stage s.

Lemma 4.3 *lim inf $R_B(\alpha_s\restriction e, s) < \infty$ for all e. Indeed, we will show that* *there is an infinite set* T *of* true stages *such that* $\lim_{t \in T} R_B(\alpha_t \restriction e, t) < \infty$ *for all* e.

Proof. Let $f \in 2^\omega$ be the "true path" on our tree of guesses at the total partial functions. That is, $f(i) = 0$ iff φ_i is total. We'll say a stage t is a *true stage* if the length of agreement between f and α_t is longer than it has been at any previous stage.

We will show that for each e, $\lim_s r_B(f \restriction e, s) < \infty$. This will show that $\lim_{t \in T} R_B(\alpha_t \restriction e, t) < \infty$ since for β off the true path, either β will only be visited finitely often (there will be only finitely many β-stages), or $r_B(\beta, t)$ will be reset to 0 at every true stage.

Let s_0 be a true stage such that for all $j < e$, if φ_j is not total, then φ_j will never appear total after stage s. That is, $\alpha_{s_0} \restriction e = f \restriction e$ and $\alpha_s \restriction e \geq f \restriction e$ for all $s \geq s_0$. Let s_0 also be such that for all $j < e$, $r_B(f \restriction j, s_0) = \lim_s r_B(f \restriction j, s)$. Let $s_1 > s_0$ be a stage such that $r_B(f \restriction e, s_1) \neq r_B(f \restriction e, s_1 - 1) = 0$. If no such s_1 exists then $\lim_s r_B(f \restriction e, s) = r_B(f \restriction e, s_0)$ or $\lim_s r_B(f \restriction e, s) = 0$ and we are done. So assume s_1 exists, i.e. that $N_e^{A >_T B}$ received attention at stage s_1. The only way this could be injured is if at some stage $s > s_1$, $\alpha_s \restriction i < f \restriction e$, and either $P_i^{B >_{st} C}$ or $N_i^{A >_T B}$ received attention. We must have $i < e$ by the assumption on s_0. So $N_i^{A >_T B}$ cannot receive attention by induction hypothesis, and $P_i^{B >_{st} C}$ will not desire to enumerate below $r_B(f \restriction e, s_1)$ as if φ_i were total it would have appeared so at s_1 and there would be no $c_i < r_B(f \restriction e, s_1)$ that wasn't already enumerated into B. Thus $\lim_s r_B(\alpha_t \restriction e, s) = r_B(f \restriction e, s_1)$. ■

Note that it follows that the $N_e^{A >_T B}$ are satisfied. Indeed, this is clear if $\lim_{t \in T} r_B(\alpha_t \restriction e, t) \neq 0$. Suppose $\lim_{t \in T} r_B(\alpha_t \restriction e, t) = 0$. Assume for a contradiction that $A = \Phi_e^B$. Let $[A]_{f \restriction e}^k$ be least such that $[A]_{f \restriction e}^k \notin A$. Let t be a true stage such that A has settled up to $[A]_{f \restriction e}^k$ and B has settled on $\varphi_e^B([A]_{f \restriction e}^k)$, and $\alpha_t \restriction e = f \restriction e$. Now if $j < e$ and $\alpha_t(j) = 0$, then φ_j really was total, and so any $c_j^B > r_B(f \restriction j, s)$ would be enumerated into B. If $c_j^B \leq \varphi_e^B([A]_{f \restriction e}^k)$ then this must have happened by stage t since we know that the stage t approximation is correct. So at stage t, $[A]_{f \restriction e}^k$ would be enumerated into A, a contradiction.

To see that the $P_e^{B >_{st} C}$ are met, note that any $d_e^B > \liminf R_B(\alpha_s \restriction e, s)$ that wants to be will eventually be enumerated, and that if something is enumerated later than it first wanted to be, that doesn't hurt anything.

For the $N_e^{B >_T C}$, note that if any $[B]_e^k$ is ever enumerated, the diagonalization will be preserved (we won't change C). So suppose for a contradiction that $B = \Phi_e^C$. Let $[B]_e^k > \liminf R(\alpha_s \restriction e, s)$. Then at the next true stage after Φ_e^C converged up to $[B]_e^k$, $[B]_e^k$ would have been enumerated into B, a contradiction.

■

5 Proof of Main Theorem

We are now ready to prove the main theorem:

Theorem 5.1 (Main Theorem) *There is a dominating sequence* $\{A_n\}_{n\in\omega}$ *such that* $A_n >_{\mathrm{T}} A_{n+1}$ *for all* n. *(Furthermore, for every computable function* g*, the sequence may be chosen to be* g*-dominating.)*

Proof. For each $\langle e, n\rangle \in \omega$ we meet the requirements:

$$P_{\langle e,n\rangle}: \quad \varphi_e \text{ total } \implies (a.e.\ x)[\ m_{A_n}(x) \geq \varphi_e(m_{A_{n+1}}(x))\ ]$$
$$N_{\langle e,n\rangle}: \quad A_n \neq \Phi_e^{A_{n+1}}$$

At stage s, α_s will be a string of length s guessing at whether the requirements $P_{\langle e,n\rangle}$ will require infinite action. As before, for $\beta \in 2^\omega$ we call s a β-stage if $\beta \restriction s \subset \alpha_s$. Let $\alpha_0 = \emptyset$, so 0 is a β-stage for all β. For $s > 0$, we define $\alpha_s(i)$ by induction on i for $0 \leq i < s$ as follows. Let $\alpha_s(i) = 0$ iff $i = \langle e, n\rangle$ and φ_e increased its length of totality since t_s where $t_s = \max\{t \mid t < s\ \ \&\ \ t \text{ is an } \alpha_s{\restriction} i\text{-stage}\}$.

For $\beta \in 2^{<\omega}$, let $r_n^i(\beta, 0) = 0$. Let

$$R_n(\delta, s) = \max\{r_n^i(\beta, s) \mid \beta \leq \delta \quad \& \quad i \in \omega\}.$$

These will be the restraint functions for A_n.

Stage $s+1$: We consider the sets $A_0, ..., A_s$. For each $0 \leq n \leq s$ we appoint witnesses $[n]_\tau^{s+1}$ where $e \leq s$ and $\tau \in 2^{\langle e,n\rangle - 1}$, which may later enter A_n for the sake of $N_{\langle e,n\rangle}$. Also, for each $0 < n \leq s$ and for each number c we appoint to possibly enter A_{n+1}, we appoint guards $c_0^n, ..., c_s^n$, all less than c, to possibly enter A_n for the sake of $P_{\langle e,n\rangle}$. All the numbers appointed at this stage should be larger than any numbers mentioned so far in the construction. Note that since we are considering only finitely many sets, we can certainly arrange to appoint the possible future entrants in this fashion. Also note that any possible entrant of A_n of the form c_i^n is linked to a possible entrant of A_m of the form $[m]_\tau^{s+1}$ for some $n < m \leq s$.

Action for "P" requirements: If $\varphi_{e,s+1}(s_d^{n+1})\downarrow$, where d was enumerated into A_{n+1} at stage s_d^{n+1}, then if d_e^n was appointed, and $d_e^n > R_n(\alpha_s\restriction\langle e,n\rangle, s)$, then if not already done so, enumerate d_e^n into A_n, and set $r_n^i(\beta, s+1) = 0$ for those $\beta > m_s\restriction\langle e,n\rangle$, $i \in \omega$ for which $d_e^n \leq r_n^i(\beta, s)$.

Action for "N" requirements:

When we enumerate into A_n for the sake of some requirement $N_{\langle e,n\rangle}$, we can only restrain with priority $\langle e,n\rangle$. Hence we must be careful to put restraint onto the sets A_m with $m \geq n$, since if something of low priority is enumerated into A_m it could cause something of high priority to want to enter A_n at a later stage. Also, we wish to only believe computations that we don't think will be injured by higher priority requirements.

Choose the least $\langle e,n\rangle \leq s$ such that $r_{n+1}^{\langle e,n\rangle}(\alpha_s \restriction \langle e,n\rangle, s) = 0$ and there is an $[n]^k_{\alpha_s\restriction\langle e,n\rangle} > R_n(\alpha_s\restriction\langle e,n\rangle, s)$ with

$$A_n\Uparrow[n]^k_{\alpha_s\restriction\langle e,n\rangle} = \Phi_e^{A_{n+1}}\Uparrow[n]^k_{\alpha_s\restriction\langle e,n\rangle},$$

and a sequence $r_{n+1}, ..., r_s$ with the following properties:

1. $r_{n+1} = \varphi_e^{A_{n+1}}([n]^k_{\alpha_s\restriction\langle e,n\rangle})$
2. For $n+1 \leq m < s$, if $d_i^m < r_m$ then $r_{m+1} > d$.
3. For $n+1 \leq m < s$, if $d_i^m < r_m$, $\langle i,m\rangle \leq \langle e,n\rangle$, d has already entered A_{m+1}, and $\alpha_s(\langle i,m\rangle) = 0$, and $d_i^m > R_m(\alpha_s\restriction\langle i,m\rangle, s)$, then d_i^m has already entered A_m.

If such $\langle e,n\rangle$ exists, choose the least such k for that $\langle e,n\rangle$, enumerate $[n]^k_{\alpha_s\restriction\langle e,n\rangle}$ into A_n, and set $r_m^{\langle e,n\rangle}(\alpha_s\restriction\langle e,n\rangle, s+1) = r_m$ for $n < m \leq s$, $r_n^i(\beta, s+1) = 0$ for $\beta > \alpha_s\restriction\langle e,n\rangle$, and $r_m^i(\beta, s+1) = r_m^i(\beta, s)$ for all other β, m, i.

If no such $\langle e,n\rangle$ exists, set $r_m^i(\beta, s+1) = r_m^i(\beta, s)$ for all β, m, i.

Verification

Let $f \in 2^{<\omega}$ be the true path. That is, $f(\langle e,n\rangle) = 0$ iff φ_e is total. We call t a *true stage* if the length of agreement between α_t and f is longer than at any previous stage. We let T denote the set of true stages.

Lemma 5.2 *For all $\langle e,n\rangle$, $\lim r_n^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s) < \infty$. Therefore, for all $\langle e,n\rangle$, $\lim r_m^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s) < \infty$ and $\lim_{t\in T} R_n(\alpha_t\restriction\langle e,n\rangle, t) < \infty$.*

Proof. By induction. Suppose holds for all $\langle i,m\rangle < \langle e,n\rangle$. Let s_0 be a stage by which $\alpha_s\restriction\langle e,n\rangle \geq f\restriction\langle e,n\rangle$ for all $s \geq s_0$ and by which $r_m^{\langle i,m\rangle}(f\restriction\langle i,m\rangle, s)$

has reached its limit for all $s \geq s_0$, $\langle i,m\rangle < \langle e,n\rangle$. Let $s_1 > s_0$ be a stage such that $r_n^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s_1) \neq r_n^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s_1-1) = 0$. If no such s_1 exists then $\lim r_n^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s) = r_n^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s_0)$ or $\lim r_n^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s) = 0$ and we are done. So assume s_1 exists, i.e. that $N_{\langle e,n\rangle}$ received attention at stage s_1. We show that the restraints $r_m = r_m^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s_1)$ for $n \leq m \leq s_1$ are never injured. First note that nothing of the form $[m]^k_\tau$ will injure r_m since if $\langle i,m\rangle < \langle e,n\rangle$ then $[m]^k_\tau$ will not enter by induction hypothesis and if $\langle i,m\rangle > \langle e,n\rangle$ then $\alpha_s\restriction\langle i,m\rangle \geq \alpha_s\restriction\langle e,n\rangle \geq f\restriction\langle e,n\rangle$ for all $s \geq s_0$ and so if $\beta = \alpha_s\restriction\langle i,m\rangle$, $R_m(\beta,s) \geq r_m^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s)$. Hence r_{s_1} is never injured. Assume r_{m+1} is never injured. Suppose $d_i^m < r_m$. Then by (2), $d < r_{m+1}$. Since r_{m+1} is never injured, d_i^m is only a threat if d was enumerated into A_{m+1} by stage s_1. If $\langle i,m\rangle > \langle e,n\rangle$ then $\alpha_s\restriction\langle i,m\rangle \geq \alpha_s\restriction\langle e,n\rangle \geq f\restriction\langle e,n\rangle$ for all $s \geq s_0$ and so $R_m(\alpha_s\restriction\langle i,m\rangle, s) \geq r_m^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s)$. If $\langle i,m\rangle \leq \langle e,n\rangle$ then $\alpha_s\restriction\langle i,m\rangle \geq \alpha_{s_1}\restriction\langle i,m\rangle$, and so if $d_i^m > R_m(\alpha_s\restriction\langle i,m\rangle, s)$ then $d_i^m > R_m(\alpha_{s_1}\restriction\langle i,m\rangle, s_1)$ and so if $\alpha_s(\langle i,m\rangle) = 0$ then d_i^m has already been enumerated into A_m by (3). If $\alpha_{s_1}(\langle i,m\rangle) = 1$ then since $\langle i,m\rangle \leq \langle e,n\rangle$ and $\alpha_{s_1}\restriction\langle e,m\rangle = f\restriction\langle e,m\rangle$, φ_i won't converge on s_d and so d_i^m won't be enumerated into A_m. Hence r_m will never be injured. ■

Lemma 5.3 *The requirements $N_{\langle e,n\rangle}$ are met.*

Proof. By construction, certainly if $\lim r_{n+1}^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s) \neq 0$ then, by the previous lemma, requirement $N_{\langle e,n\rangle}$ was met. Assume for a contradiction that $A_n = \Phi_e^{A_{n+1}}$ and hence that $\lim r_{n+1}^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s) = 0$. Let s be a true stage such that for all $t \in T$, $t \geq s$, $\alpha_t\restriction\langle e,n\rangle = f\restriction\langle e,n\rangle$ and $R_n(f\restriction\langle e,n\rangle, t) = R_n(f\restriction\langle e,n\rangle, s)$. Let k be least such that $[n]^k_{\alpha_s\restriction\langle e,n\rangle} \notin A_n$, $[n]^k_{\alpha_s\restriction\langle e,n\rangle} > R_n(f\restriction\langle e,n\rangle, s)$ and $A_n \Vert [n]^k_{\alpha_s\restriction\langle e,n\rangle} = \Phi_e^{A_{n+1}} \Vert [n]^k_{\alpha_s\restriction\langle e,n\rangle}$. Such k must exist since $A_n = \Phi_e^{A_{n+1}}$ and $\lim r_{n+1}^{\langle e,n\rangle}(f\restriction\langle e,n\rangle, s) = 0$. Now let $t > s$ be a true stage such that $A_n \Vert [n]^k_{f\restriction\langle e,n\rangle}[t] = A_n \Vert [n]^k_{f\restriction\langle e,n\rangle}$ and

$$A_{n+1} \Vert \varphi_e^{A_{n+1}}([n]^k_{f\restriction\langle e,n\rangle})[t] \;=\; A_{n+1} \Vert \varphi_e^{A_{n+1}}([n]^k_{f\restriction\langle e,n\rangle}).$$

Let $r_{n+1} = \varphi_e^{A_{n+1}}([n]^k_{f\restriction\langle e,n\rangle})$. For $n+1 < n \leq t$ let r_{m+1} be minimal such that if $d_i^m < r_m$ then $r_{m+1} > d$. Let $t' > t$ be a true stage with $A_m \Vert r_m[t'] = A_m \Vert r_m$ for $n+1 \leq m \leq t$. For $t < m \leq t'$, let $r_m = r_t$. Then the sequence $r_{n+1}, ..., r_{t'}$ clearly satisfies (1) and (2) (since by construction any future member of A_m must be greater than r_t for $m > t$). The sequence also satisfies (3). Indeed, suppose $d_i^m < r_m$, $\langle i,m\rangle \leq \langle e,n\rangle$, d has already entered A_{m+1}, and $\alpha_t(\langle i, m\rangle) = 0$, and $d_i^m > R_m(\alpha_t \restriction \langle i, m\rangle, t)$. Note that $n + 1 \leq m < t$. Since $t > s$ is a true stage and $\alpha_t(\langle i, m\rangle) = 0$, φ_i is total. If $d_i^m > R_m(\alpha_t \restriction \langle i, m\rangle, t)$ then d_i^m will be enumerated into A_m at the next true stage after $\varphi_i(s_d^{m+1}) \downarrow$. But A_m has settled up to r_m, so $\varphi_i(s_d^{m+1})$ must already have converged and d_i^m must have been enumerated into A_m by stage t'. ■

Lemma 5.4 *The requirements $P_{\langle e,n\rangle}$ are met.*

Proof. For every possible entrant d of A_{n+1} appointed after stage e there was a $d_e^n < d$ appointed to possibly enter A_n. So for a.e. d which entered A_{n+1}, there was some $d_e^n < d$ appointed with $d_e^n > \lim_{t \in T} R(m_t \restriction \langle e, n\rangle, t)$. Hence, d_e^n was enumerated in A_n at the first stage after $\varphi_e(s_d) \downarrow$ that $R(\alpha_s \restriction \langle e, n\rangle, s) < d_e^n$. ■

If g is any computable function, the above proof can be modified to give a settling-time g-dominating sequence by ensuring that the guards $c_0^n, ..., c_s^n$ are such that $g(c_0^n), ..., g(c_s^n)$ are all less than c (this works if g is non-decreasing, which we may assume without loss of generality).

■

Question 6 *Does there exist a uniformly computably enumerable sequence $\{A_n\}_{n \in \omega}$ of c.e. sets that is settling-time g-dominating for all computable functions g?*

References

[1] [BL, ta]
G. Barmpalias and E.M. Lewis, The *ibT* degrees of computably enumerable sets are not dense, *Annals of Pure and Applied Logic*, to appear.

[2] [BL2, ta]
G. Barmpalias and E.M. Lewis, Randomness and the Lipschitz degrees of computability, to appear.

[3] [Cooper, 2004]
S.B. Cooper, *Computablility Theory*, Chapman & Hall/CRC Mathematics, London, New York, 2004.

[4] [Csima, 2003]
B. F. Csima, Applications of computability theory to prime models and differential geometry, Ph.D. thesis, The University of Chicago, 2003.

[5] [CS, ta]
B. F. Csima and R. A. Shore, The settling-time reducibility ordering, to appear.

[6] [DHL1, 2001]
R. Downey, D. Hirschfeldt, and G. LaForte, Randomness and Reducibility. Mathematical foundations of computer science, (2001), 316–327, *Lecture Notes in Comput. Sci.*, 2136, Springer, Berlin, 2001.

[7] [DHL2, 2004]
R. Downey, D. Hirschfeldt, and G. LaForte, Randomness and Reducibility, *Journal of Computer and System Sciences* **68** (2004) 96–114.

[8] [NW, 2000]
A. Nabutovsky and S. Weinberger, Variational problems for Riemannian functionals and arithmetic groups, Publications Mathématiques, Institut des Hautes Études Scientifiques, no. 92, (2000), 5–62.

[9] [NW, 2003]
A. Nabutovsky and S. Weinberger, The Fractal Nature of Riem/Diff I, *Geometrica Dedicata* **101** (2003), 1–54.

[10] [Soare, 1987]
R. I. Soare, *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets,* Springer-Verlag, Heidelberg, 1987.

[11] [Soare, 2004]
R. I. Soare, Computability theory and differential geometry, *Bull. Symb. Logic* **10** (2004), 457–486.

[12] [Soare, cta]
R. I. Soare, *Computability Theory and Applications,* Springer-Verlag, Heidelberg, to appear.

[13] [Weinberger, 2005]
S. Weinberger, *Computers, rigidity, and moduli. The large-scale frcatal geometry of Riemannian moduli space*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2005.

Department of Pure Mathematics
University of Waterloo
200 University Ave. W.
Waterloo, ON
Canada N2L 3G1
E-mail: csima@math.uwaterloo.ca
Internet: http://www.math.uwaterloo.ca/~csima

Department of Mathematics
University of Chicago
Chicago, Illinois 60637-1546
E-mail: soare@uchicago.edu
Internet: http://www.people.cs.uchicago.edu/~soare/