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Computable measure of entanglement

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We present a measure of entanglement that can be computed effectively for any mixed state of an arbitrary bipartite system. We show that it does not increase under local manipulations of the system, and use it to obtain a bound on the teleportation capacity and on the distillable entanglement of mixed states.

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I. INTRODUCTION

In recent years it has been realized that quantum mechanics offers unexpected possibilities in information transmission and processing, and that quantum entanglement of composite systems plays a major role in many of them. Since then, a remarkable theoretical effort has been devoted both to classifying and quantifying entanglement.

Pure-state entanglement of a bipartite system is presently well understood, in that the relevant parameters for its optimal manipulation under local operations and classical communication (LOCC) have been identified, in some asymptotic sense [1] as well as for the single-copy case [2]. Given an arbitrary bipartite pure state $|\psi_{AB}\rangle$, the *entropy of entanglement* $E(\psi_{AB})$ [1], namely, the von-Neumann entropy of the reduced density matrix $\rho_A \equiv \mathrm{Tr}_B|\psi_{AB}\rangle\langle\psi_{AB}|$, tells us exhaustively about the possibilities of transforming, using LOCC, $|\psi_{AB}\rangle$ into other pure states, in an asymptotic sense. When manipulating a single copy of $|\psi_{AB}\rangle$, this information is provided by the n entanglement monotones $E_l = \sum_{i=l}^{n}\lambda_i$ $(l=1,\ldots,n)$ [2], where λ_i are the eigenvalues of ρ_A in decreasing order.

Many efforts have also been devoted to the study of the mixed-state entanglement. In this case several measures have been proposed. The entanglement of formation $E_F(\rho)$ [3] —or, more precisely, its renormalized version, the entanglement cost $E_C(\rho)$ [4]— and the distillable entanglement $E_D(\rho)$ [3] quantify, respectively, the asymptotic pure-state entanglement required to create ρ, and that which can be extracted from ρ, by means of LOCC. The relative entropy of entanglement [5] appears as a third, related measure [6] that interpolates between E_C and E_D [7].

However, in practice, it is not known how to effectively compute these measures, nor any other, for a generic mixed state, because they involve variational expressions. To our knowledge, the only exceptions are Wootter's closed expression for the entanglement of formation $E_F(\rho)$ [and *concurrence* $C(\rho)$] of two-qubit states [8], and its single-copy analog $E_2(\rho)$ also for two qubits [9].

Multipartite pure-state entanglement represents the next order of complexity in the study of entanglement, and is of interest, because one hopes to gain a better understanding of the correlations between different registers of a quantum computer. Consider a tripartite state $|\psi_{ABC}\rangle$. Some of its entanglement properties depend on those of the two-party reduced density matrices, which are in a mixed state. For instance, the relative entropy of $\rho_{AB} \equiv \mathrm{tr}_C|\psi_{ABC}\rangle\langle\psi_{ABC}|$ has been used to prove that bipartite and tripartite pure-state entanglements are asymptotically inequivalent [10]. Thus, the lack of an entanglement measure that can be easily computed for bipartite mixed states is not only a serious drawback in the study of mixed-state entanglement, but also a limitation for understanding multipartite pure-state entanglement.

The aim of this paper is to introduce a *computable* measure of entanglement [11], and thereby fill an important gap in the study of entanglement. It is based on the trace norm of the partial transpose ρ^{T_A} of the bipartite mixed state ρ, a quantity whose evaluation is completely straightforward using standard linear algebra packages. It essentially measures the degree to which ρ^{T_A} fails to be positive, and therefore it can be regarded as a quantitative version of Peres' criterion for separability [12]. From the trace norm of ρ^{T_A}, denoted by $||\rho^{T_A}||_1$, we will actually construct two useful quantities. The first one is the *negativity*

$$\mathcal{N}(\rho) \equiv \frac{\|\rho^{T_A}\|_1 - 1}{2}, \tag{1}$$

which corresponds to the absolute value of the sum of negative eigenvalues of ρ^{T_A} [13], and which vanishes for unentangled states. As we will prove here, $\mathcal{N}(\rho)$ does not increase under LOCC, i.e., it is an entanglement monotone [14], and as such it can be used to quantify the degree of the entanglement in composite systems. We will also consider the *logarithmic negativity*

$$E_{\mathcal{N}}(\rho) \equiv \log_2 ||\rho^{T_A}||_1, \tag{2}$$

which again exhibits some form of monotonicity under LOCC (it does not increase during deterministic distillation protocols) and is, remarkably, an additive quantity.

The importance of $\mathcal{N}$ and $E_{\mathcal{N}}$ is boosted, however, beyond their practical computability by two results that link these measures with relevant parameters characterizing entangled mixed states. The negativity will be shown to bound the extent to which a single copy of the state ρ can be used, together with LOCC, to perform quantum teleportation [15].

In turn, the logarithmic negativity bounds the distillable entanglement E_D^ϵ contained in ρ, that is, the amount of "almost pure"-state entanglement that can be asymptotically distilled from $\rho^{\otimes N}$, where "almost" means that some small degree ϵ of imperfection is allowed in the output of the distillation process.

Remarkably, this last result has already found an application in the context of asymptotic transformations of bipartite entanglement [16], as a means to prove that [positive partial transposition (PPT)] bound entangled states [17] cannot be distilled into entangled pure states even if loaned (i.e., subsequently recovered for replacement) pure-state entanglement is used to assist the distillation process. In this way, the bound on distillability implied by $E_{\mathcal{N}}$ has contributed to prove that, in a bipartite setting, asymptotic local manipulation of the mixed-state entanglement is sometimes, in contrast to its pure-state counterpart, an inherently irreversible process.

We have divided this paper into seven sections. In Sec. II some properties of the negativity $\mathcal{N}$, such as its monotonicity under LOCC, and of the logarithmic negativity $E_{\mathcal{N}}$ are proved. We also discuss a more general construction leading to several other (nonincreasing under LOCC) negativities. In Secs. III and IV we derive, respectively, the bounds on teleportation capacity and on asymptotic distillability. Then in Sec. V we calculate the explicit expression of $\mathcal{N}$ and $E_{\mathcal{N}}$ for pure states and for some highly symmetric mixed states, also for Gaussian states of light field. In Sec. VI extensions of these quantities to multipartite systems are briefly considered, and Sec. VII contains some discussion and conclusions.

II. MONOTONICITY OF $\mathcal{N}(\rho)$ UNDER LOCC

In this section we show that the negativity $\mathcal{N}(\rho)$ is an entanglement monotone. We first give a rather detailed proof of this result. Then we sketch an argument extending this observation to several other similarly constructed negativities—e.g., the *robustness of entanglement* [18].

A. Definition and basic properties

From now on we will denote by ρ a generic state of a bipartite system with finite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ shared by two parties, Alice and Bob. ρ^{T_A} denotes the *partial transpose* of ρ with respect to Alice's subsystem, that is the Hermitian, trace-normalized operator defined to have matrix elements

$$\langle i_A, j_B|\rho^{T_A}|k_A, l_B\rangle \equiv \langle k|_A, j_B\rho|i_A, l_B\rangle \tag{3}$$

for a fixed but otherwise arbitrary orthonormal product basis $|i_A, j_B\rangle \equiv |i\rangle_A \otimes |j\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B$. The *trace norm* of any Hermitian operator A is $\|A\|_1 \equiv \mathrm{tr}\sqrt{A^\dagger A}$ ([19] Sec.VI 6), which is equal to the sum of the absolute values of the eigenvalues of A, when A is Hermitian [20]. For density matrices, all eigenvalues are positive and thus $\|\rho\|_1 = \mathrm{tr}\rho = 1$. The partial transpose ρ^{T_A} also satisfies $\mathrm{tr}[\rho^{T_A}] = 1$, but since it may have negative eigenvalues $\mu_i < 0$, its trace norm reads in general

$$\|\rho^{T_A}\|_1 = 1 + 2\left|\sum_i \mu_i\right| \equiv 1 + 2\mathcal{N}(\rho). \tag{4}$$

Therefore, the negativity $\mathcal{N}(\rho)$—the sum $|\Sigma_i \mu_i|$ of the negative eigenvalues μ_i of ρ^{T_A}—measures by how much ρ^{T_A} fails to be positive definite. Notice that for any separable or unentangled state ρ_s [21],

$$\rho_s = \sum_k p_k |e_k, f_k\rangle\langle e_k, f_k|; \quad p_k \geq 0, \sum_k p_k = 1, \tag{5}$$

its partial transposition is also a separable state [12]

$$\rho_s^{T_A} = \sum_k p_k |e_k^*, f_k\rangle\langle e_k^*, f_k| \geq 0, \tag{6}$$

and therefore $\|\rho_s^{T_A}\|_1 = 1$ and $\mathcal{N}(\rho_s) = 0$.

The practical computation of $\mathcal{N}(\rho)$ is straightforward, using standard linear algebra packages for eigenvalue computation of Hermitian matrices. On the other hand, this representation is not necessarily the best for proving estimates and general properties of $\mathcal{N}(\rho)$. To begin with a simple example, consider the property that $\mathcal{N}(\rho)$ does not increase under mixing

Proposition 1. $\mathcal{N}$ is a convex function, i.e.,

$$\mathcal{N}\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i \mathcal{N}(\rho_i), \tag{7}$$

whenever the ρ_i are Hermitian, and $p_i \geq 0$ with $\Sigma_i p_i = 1$.

There is nothing to prove here, when we write $\mathcal{N}(\rho) = (\|\rho^{T_A}\|_1 - 1)/2$, and observe that $\|\cdot\|_1$, as any norm, satisfies the triangle inequality and is homogeneous of degree 1 for positive factors, hence convex.

However, the fact that $\|\rho\|_1$ is indeed a norm is not so obvious, when it is defined in terms of the eigenvalues. This is shown best by rewriting it as a variational expression. Our reason for recalling this standard observation from the theory of the trace norm is that the same variational expression will be crucial for showing monotonicity under LOCC operations. The variational expression is simply the representation of a general Hermitian matrix A as a *difference* of positive operators: Since we are in finite dimension we can always write

$$A = a_+\rho^+ - a_-\rho^-, \tag{8}$$

where $\rho^\pm \geq 0$ are density matrices ($\mathrm{tr}[\rho^\pm] = 1$) and $a_\pm \geq 0$ are positive numbers. Note that by taking the trace of this equation we simply have $\mathrm{tr}[A] = a_+ - a_-$.

Lemma 2. For any Hermitian matrix A there is a decomposition of the form (8) for which $a_+ + a_-$ is minimal. For this decomposition, $\|A\|_1 = a_+ + a_-$, and a_- is the absolute sum of the negative eigenvalues of A.

Proof. Let P^- be the projector onto the negative eigenvalued subspace of A, and $\mathcal{N} = -\mathrm{tr}[AP^-]$ the absolute sum of the negative eigenvalues. We can reverse the decomposition (8) to obtain that $A + a_-\rho^-$ is positive semidefinite. This implies that

$$0 \leqslant \mathrm{tr}[(A + a_- \rho^-)P^-] = -\mathcal{N} + a_- \mathrm{tr}[\rho^- P^-]. \tag{9}$$

But $\mathrm{tr}[\rho^- P^-] \leqslant 1$, that is $a_- \geqslant \mathcal{N}$. This bound can be saturated with the choice $a_- \rho^- \equiv -P^- A P^-$ (corresponding to the Jordan decomposition of A, where ρ^- and ρ^+ have disjoint support), which ends the proof. □

For the negativity we, therefore, get the formula

$$\mathcal{N}(A) = \inf\{a_- | A^{T_A} = a_+ \rho^+ - a_- \rho^-\}, \tag{10}$$

where the infimum is over all density matrices $\rho^\pm$ and $a_\pm \geqslant 0$.

Another remarkable property of $\mathcal{N}(\rho)$ is the easy way in which $\mathcal{N}(\rho_1 \otimes \rho_2)$ relates to the negativity of ρ_1 and that of ρ_2. This relationship is an important, but notoriously difficult issue for discussing asymptotic properties of entanglement measures (see, e.g., [22] for a discussion and a counterexample to the conjectured additivity of the relative entropy of entanglement).

For the entanglement measure proposed in this paper we get additivity for free. We start from the identity $\|\rho_1 \otimes \rho_2\|_1 = \|\rho_1\|_1 \|\rho_2\|_1$, which is best shown by using the definition of the trace norm via eigenvalues, and we observe that partial transposition commutes with taking tensor products. After taking logarithms, we find for the logarithmic negativity

$$E_{\mathcal{N}}(\rho_1 \otimes \rho_2) = E_{\mathcal{N}}(\rho_1) + E_{\mathcal{N}}(\rho_2). \tag{11}$$

It might seem from this that $E_{\mathcal{N}}$ is a candidate for the much sought for canonical measure of entanglement. However, it has other drawbacks. For instance, it is *not* convex, as is already suggested by the combination of a convex functional (the trace norm) with the concave log function, which implies that it increases under some LOCC. And although it has an interesting, monotonic behavior during asymptotic distillation (as shown in Sec. IV), it does not correspond to the entropy of entanglement for pure states (see Sec. V).

B. Negativity as a mixed-state entanglement monotone

By definition, a LOCC operation (possibly for many parties) consists of a sequence of steps, in each of which one of the parties performs a local measurement and broadcasts the result to all other parties. In each round the local measurement chosen is allowed to depend on the results of all prior measurements. If at the end of a LOCC operation with initial state ρ the classical information available is "i," which occurs with probability p_i, and final state conditional on this occurrence is ρ_i', we require of an entanglement monotone [14] E that

$$E(\rho) \geqslant \sum_i p_i E(\rho_i'). \tag{12}$$

It is clear by iteration that this may be proved by looking at just one round of a LOCC protocol, consisting of a single local operation. In the present case, since $\mathcal{N}$ makes no distinction between Alice and Bob, it suffices to consider just one local measurement by Bob.

Now the most general local measurement is described by a family $\mathcal{M}_i$ of completely positive linear maps such that, in the notation used in the previous paragraph, $\mathcal{M}_i(\rho) = p_i \rho_i'$. These maps satisfy the normalization condition $\Sigma_i \mathrm{tr}[\mathcal{M}_i(\rho)] = \mathrm{tr}(\rho)$. This can be further simplified [14] when some $\mathcal{M}_i$ can be decomposed further into completely positive maps, e.g., $\mathcal{M}_i = \mathcal{M}_i' + \mathcal{M}_i''$. Then we may simply consider the finer decomposition as a finer measurement, with the result i replaced by two others, i' and i''. Using the convexity already established it is clear that it suffices to prove Eq. (12) for the finer measurement. That is, we can assume that there are no proper decompositions of the $\mathcal{M}_i$, or that $\mathcal{M}_i$ is "pure." This is equivalent to $\mathcal{M}_i$ taking pure states to pure states, or to the property [23] that it can be written with a single Kraus summand. Taking into account that this describes a local measurement by Bob, we can write

$$\mathcal{M}_i(\rho) = (I_A \otimes M_i)\rho(I_A \otimes M_i^\dagger), \tag{13}$$

where the Kraus operators M_i must satisfy the normalization condition $\Sigma_i M_i^\dagger M_i \leqslant I_B$. For computing the right-hand side of Eq. (12) we need that

$$\mathcal{M}_i(\rho)^{T_A} = \mathcal{M}_i(\rho^{T_A}), \tag{14}$$

which immediately follows from Eq. (13) by expanding ρ as a sum of (not necessarily positive) tensor products. A similar formula holds for Alice's local operations, but with a modified operation $\mathcal{M}_i$ on the (rhs) right-hand side, in which the Kraus operators have been replaced by their complex conjugates. Consider the decomposition

$$\rho^{T_A} = (1+N)\rho^+ - N\rho^- \tag{15}$$

with density operators $\rho^\pm$ and $N = \mathcal{N}(\rho)$. Then we can also decompose the partially transposed output states

$$\begin{aligned} p_i(\rho_i')^{T_A} &= \mathcal{M}_i(\rho)^{T_A} = \mathcal{M}_i(\rho^{T_A}) \\ &= (1+N)\mathcal{M}_i(\rho^+) - N\mathcal{M}_i(\rho^-). \end{aligned} \tag{16}$$

Dividing by p_i we get a decomposition of precisely the sort, Eq. (10), defining $\mathcal{N}(\rho_i')$. The coefficient $a_- = N/p_i$ must be larger than the infimum, i.e., $\mathcal{N}(\rho_i') \leqslant N/p_i$. Multiplying by p_i and summing, we find the following inequality.

Proposition 3.

$$\sum_i p_i \mathcal{N}(\rho_i') \leqslant \mathcal{N}(\rho), \tag{17}$$

i.e., $\mathcal{N}(\rho)$ is indeed an entanglement monotone.

C. Other negativities

Both the proofs, of convexity and of monotonicity, are based on the variational representation of the trace norm in lemma 2. The abstract version of this lemma is the definition of the so-called *base norm* $\|\cdot\|_S$ associated with a compact set S in a real vector space [24]. The negativity introduced above then corresponds to a special choice of S, and we can easily find the property of S required for proving LOCC monotonicity in the abstract setting. Other choices of S then lead to other entanglement monotones, some of which have been proposed in the literature.

For our purposes, we can take S as an arbitrary compact convex subset of the Hermitian operators with unit trace, whose real linear hull equals all Hermitian operators. Then, in analogy to lemma 2, we define the associated base norm and "S negativity" as

$$\|A\|_S = \inf\{a_+ + a_- | A = a_+\rho^+ - a_-\rho^-, a_\pm \geq 0,\ \rho^\pm \in S\}, \tag{18}$$

$$\mathcal{N}_S(A) = \inf\{a_- | A = a_+\rho^+ - a_-\rho^-, a_\pm \geq 0, \rho^\pm \in S\}. \tag{19}$$

Note that once again, if A has trace 1 we have that $\|A\|_S = 1 + 2\mathcal{N}_S(A)$. Then norm and convexity properties of $\mathcal{N}_S$ and $\|\cdot\|_S$ follow exactly as before.

Taking S as the set of all density matrices, we get $\|A\|_S = \|A\|_1$, for all Hermitian A, and a totally uninteresting entanglement quantity, as $\mathcal{N}_S(\rho)$ vanishes for all density matrices. The negativity of the preceding section corresponds to the choice of S equal to the set of all matrices A such that $A = A^\dagger$, tr$A = 1$, and $A^{T_A} \geq 0$ [additionally, we have replaced A^{T_A} with A in the lhs of Eq. (10) A, so that we can write $\mathcal{N}(\rho)$ instead of $\mathcal{N}(\rho^{T_A})$].

We could have also taken S as the subset of density matrices with positive partial transpose, $\rho^\pm \geq 0$ and $\rho^{\pm T_A} \geq 0$. In this case S corresponds to all states such that its partial transpose is also a state. The resulting quantity we will denote by $\mathcal{N}_{\rm PPT}$. Even more restrictively, if we take for S the set of *separable* density operators, i.e., we take $\rho^\pm$ (and therefore also $\rho^{\pm T_A}$) in Eqs. (18) and (19) to be separable, the corresponding quantity $\mathcal{N}_{\rm SS}$ amounts to the *robustness of the entanglement*, originally introduced in [18] (see also [25]) as the minimal amount of separable noise needed to destroy the entanglement of ρ. From the inclusions between the respective sets S we immediately get the inequalities

$$\mathcal{N}_{\rm SS}(\rho) \geq \mathcal{N}_{\rm PPT}(\rho) \geq \mathcal{N}(\rho) \geq 0. \tag{20}$$

In general, all these inequalities are strict. For example, $\mathcal{N}_{\rm SS}(\rho)$ vanishes only on separable states (SS), whereas $\mathcal{N}_{\rm PPT}(\rho)$ and $\mathcal{N}(\rho)$ vanish for all PPT states.

We claim that also $\mathcal{N}_{\rm SS}$ and $\mathcal{N}_{\rm PPT}$ are entanglement monotones. The proof is quite simple. An analysis of the arguments given in the preceding section shows that we really used only one property of S, namely, for all operations $\mathcal{M}_i$ appearing in a LOCC protocol, we have $\mathcal{M}_i(\rho) \in S^\vee$, whenever $\rho \in S^\vee$, where $S^\vee$ notes the cone generated by S (equivalently the set of $\lambda\rho$ with $\lambda \geq 0, \rho \in S$). But this is obvious for both separable states and PPT states.

III. UPPER BOUND TO TELEPORTATION CAPACITY

Sections III and IV are devoted to discuss applications of the previous results. More specifically, we derive bounds to some properties characterizing the entanglement both of a single copy of a mixed state ρ (this section) and of asymptotically many copies of it (following section).

For a single copy of a bipartite state ρ acting on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, where we set $d_1 = d_2 \equiv m$ for simplicity, an important question in quantum-information theory is to what extent this state can be used to implement some given tasks requiring entanglement, such as teleportation. The best approximation $P_{opt}(\rho)$ to a maximally entangled state

$$|\Phi^+\rangle \equiv \frac{1}{\sqrt{m}} \sum_{\alpha=1}^{m} |\alpha_A \otimes \alpha_B\rangle \tag{21}$$

that can be obtained from ρ by means of LOCC is then interesting, because it determines, for instance, how useful the state ρ is to *approximately* teleport $\log_2 m$ qubits of information. In this section we will show that the negativity $\mathcal{N}(\rho)$ provides us with an explicit lower bound on how close ρ can be taken, by means of LOCC, to the state Φ^+. From here a lower bound on the teleportation distance (i.e., an upper bound on how good teleportation results from ρ) will also follow.

A. Singlet distance

In order to characterize the optimal state $P_{opt}(\rho)$ achievable from ρ by means of LOCC, we need to quantify its closeness to the maximally entangled state $P_+ \equiv |\Phi^+\rangle\langle\Phi^+|$. Let ρ_1 and ρ_2 be two density matrices. The trace norm of $\rho_1 - \rho_2$, (or *absolute distance* [26]), is a measure of the degree of distinguishability of ρ_1 and ρ_2, and it is, therefore, reasonable to use it to measure how much $P(\rho)$—the state resulting from applying a local protocol P to state ρ—resembles P_+. In what follows we will prove that the negativity is a lower bound to the *singlet distance* of ρ,

$$\Delta(P_+, \rho) \equiv \inf_P ||P_+ - P(\rho)||_1, \tag{22}$$

where the infimum is taken over local protocols P.

We start by recalling that the absolute distance $D(\rho_1, \rho_2) \equiv ||\rho_1 - \rho_2||_1$ is a convex function [26]

$$\sum_i p_i D(\sigma, \rho_i) \geq D\left(\sigma, \sum_i p_i \rho_i\right), \tag{23}$$

which confirms, as already assumed, that the optimal approximation $P(\rho)$ to P_+ can always be chosen to be a single state—as opposed to a distribution of states $\{p_i, \rho_i\}$ corresponding to the output of a probabilistic transformation. Therefore, in Eq. (22) we need only consider *deterministic* protocols P based on LOCC.

A second feature of the absolute distance that we need is that

$$D(W\rho_1 W^\dagger, W\rho_2 W^\dagger) = D(\rho_1, \rho_2), \tag{24}$$

for any unitary transformation W. Properties (23) and (24) together imply that the best approximation to the maximally entangled state P_+ can always be "twirled" without losing optimality. Consider the state

$$\int dU \quad U\otimes U^{*}P_{opt}(\rho)U^{\dagger}\otimes U^{\dagger *}, \tag{25}$$

which the parties can locally obtain from $P_{opt}(\rho)$ by Alice applying an arbitrary unitary U, by Bob applying U^*, and then by deleting the classical information concerning which unitary has been applied. It follows from the invariance of P_+ under $U\otimes U^*$ and from property (24) that $D(U\otimes U^*P_{opt}(\rho)U^{\dagger}\otimes U^{\dagger *},P_+)=D(P_{opt}(\rho),P_+)$ for any U. Then property (23) implies that the mixture in Eq. (25) is not further away from P_+ than $P_{opt}(\rho)$. But $P_{opt}(\rho)$ was already minimizing Eq. (22), and therefore state (25) must also be optimal.

We can then assume that $P_{opt}(\rho)$ has already undergone a twirling operation. This means that it is a *noisy singlet* [27]

$$\rho_p = pP_+ + (1-p)\frac{I\otimes I}{m^2}, \tag{26}$$

from which the absolute distance to P_+ can be easily computed, $D(P_+,\rho_p)=2(1-p)(m^2-1)/m^2$. Similarly, the trace norm of $\rho_p^{T_A}$ reads $||\rho_p^{T_A}||_1 = mp+(1-p)/m$, and therefore

$$D(P_+,\rho_p)=2\left(1-\frac{||\rho_p^{T_A}||_1}{m}\right). \tag{27}$$

The lower bound to the singlet distance (22) follows now straightforwardly from the monotonicity of $||\rho^{T_A}||_1$ [or $\mathcal{N}(\rho)$] under LOCC, that is, $||\rho^{T_A}||_1 \geq ||P_{opt}(\rho)^{T_A}||_1$, and reads

$$\Delta(P_+,\rho)\geq 2\left(1-\frac{||\rho^{T_A}||_1}{m}\right). \tag{28}$$

Therefore, we have proved the following bound for the singlet distance.

Proposition 4.

$$\Delta(P_+,\rho)\geq 2\left(1-\frac{1+2\mathcal{N}(\rho)}{m}\right). \tag{29}$$

B. Teleportation distance

A quantum state ρ shared by Alice and Bob can be used as a teleportation channel Λ [15]. That is, given the shared state ρ and a classical channel between the parties, Alice can transmit an arbitrary (unknown) state $\phi\in\mathcal{C}^m$ to Bob with some degree of approximation. Let $\Lambda_{T,\rho}(\phi)$ be the state that Bob obtains when Alice sends ϕ using ρ and some protocol T involving LOCC only. The teleportation distance

$$d(\Lambda)\equiv\int d\phi \quad D(\phi,\Lambda(\phi)), \tag{30}$$

where $D(\phi,\Lambda(\phi))\equiv|||\phi\rangle\langle\phi|-\Lambda(\phi)||_1$, can be used to quantify the degree of performance of the channel. The measure $d\phi$ is consistent with the Haar measure dU in SU(m), and thus $d(\Lambda)$ is invariant under the twirling of the channel, that is the application of an arbitrary unitary U to ϕ previous to the teleportation, followed by the application of $U^{\dagger}$ after the teleportation scheme. Indeed,

$$d(\Lambda)=\int dW \quad D(WP_0W^{\dagger},\Lambda(WP_0W^{\dagger})), \tag{31}$$

for some reference state $P_0\equiv|\phi_0\rangle\langle\phi_0|$, and using property (24) of the trace norm, Eq. (31) is also equal to

$$\int dW\, d(WP_0W^{\dagger},U^{\dagger}\Lambda(UWP_0W^{\dagger}U^{\dagger})U). \tag{32}$$

We can now average over U to obtain

$$d(\Lambda)=\int dU\int d\phi D(\phi,U^{\dagger}\Lambda(U\phi)U), \tag{33}$$

where the right side of the equation corresponds to the teleportation distance of the twirled channel.

We next adapt a reasoning of the Horodecki [27] to our present situation. It uses an isomorphism between states ρ_Λ and channels Λ due to Jamiołkowski [28] and first exploited by Bennett *et al.* [3]. Let us ascribe the channel Λ to the state $\rho_\Lambda=(I\otimes\Lambda)P_+$. The state ρ_Λ can be produced by sending Bob's part of the bipartite system in state P_+ down the channel Λ. Conversely, the standard teleportation protocol [15] (or a slight and obvious modification of it) applied to state ρ_Λ reproduces the channel Λ with probability $1/m^2$. However, if the state ρ_Λ is a noisy singlet ρ_p, then the corresponding channel is the depolarizing channel

$$\Lambda_p^{dep}(\varrho)=p\varrho+(1-p)\frac{I}{m}, \tag{34}$$

which the standard teleportation scheme reproduces with *certainty* using state ρ_p. For this case $d(\Lambda_p^{dep})=2(1-p)(m-1)/m$. Therefore, there is a complete physical equivalence between noisy singlets and depolarizing teleportation channels. In addition,

$$d(\Lambda_p^{dep})=\frac{m}{m+1}D(P_+,\rho_p). \tag{35}$$

Now, since both quantities d and D are invariant under twirling, and any channel (state) can be taken into the depolarizing (noisy singlet) form, this equality holds for any channel Λ and state ρ_Λ.

Lemma 5. (adapted from [27]). The minimal distance $d_{min}(\rho)$ that can be achieved when using the bipartite state ρ to construct an arbitrary teleportation channel is given by

$$d_{min}(\rho)=\frac{m}{m+1}\Delta(P_+,\rho). \tag{36}$$

Proof. $d_{min}(\rho)\leq m\Delta(P_+,\rho)/(m+1)$, because a possible way to use ρ as a teleportation channel is by using a twirled version of an optimal state $P(\rho)$ and the standard teleportation scheme, which produces a depolarizing teleportation channel with $d=mD(P_+,P(\rho))/(m+1)$ [recall Eq. (35)].

On the other hand $d_{min}(\rho)$ is at least $m\Delta(P_+,\rho)/(m+1)$. Indeed, we take an optimal teleportation scheme employing the state ρ and LOCC only. It will produce some optimal teleportation channel Λ, that we can turn into a depolarizing channel without increasing $d(\Lambda_p^{dep})=d_{min}(\rho)$. Then we can send half of P_+ through the channel to obtain a noisy singlet ρ_p that satisfies Eq. (35). The desired inequality follows then from the fact that $D(P_+,\rho_p)\geqslant\mathcal{D}(P_+,P_{opt}(\rho))$.

Therefore, using Eq. (28) we can announce the following upper bound to the optimal teleportation distance $d_{min}(\rho)$ achievable with state ρ and LOCC

Proposition 6.

$$d_{min}(\rho)\geqslant\frac{2}{m+1}[m-1+2\mathcal{N}(\rho)]. \tag{37}$$

The two results of this section can also be derived in terms of fidelities (the so-called singlet and channel fidelities, see, for instance, [27]). The upper bounds one obtains read

$$F_{opt}\equiv\max_P\langle\Phi^+|P(\rho)|\Phi^+\rangle\leqslant\frac{1+2\mathcal{N}(\rho)}{m}; \tag{38}$$

$$f_{opt}(\rho)\equiv\max_{\Lambda_\rho}\int d\phi\langle\phi|\Lambda(|\phi\rangle\langle\phi|)|\phi\rangle\leqslant\frac{2d(\mathcal{N}(\rho)+1)}{m+1}. \tag{39}$$

IV. UPPER BOUND TO DISTILLATION RATES

We now move to consider a second application of the previous measures, namely, a bound on the asymptotic distillability of a mixed state ρ in terms of $E_{\mathcal{N}}(\rho)$.

The distillation rate of a bipartite state ρ is the best rate at which we can extract near-perfect singlet states from multiple copies of the state by means of LOCC. The asymptotic (in the number of copies) distillation rate is the so-called *entanglement of distillation* $E_D(\rho)$ [3], one of the fundamental measures of the entanglement. In this section we will show that the logarithmic negativity $E_{\mathcal{N}}$ is always at least as great as the entanglement of distillation $E_D^{\epsilon}(\rho)$, where ϵ denotes the degree of imperfection allowed in the distilled singlets.

Let Υ denote a maximally entangled state of two qubits, and consider, for some number n_α of copies of ρ, the best approximation to m_α copies of Υ that can be obtained from $\rho^{\otimes n_\alpha}$ by means of LOCC. As in the preceding section, we define

$$\Delta(\Upsilon^{\otimes m_\alpha},\rho^{\otimes n_\alpha})\equiv\inf_P\|\Upsilon^{\otimes m_\alpha}-P(\rho^{\otimes n_\alpha})\|_1, \tag{40}$$

where P runs over all deterministic protocols built from LOCC. We say that c is an achievable distillation rate for ρ, if for any sequences $n_\alpha,m_\alpha\to\infty$ of integers such that $\limsup_\alpha(n_\alpha/m_\alpha)\leqslant c$ we have

$$\lim_\alpha\Delta(\Upsilon^{\otimes n_\alpha},\rho^{\otimes m_\alpha})=0. \tag{41}$$

The distillable entanglement $E_D(\rho)$ corresponds then to the supremum of all achievable distillation rates. Several variants of this definition are available in the literature, which are however, equivalent to the one given here. In particular, we may replace "$\Delta\to 0$" by "fidelity $\to 1$," and we may consider selective protocols, in which operations produce variable numbers of output systems on the same input, and the expected rate is optimized. Of course, restricting the amount of classical communication between Alice and Bob will in general change the rate.

The above definition requires that the errors go to zero, but in many applications one can live with a small but finite error level. Therefore, we introduce $E_D^{\epsilon}(\rho)$, the distillable entanglement at error level ϵ, which is defined exactly as above, but Eq. (41) is replaced by

$$\limsup_\alpha\Delta(\Upsilon^{\otimes n_\alpha},\rho^{\otimes m_\alpha})\leqslant\epsilon. \tag{42}$$

Of course, $E_D^0(\rho)=E_D(\rho)$, and $\epsilon\mapsto E_D^{\epsilon}(\rho)$ is a nondecreasing function. The main result of this section is the following bound.

Proposition 7.

$$E_D^{\epsilon}(\rho)\leqslant E_{\mathcal{N}}(\rho), \tag{43}$$

for all $0\leqslant\epsilon<1$.

Proof. The only property of LOCC operations used in the proof is that for any such operation P, there is another, P' such that $P(\rho)^{T_A}=P'(\rho^{T_A})$. We denote by Υ_d the maximally entangled state on a pair of d-dimensional spaces. Then, as shown below, we have $\|\Upsilon_d^{T_A}\|_1=d$. In some sense this is the worst case: for general Hermitian operators we have $\|A^{T_A}\|_1\leqslant d\|A\|_1$.

Now suppose that P is the transformation for which the infimum (40) for $\Delta(\Upsilon_d,\rho)$ is attained. Then

$$\|\rho^{T_A}\|_1\geqslant\|P'(\rho^{T_A})\|_1=\|P(\rho)^{T_A}\|_1, \tag{44}$$

where the first estimate holds, because P', as a bona fide LOCC operation, does not increase the trace norm [recall the monotonicity of $\mathcal{N}(\rho)$]. On the other hand,

$$\|P(\rho)^{T_A}\|_1\geqslant\|\Upsilon_d^{T_A}\|_1-\|[\Upsilon_d-P(\rho)]^{T_A}\|_1\geqslant d-d\Delta(\Upsilon_d,\rho). \tag{45}$$

Taking the logarithm, we find

$$E_{\mathcal{N}}(\rho)\geqslant\log_2(d)+\log_2[1-\Delta(\Upsilon_d,\rho)]. \tag{46}$$

Now let n_α,m_α be diverging integer sequences as in the definition of achievable rate c. Then, using the additivity of $E_{\mathcal{N}}$, and the last inequality with $d=2^{n_\alpha}$, we find

$$E_{\mathcal{N}}(\rho)=\frac{1}{m_\alpha}E_{\mathcal{N}}(\rho^{\otimes m_\alpha})$$

$$\geqslant\frac{1}{m_\alpha}\{n_\alpha+\log_2[1-\Delta(\Upsilon^{\otimes n_\alpha},\rho^{\otimes m_\alpha})]\}.$$

We now go to the limit superior with respect to α, observing that the error Δ is uniformly bounded away from 1, and $m_\alpha \to \infty$. Hence $E_{\mathcal{N}}(\rho) \geqslant c$ for every achievable rate c, which concludes the proof.

V. EXPLICIT EXAMPLES

In this section, we display explicit expressions for the negativity for some particular classes of bipartite states, namely, for arbitrary pure states, for mixed states with a high degree of symmetry, and finally also for Gaussian states of a light field.

A. Pure states

All entanglement measures based on asymptotic distillation and dilution of pure-state entanglement, in particular, the entanglement of formation E_F and the distillable entanglement E_D [3], but also the relative entropy of entanglement [5] agree on pure states, where they give the von Neumann entropy of the restricted states. Negativity gives a larger value: Let $\rho = |\Phi\rangle\langle\Phi|$ be a pure state, and write the wave vector in its Schmidt decomposition $\Phi = \Sigma_\alpha c_\alpha e'_\alpha \otimes e''_\alpha$, where $c_\alpha > 0$ are the Schmidt coefficients of Φ, and the $e^{(i)}_\alpha$ are suitable orthonormal basis. Then we get the following result.

Proposition 8.

$$\mathcal{N}(\rho) = \frac{1}{2}\left[\left(\sum_\alpha c_\alpha\right)^2 - 1\right]. \tag{47}$$

This is precisely $\mathcal{N}_{\mathrm{SS}}/2$, i.e., half of the robustness of the entanglement, as computed in [18].

Proof. Introducing the operators "flip" $\mathbf{F} e'_\alpha \otimes e''_\beta = e'_\beta \otimes e''_\alpha$, and $C' = \Sigma_\alpha c_\alpha |e'_\alpha\rangle\langle e'_\alpha|$, and a similar C'' for the second tensor factor, we find

$$(|\Phi\rangle\langle\Phi|)^{T_A} = \sum_{\alpha\beta} c_\alpha c_\beta |e'_\alpha \otimes e''_\beta\rangle\langle e'_\beta \otimes e''_\alpha| = \mathbf{F}(C' \otimes C''). \tag{48}$$

From the trace norm $\|X\|_1 = \mathrm{tr}\sqrt{X^\dagger X}$ we may omit unitary factors, such as $\mathbf{F}$, so the trace norm is equal to the trace of the positive operator $(C' \otimes C'')$, namely, $(\Sigma_\alpha c_\alpha)^2$.

Since $E_{\mathcal{N}}$ is an upper bound on the distillation rate, and that rate is known to be $E(\rho)$, the von Neumann entropy of the restricted state, we know that $E_{\mathcal{N}}(\rho) \geqslant E(\rho)$. But, of course, we can get this more directly: using the concavity of the logarithm, we get

$$E(\rho) = 2\sum_\alpha c_\alpha^2 \log_2\left(\frac{1}{c_\alpha}\right) \leqslant 2\log_2\left(\sum_\alpha c_\alpha\right) = E_{\mathcal{N}}(\rho). \tag{49}$$

This derivation also allows the characterization of the cases of equality: Since the logarithm is strictly concave, we get equality if and only if all nonzero c_α are equal. Hence equality for pure states holds exactly for maximally entangled states (which may have been expanded by zeros to live on a larger Hilbert space).

B. States with symmetry

All entanglement measures can be computed more easily for states that are invariant under some large group of local unitary transformations [22,29]. The negativity is no exception. The main gain from local symmetries is that the partial transpose lies in a low-dimensional algebra, and is hence easily diagonalized. For this background we refer to Ref. [22]. But often a direct computation is just as easy.

Consider, for example, the states ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$, which commute with all unitaries of the form $U \otimes U$, where U is real orthogonal. These can be written as

$$\rho = ad|\Phi^+\rangle\langle\Phi^+| + b\mathbf{F} + c\mathbb{I}, \tag{50}$$

where $|\Phi^+\rangle = (\Sigma_{\alpha=1}^d |\alpha \otimes \alpha\rangle)/\sqrt{d}$ is again the standard maximally entangled vector, and a,b,c are suitable real coefficients. This family includes both the so-called Werner states [21] with $a=0$ and, with $b=0$, the so-called isotropic states [30] [or noisy singlets, compare Eq. (26) above]. The three operators in this expansion commute, so all operators of the form (50) can be diagonalized simultaneously, with spectral projections

$$p_0 = |\Phi^+\rangle\langle\Phi^+|,$$

$$p_1 = (\mathbb{I} - \mathbf{F})/2,$$

$$p_2 = (\mathbb{I} + \mathbf{F})/2 - |\Phi^+\rangle\langle\Phi^+|.$$

We parametrize the states of the form (50) by the two expectation values $f = d\,\mathrm{tr}(\rho|\Phi^+\rangle\langle\Phi^+|)$ and $g = \mathrm{tr}(\rho\mathbf{F})$, the third parameter for determining a,b,c being given by the normalization. Then the states correspond to the triangle $0 \leqslant f \leqslant d$, $-1 \leqslant g \leqslant 1$, $f \leqslant d(1+g)/2$.

Since partial transposition simply swaps the operators $\mathbf{F}$ and $d|\Phi^+\rangle\langle\Phi^+|$, leaving $\mathbb{I}$ unchanged, we can apply the same method to compute the trace norm of the partial transpose, and hence $\mathcal{N}(\rho)$. Explicitly, we get

$$\mathcal{N}(\rho) = \frac{1}{4}|1-f| + \frac{1}{4}|1+f-2g/d| + \frac{1}{2}|g/d| - \frac{1}{2}. \tag{51}$$

It turns out [22] that in this class of states the Peres-Horodecki separability criterion holds (in spite of the arbitrary dimension d), i.e., the set of PPT states is the same as the set of separable states, and in the parametrization chosen equal to the square $f,g \in [0,1]$. Hence $\mathcal{N}_{\mathrm{SS}}(\rho) = \mathcal{N}_{\mathrm{PPT}}(\rho)$. Evaluating a simple variational expression, we get

$$\mathcal{N}_{\mathrm{SS}}(\rho) = \frac{1}{2}\max\{|2f-1|-1, |2g-1|-1, 0\}. \tag{52}$$

C. Gaussian states

Gaussian states frequently occur in applications in quantum optics, where they describe the light field. Alice's and Bob's systems are then described by a certain number of canonical degrees of freedom, such as field quadratures of suitable modes. However, the same formalism applies when the canonical operators are positions and momenta of a certain number of harmonic oscillators. Gaussian states are then defined as those (possibly mixed) states with Gaussian Wigner function.

For simplicity we denote the full collection of canonical operators by $R_\alpha, \alpha=1,\ldots,2n$, where Alice holds n_A oscillators and Bob holds n_B, and $n=n_A+n_B$. These are either position or momentum operators, whose commutation relations are of the form

$$[R_\alpha, R_\beta]=i\sigma_{\alpha\beta}\mathbb{I}, \tag{53}$$

with an antisymmetric scalar matrix σ, called the symplectic matrix, which has the block matrix decomposition

$$\Delta=\begin{pmatrix}\sigma_A & 0\\ 0 & \sigma_B\end{pmatrix} \tag{54}$$

with respect to a decomposition of the set of indices into Alice's and Bob's. This form expresses the fact that all variables of Alice commute with all of Bob's.

A Gaussian state is determined by its first two moments $m_\alpha=\mathrm{tr}(\rho R_\alpha)$ and

$$\gamma_{\alpha\beta}=\mathrm{tr}(\rho R_\alpha R_\beta)-\frac{i}{2}\sigma_{\alpha\beta}, \tag{55}$$

where the subtraction is chosen as the antisymmetric part of $\mathrm{tr}(\rho R_\alpha R_\beta)$, which is fixed by the commutation relations, independently of the state. γ is then a real symmetric matrix. Since the mean m_α can be made zero by a local unitary transformation (a translation in phase space), it is irrelevant for entanglement, and we choose it to be zero. The second moment γ is then the same as the *covariance matrix* of the state. The uncertainty relation, the universal lower bound on variances, is then expressed as the positive definiteness of $\gamma_{\alpha\beta}+(i/2)\sigma_{\alpha\beta}$. It will be crucial for the later that for a classical Gaussian it is only necessary that γ itself is positive definite. Hence we can have nonpositive operators, whose Wigner function is an ordinary, if somewhat sharply peaked Gaussian.

In order to compute the trace norm of such an operator or, more generally, to compute the spectrum or other characteristics not depending on the Alice-Bob partition of the system, we can bring γ into a standard form by a process known as symplectic diagonalization or *normal-mode decomposition*. This means choosing a suitable canonical linear transformation (i.e., a transformation leaving the symplectic form σ invariant), which can be implemented on the Hilbert space level by unitary operators (known as the metaplectic representation). Assuming σ to be in standard form, i.e., block diagonal with n 2×2 blocks of the form $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ this results in a diagonal γ, with equal eigenvalues for each block, i.e., $\gamma=\mathrm{diag}(c_1,c_1,c_2,c_2,\ldots,c_n,c_n)$. We call $(c_1,c_2,\ldots,c_n)$ the *symplectic spectrum* of γ. The fast way to compute it is via the eigenvalues of the matrix $\sigma^{-1}\gamma$, which are $\pm ic_1,\ldots,\pm ic_n$. At the Hilbert space level the normal-mode decomposition transforms the state into a tensor product of independent harmonic oscillators, each of which is in a thermal oscillator state, the temperature being a function of the c_α. The smallest value allowed by uncertainty is $c_\alpha=1/2$, which gives the oscillator ground state.

In this context *transposition* is best identified with time reversal. Indeed, in the general scheme we can choose a basis in which the transpose is computed, but all these choices are equivalent via a local unitary transformation. In this case we choose the position representation, in which transposition is the same as reversing all momenta, keeping all positions, and to lift this to products by observing that transposition reverses operator products. Partial transposition T_A is completely analogous. Only in this case just Alice's momenta are reversed and Bob's are left unchanged. On the level of Wigner functions and covariance matrices of Gaussians, we just have to apply the corresponding linear transformations on phase space. That is, γ^{T_A}, the covariance matrix of the partial transpose of a Gaussian state with covariance matrix γ is constructed by multiplying by -1 all matrix elements, which connect one of Alice's momenta to either a position, or a degree of freedom belonging to Bob, and leaving all other matrix elements unchanged.

The point is, of course, that while this transformation preserves the positive definiteness of γ, and hence we get another Gaussian Wigner function, it does *not* respect the uncertainty relation, so the partially transposed operator may fail to be positive. However, the whole formalism of the normal-mode decomposition for γ works exactly as before: we get a representation of the partial transpose as a tensor product of (not necessarily positive) trace class operators. The trace norm of this operator is just the product of the trace norms, so we have completely reduced the computation of the trace norm to the single-mode case. To summarize the results so far, we proceed as follows.

Let ρ be a Gaussian density operator with covariance matrix γ, and let $(\tilde{c}_1,\ldots,\tilde{c}_n)$ be the symplectic spectrum of γ^{T_A}. Then

$$E_{\mathcal{N}}(\rho)=\sum_{\alpha=1}^{n}F(\tilde{c}_\alpha), \tag{56}$$

where $F(c)=\log_2\|\rho_c\|_1$, and ρ_c is the operator whose Wigner function is a Gaussian with covariance diag (c,c).

Of course, the function F vanishes for $c\geq(1/2)$. It is easily determined by looking at Gaussian states for oscillators as the temperature states of the oscillator. With $z=e^{-\beta}$, and $|n\rangle$ the nth eigenstate of the oscillator, a general Gaussian is of the form

$$\rho=(1-z)\sum_{n=0}z^n|n\rangle\langle n|, \tag{57}$$

where $z\geq 0$ corresponds to density operators, and $-1<z<0$ to Gaussians whose Wigner functions have sub-Heisenberg variance. Then we get

$$\|\rho\|_1=(1-z)(1-|z|)^{-1}$$

$$c=\operatorname{tr}(\rho P^2)=\operatorname{tr}\left[\rho\frac{1}{2}(P^2+Q^2)\right] \tag{58}$$

$$=(1-z)\sum_n z^n\left(n+\frac{1}{2}\right)$$

$$=(1-z)^{-1}-\frac{1}{2}. \tag{59}$$

Solving Eq. (59) for z and substituting into Eq. (58) we find

$$F(c)=\begin{cases} 0 & \text{for } 2c\geqslant 1 \\ -\log_2(2c) & \text{for } 2c<1. \end{cases} \tag{60}$$

Together with Eq. (56) and the process of normal-mode decomposition this is an efficient procedure for determining $E_{\mathcal{N}}(\rho)$.

In the simplest case of one oscillator each for Alice and Bob we may go even further, by expressing the symplectic spectrum of the partial transpose directly in terms of the covariance matrix. Suppose that

$$\gamma=\begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{61}$$

with 2×2 matrices A,B,C. Then, as shown in Ref. [31], the numbers $\det A,\det B,\det C$, and $\det\gamma$ are a complete set of invariants for γ with respect to local symplectic transformations. Moreover, when passing from γ to γ^{T_A} only $\det C$ changes sign, and the others remain unchanged. Pure states are characterized by the conditions $\det\gamma=1/16$, and $\det A+\det B+2\det C=1/2$, and can be brought into the normal form

$$\gamma=\begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{pmatrix}, \tag{62}$$

where $a^2=c^2+1/4$.

Coming back to the general case of Eq. (61), the characteristic equation of $\sigma^{-1}\gamma^{T_A}$, whose solutions are the $\pm\tilde{c}_\alpha$, takes the form

$$\xi^4+(\det A+\det B-2\det C)\xi^2+\det\gamma=0. \tag{63}$$

Together with Eq. (56) this amounts to an explicit formula. For the particular case of a pure state we find

$$E_{\mathcal{N}}(\rho)=-2\log_2(\sqrt{a-1/2}-\sqrt{a+1/2}), \tag{64}$$

which is readily seen to agree with Eq. (47).

VI. MULTIPARTITE SYSTEMS

As argued in the Introduction, a computable measure of the entanglement for bipartite mixed states is also very convenient for the quantification of the multipartite entanglement. In this section we describe a whole set of computable parameters related to the negativity that can be associated to a multipartite state to make quantitative statements about its entanglement.

A. Multipartite negativities

Consider a quantum system consisting of, say, three parts, associated to Alice, Bob, and Charlie. Let ρ_{ABC} be the (either pure or mixed) state of the system. A possible way to classify the entanglement properties of such a state is by looking at the different bipartite splittings [32] of the system.

First, we can join two of the three parts, say those of Alice and Bob, and compute the sum of negative eigenvalues of $\rho_{ABC}^{T_C}$, $\mathcal{N}_{(AB)\text{-}C}(\rho_{ABC})$. This is automatically an entanglement monotone [33], which quantifies the strength of quantum correlations between Charlie and the other two parties. Similarly, the negativities $\mathcal{N}_{(AC)\text{-}B}(\rho_{ABC})$ and $\mathcal{N}_{(BC)\text{-}A}(\rho_{ABC})$ are two other monotonic functions under LOCC with analogous meaning.

We can also consider the entanglement properties of two-party reduced density matrices. Suppose, for instance, that Charlie decides not to cooperate with the two other parties in the manipulation of the tripartite system according to LOCC. Alice and Bob's effective density matrix, $\sigma_{AB}\equiv\operatorname{Tr}_C[\rho_{ABC}]$, may still retain some of the original entanglement. The negativity of σ_{AB}, $\mathcal{N}_{A\text{-}B;\mathbb{C}}(\rho_{ABC})$, can be used to quantify this residual entanglement. Analogous quantities can be used to quantify the entanglement of σ_{AC} and σ_{BC}.

Thus, altogether we have obtained six computable functions to quantify the entanglement of any state of a tripartite system. In a four-partite setting the number of possible splittings is much greater (see [32] for a more detailed description), and thus, we obtain up to 26 inequivalent measures, namely: (i) $\mathcal{N}_{A\text{-}BCD}(\rho_{ABCD})$ and the corresponding permutations, i.e., four inequivalent measures; (ii) $\mathcal{N}_{AB\text{-}CD}(\rho_{ABCD})$ and permutations (four measures); (iii) $\mathcal{N}_{A\text{-}BC;\mathbb{D}}(\rho_{ABCD})$ and permutations (12 measures); (iv) $\mathcal{N}_{A\text{-}B;\mathbb{C}\mathbb{D}}(\rho_{ABCD})$ and permutations (six measures).

B. Hierarchy

Notice that although all these measures are independent functions of the multipartite state, there is a strength hierarchy between them when corresponding to related bipartite splittings with different number of parties. In the four-party case we have that, for instance,

$$\mathcal{N}_{A\text{-}BCD}\geqslant\mathcal{N}_{A\text{-}BC;\mathbb{D}}\geqslant\mathcal{N}_{A\text{-}B;\mathbb{C}\mathbb{D}}, \tag{65}$$

which follows from the fact that to trace out a part of a local system is an operation of the set LOCC, under which the negativity can only decrease. Of course, the same inequalities hold for the corresponding logarithmic negativities, and thus also for the several bounds on distillability—of different kinds of multipartite entanglement—implied by the later.

It should be noted, however, that in this way one can quantify only some aspects of the multiparticle entanglement: there are tripartite states that are separable with respect to every splitting of the system, but are nevertheless not a convex combination of triple tensor products of density operators [32,34]. States that have positive partial transpose with respect to every subsystem satisfy a large class of Bell inequalities [35].

VII. DISCUSSION AND CONCLUSIONS

In this paper we have presented a computable measure of the entanglement for bipartite mixed states, the negativity $\mathcal{N}(\rho)$, which we have proved not to increase under LOCC. Although it lacks a direct physical interpretation, we have shown that it bounds two relevant quantities characterizing the entanglement of mixed states: the channel capacity and the distillable entanglement E_D^ϵ.

Ideally, quantum correlations would be best quantified by measures with a given physical meaning. Which measure are to be used, would depend on which question we want to answer. For instance, if we want to know how much pure-state entanglement the parties can extract from (infinitely) many copies of the state ρ, then the proper measure to be used is the entanglement of the distillation $E_D(\rho)$.

In practice, however, the value of these measures is not known. Recent studies of entangled systems, such as those of *entangled chains, entanglement molecules, entangled rings, entangled Heisenberg models*, and *cluster states* [36], which are N-qubit systems in some global entangled state, are focused on the two-qubit quantum correlations associated to the global state, as measured by the entanglement of formation (or the related concurrence) [8]. This choice of measure of the entanglement is somehow arbitrary—it is often forced simply by the lack of an alternative measure that can also be computed for two-qubit mixed states—, because it does not reflect in anyway the entanglement cost of formation of the N-qubit state. We envisage that in these and similar contexts it will pay off to use a computable entanglement measure, such as the negativity, whose evaluation is not restricted to two-qubit mixed states. The negativity will allow, for instance, to generalize the previous investigations to analogous constructions with l-level systems ($l>2$) instead of qubits, as also to analyze quantitatively the entanglement between subsets of these l-level systems.

Finally, in a similar way as the negativity has recently played a role in proving the irreversibility of asymptotic local manipulation of bipartite mixed-state entanglement [16], we hope that this computable measure will also be a useful tool to answer other fundamental questions of the entanglement theory.

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[20] The notation for adjoints and complex conjugates of matrices is unfortunately not uniform in the literature. The notations used here are those usually employed for the first author, i.e., $U^\dagger$ for the adjoint of U, and U^* for the matrix elementwise complex conjugate. The second author would normally follow the usage of another branch of the physics literature, as well as a majority of mathematicians, using U^* for the adjoint, and $\bar{U}$ for the complex conjugate. It seems the quantum-information

community will have to live with this ambiguity.

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